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QUANTITATIVE FORM OF BALL'S CUBE SLICING IN $\mathbb{R}^n$ AND EQUALITY CASES IN THE MIN-ENTROPY POWER INEQUALITY

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Abstract. We prove a quantitative form of the celebrated Ball's theorem on cube slicing in $\mathbb{R}^n$ and obtain, as a consequence, equality cases in the min-entropy power inequality. Independently, we also give a quantitative form of Khintchine's inequality in the special case $p = 1$.

1. Introduction

In his seminal paper [1], Keith Ball proved that the maximal $(n-1)$-dimensional volume of the section of the cube $C_n := [-\frac{1}{2}, \frac{1}{2}]^n$ by an hyperplane is $\sqrt{2}$. Therefore proving a conjecture by Hensley [13].

More precisely, for $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ with $|a| := \sqrt{a_1^2 + \cdots + a_n^2} = 1$, put $\sigma(a,t) = |C_n \cap H_{a,t}|_{n-1}$ for the volume of the intersection of the cube with the hyperplane $H_{a,t} = \{x \in \mathbb{R}^n : \langle x, a\rangle = t\}$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$ and $|\cdot|_{n-1}$ stands for the ($(n-1)$-dimensional) volume.

Theorem 1 (Ball [1])**.** *For all unit vector a and all $t \in \mathbb{R}$, it holds $\sigma(a,t) \leq \sqrt{2}$. Moreover, equality holds only if $t = 0$ and a has only two non-zero coordinates having absolute value $\frac{1}{\sqrt{2}}$.*

Ball's result means that the maximal volume of the sections of the cube by hyperplanes is achieved when the section is a product of a $(n-2)$-dimensional cube C_{n-2} with the diagonal of a 2-dimensional cube C_2. The original proof is based on Fourier transform and series expansion. Alternative proofs can be found in [28] (based on distribution functions) and very recently in [27] (by mean of a transport argument).

Ball used Theorem 1 to give a negative answer to the famous Busemann-Petty problem in dimension 10 and higher [2]. His paper has inspired significant research in convex geometry and is still very current. We refer to [7, 8, 14, 17–19] to quote just a few of the most recent papers in the field and refer to the reference therein for a more detailed description of the literature.

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Our first main result is the following quantitative version of Ball's theorem.

Theorem 2. *Fix* $\varepsilon \in (0, \frac{1}{75})$. *Let* $a \in \mathbb{R}^n$ *with* $|a| = 1$ *and* $t \in \mathbb{R}$ *be such that* $\sigma(a,t) \geq (1-\varepsilon)\sqrt{2}$. *Then, there exists two indices* j_o, j_1 *such that*

$$\frac{1}{\sqrt{2}}(1 - 37.5\varepsilon) \leq |a_{j_o}|, |a_{j_1}| \leq \frac{1}{\sqrt{2}}(1 + 2\varepsilon).$$

Moreover, $\sum_{j \neq j_o, j_1} a_j^2 \leq 50\varepsilon$ *and in particular, for all* $j \neq j_o, j_1$, $|a_j| \leq \sqrt{50\varepsilon}$.

Ball's slicing theorem, combined with a result of Rogozin [34], was used by Bobkov and Chistyakov [3] to derive an optimal inequality for min-entropy power. Namely, they proved that

$$N_\infty(X_1 + \cdots + X_n) \geq \frac{1}{2} \sum_{i=1}^{\infty} N_\infty(X_i) \tag{1}$$

for any independent ($\mathbb{R}$-valued) random variables $X_1, \ldots, X_n$, with N_∞ the min-entropy power we now define. We may call the latter *Bobkov-Chistyakov's min-entropy power inequality.*

For a ($\mathbb{R}$-valued) random variable X, the *min-Entropy power* is defined as

$$N_\infty(X) = M^{-2}(X),$$

when

$$M(X) := \inf \{c : \mathbb{P}(X \in A) \leq c\ |A| \text{ for all Borel } A\} < \infty$$

and $N_\infty(X) = 0$ otherwise. When X is absolutely continuous with respect to the Lebesgue measure, with density f, then $M(X) = \|f\|_\infty$ is the essential supremum of f with respect to the Lebesgue measure.

The nomenclature "min-entropy power" is information theoretic. In that field the entropy power inequality refers to the fundamental inequality due to Shannon [35] which demonstrates that X_i independent random variables with densities f_i satisfy

$$N(X_1 + \cdots + X_n) \geq \sum_i N(X_i),$$

where $N(X) = e^{2h(X)}$ denotes the "entropy power", with the Shannon entropy $h(X) = -\int f(x) \log f(x) dx$. The Rényi entropy [32], for $\alpha \in [0, \infty]$ defined as $h_\alpha(X) = \frac{\int f^\alpha(x) dx}{1-\alpha}$ for $\alpha \in (0,1) \cup (1,\infty)$ and through continuous limits otherwise, gives a parameterized family of entropies that includes the usual Shannon entropy as a special case (by taking $\alpha = 1$). It can be easily seen (through Jensen's inequality, and the expression $h_\alpha(X) = \left(\mathbb{E} f^{\alpha-1}(X)\right)^{\frac{1}{1-\alpha}}$) that for a fixed variable X, the Rényi entropy is decreasing in α. Thus for a fixed variable X, the parameter $\alpha = \infty$, $h_\infty(X) = -\log \|f\|_\infty$, furnishes the minimizer of the family $\{h_\alpha(X)\}_\alpha$, and is often referred to as the "min-entropy". Hence the notation $N_\infty(X) = e^{2h_\infty(X)}$ and the terminology "min-entropy power" is in analogy with the Shannon entropy power $N(X) = e^{2h(X)}$. Entropy power inequalities for the full class of Rényi entropies have been a topic of recent interest in information theory, see e.g. [4–6, 20, 21, 23, 26, 31, 33], and for more background we refer to [24] and references therein.

In [3] it was observed in a closing remark that the constant $\frac{1}{2}$ in (1) is sharp. Indeed by taking $n = 2$ and X_1 and X_2 to be *i.i.d.* uniform on an interval (1) is seen to hold with equality. In Theorem 3, we demonstrate that this is (essentially) the only equality case. In fact, thanks to the quantitative form of Ball's slicing theorem above, we can derive a quantitative form of Bobkov-Chistyakov's min entropy power inequality, see Corollary 6, that, in turn, allows us to characterize equality cases in (1) which constitutes our second main theorem.

Theorem 3. *For $X_1, \dots, X_n$ independent random variables,*

$$N_\infty(X_1 + \cdots + X_n) \geq \frac{1}{2} \sum_{i=1}^{n} N_\infty(X_i) \tag{2}$$

with equality if and only if there exists i_1 and i_2 and $x \in \mathbb{R}$ such that X_{i_1} is uniform on a set A, and X_{i_2} is a uniform distribution on $x - A$ and for $i \neq i_1, i_2$, X_i is a point mass.

Note that this is distinct from the d-dimensional case, see [25], where sharp constants can be approached asymptotically for X_i *i.i.d.* and uniform on a d-dimensional ball. More explicitly, for $d \geq 2$, if Λ denotes all finite collections of independent $\mathbb{R}^d$-valued random variables

$$\sup_{X \in \Lambda} \frac{N_\infty(X_1 + \cdots + X_m)}{\sum_{i=1}^{m} N_\infty(X_i)} = \lim_{n \to \infty} \frac{N_\infty(Z_1 + \cdots + Z_n)}{\sum_{i=1}^{n} N_\infty(Z_i)},$$

where Z_i are *i.i.d.* and uniform on a d-dimensional Euclidean unit ball.

We end with a quantitative Khintchine's inequality. Though our result is independent, we stress that, as it is well known in the field and as it was pointed out by Ball himself in [1, Additional remarks], the inequality $\sigma(a,t) \leq \sqrt{2}$ of Theorem 1 is related to Khintchine's inequalities.

Denote by $B_1, B_2, \dots$ symmetric $-1, 1$-Bernoulli variables. Khintchine's inequalities assert that, for any $p \in (0, \infty)$ there exist some constant A_p, A_p' such that for all n and all $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ it holds

$$A_p \left(\sum_{i=1}^{n} a_i^2 \right)^{\frac{p}{2}} \leq R_p(a) := \mathbb{E}\left[\left| \sum_{i=1}^{n} a_i B_i \right|^p \right] \leq A_p' \left(\sum_{i=1}^{n} a_i^2 \right)^{\frac{p}{2}}. \tag{3}$$

Such inequalities were proved by Khintchine in a special case [16], and studied in a more systematic way by Littlewood [22] and Paley and Zygmund [29, 30].

The best constants in (3) are known. This is due to Haagerup [11], after partial results by Stečkin [36], Young [38] and Szarek [37]. In particular, Szarek proved that $A_1 = 1/\sqrt{2}$, that was a long outstanding conjecture of Littlewood, see [12].

The connection between Theorem 1 and Khintchine's inequalities goes as follows: as fully derived in [7], Ball's theorem can be rephrased as

$$\mathbb{E}\left[\left| \sum_{i=1}^{n} a_i \xi_i \right|^{-1} \right] \leq \sqrt{2} \left(\sum_{i=1}^{n} a_i^2 \right)^{-\frac{1}{2}},$$

where ξ_i are i.i.d. random vectors in $\mathbb{R}^3$ uniform on the centered Euclidean unit sphere S^2. As a result Ball's slicing of the cube can be seen as a sharp $L_{-1} - L_2$ Khintchine-type inequality.

Our last main result is a quantitative version of (the lower bound in) Khintchine's inequality for $p = 1$, that has the same flavour as Theorem 2 (though being independent).

Theorem 4. *Fix $\varepsilon \in (0, 1/100)$, an integer n and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ such that $|a| = 1$, satisfying*

$$R_1(a) \le \frac{1+\varepsilon}{\sqrt{2}}.$$

Then, there exists two indices i_1, i_2 such that

$$\frac{1-30\varepsilon}{\sqrt{2}} \le |a_{i_1}|, |a_{i_2}| \le \frac{1+\varepsilon}{\sqrt{2}}.$$

Also, it holds $\sum a_i^2 \le 57\varepsilon$ and in particular, for any $i \neq i_1, i_2$, $|a_i| \le \sqrt{57\varepsilon}$.

The proofs of Theorem 2 and Theorem 4 are based on a careful analysis of Ball's integral inequality

$$\int_{-\infty}^{\infty} \left| \frac{\sin(\pi u)}{\pi u} \right|^s du \le \sqrt{\frac{2}{s}}, \qquad s \ge 2$$

and, respectively, Haagerup's integral inequality

$$\int_0^{\infty} \left(1 - \left|\cos\left(\frac{u}{\sqrt{s}}\right)\right|^s\right) \frac{du}{u^{p+1}} \ge \int_{-\infty}^{\infty} \left(1 - e^{-u^2/2}\right) \frac{du}{u^{p+1}}, \qquad s \ge 2$$

in the special case $p = 1$. It is worth mentioning that our proof of Theorem 4 is restricted to $p = 1$ because the latter integrals can be made explicit only in that case. In order to deal with general p (at least $p \in [p_o, 2)$, say, with $p_o \simeq 1.85$ implicitly defined through the Gamma function, see [11]), one would need to study very carefully the map $F_p\colon s \mapsto \int_0^\infty \left(1 - \left|\cos\left(\frac{u}{\sqrt{s}}\right)\right|^s\right) \frac{du}{u^{p+1}}$ and prove that it is increasing and then decreasing on $[2, \infty)$ with careful control of its variations. The difficulty is also coming from the fact that, at $p = p_o$, $F_p(2) = F_p(\infty)$. This in particular makes the quantitative version difficult to state properly. Indeed, for $0 < p < p_o$, the extremizers in the lower bound of (3) are those a with two indices equal to $1/\sqrt{2}$ and the others vanishing. While for $p > p_o$, there are no extremizers for finite n (the "extremizer" is $a = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})$ in the limit (by the central limit theorem)). At $p = p_o$ the two "extremizers" coexist. Theorem 4 is therefore only an attempt in the understanding of quantitative forms of Khintchine's inequalities.

We end this introduction with two remarks, pointed out to us by Tomasz Tkocz after the manuscript was submitted to the journal, related to the current literature.

First, Theorem 2 was independently proved by Chasapis, Nayar, and Tkocz, see [8, Theorem 2, (4)]. The authors were motivated by quantifying all known extremal-volume hyperplane section results for ℓ_p-balls. Both their motivations and proof are different from ours, making both approaches of possible interest for the community.

On the other hand, Theorem 4 is reminiscent of a result of De, Diakonikolas and Servedio [9]. Indeed, the authors prove the following quantitative form of Khintchine's inequality (they call it "robust"): for $a = (a_1, \dots, a_n)$, a unit vector with $a_1 \ge a_2 \ge \cdots \ge a_n$,

$$R_1(a) \ge \frac{1}{\sqrt{2}} + c|a - a^*|$$

for some constant c and $a^* = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)$. Under the assumption of Theorem 4 the latter leads (after few simplification) to $|a_1 - \frac{1}{\sqrt{2}}|^2 + |a_2 - \frac{1}{\sqrt{2}}|^2 + 1 - a_1^2 - a_2^2 \le \frac{\varepsilon^2}{2c^2}$. Therefore, $|a_1 - \frac{1}{\sqrt{2}}| \le c'\varepsilon$, and similarly for a_2, for some universal constant c'. Theorem 4, with a non explicit constant, is in turn a consequence of

De, Diakonikolas and Servedio's result. We observe however that our proof of the quantitative Khintchine's inequality is very different from that of [9], making our approach again of potential interest for the readers.

The next sections are devoted to the proof of Theorem 2, Theorem 3 and Theorem 4.

2. Quantitative slicing: Proof of Theorem 2

In this section, we give a proof of Theorem 2. We need first to recall part of the original proof by Ball, based on Fourier and anti-Fourier transform. We may omit some details that can be found in [1].

By symmetry we can assume without loss of generality that $a_j \geq 0$ for all j. Reducing the dimension of the problem if necessary, we will further reduce it to $a_j \neq 0$ for all j.

In [1] it is proved that $\sigma(a,t) \leq \frac{1}{a_j}$ for all j (see also [28, step 1]). The argument is geometric. Put $e_j := (0, \dots, 0, 1, 0, \dots 0)$ for the j-th unit vector of the canonical basis. Then it is enough to observe that the volume of $C_n \cap H_{a,t}$ equals the volume of its projection to the hyperplane $H_{e_j,0}$ (orthogonal to the j-th direction) divided by the cosine of the angle of a and e_j, that is precisely a_j, while the projection of C_n on $H_{e_j,0}$ has volume 1. Therefore $a_j \leq \frac{1}{\sqrt{2}(1-\varepsilon)} \leq \frac{1}{\sqrt{2}}(1+2\varepsilon)$ for all j, which proves one inequality of Theorem 2.

We follow the presentation of [28, step 2]. Let $\hat{S}$ be the Fourier transform of $S\colon t \mapsto \sigma(a,t)$. By definition, we have

$$\begin{aligned}
\hat{S}(u) &= \int_{\mathbb{R}} \sigma(a,t) e^{-2i\pi ut} dt \\
&= \int_{C_n} e^{-2i\pi u <x,a>} dx \\
&= \prod_{j=1}^{n} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2i\pi u a_j a_j} dx_j \\
&= \prod_{j=1}^{n} \frac{\sin(\pi a_j u)}{\pi a_j u}.
\end{aligned}$$

Therefore, taking the anti-Fourier transform, Ball obtained the following explicit formula[1] for $\sigma(a,t)$:

$$\begin{aligned}
\sigma(a,t) &= \int_{-\infty}^{\infty} \hat{S}(u) e^{2i\pi ut} du \\
&= \int_{-\infty}^{\infty} e^{2\pi iut} \prod_{j=1}^{n} \frac{\sin(\pi a_j u)}{\pi a_j u} du.
\end{aligned}$$

[1] An alternative explicit formula is given by Franck and Riede [10] (with different normalization). The authors ask if there could be an alternative proof of Ball's theorem based on their formula.

Applying Hölder's inequality, since $a_1^2 + \cdots + a_n^2 = 1$, one gets

$$\sigma(a,t) \le \int_{-\infty}^{\infty} \prod_{j=1}^{n} \left| \frac{\sin(\pi a_j u)}{\pi a_j u} \right| du$$

$$\le \prod_{j=1}^{n} \left(\int_{-\infty}^{\infty} \left| \frac{\sin(\pi a_j u)}{\pi a_j u} \right|^{1/a_j^2} du \right)^{a_j^2}. \tag{4}$$

Ball's theorem follows from the fact that $I(a_j) := \int_{-\infty}^{\infty} \left| \frac{\sin(\pi a_j u)}{\pi a_j u} \right|^{1/a_j^2} du \le \sqrt{2}$ with equality only if $a_j = 1/\sqrt{2}$. Changing variable, this is equivalent to proving that

$$\int_{-\infty}^{\infty} \left| \frac{\sin(\pi u)}{\pi u} \right|^{s} du < \sqrt{\frac{2}{s}} \tag{5}$$

for every $s > 2$ (for $s = 2$ this is an identity). The latter is known as Ball's integral inequality and was proved in [1][2] (see [27, 28] for alternative approaches).

One key ingredient in the proof of Theorem 2 is a reverse form of Ball's integral inequality given in Lemma 5.

Turning to our quantitative question, observe that if for all $j = 1, \ldots, n$, $I(a_j) < (1-\varepsilon)\sqrt{2}$, then (4) would imply that $\sigma(a,t) < (1-\varepsilon)\sqrt{2}$, a contradiction. Therefore, there must exist j_o such that $I(a_{j_o}) \ge (1-\varepsilon)\sqrt{2}$. The aim is now to prove that a_{j_o} is close to $1/\sqrt{2}$. In fact, changing variables ($s = 1/a_{j_o}^2 \ge 2(1-\varepsilon)$), we observe that

$$I(a_{j_o}) = \int_{-\infty}^{\infty} \left| \frac{\sin(\pi a_j u)}{\pi a_j u} \right|^{1/a_j^2} du$$

$$= \sqrt{s} \int_{-\infty}^{\infty} \left| \frac{\sin(\pi u)}{\pi u} \right|^{s} du.$$

Hence, $I(a_{j_o}) \ge (1-\varepsilon)\sqrt{2}$ is equivalent to saying that

$$\int_{-\infty}^{\infty} \left| \frac{\sin(\pi u)}{\pi u} \right|^{s} du \ge (1-\varepsilon)\sqrt{\frac{2}{s}}.$$

Lemma 5 guarantees that, if $s \ge 2$, then $s = \frac{1}{a_{j_o}^2} \le 2 + 50\varepsilon$. If $s \le 2$ then $\frac{1}{a_{j_o}^2} \le 2$ which amounts to $a_{j_o} \ge \frac{1}{\sqrt{2}}$. In any case

$$a_{j_o} \ge \frac{1}{\sqrt{2+50\varepsilon}}$$

$$\ge \frac{1}{\sqrt{2}}(1 - \frac{25}{2}\varepsilon)$$

since $\frac{1}{\sqrt{1+t}} \ge 1 - \frac{1}{2}t$ for any $t \in (0,1)$.

[2] An asymptotic study of such integrals can be found in [15].

Iterating the argument, assume that for all $j \neq j_o$, $I(a_j) < (1-3\varepsilon)\sqrt{2}$. Since $I(a_{j_o}) \leq \sqrt{2}$, (4) would imply that

$$\begin{aligned}\sigma(a,t) &< (1-3\varepsilon)^{1-a_{j_o}^2}\sqrt{2}\\ &\leq (1-3\varepsilon)^{1-\frac{1}{2(1-\varepsilon)^2}}\sqrt{2}\\ &\leq (1-\varepsilon)\sqrt{2},\end{aligned}$$

where we used that $a_{j_o} \leq 1/(\sqrt{2}(1-\varepsilon))$ and some algebra. This is a contradiction. Therefore, there exists a second index $j_1 \neq j_o$ such that $I(a_{j_1}) \geq (1-3\varepsilon)\sqrt{2}$. Proceeding as for j_o, we can conclude that necessarily

$$a_{j_1} \geq \frac{1}{\sqrt{2}}(1-\frac{75}{2}\varepsilon).$$

The expected result concerning a_{j_o}, a_{j_1} follows.

Since $a_1^2+\cdots+a_n^2 = 1$ we can conclude that

$$\sum_{j\neq j_0,j_1} a_j^2 \leq 1-\frac{1}{2}(1-\frac{25}{2}\varepsilon)^2-\frac{1}{2}(1-\frac{75}{2}\varepsilon)^2 \leq 50\varepsilon.$$

Thus, $a_j^2 \leq 50\varepsilon$ for all $j \neq j_o, j_1$. This ends the proof of the theorem.

Lemma 5. *Let $s \geq 2$ be such that*

$$\int_{-\infty}^{\infty}\left|\frac{\sin(\pi u)}{\pi u}\right|^s du \geq (1-\delta)\sqrt{\frac{2}{s}}$$

for some small $\delta > 0$. Then, $s \leq 2+50\delta$.

Proof. Set $\sigma = \frac{s}{2}-1$. We use the technology developed in [1] where it is proved that

$$\int_{-\infty}^{\infty}\left|\frac{\sin(\pi u)}{\pi u}\right|^s du = \frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{\sin^2(t)}{t^2}\right|^{1+\sigma} dt = 1-\sum_{n=1}^{\infty}\frac{|\sigma(\sigma-1)\ldots(\sigma-n+1)|}{n!}\beta_n$$

and

$$\sqrt{\frac{2}{s}} = \sqrt{\frac{1}{1+\sigma}} = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(e^{-t^2/\pi}\right)^{1+\sigma} dt = 1-\sum_{n=1}^{\infty}\frac{|\sigma(\sigma-1)\ldots(\sigma-n+1)|}{n!}\alpha_n$$

with

$$\alpha_n := \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-t^2/\pi}\left(1-e^{-t^2/\pi}\right)^n dt, \qquad \beta_n := \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2(t)}{t^2}\left(1-\frac{\sin^2(t)}{t^2}\right)^n dt.$$

Therefore, the assumption

$$\int_{-\infty}^{\infty}\left|\frac{\sin(\pi u)}{\pi u}\right|^s du \geq (1-\delta)\sqrt{\frac{2}{s}}$$

can be recast

$$\sum_{n=1}^{\infty}\frac{|\sigma(\sigma-1)\ldots(\sigma-n+1)|}{n!}(\beta_n-\alpha_n) \leq \delta\sqrt{\frac{1}{1+\sigma}}.$$

Note that, in [1], it is proved that $\alpha_n < \beta_n$ so that the left hand side of the latter is positive and in fact an infinite sum of positive terms. Hence, the first term of the sum must not exceed the right hand side. Since $\beta_1 = \frac{1}{3}$ and $\alpha_1 = \frac{\sqrt{2}-1}{\sqrt{2}}$, it holds

$$\sigma \frac{3-2\sqrt{2}}{3\sqrt{2}} = \sigma(\beta_1 - \alpha_1) \leq \delta \sqrt{\frac{1}{1+\sigma}} \leq \delta.$$

Returning to the variable s it follows that $s \leq 2 + \delta \frac{6\sqrt{2}}{3-2\sqrt{2}}$ from which the expected result follows since $\frac{6\sqrt{2}}{3-2\sqrt{2}} \simeq 49.46 \leq 50$. $\square$

3. MIN-ENTROPY POWER INEQUALITY

In this section we extend the quantitative slicing results for the unit cube to a quantitative version (Corollary 6) of Bobkov and Chistyakov's min-entropy power inequality (Inequality (1)) for random variables in $\mathbb{R}$. Then we prove the full characterization of extremizers of this min-entropy power inequality, i.e. we prove Theorem 3.

The quantitative version of Bobkov and Chistyakov's min-entropy power inequality reads as follows.

Corollary 6. *For X_i independent random variables and $\varepsilon \in (0, 1/75)$ if*

$$N_\infty\left((1-\varepsilon)\sum_{i=1}^n X_i\right) \leq \frac{1}{2}\sum_{i=1}^n N_\infty(X_i), \tag{6}$$

then there exists indices i_o and i_1 such that

$$(1-37.5\varepsilon)^2 \left(\frac{1}{2}\sum_{i=1}^n N_\infty(X_i)\right) \leq N_\infty(X_{i_o}), N_\infty(X_{i_1}) \leq (1+2\varepsilon)^2 \left(\frac{1}{2}\sum_{i=1}^n N_\infty(X_i)\right)$$

while

$$\sum_{i \neq i_o, i_1} N_\infty(X_i) \leq 50\varepsilon \sum_{i=1}^n N_\infty(X_i).$$

Its proof relies on the following result by Rogozin.

Theorem 7 (Rogozin [34]). *For X_i independent random variables, let Z_i be independent random variables uniform on an origin symmetric interval chosen such that $N_\infty(X_i) = N_\infty(Z_i)$, with the interpretation that Z_i is deterministic, and equal to zero, in the case that $N_\infty(X_i) = 0$. Then,*

$$N_\infty(X_1 + \cdots + X_n) \geq N_\infty(Z_1 + \cdots + Z_n). \tag{7}$$

Note that though our frame work here is formally more general than [34] and [3], there is no difficulty extending these results to the measure theoretic setting. Indeed, for independent $X \sim \mu$ and $Y \sim \nu$, $\mathbb{P}(X+Y \in A) = \int_{\mathbb{R}^2} \mathbb{1}_A(x+y)d\mu(x)d\nu(y) = \int_{\mathbb{R}} \mu(A-y)d\nu(y) \leq M(X)|A|$. Thus adding variables without density can only increase the left hand side of both (1) and (7) while preserving the right hand sides, so the analogous results in the relaxed formulation follow.

Proof of Corollary 6. Suppose that, for $\delta > 1$

$$N_\infty(X_1 + \cdots + X_n) \leq \frac{\delta}{2}\sum_{i=1}^n N_\infty(X_i), \tag{8}$$

then by Theorem 7,

$$N_\infty(Z_1 + \cdots + Z_n) \leq \frac{\delta}{2} \sum_{i=1}^n N_\infty(X_i).$$

Writing $U_i = \frac{Z_i}{\sqrt{N_\infty(Z_i)}}$ and $\theta_i = \frac{N_\infty(X_j)}{\sum_j N_\infty(Z_j)}$ we can re-write this inequality as

$$N_\infty(\theta_1 U_1 + \cdots + \theta_n U_n) \leq \frac{\delta}{2},$$

where we observe that $\theta = (\theta_1, \ldots, \theta_n)$ is a unit vector and $U = (U_1, \ldots, U_n)$ is the uniform distribution on the unit cube. Moreover since U_i are log-concave and symmetric, $\sum_i \theta_i U_i = \langle \theta, U \rangle$ is as well, and hence $N_\infty(\theta_1 U_1 + \cdots + \theta_n U_n) = f^{-2}_{\langle \theta, U \rangle}(0) = \sigma^{-2}(\theta, 0)$. Thus, we have

$$\sigma(\theta, 0) \geq \sqrt{\frac{2}{\delta}}.$$

Now observe that the min-entropy is 2-homegeneous, i.e. $N_\infty(\lambda X) = \lambda^2 N_\infty(X)$. Therefore, (6) gives (8) with $\delta = (1-\varepsilon)^{-2}$. Hence

$$\sigma(\theta, 0) \geq (1-\varepsilon)\sqrt{2}.$$

Thus by Theorem 2, there exist i_o and i_1 such that

$$\frac{1}{\sqrt{2}}(1 - 37.5\varepsilon) \leq \theta_{i_o}, \theta_{i_1} \leq \frac{1}{\sqrt{2}}(1 + 2\varepsilon)$$

while

$$\sum_{i \neq i_o, i_1} \theta_i^2 \leq 50\varepsilon.$$

Interpreting this in terms of the definition $\theta_j = \sqrt{N_\infty(X_j)/\sum_i N_\infty(X_i)}$. This gives

$$\begin{aligned}(1-37.5\varepsilon)^2 \left(\frac{1}{2}\sum_{i=1}^n N_\infty(X_i)\right) &\leq N_\infty(X_{i_o}), N_\infty(X_{i_1})) \\ &\leq (1+2\varepsilon)^2 \left(\frac{1}{2}\sum_{i=1}^n N_\infty(X_i)\right),\end{aligned}$$

while

$$\sum_{i \neq i_o, i_1} N_\infty(X_i) \leq 50\varepsilon \sum_{i=1}^n N_\infty(X_i).$$

This ends the proof of the Corollary. ☐

Proof of Theorem 3. We distinguish between sufficiency and necessity. The former being simpler.

Necessity. Writing for convenience $N_\infty(X_1) \geq N_\infty(X_2) \geq \cdots \geq N_\infty(X_n)$, by Corollary 6 when $N_\infty(X_1) > 0$, and the equality in (2) imply that

$$N_\infty(X_1) = N_\infty(X_2), \qquad N_\infty(X_k) = 0 \text{ for } k \geq 3.$$

That is

$$N_\infty(X_1 + X_2 + X_3 \cdots + X_n) = N_\infty(X_1 + X_2) = N_\infty(X_1),$$

and since symmetric rearrangement preserves min-entropy and reduces the entropy of independent sums, $N_\infty(X_1 + X_2) \geq N_\infty(X_1^* + X_2^*) \geq \frac{1}{2}(N_\infty(X_1^*) + N(X_2^*)) = N_\infty(X_1) = N_\infty(X_1 + X_2)$. Letting f, g represent the densities of X_1^* and X_2^* respectively, this implies

$$\begin{aligned} \|f * g\|_\infty &= f * g(0) \\ &= \int f(y)g(y)dy \\ &= \int_{\{f = \|f\|_\infty\}} \|f\|_\infty g(y)dy + \int_{\{f < \|f\|_\infty\}} f(y)g(y)dy \\ &= \|f\|_\infty \end{aligned}$$

which can only hold if $\{g > 0\} \subseteq \{f = \|f\|_\infty\}$. Reversing the roles of f and g, we must also have $\{f > 0\} \subseteq \{g = \|g\|_\infty\}$. Since $\{f = \|f\|_\infty\} \subseteq \{f > 0\}$ obviously holds, we have the following chain of inclusions,

$$\{g > 0\} \subseteq \{f = \|f\|_\infty\} \subseteq \{f > 0\} \subseteq \{g = \|g\|_\infty\} \subseteq \{g > 0\}.$$

For this it follows that X_1^* and X_2^* are *i.i.d.* uniform distributions.

Thus, X_1 and X_2 are uniform distributions as well. Without loss of generality we may assume that X_1 and X_2 are uniform on sets of measure 1, K_1 and K_2. Denote $f_i = \mathbb{1}_{K_i}$. Then $f_1 * f_2$ is uniformly continuous and $f_1 * f_2(x) \to 0$ with $|x| \to \infty$. Indeed, because continuous compactly supported functions are dense in L^2, it follows[3] that for $g_{\tau_y}(x) \coloneqq g(x+y)$, $\|g_{\tau_y} - g\|_2 \to 0$ for $y \to 0$. Further $\|g_{\tau_{y_1}} - g_{\tau_{y_2}}\|_2 = \|g_{\tau_{y_1 - y_2}} - g\|_2$, so that for $|y_1 - y_2|$ sufficiently small, $\|g_{\tau_{y_1}} - g_{\tau_{y_2}}\|_2$ can be made arbitrarily small as well. Thus,

$$\begin{aligned} |f_1 * f_2(x) - f_1 * f_2(x')| &\leq \int |f_1(-y)||f_2(x+y) - f_2(x'+y)|dy \\ &\leq \|f_1\|_2 \|(f_2)_{\tau_x} - (f_2)_{\tau_{x'}}\|_2 \\ &= \|(f_2)_{\tau_{x-x'}} - f_2\|_2 \end{aligned}$$

hence $f_1 * f_2$ is indeed uniformly continuous.

Taking φ_i to be continuous, compactly supported functions approximating f_i in L^2, we have

$$\begin{aligned} \|\varphi_1 * \varphi_2 - f_1 * f_2\|_\infty &\leq \|f_1 * (\varphi_2 - f_2)\|_\infty + \|\varphi_2 * (\varphi_1 - f_1)\|_\infty \\ &\leq \|f_1\|_2 \|\varphi_2 - f_2\|_2 + \|\varphi_2\|_2 \|\varphi_1 - f_1\|_2. \end{aligned}$$

Since the right hand side goes to zero, and $\varphi_1 * \varphi_2$ is compactly supported, it must be true that $f_1 * f_2(x)$ tends to zero for large $|x|$. Thus $f_1 * f_2$ attains its maximum value at some point x, and thus we can rewrite the equality of the min-entropies

[3]Given an $\varepsilon > 0$, there exists φ continuous and compactly supported such that $\|\varphi - g\|_2 < \varepsilon/3$. Since φ is continuous and compactly supported, it is uniformly continuous, and hence for small enough y, $\|\varphi_{\tau_y} - y\|_2 < \varepsilon/3$, Thus $\|g_{\tau_y} - g\| \leq \|g_{\tau_y} - \varphi_{\tau_y}\| + \|\varphi_{\tau_y} - \varphi\| + \|\varphi - g\| < \varepsilon$.

of X_1+X_2, X_1 and X_2, as $f_1 * f_2(x) = |K_1 \cap (x-K_2)| = |K_1| = |K_2| = 1$. Thus almost surely $x - K_1 = K_2$.

Put $Y = X_2 + \cdots + X_n$. By the same argument, since $N_\infty(X_1+Y) = \frac{1}{2}(N_\infty(X_1)+N_\infty(Y))$, Y is uniform on a set $x'-K_1$. Thus, $\mathrm{Var}(Y) = \sum_{i=2}^n \mathrm{Var}(X_i) = \mathrm{Var}(X_2)$. Hence, for $i>2$, $\mathrm{Var}(X_i)=0$ and the X_i are deterministic. Letting $A = K_1$, the proof of necessity is complete.

Sufficiency. To prove sufficiency, assume that X_1 is uniform on a set A, X_2 uniform on $x-A$ and X_i a point mass for $i \geq 3$ then,

$$\begin{aligned} N_\infty(X_1+X_2+X_3+\cdots+X_n) &= N_\infty(X_1+X_2) \\ &= \left\| \frac{\mathbb{1}_A}{|A|} * \frac{\mathbb{1}_{x-A}}{|A|} \right\|_\infty^{-2}. \end{aligned}$$

Observe that

$$\begin{aligned} \frac{\mathbb{1}_A}{|A|} * \frac{\mathbb{1}_{x-A}}{|A|}(x) &= \frac{1}{|A|^2} \int \mathbb{1}_A(y) \mathbb{1}_{x-A}(x-y)dy \\ &= \frac{1}{|A|}. \end{aligned}$$

Thus $|A|^2 \geq N_\infty(X_1+X_2)$ and it follows that $|A|^2 = N_\infty(X_1+X_2) = N_\infty(X_1) = N_\infty(X_2)$. □

4. Quantitative khintchine's inequality

In this section we prove Theorem 4 that resembles the proof of Theorem 2. We need to recall some results from [11].

Assume without loss of generality that $a_k \neq 0$ for all k. Put

$$F(s) = \frac{2}{\pi} \int_0^\infty \left(1 - \left|\cos\left(\frac{t}{\sqrt{s}}\right)\right|^s\right) \frac{dt}{t^2}, \qquad s > 0.$$

From [11, Lemma 1.4 (and its proof)], we can extract that

$$F(s) = \frac{2}{\sqrt{\pi s}} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = \sqrt{\frac{2}{\pi}} \prod_{k=0}^\infty \left(1 - \frac{1}{(s+2k+1)^2}\right)^{\frac{1}{2}}$$

is an increasing function of s, with $F(2) = 1/\sqrt{2}$ and $\lim_{s\to\infty} F(s) = \sqrt{\frac{2}{\pi}}$. Haagerup also proved [11, Lemma 1.3] that

$$R_1(a) \geq \sum_{k=1}^n a_k^2 F\left(\frac{1}{a_k^2}\right) \tag{9}$$

with the convention that $a_k^2 F\left(\frac{1}{a_k^2}\right) = 0$ if $a_k = 0$ (recall the definition of R_1 from (3)). For completeness, let us reproduce the argument using Nazarov and Podkorytov's presentation [28]. From the identity

$$|s| = \frac{2}{\pi} \int_0^\infty (1-\cos(st)) \frac{dt}{t^2}$$

applied to $s = \sum_{k=1}^n a_k B_k$, we have

$$\begin{aligned} R_1(a) &= \mathbb{E}\left(\left|\sum_{k=1}^n a_k B_k\right|\right) \\ &= \frac{2}{\pi}\int_0^\infty \left(1 - \mathbb{E}\left(\cos\left(t\sum_{k=1}^n a_k B_k\right)\right)\right)\frac{dt}{t^2} \\ &= \frac{2}{\pi}\int_0^\infty \left(1 - \prod_{k=1}^n \cos(a_k t)\right)\frac{dt}{t^2} \end{aligned}$$

where at the last line we used that

$$\begin{aligned} \mathbb{E}\left(\cos\left(t\sum_{k=1}^n a_k B_k\right)\right) &= \mathrm{Re}\left(\mathbb{E}\left(e^{it\sum_{k=1}^n a_k B_k}\right)\right) \\ &= \prod_{k=1}^n \cos(a_k t). \end{aligned}$$

Since $\sum a_k^2 = 1$, the following Young's inequality $\prod_{k=1}^n s_k^{a_k^2} \leq \sum a_k^2 s_k$ holds for any non-negative $s_1, \ldots, s_n$. Therefore, (take $s_k = |\cos(a_k t)|^{a_k^{-2}}$), it holds

$$\begin{aligned} R_1(a) &\geq \frac{2}{\pi}\int_0^\infty \left(1 - \prod_{k=1}^n |\cos(a_k t)|\right)\frac{dt}{t^2} \\ &\geq \frac{2}{\pi}\int_0^\infty \left(1 - \sum_{k=1}^n a_k^2 |\cos(a_k t)|^{a_k^{-2}}\right)\frac{dt}{t^2} \\ &= \sum_{k=1}^n a_k^2 \frac{2}{\pi}\int_0^\infty \left(1 - |\cos(a_k t)|^{a_k^{-2}}\right)\frac{dt}{t^2} \end{aligned}$$

which amounts to (9).

Now observe that $R_1(a) \geq \max_k |a_k|$. Indeed, given k_o, multiplying by B_{k_o}, that satisfies $|B_{k_o}| = 1$, it holds

$$\begin{aligned} R_1(a) &= \mathbb{E}\left(|B_{k_o}|\left|\sum_{k=1}^n a_k B_k\right|\right) \\ &= \mathbb{E}\left(\left|a_{k_o} + \sum_{\substack{k=1 \\ k\neq k_o}}^n a_k B_k B_{k_o}\right|\right) \\ &\geq \left|\mathbb{E}\left(a_{k_o} + \sum_{\substack{k=1 \\ k\neq k_o}}^n a_k B_k B_{k_o}\right)\right| \\ &= |a_{k_o}|. \end{aligned}$$

It follows by assumption that $|a_k| \leq \frac{1+\varepsilon}{\sqrt{2}}$ for any k.

Assume that $F(1/a_k^2) > \frac{1+\varepsilon}{\sqrt{2}}$ for all k. Then, by (9) and monotonicity of F, it would hold

$$R_1(a) \geq \sum_{k=1}^{n} a_k^2 F\left(\frac{1}{a_k^2}\right) > \frac{1+\varepsilon}{\sqrt{2}}.$$

This contradicts the starting hypothesis $R_1(a) \leq \frac{1+\varepsilon}{\sqrt{2}}$. Therefore, there exists at least one index, say k_o, such that $F(1/a_{k_o}^2) \leq \frac{1+\varepsilon}{\sqrt{2}}$. Using Lemma 8 we can conclude that

$$|a_{k_o}| \geq \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1+20\varepsilon}} \geq \frac{1-10\varepsilon}{\sqrt{2}}$$

since $1/\sqrt{1+t} \geq 1 - \frac{t}{2}$ for any $t \in (0,1)$.

We iterate the argument. Assume that $F(1/a_k^2) > \frac{1+3\varepsilon}{\sqrt{2}}$ for all $k \neq k_o$. From (9) and monotonicity of F, it would hold (recall that $|a_k| \leq \frac{1+\varepsilon}{\sqrt{2}}$ for any k and in particular for k_o)

$$\begin{aligned} R_1(a) &\geq \sum_{k=1}^{n} a_k^2 F\left(\frac{1}{a_k^2}\right) \\ &> \frac{1+3\varepsilon}{\sqrt{2}} \sum_{k\neq k_o} a_k^2 + a_{k_o}^2 F\left(\frac{1}{a_{k_o}^2}\right) \\ &\geq \frac{1+3\varepsilon}{\sqrt{2}} \sum_{k\neq k_o} a_k^2 + a_{k_o}^2 F\left(\frac{2}{(1+\varepsilon)^2}\right). \end{aligned}$$

Now Lemma 9 guarantees that $F\left(\frac{2}{(1+\varepsilon)^2}\right) \geq \frac{1-\alpha\varepsilon}{\sqrt{2}}$, with $\alpha = \frac{\pi^2}{12}$, so that, since $\sum_{k\neq k_o} a_k^2 = 1 - a_{k_o}^2$ and $|a_{k_o}| \leq \frac{1+\varepsilon}{\sqrt{2}}$, it holds

$$\begin{aligned} R_1(a) &> \frac{1+3\varepsilon}{\sqrt{2}} \sum_{k\neq k_o} a_k^2 + a_{k_o}^2 \frac{1-\alpha\varepsilon}{\sqrt{2}} \\ &= \frac{1+3\varepsilon}{\sqrt{2}} + a_{k_o}^2 \left(\frac{1-\alpha\varepsilon}{\sqrt{2}} - \frac{1+3\varepsilon}{\sqrt{2}}\right) \\ &\geq \frac{1+3\varepsilon}{\sqrt{2}} - \left(\frac{1+\varepsilon}{\sqrt{2}}\right)^2 \frac{(3+\alpha)\varepsilon}{\sqrt{2}} \\ &= \frac{1+\varepsilon}{\sqrt{2}} + \frac{\varepsilon}{4\sqrt{2}} \left(4 - (3+\alpha)(1+\varepsilon)^2\right) \\ &> \frac{1+\varepsilon}{\sqrt{2}} \end{aligned}$$

since for $\varepsilon \in (0, 1/100)$, $4 > (3+\alpha)(1+\varepsilon)^2$. This again contradicts the hypothesis $R_1(a) \leq \frac{1+\varepsilon}{\sqrt{2}}$. Therefore, there exists a second index $k_1 \neq k_o$, such that $F(1/a_{k_1}^2) \leq \frac{1+3\varepsilon}{\sqrt{2}}$. Lemma 8 then implies that

$$|a_{k_1}| \geq \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1+60\varepsilon}} \geq \frac{1-30\varepsilon}{\sqrt{2}}$$

(since, again, $1/\sqrt{1+t} \geq 1 - \frac{t}{2}$). This proves the first part of the theorem.

For the second part we use the previous results together with $\sum_{k=1}^n a_k^2 = 1$ to ensure that

$$\sum_{k\neq k_o,k_1} a_k^2 = 1 - a_{k_o}^2 - a_{k_1}^2 \leq 1 - \left(\frac{1-10\varepsilon}{\sqrt{2}}\right)^2 - \left(\frac{1-30\varepsilon}{\sqrt{2}}\right)^2 \leq \frac{80}{\sqrt{2}}\varepsilon \leq 57\varepsilon.$$

This ends the proof of the theorem.

Lemma 8. *Fix* $\varepsilon \in (0, 3/100)$ *and* $s > 0$ *such that* $F(s) \leq \frac{1+\varepsilon}{\sqrt{2}}$. *Then*

$$s \leq 2(1+20\varepsilon).$$

Proof. Assume that $s \geq 2$ (otherwise there is nothing to prove). By expansion, $F(s) = F(2) + \int_2^s F'(t)dt \leq \frac{1+\varepsilon}{\sqrt{2}}$. Therefore, since $F(2) = 1/\sqrt{2}$,

$$\int_2^s F'(t)dt \leq \frac{\varepsilon}{\sqrt{2}}.$$

Observe that $F(3) = \frac{4}{\pi\sqrt{3}} \simeq 0.74 \geq 0.71 \simeq \frac{1.01}{\sqrt{2}} \geq \frac{1+\varepsilon}{\sqrt{2}}$. Hence, since F is increasing, necessarily $s \leq 3$. It follows that

$$(s-2) \inf_{2\leq t\leq 3} F'(t) \leq \frac{\varepsilon}{\sqrt{2}}$$

and we are left with estimating $\inf_{2\leq t\leq 3} F'(t)$. Using the expression of F above as a product, we deduce that, for $t \in (2,3)$

$$\begin{aligned} F'(t) &= F(t) \sum_{k=0}^\infty \frac{1}{(t+2k)(t+2k+1)(t+2k+2)} \\ &\geq F(2) \sum_{k=0}^\infty \frac{1}{(2k+3)(2k+4)(2k+5)} \\ &\geq \frac{1}{40\sqrt{2}} \end{aligned}$$

where in the last inequality we used that $F(2) = 1/\sqrt{2}$ and estimated from below the infinite sum by the first 5 terms.[4] The expected result follows. ☐

Lemma 9. *Fix* $\varepsilon \in (0, 1/100)$. *Then*

$$F\left(\frac{2}{(1+\varepsilon)^2}\right) \geq \frac{1-\alpha\varepsilon}{\sqrt{2}}$$

with $\alpha = \pi^2/12$.

Proof. By expansion,

$$F\left(\frac{2}{(1+\varepsilon)^2}\right) = F(2) - \int_{\frac{2}{(1+\varepsilon)^2}}^2 F'(t)dt \geq \frac{1}{\sqrt{2}} - \left(2 - \frac{2}{(1+\varepsilon)^2}\right) \sup_{\frac{2}{(1+\varepsilon)^2}\leq t\leq 2} F'(t).$$

Now, as in the proof of Lemma 8, for any $t \in (\frac{2}{(1+\varepsilon)^2}, 2)$, it holds

$$F'(t) = F(t) \sum_{k=0}^\infty \frac{1}{(t+2k)(t+2k+1)(t+2k+2)} \leq F(2) \sum_{k=0}^\infty \frac{1}{8(k+1)^2} = \frac{\pi^2}{48\sqrt{2}},$$

[4] Alternatively one can argue that $\sum_{k=0}^\infty \frac{1}{(2k+3)(2k+4)(2k+5)} \geq \sum_{k=0}^\infty \frac{1}{(2k+4)^3} = \frac{1}{8}(\zeta(3)-1)$ where $\zeta(3) \simeq 1.202 \geq 1.2$ is the Riemann zeta function, from which we deduce that the infinite series is bounded below by $1/40$.

where the inequality follows from the rough estimate $(t+2k)(t+2k+1)(t+2k+2) \geq 8(k+1)^2$, valid for any k and any $t \in (\frac{2}{(1+\varepsilon)^2}, 2)$ (this is trivial for $t \geq 1$ and $k \geq 1$, the case $k = 0$ has to be treated separately, details are left to the reader).

Combining this with the previous estimate, we get

$$\begin{aligned} F\left(\frac{2}{(1+\varepsilon)^2}\right) &\geq \frac{1}{\sqrt{2}}\left(1 - \frac{\pi^2}{48}\left(2 - \frac{2}{(1+\varepsilon)^2}\right)\right) \\ &= \frac{1}{\sqrt{2}}\left(1 - \frac{\pi^2}{24}\frac{2\varepsilon+\varepsilon^2}{(1+\varepsilon)^2}\right) \\ &\geq \frac{1}{\sqrt{2}}\left(1 - \frac{\pi^2}{12}\varepsilon\right) \end{aligned}$$

which is the result we claimed. □

Remark 10. The range $\varepsilon \in (0, 1/100)$ in Theorem 4 is technical and here to guarantee that $(1+\varepsilon)/\sqrt{2} \leq \sqrt{2}/\sqrt{\pi} = \lim_{s\to\infty} F(s)$ and also that $F(3) \geq (1+\varepsilon)/\sqrt{2}$ (see the proof of Lemma 8).

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