Isoperimetry for Product of Probability Measures: Recent Results[*]

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Abstract. We present recent results on the isoperimetric problem for product of probability measures. For distributions with tails between exponential and Gaussian we state a dimension free result. We sketch its proof that relies on a functional inequality of Poincaré type and on a semi-group argument. Also, we give isoperimetric and concentration results, depending on the dimension, for measures with heavy tails.

1. Introduction

The well known classical isoperimetric inequality in $\mathbb{R}^n$ (see [59] for a survey) asserts that among sets of prescribed Lebesgue measure, balls are those with minimal boundary measure. Mathematically, for any Borel set $A \subset \mathbb{R}^n$,

$$|\partial A|_{n-1} \geq n\omega_n |A|_n^{1-1/n}$$

where ω_n is the measure of the Euclidean ball. The above constant is optimal since there is equality for balls.

In this note[1] we are interested in isoperimetric inequalities for product probability measures. Even if most of the results are available in more general settings, we shall focus only on $\mathbb{R}^n$. We start with some notations. Let μ be a one

[1]As one of the participants of the conference "Inhomogeneous Random Systems" held in Paris on January 27th and 28th 2009, I was asked to write the following article. I would like

dimensional probability measure and μ^n its n-fold product in $\mathbb{R}^n$. Then, the boundary measure of any Borel set $A \subset \mathbb{R}^n$ is given by the following Minkowski content

$$\mu_s^n\,(\partial A) = \liminf_{h \to 0} \frac{\mu^n(A_h \setminus A)}{h}$$

where

$$A_h = \{x \in \mathbb{R}^n : d(x, A) \leq h\}$$

is the h-enlargement of A for the Euclidean distance d. Given the isoperimetric profile

$$I_{\mu^n}(a) = \inf_{A:\mu^n(A)=a} \{\mu_s^n(\partial A)\}, \qquad a \in [0, 1],$$

the isoperimetric inequality we shall deal with reads as

$$\mu_s^n(\partial A) \geq I_{\mu^n}\left(\mu^n(A)\right), \qquad A \subset \mathbb{R}^n. \tag{1.1}$$

Having the classical isoperimetric inequality in mind, two natural questions arise: is it possible to find the isoperimetric profile $a \mapsto I_{\mu^n}(a)$ as a function of a *and* of n? Also, is it possible to find the extremal sets in the latter, i.e. sets for which there is equality in (1.1)?

Let us already quote that the second question is too hard in general. Hence we shall focus on the first one.

As a guideline the reader might keep in mind the family of probability measures on $\mathbb{R}$,

$$d\mu_\alpha(x) = \frac{\exp\{-|x|^\alpha/\alpha\}}{2\alpha^{1/\alpha}\Gamma(1 + (1/\alpha))}\,dx \qquad \alpha > 0.$$

Thanks to the factor $1/\alpha$ inside the exponential we have $\mu_2 = \gamma$, the standard Gaussian measure, while $\mu_1 = (1/2)\exp\{-|x|\}\,dx$ is the two-sided exponential distribution. As we shall see there are three different behaviors of the isoperimetric profile, as a function of n, corresponding to three different regimes, $\alpha < 1$ (distributions with heavy tails), $\alpha \in [1, 2]$ (between exponential and Gaussian) and $\alpha > 2$.

This note does not intend to give a complete overview on the isoperimetric problem. Our aim is much more modest and we refer to [44] for a detailed account on this topic and some of its applications and to [9,55] for its geometrical aspects and its link with convex geometry. In fact we merely would like to collect from [11–13, 31] some of our recent results on the isoperimetric inequality for product of probability measures.

This paper is divided into 6 sections. In the first one we study the isoperimetric inequality on the line where extremal sets are known. This will allow us to give an explicit expression of the isoperimetric profile in dimension 1, as a function of a. Then, we deal with product measures in $\mathbb{R}^n$. Section 4 is dedicated to the analytic proof of one of our theorem, while in Section 5 we make the connection with the concentration of measure phenomenon. We end this note with few remarks and perspectives.

2. Isoperimetric inequalities in $\mathbb{R}$

We start with the one dimensional case of $\mathbb{R}$ for which extremal sets are known. In turn we shall deduce the isoperimetric profile explicitly. We distinguish between two cases, log-concave probability measures and distributions with heavy tails.

2.1. Log concave probability measures

For log concave probability measures Bobkov computed the extremal sets in the isoperimetric inequality and deduced the isoperimetric profile.

Theorem 2.1 (Bobkov [19] Proposition 2.1). *Let $d\mu(x) = Z_\Phi^{-1} e^{-\Phi(x)} dx$, $x \in \mathbb{R}$, be a probability measure with $\Phi : \mathbb{R} \to \mathbb{R}$ convex and even. Then half-lines are extremal sets in the isoperimetric inequality.*

The two-sided exponential $d\mu_1(x) = (1/2)\exp\{-|x|\}\, dx$ has been studied in [58].

Denote by $F_\mu : t \mapsto \mu((-\infty, t])$ the distribution function of μ (μ as in the theorem). Then for $t \leq 0$

$$\mu_s(\partial(-\infty, t]) = \liminf_{h \to 0} \frac{1}{h} \int\limits_h^{t+h} d\mu(x) = Z_\Phi^{-1} e^{-\Phi(t)} = F_\mu'(t).$$

Hence, by Bobkov's Theorem, we have

$$I_\mu(F_\mu(t)) = I_\mu(\mu(-\infty, t]) = \mu_s(\partial(-\infty, t]) = F_\mu'(t).$$

A similar result holds for $t \geq 0$. This readily leads to an explicit expression of I_μ in the log-concave case:

$$I_\mu(a) = F_\mu' \circ F_\mu^{-1}(a), \qquad a \in [0, 1].$$

Note that by symmetry $I_\mu(a) = I_\mu(1 - a)$. Bobkov also proved that I_μ is concave and that the concavity of I_μ is actually equivalent to the fact that Φ is convex (see [19, Appendix]). Moreover, for any continuous concave function $I : [0, 1] \to \mathbb{R}^+$ such that $I(0) = 0$ and $I(1 - a) = I(a)$, there exists a symmetric log-concave probability measure μ on $\mathbb{R}$ for which $I_\mu = I$ [26].

2.2. Heavy tails probability measures

For heavy tails probability measures Bobkov and Houdré computed the extremal sets in the isoperimetric inequality. As for the log-concave case we shall deduce the isoperimetric profile from their result.

Theorem 2.2 (Bobkov – Houdré [26, Corollary 13.10]). *Consider the probability measure* $d\mu(x) = Z_\Phi^{-1} \exp\{-\Phi(x)\}\, dx$, $x \in \mathbb{R}$, *with* $\Phi : \mathbb{R} \to \mathbb{R}$ *even. Then extremal sets in the isoperimetric inequality can be found among half-lines, symmetric segments and their complements.*

The same type of computation as for the log-concave case leads to

$$I_\mu(a) = \min\left(F'_\mu \circ F_\mu^{-1}(a), 2F'_\mu \circ F_\mu^{-1}\left(\frac{\min(a, 1-a)}{2} \right) \right).$$

Again by symmetry $I_\mu(a) = I_\mu(1-a)$.

2.3. Examples

Computations are easy for the two-sided exponential measure $d\mu_1(x) = 1/2 \exp\{-|x|\}dx$. One has

$$I_{\mu_1}(a) = \min(a, 1-a).$$

For the exponential type distribution

$$d\mu_\alpha(x) = \frac{\exp\{-|x|^\alpha/\alpha\}}{2\alpha^{1/\alpha}\Gamma(1 + (1/\alpha))}\, dx$$

with $\alpha \neq 1$ the computation is not explicit, but the following asymptotics holds

$$\lim_{a\to 0} \frac{I_{\mu_\alpha}(a)}{a\,(\alpha\log(1/a))^{1-1/\alpha}} = 1$$

and similarly when $a \to 1$ (see [6, 29]). In particular, for any $\alpha > 0$, there exist two constants $c, c' > 0$ (that might depend on α) such that

$$cp\left(\log\frac{1}{p}\right)^{1-1/\alpha} \leq I_{\mu_\alpha}(a) \leq c'p\left(\log\frac{1}{p}\right)^{1-1/\alpha} \qquad \text{with } p = \min(a, 1-a).$$

For Cauchy-type distributions of the form

$$dm_\alpha(x) = \frac{\alpha}{2(1 + |x|)^{1+\alpha}}\, dx$$

with $\alpha > 0$ one gets $F'_{m_\alpha} \circ F_{m_\alpha}^{-1}(a) = \alpha 2^{1/\alpha} \min(a, 1-a)^{1+1/\alpha}$ and thus,

$$I_{m_\alpha}(a) = \alpha \min(a, 1-a)^{1+1/\alpha}.$$

Similar computations can be done for the more general probability measure $d\mu_\Phi(x) = Z_\Phi^{-1} e^{-\Phi(x)} dx$. Assume that Φ is $\mathcal{C}^2$ and even. If in addition either Φ is convex and $\sqrt{\Phi}$ is concave on $(0, \infty)$ or Φ is concave on $(0, \infty)$ satisfying $\Phi(x)/x \to 0$ and Φ^θ convex for some $\theta > 1$, then $I_{\mu_\Phi}(a)$ compares to $p\Phi' \circ \Phi^{-1}(\log(1/p))$ with $p = \min(a, 1 - a)$. See [13, Proposition 13] and [31, Proposition 3.18] for more details and the proofs.

3. Isoperimetric inequalities in $\mathbb{R}^n$

We start with a simple observation. Given a Borel set $A \subset \mathbb{R}^n$, we have on one hand $\mu^{n+1}(A \times \mathbb{R}) = \mu^n(A)$, and on the other hand $\mu_s^{n+1}(\partial A \times \mathbb{R}) = \mu_s^n(\partial A)$. Hence, by the very definition of the isoperimetric profile, $I_{\mu^{n+1}} \leq I_{\mu^n}$. In particular

$$I_\mu \geq I_{\mu^2} \geq \cdots \geq I_{\mu^n} \geq \cdots \geq I_{\mu^\infty} := \inf_{n \geq 1} I_{\mu^n}.$$

Quite remarkable is the fact that for the Gaussian measure μ_2 there is equality in the previous sequence. This is a direct consequence of the following theorem.

Theorem 3.1 (Sudakov–Tsirel'son [57], Borell [30]). *For any dimension n, half-spaces are extremal sets in the isoperimetric problem for the standard Gaussian measure μ_2^n. In other words $I_{\mu_2} = I_{\mu_2^n}$ for any n.*

The fact that $I_{\mu^\infty} = I_\mu$ is actually characteristic of the Gaussian measures. Indeed, it can be shown that Gaussian measures are the only symmetric measures on the line such that for any dimension n the coordinate half-spaces $\{x \in \mathbb{R}^n : x_1 \leq t\}$ are extremal sets in the isoperimetric inequality for the corresponding product measure. See [27, 41, 54] for more precise (and stronger) statements.

An alternative proof of the latter Gaussian isoperimetric result was given by Bobkov [20]. He proved the following functional inequality

$$I_{\mu_2}\left(\int f d\mu\right) \leq \int \sqrt{I_{\mu_2}(f)^2 + |\nabla f|^2} d\mu \qquad \forall f$$

first for μ the Bernoulli measure on two points, then by tensorization on the hypercube (and μ a product of Bernoulli) and finally by the central limit theorem on $\mathbb{R}^n$ for the standard Gaussian measure $\mu = \mu_2^n$. One of the nice feature of the latter is that it tensorizes, i.e. the inequality in dimension one implies the same inequality in any dimension. However it appeared to be very hard to generalize to other measures, at the notable exception of the two-sided exponential μ_1 [25]. The former Gaussian result is also established using a symmetrization argument by Ehrhard [33] and a martingale approach by Barthe and Maurey [14]. An other proof was given, using analytic techniques, by Bakry and Ledoux [5]. Those authors also established an isoperimetric inequality of Gaussian type

for general measures under a curvature condition. Moreover, their technique allowed us to prove the following result.

Theorem 3.2 (Barthe$-$Cattiaux$-$Roberto [12, 13]). *There exists $K > 0$ such that for any $\alpha \in [1, 2]$,*

$$KI_{\mu_\alpha} \leq I_{\mu_\alpha^\infty} \leq I_{\mu_\alpha}.$$

This result was proved by Bobkov and Houdré [25] for the two-sided exponential measure μ_1, with $K = 1/(2\sqrt{6})$. A more general version can be found in [13, Theorem 15] for $d\mu(x) = Z_\Phi^{-1} e^{-\Phi(x)} dx$ with Φ convex, $\sqrt{\Phi}$ concave and few mild technical assumptions. Sodin exploited Theorem 3.2 in his study of the isoperimetric problem on ℓ_p balls [56]. We shall sketch the proof of this theorem in the next section. But before that, let us complete the picture: what happens for $\alpha > 2$ and $\alpha < 1$?

In [14] (see [8, 55] for extensions), it is proved that if $I_\mu \geq cI_{\mu_2}$ for some $c > 0$, then

$$I_{\mu^n} \geq cI_{\mu_2^n} = cI_{\mu_2}. \tag{3.1}$$

This has applications in the isoperimetric problem for product of spheres [7]. Using an argument based on the central limit theorem, it is possible to prove that, starting from $I_\mu \geq cI_{\mu_2}$, in the limit $I_{\mu^\infty} \leq c_\mu I_{\mu_2}$ (i.e. the other direction than (3.1)). Hence, the sequence $(I_{\mu^k})_k$ is decreasing and its limit compares to the Gaussian isoperimetric profile. See [8] for a control on the speed of convergence in terms of what is called the isoperimetric dimension.

Using an extension of (3.1) together with a nice refinement of the analytic technique of Bakry and Ledoux [5] (first used in [50]) and capacity arguments, E. Milman proved the following result.

Theorem 3.3 (E. Milman [51]). *Let $I : [0, 1] \to \mathbb{R}^+$ be a continuous concave function satisfying $I(0) = 0$ and $I(1 - a) = I(a)$ for any $a \in [0, 1]$. Assume that there exists $c > 0$ such that for any $0 < s \leq t \leq 1/2$*

$$\frac{I(s)}{I_{\mu_2}(s)} \leq c\frac{I(t)}{I_{\mu_2}(t)}. \tag{3.2}$$

Let μ be a probability measure on $\mathbb{R}$ such that $I_\mu \geq I$ on $[0, 1]$. Then, there exists a constant c' such that for any n,

$$I_{\mu^n} \geq c'I \quad on \ [0, 1].$$

This recovers the previous theorem. Indeed it is easy to check that μ_α satisfies (3.2) with $I = I_{\mu_\alpha}$. Note that it is probably feasible to approach the proof of Theorem 3.3 by pushing further the techniques used in [13] but at the price of some more assumptions and technicalities. In this sense Theorem 3.3 is somehow more intrinsic than Theorem 3.2.

Observe that Milman's theorem deals with measures "between" exponential and Gaussian, in the sense that Assumption (3.2) imposes that the measure μ is below the Gaussian and the concavity assumption on the isoperimetric profile forces μ to be above the exponential.

In fact, dimension free results, as in the two previous theorems, can only be obtained for measures "between" exponential and Gaussian. That the distribution has tails heavier than Gaussian was explained by the central limit argument above. On the other hand, as observed by Talagrand [58]: if μ is a probability measure on $\mathbb{R}$ such that there exists $h > 0$ and $\varepsilon > 0$ for which for any $n \geq 1$, and all $A \subset \mathbb{R}^n$ with $\mu^n(A) \geq 1/2$, one has

$$\mu^n\left(A + [-h, h]^n\right) \geq \frac{1}{2} + \varepsilon$$

then μ has at least exponential tails, that is there exist $C_1, C_2 > 0$ such that $\mu([x, \infty)) \leq C_1 e^{-C_2 x}$, $x \in \mathbb{R}$. This implies that for measures with heavy tails, the isoperimetric profile of μ^n has to depend on the dimension and goes to 0 as n goes to infinity. Using transportation of mass techniques, we have the following result in this direction.

Theorem 3.4 (Cattiaux – Gozlan – Guillin – Roberto [31]). *Consider a symmetric probability measure $d\mu(x) = Z_\Phi^{-1} e^{-\Phi(x)} dx$, on $\mathbb{R}$, with Φ concave on $[0, \infty)$. Then, for any n and any Borel set $A \subset \mathbb{R}^n$,*

$$I_{\mu^n}(a) \geq 2n F_\mu' \circ F_\mu^{-1}\left(\frac{\min(a, 1-a)}{24n}\right).$$

This theorem gives a lower bound on the isoperimetric profile of I_{μ^n}. An upper bound can be found in [11] using the following strategy: since $(A^n)_h \subset (A_h)^n$ (where A^n is the Cartesian product of A), we have

$$\mu_s^n(\partial A) \leq n\mu(A)^{n-1}\mu_s(\partial A).$$

Hence, by the very definition of the boundary measure, for all $a \in (0, 1)$, $I_{\mu^n}(a^n) \leq na^{n-1} I_\mu(a)$.

Now we illustrate this on two explicit examples. For the sub-exponential type law

$$d\mu_\alpha(x) = \frac{\exp\{-|x|^\alpha/\alpha\}}{2\alpha^{1/\alpha}\Gamma(1 + (1/\alpha))}\, dx, \quad \alpha \leq 1,$$

we get from Theorem 3.4 (for the lower bound) and the above strategy (for the upper bound) that there exist two constants $c_1, c_2 > 0$ such that for any n

$$c_1 p\left(\log \frac{n}{p}\right)^{1-1/\alpha} \leq I_{\mu_\alpha^n}(a) \leq c_2 p \log \frac{1}{p}\left(\log \frac{n}{\log(1/p)}\right)^{1-1/\alpha}, \quad p = \min(a, 1-a).$$

For Cauchy-type distributions of the form

$$dm_\alpha(x) = \frac{\alpha}{2(1+|x|)^{1+\alpha}}\, dx, \quad \alpha > 0,$$

we get that there exist two constants $c_1, c_2 > 0$ such that for any n

$$c_1 \frac{p^{1+1/\alpha}}{n^{1/\alpha}} \leq I_{m_\alpha^n}(a) \leq c_2 \frac{p\,(\log(1/p))^{1-1/\alpha}}{n^{1/\alpha}}, \quad p = \min(a, 1-a).$$

Note that in both cases the dependence in n is of the correct order.

As a summary we end this section with a "graph" suggested to us by Franck Barthe [10]. The first coordinate is the dimension n, the second corresponds to $\alpha > 0$. We represent the behavior of $I_{\mu_\alpha^n}$ as a function of n. For any $\alpha > 0$, $I_{\mu_\alpha^n}$ is non-increasing (as a function of n). It decays to the Gaussian isoperimetric profile (up to constants) when $\alpha > 2$, it is constant for $\alpha = 2$, almost constant for $\alpha \in (1, 2)$ and goes to 0 when $\alpha < 1$.
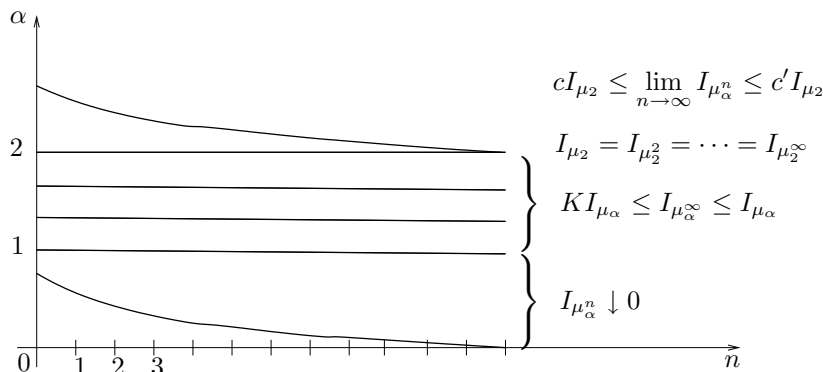
4. Analytic proof of Theorem 3.2

In this section we sketch the proof of Theorem 3.2. It mainly relies on two ingredients, one based on the celebrated Γ_2 calculus of Bakry–Emery [3,4], and the other one on functional inequalities.

The diffusion operator $\mathbf{L}f(x) = \Delta f(x) - \sum_{i=1}^n \Phi'(x_i)\partial_i f$, acting on smooth functions on $\mathbb{R}^n$, is symmetric in $\mathbb{L}^2(\mu^n)$, where $d\mu(x) = Z_\Phi^{-1} e^{-\Phi(x)} dx$. Let $(\mathbf{P}_t)_{t \geq 0}$ be its associated Markov semi-group. The following lemma [5] shows how the semi-group encodes some of the geometrical aspects of the measure. Its proof heavily relies on the commutation property of the semi-group and the gradient of a function, which in turn comes from Γ_2 calculus.

Lemma 4.1 (Bakry$-$Ledoux [5, Lemma 4.2]). *Let $d\mu(x) = Z_\Phi^{-1} e^{-\Phi(x)} dx$ be a probability measure on $\mathbb{R}$. Assume that Φ is convex and $\mathcal{C}^2$. Then, for any n, any Borel set $A \subset \mathbb{R}^n$ with $\mu^n(A) \leq 1/2$,*

$$\mu_s^n(\partial A) \geq \frac{\sqrt{2}}{\sqrt{t}} \left(\mu^n(A) - \int (\mathbf{P}_t \mathbb{1}_A)^2 \, d\mu^n \right). \tag{4.1}$$

A similar result holds when $\mu_\alpha^n(A) \geq 1/2$. The next step in the proof consists in controlling $\int (\mathbf{P}_t \mathbb{1}_A)^2 \, d\mu^n$. This will be achieved using functional inequalities. Let us explain the strategy on the simple example of the two-sided exponential measure μ_1 and the Poincaré inequality, as observed by Ledoux [43]. It is well known that the two-sided exponential satisfies the following (optimal) Poincaré inequality

$$\int \left(f - \int f \, d\mu_1^n \right)^2 d\mu_1^n \leq 4 \int |\nabla f|^2 d\mu_1^n. \tag{4.2}$$

Here $|\nabla g|^2 := \sum_{i=1}^n (\partial_i g)^2$ is the square of the Euclidean norm of the gradient. Consider $u(t) = \int (\mathbf{P}_t f)^2 \, d\mu_1^n$ with f satisfying $\int f \, d\mu_1^n = 0$. Then, using the integration by parts formula and the Poincaré inequality, we have

$$u'(t) = 2 \int \mathbf{P}_t f \mathbf{L} \mathbf{P}_t f \, d\mu_1^n = -2 \int |\nabla \mathbf{P}_t f|^2 d\mu_1^n \leq -\frac{1}{2} u(t).$$

Hence, after integration, and by approximation of $\mathbb{1}_A$, we get

$$\int \left(\mathbf{P}_t \mathbb{1}_A - \int \mathbb{1}_A d\mu_1^n \right)^2 d\mu_1^n \leq e^{-t/2} \int \left(\mathbb{1}_A - \int \mathbb{1}_A d\mu_1^n \right)^2 d\mu_1^n.$$

After few rearrangements we arrive at

$$\int (\mathbf{P}_t \mathbb{1}_A)^2 \leq \mu_1^n(A)^2 + e^{-t/2} \mu_1^n(A) \left(1 - \mu_1^n(A) \right).$$

Hiding some technicalities (the potential $\Phi(x) = |x|$ is not twice differentiable and should be replaced by some smooth version) and applying Bakry and Ledoux's Lemma we finally get that for any A with $\mu_1^n(A) \leq 1/2$,

$$(\mu_1)_s^n(\partial A) \geq \frac{c \left(1 - \exp\{-c't\} \right)}{\sqrt{t}} \mu_1^n(A) \left(1 - \mu_1^n(A) \right)$$

for some constants $c, c' > 0$ (coming from the technical approximations). Optimizing over t leads to Theorem 3.2 for μ_1. The latter is known as the Cheeger inequality [32].

One of the main feature of this approach is that inequality (4.1) and the Poincaré inequality (4.2) are dimension free.

Note that isoperimetric inequalities imply (without any assumptions) $\mathbb{L}^2$ functional inequalities of Poincaré type, by the co-area formula [32, 49]. The converse is false in general and one has to add some (e.g. convexity) assumptions.

In order to prove Theorem 3.2 we have to find an appropriate dimension free functional inequality associated to the family of measures μ_α, and then to derive a bound on $\int \left(\mathbf{P}_t \mathbb{1}_A\right)^2 d\mu_\alpha^n$. This will be achieved using the so-called super Poincaré inequalities. Unlike the standard Poincaré inequality, those inequalities do not tensorize in general. So one has to prove first an intermediate family of inequalities, called Latała and Oleszkiewicz inequalities, that do tensorize, and then derive, via capacity arguments, the super Poincaré inequality along the scheme:

Latała and Oleszkiewicz inequality in dimension 1

$\Rightarrow$ Latała and Oleszkiewicz inequality in dimension n

$\Rightarrow$ Capacity-measure inequality

$\Rightarrow$ Super-Poincaré inequality in dimension n.

Now we introduce those inequalities. A measure μ^n is said to satisfy a Latała and Oleszkiewicz inequality (respectively a super Poincaré inequality) if for some $C > 0$, every smooth $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\sup_{p\in(1,2)} \frac{\int f^2 d\mu^n - \left(\int |f|^p d\mu^n\right)^{2/p}}{T(2-p)} \leq \int |\nabla f|^2 d\mu^n \qquad (4.3)$$

with $T(x) = Cx^{2(1-1/\alpha)}$, respectively

$$\int f^2 d\mu^n \leq \beta(s) \int |\nabla f|^2 d\mu^n + s\left(\int |f| d\mu^n\right)^2 \qquad \forall s \geq 1 \qquad (4.4)$$

with $\beta(s) = C/\log(1+s)^{2(1-1/\alpha)}$.

The case $T(x) = x$ in (4.3) was proved by Beckner [18] for the measure μ_2. This corresponds to the stronger logarithmic Sobolev inequality of Gross [38] (see also [2, 37]). Latała and Oleszkiewicz proved that inequality (4.3) holds with $T(x) = Cx^{2(1-1/\alpha)}$ for μ_α^n and deduced some concentration results. This was then re-proved using Hardy type inequalities in [15].

The two first steps of the previous scheme are thus a consequence of [42] (or [15]).

Using the notion of capacity introduced by Maz'ja [49] and revisited in the setting of probability measures in [15], the two last steps of the previous scheme follow (see more precisely [12, Theorem 18 and Lemma 19] and [13, Corollary 6], see also Wang [60, 61]). Hence the measure μ_α^n, $\alpha \in [1, 2]$, satisfies inequality (4.4) with a constant C independent of n.

Now let $u(t) = \int \left(\mathbf{P}_t f\right)^2 d\mu_\alpha^n$. As for the Poincaré inequality, after differentiation, thanks to (4.4) we have for any $s \geq 1$,

$$u'(t) = -2\int |\nabla \mathbf{P}_t f|^2 d\mu_\alpha^n \leq -\frac{2}{\beta(s)}\left(u(t) - s\left(\int |f| d\mu_\alpha^n\right)^2\right).$$

Hence, by approximation of $\mathbb{1}_A$ (with $A \subset \mathbb{R}^n$ a Borel set), we get

$$\int \left(\mathbf{P}_t \mathbb{1}_A\right)^2 d\mu_\alpha^n \leq e^{-2t/\beta(s)} \mu_\alpha^n(A) + s\left(1 - e^{-2t/\beta(s)}\right)\mu_\alpha^n(A)^2.$$

Hiding again some technicalities, we can use Lemma 4.1 to obtain that

$$(\mu_\alpha^n)(\partial A) \geq \frac{\sqrt{2}}{\sqrt{t}}\, \mu_\alpha^n(A)\,(1 - s\mu_\alpha^n(A))\left(1 - e^{-2t/\beta(s)}\right) \qquad \forall t > 0, \quad \forall s \geq 1.$$

Choosing $s = 1/(2\mu_\alpha^n(A))$ and $t = \beta(s)/2$ leads to

$$(\mu_\alpha^n)(\partial A) \geq c\mu_\alpha^n(A)\left(\log \frac{1}{\mu_\alpha^n(A)}\right)^{1-1/\alpha} \geq c' I_{\mu_\alpha}\left(\mu_\alpha^n(A)\right)$$

for every A such that $\mu_\alpha^n(A) \leq 1/2$ and some constant $c' > 0$ that does not depend on n. Since a similar result holds true for A with $\mu_\alpha^n(A) \geq 1/2$, we achieve the proof of Theorem 3.2.

5. Concentration of measure phenomenon

In this section we derive a somehow new concentration result from the isoperimetric inequality for the Cauchy distributions. We refer to [46] for a detailed account on the topic of concentration of measure phenomenon and its applications. The connection between isoperimetry and concentration is given in the following result.

Proposition 5.1. *Let $d\mu(x) = Z_\Phi^{-1} e^{-\Phi(x)} dx$ be a probability measure on $\mathbb{R}$. Let $v : \mathbb{R} \to [0,1]$ be an increasing differentiable function. The following are equivalent*

(i) $I_{\mu^n} \geq v' \circ v^{-1}$.

(ii) *For every $h > 0$ and every Borel set $A \subset \mathbb{R}^n$,*

$$\mu^n(A_h) \geq v\left(v^{-1}\left(\mu^n(A)\right) + h\right).$$

See [46, Proposition 2.1] for the proof. For the Gaussian measure μ_2^n, since $F_{\mu_2}(0) = 1/2$ and $F_{\mu_2}(h) \geq 1 - \exp\{-h^2/2\}$, Sudakov–Tsirel'son and Borell's

Theorem together with the latter proposition ensure that, for any n and any Borel set $A \subset \mathbb{R}^n$ with $\mu_2^n(A) \geq 1/2$,

$$\mu_2^n(A_h) \geq F_{\mu_2}\left(F_{\mu_2}^{-1}(1/2) + h\right) \geq F_{\mu_2}(h) \geq 1 - \exp\{-h^2/2\}. \qquad (5.1)$$

Such a bound for the measures μ_α^n, $\alpha \geq 2$, can be obtained using inf-convolution method [48] (see also [23, 24]). Also, the celebrated Herbst argument implies the former Gaussian concentration result via the logarithmic Sobolev inequality, see [2, 45]. Bobkov and Ledoux introduced a modified version of the logarithmic Sobolev inequality [28] in order to improve an exponential concentration result of Talagrand [58] for μ_1^n. Those modified inequalities were generalized by Gentil, Guillin and Miclo [34, 35] and re-proved, using Hardy type inequalities, in [16] and by transportation of mass technique [36] (see also [47]), leading to new concentration results for μ_α, $\alpha \geq 1$.

A concentration result of the type (5.1) is, *a priori*, weaker than the isoperimetric inequality. However, under a convexity assumption, very recently Milman [52] proved that (5.1) (with a general function replacing $h^2/2$) is equivalent to an isoperimetric inequality.

Here we would like to focus on measures with heavy tails. In [11], some concentration results of Lipschitz functions for product of measures with heavy tails are obtained, using an induction technique of Aida, Masuada and Shigekawa [1]. Here we would like to take advantage of Theorem 3.4.

Proposition 5.2. *Let* $d\mu(x) = Z_\Phi^{-1} e^{-\Phi(x)} dx$ *be a symmetric probability measure on* $\mathbb{R}$ *with* Φ *concave on* $[0, \infty)$. *Then, for any* n, *any Borel set* $A \subset \mathbb{R}^n$ *and any* $h > 0$ *with* $\mu^n(A) \geq 1/2$,

$$\mu^n(A_h) \geq 1 - 24n F_\mu\left(F_\mu^{-1}\left(\frac{1}{48n}\right) - \frac{h}{12}\right).$$

Proof. Let

$$\Psi(h) = F_\mu^{-1}\left(\frac{1 - \mu^n(A_h)}{24n}\right)$$

with $A \subset \mathbb{R}^n$ a Borel set satisfying $\mu^n(A) \geq 1/2$. Thanks to Theorem 3.4 we have

$$\Psi'(h) = -\frac{\mu_s^n(\partial A_h)}{24n F_\mu' \circ F_\mu^{-1}\left(\frac{1 - \mu^n(A_h)}{24n}\right)} \leq -\frac{1}{12}.$$

Hence,

$$\Psi(h) = \Psi(0) + \int_0^h \Psi'(u) du \leq \Psi(0) - \frac{h}{12} = F_\mu^{-1}\left(\frac{1 - \mu^n(A)}{24n}\right) - \frac{h}{12}.$$

The expected result follows by monotony of F_μ. $\qquad\qquad\square$

We illustrate this proposition on the simple example of the Cauchy distributions

$$dm_\alpha(x) = \frac{\alpha}{2(1+|x|)^{1+\alpha}} dx, \quad \alpha > 0.$$

For $r \leq 0$, we have

$$F_{m_\alpha}(r) = \frac{1}{2(1-r)^\alpha}.$$

Hence, after simple computations (left to the reader), using the fact that $\mu^n(A) \geq 1/2$, we get that for $h = tn^{1/\alpha}$ (which is the correct scaling),

$$m_\alpha^n\left(A_{tn^{1/\alpha}}\right) \geq 1 - \frac{12^{1+\alpha}}{t^\alpha}.$$

This improves [11, inequality (5.12)] of a logarithmic factor. As explained in [11, inequality (5.16)] on the explicit example $A = (-\infty, r]^n$ with $r = F_{m_\alpha}^{-1}(a^{1/n})$ determined so as to have $m_\alpha^n(A) = a \geq 1/2$, the latter estimate is of correct order, since for this particular choice of A,

$$m_\alpha^n\left(A_{tn^{1/\alpha}}\right) \leq 1 - \frac{C_\alpha}{t^\alpha}$$

for some constant $C_\alpha > 0$.

With more efforts, similar computations can be done for concave potentials of power type, recovering the concentration result [11, Proposition 6.4]: under mild assumptions on $\Phi : [0,\infty) \to \mathbb{R}^+$ concave, the measure $d\mu(x) = Z_\Phi^{-1} e^{-\Phi(|x|)} dx$ satisfies

$$\mu^n(A_h) \geq 1 - C\exp\{-c\Phi' \circ \Phi^{-1}\left(\max(\Phi(h), \log n)\right)\} \quad \forall h > 0$$

for any Borel set $A \subset \mathbb{R}^n$ with $\mu^n(A) \geq 1/2$ and some constants $c, C > 0$.

6. Final remarks and comments

We end this note with few comments on some recent works related to the isoperimetric problem. Also we give some open directions.

Remark first that the isoperimetric problem changes if one changes the distance in the definition of the enlargement A_h. See [21] for a use of the ℓ^∞ distance, [29, 51, 53] for a use of the ℓ^p distance.

To deal with isoperimetric inequalities for non-product measures in not an easy task. The analytic technique presented in this note is one way in this direction. We mention the paper by Milman and Sodin [53] around uniformly log-concave probability measures, Huet [39] for spherically symmetric log-concave probability measures and Bobkov [22] for convex probability measures. In [17], an isoperimetric inequality is found for a canonical Gibbs measure.

In order to go further in the study of isoperimetric inequalities, it would be certainly very interesting, and also very challenging, to obtain some functional inequalities. Indeed, it is known [25] that on the one hand the measure μ_1 satisfies

$$\int \sqrt{1+f^2}\,d\mu_1 \leq \int \sqrt{1+C(f')^2}\,d\mu_1$$

for all smooth $f : \mathbb{R} \to \mathbb{R}$, and on the other hand that the Gaussian measure μ_2 satisfies [5, 20]

$$I_{\mu_2}\left(\int f d\mu_2\right) \leq \int \sqrt{I_{\mu_2}(f)^2 + (f')^2}\,d\mu_2 \quad \forall f.$$

Hence it would be natural to have an intermediate (interpolating) inequality for the measure μ_α, $\alpha \in [1,2]$, involving I_{μ_α}. In particular, the two previous inequalities tensorize, and so one could get infinite dimension results studying only a one dimensional inequality. This direction would re-prove Theorem 3.2. Also it would probably give some hints on how to deal with non-product measures such as Gibbs measures.

Also, a generalization of the semi-group technique presented in this note to other distance in the definition of the enlargement A_h would lead to very interesting problems in non-linear analysis.

Finally let us mention the very deep conjecture by Kannan, Lovász and Simonovits [40]. This conjecture asks for the existence of a universal constant $c > 0$ such that for any dimension n, any n-dimensional log-concave probability measure ν in the isotropic position (i.e. such that $\int \langle x, \theta \rangle^2 d\nu(x) = 1$ for any unit vector θ, and $\int x d\nu(x) = 0$) satisfies the following Cheeger inequality

$$I_\nu(a) \geq c \min(a, 1-a).$$

Equivalently, thanks to a result by E. Milman [50], the conjecture asks for the existence of a universal constant $C > 0$ such that for any dimension n, any n-dimensional log-concave probability measure ν in the isotropic position satisfies the following (*a priori* weaker) Poincaré inequality

$$\int f^2 d\nu - \left(\int f d\nu\right)^2 \leq C \int |\nabla f|^2 d\nu.$$

This conjecture would lead to many applications in convex geometry [9].

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