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Transport-majorization to analytic and geometric inequalities

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We introduce a transport-majorization argument that establishes a majorization in the convex order between two densities, based on control of the gradient of a transportation map between them. As applications, we give elementary derivations of some delicate Fourier analytic inequalities, which in turn yield geometric "slicing-inequalities" in both continuous and discrete settings. As a further consequence of our investigation we prove that any strongly log-concave probability density majorizes the Gaussian density and thus the Gaussian density maximizes the Rényi and Tsallis entropies of all orders among all strongly log-concave densities.

Contents

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1. Introduction

Let us introduce the notion of Majorization which will play a key role in the investigations of this paper.

Definition 1.1 *(Majorization).* For a finite signed measure σ on a vector space E, we write $0 \prec \sigma$ and say that σ *majorizes* 0 *in the convex order* when φ convex implies

$$0 \leq \int_E \varphi \, d\sigma.$$

We will be particularly interested in using the machinery of majorization to make statements about the behavior of density functions. This will correspond to the case that σ is supported on $[0, \infty)$ and is the difference of two positive measures, each a pushforward of a density. That is, when $(\mathcal{X}, \mathcal{A}, \mu)$ is a measure space with a measurable function $g : \mathcal{X} \to [0, \infty)$, and $\sigma_1 = g\#\mu$, where

$$g\#\mu(A) := \mu(g^{-1}(A)) \tag{1}$$

for measurable $A \subseteq [0, \infty)$ and similarly $\sigma_2 = f\#\nu$ with $f : \mathcal{Y} \to [0, \infty)$ measurable for $(\mathcal{Y}, \mathcal{B}, \nu)$ a measure space, and $\sigma = \sigma_2 - \sigma_1$. We say that $\bar{\mu} = g\#\mu$ is the pushforward of μ by g or that g transports μ to $\bar{\mu}$ when (1) holds for all measurable A. Observe that $0 \prec f\#\nu - g\#\mu$ is equivalent to

$$\int \varphi(g) d\mu \leq \int \varphi(f) d\nu$$

for all convex functions $\varphi : [0, \infty) \to \mathbb{R}$ and in this case we may write for brevity $g\#\mu \prec f\#\nu$ in place of $0 \prec f\#\nu - g\#\mu$. When $\mu = \nu$, we will further abbreviate to $f \prec_\mu g$.

We direct the reader to the textbooks [20,28] for further background on the convex order and majorization. We only stress here that our definition $0 \prec f\#\nu - g\#\mu$ is slightly more general and does imply $g\#\mu \prec f\#\nu$ in the commonly used sense.

We adopt a formulation with signed measures for ease of use against integrability issues. The hypothesis that σ is a finite signed measure majorizing 0 implies that the positive measures σ_+ and σ_- in the Hahn decomposition of $\sigma = \sigma_+ - \sigma_-$, possess the same finite measure, as $\int L d(\sigma_+ - \sigma_-) = 0$ for any linear function L, and in particular when $L = 1$. However our definitions do not require that $f\#\nu$ and $g\#\mu$ themselves be finite measures. For example, with the signed measure formulation, one may consider and prove $g \prec_\mu f$ even when $\int g d\mu = \infty$, so long as $f - g$ is integrable, see Lemma 1.2 below.

The notion of a distribution function, that we now introduce, will be useful in connecting our current investigations to previous literature, and for giving several equivalent formulations of Definition 1.1 (Theorem 2.1 below). Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a measure space (μ need not be a probability measure, we may often omit the σ-algebra). For a non-negative measurable function $g\colon \mathcal{X} \to [0, \infty)$, define its distribution function $G\colon [0, \infty) \to [0, \infty]$ by

$$G(\lambda) := \mu\left(\{x \in \mathcal{X} : g(x) > \lambda\}\right).$$

We will demonstrate that the concept of majorization provides a simple and systematic means for understanding important integral inequalities. In fact, though not explicitly acknowledged in the literature, majorization techniques have been of significant recent interest for proving analytic and geometric inequalities. In their seminal paper [24], Nazarov and Podkorytov introduced a very elementary but powerful lemma, that we may call Nazarov-Podkorytov's lemma in the sequel, based on distribution functions.

Lemma 1.2 *([24]). Let f and g be any two non-negative measurable functions on a measure space $(\mathcal{X}, \mathcal{A}, \mu)$. Let F and G be their distribution functions. Assume that both $F(\lambda)$ and $G(\lambda)$ are finite for every $\lambda > 0$. Assume also that at some point λ_o the difference $F - G$ changes sign from $-$ to $+$, i.e., $F(\lambda) \le G(\lambda)$ for all $\lambda \in (0, \lambda_o)$ and $F(\lambda) \ge G(\lambda)$ for all $\lambda > \lambda_o$. Let $S := \{s > 0 : f^s - g^s \in \mathbb{L}^1(\mathcal{X}, \mu)\}$. Then if $\int_{\mathcal{X}} (f^{s_o} - g^{s_o}) d\mu = 0$, $\int_{\mathcal{X}} (f^s - g^s) d\mu \ge 0$ for each $s > s_o$, $s \in S$. The equality may hold only if the functions F and G coincide.*

It has attracted attention and found utility as a tool for delivering relatively simple arguments for L^p norm comparisons between functions that would otherwise be very challenging to compare.[2]

[2] In fact, Nazarov and Podkorytov proved a stronger result, that the function

$$s \mapsto \varphi(s) := \frac{1}{s\lambda_o^s} \int_{\mathcal{X}} (f^s - g^s) d\mu$$

is increasing on S. However in applications, this monotonicity result has yet to find utility outside of the context of Lemma 1.2. Further, Nazarov-Podkorytov's lemma was used, to the best of our knowledge, only with $\mathcal{X} = (0, \infty)$, or $\mathbb{R}$.

The change of sign between the distribution functions F and G that appears in Lemma 1.2 is known in the literature as the *single crossing property*. It is hard to give a sure attribution of this terminology. We could find its definition in a paper by Diamond and Estiglitz [9, Page 3], in economy. However, such a property, with no specific name, was used earlier in probability theory, see *e.g.* Karlin's book [15]. Moreover, it appears that Nazarov-Podkorytov's lemma is essentially already contained, inter alia, in [16], though this paper does not state it as clearly. Such a lemma is essentially part of the folklore and is often re-derived on an ad-hoc basis. It is worth mentioning that Nazarov and Podkorytov themselves do no pretend at any novelty. Besides the papers already quoted above, let us mention [21,22] for a few other places where the reader can find similar statements.

In fact, [16] also holds the majorization interpretation (Lemma 1.3) that was at the starting point of our investigations, even if we realized the existence of [16] only after the writing of the present article was complete.[3]

We direct the reader to [1] for recent extensions of Nazarov-Podkorytov's lemma to interpolation spaces using majorization.

The above lemma was the starting point of Nazarov and Podkorytov's idea of unifying and re-deriving, in a very elegant way, very deep results of Ball on sections of the unit cube [2] and of Haagerup on sharp constants in Kintchin's inequalities [13].

In Ball's approach of the cube slicing problem, one key ingredient is to prove that

$$\int_{\mathbb{R}} e^{-s\pi x^2/2} - \left|\frac{\sin(\pi x)}{\pi x}\right|^s dx > 0$$

for all $s > 2$, observe that for $s = 2$ this is an identity. We may refer to it as *Ball's integral inequality*. Here and below dx stands for the integration against the Lebesgue measure. For the second statement the authors had to use a modified lemma (of the same spirit but with $F_*(y) := \mu(\{x : f(x) < y\})$ for f bounded) to prove that

$$\int_0^\infty \left(e^{-s\frac{x^2}{2}} - |\cos x|^s\right) \frac{dx}{x^{p+1}} \geq 0$$

for all $s \geq 2$, all $p \in [p_o, 2)$ (for some well defined $p_o \simeq 1.85$, see [13]), an inequality due to Haagerup that we may call *Haagerup's integral inequality*.

Besides Ball-Haagerup's integral inequalities, we mention that Nazarov and Podkorytov's lemma was exploited by different authors. Namely in a refined version of Ball's integral inequality by König and Koldobsky [17], with application to Busemann-Petty problem for the surface area, in his study of optimal Khinchin's inequality for $p \in (2, 3)$

[3] Confirming the law that mathematicians often rediscover results known for a long time... In the field of functional inequalities, the logarithmic Sobolev inequality of Shannon-Gross is a striking example.

by Mordhorst [23]. It was also used in [18] to compare the Fourier transform of a Bernoulli random variable to a Gaussian.

The next lemma, that goes back to [16], gives a majorization interpretation of Nazarov and Podkorytov's lemma. Our aim in this article is to introduce, in the Lebesgue case, an alternative based on transport arguments. Both constitute practical tools to prove majorization (between two integrable functions) that is, in many situations, a hard task.

Lemma 1.3 *([16]). Let f and g be any two non-negative functions on a measure space $(\mathcal{X}, \mathcal{A}, \mu)$ satisfying $\int(f-g)d\mu = 0$. Let F and G be their distribution functions. Assume that at some point λ_o the difference $F - G$ changes sign from $-$ to $+$, i.e., $F(\lambda) \leq G(\lambda)$ for all $\lambda \in (0, \lambda_o)$ and $F(\lambda) \geq G(\lambda)$ for all $\lambda > \lambda_o$. Then*

$$g \prec_\mu f,$$

and consequently $\int_{\mathcal{X}}(\varphi(f) - \varphi(g))d\mu \geq 0$ when φ is convex.

In other words, *the single crossing property implies majorization*. In the next lemma we will prove that *transport implies majorization* leading to the same conclusion under very different conditions (see Remark 2.15).

We anticipate on the fact that the conclusion of Nazarov and Podkorytov's lemma (Lemma 1.2, for integrable functions) is then a straightforward consequence of Lemma 1.3 thanks to a well-known equivalent formulation of the majorization (see Section 2).

In order to state our second lemma, we need to introduce some notations and definition. We use the notation $|T'(x)|$ to denote the absolute value of the determinant of the linear map $T'(x)$ obtained as the derivative of a function $T : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 1.4 *(Change of variables).* Let $\varphi, \psi \colon \mathbb{R}^n \to \mathbb{R}_+$ be measurable with $\int \varphi(x)dx < \infty$. We say that a function $T : \mathbb{R}^n \to \mathbb{R}^n$ is a change of variables between φ and ψ if T is locally-Lipschitz, injective and satisfies Lebesgue almost everywhere $\varphi = \psi(T)|T'|$.

The definition implies that, for any measurable function h, it holds $\int h(T(x))\varphi(x)dx = \int h(x)\psi(x)dx$. Change of variables are related to transport of mass. We refer the reader to Section 2 for comments on the existence and regularity of changes of variables and to the book by Villani [32] for an introduction to the field of optimal transport. We stress that the existence of a change of variables implies the conservation of mass $\int \varphi(x)dx = \int \psi(x)dx$.

In the sequel inf ess and sup ess denotes the essential infimum and essential supremum, respectively, namely for a measurable function $f \colon \mathbb{R}^n \to \mathbb{R}$, $\operatorname{sup\,ess} f = \inf\{a : f(x) \leq a$ for Lebesgue almost all $x \in \mathbb{R}^n\}$ (with $\operatorname{sup\,ess} f = \infty$ if the set is empty) and similarly for inf ess.

Lemma 1.5. *Let $\mu(dx) = u(x)dx$ be a (not necessarily finite) measure on $\mathbb{R}^n$, $f, g : \mathbb{R}^n \to \mathbb{R}_+$ be non-negative measurable such that $\int g d\mu < \infty$, and assume that there exists a change of variables $T : \mathbb{R}^n \to \mathbb{R}^n$ from ug and uf satisfying $\inf \mathrm{ess} \frac{u}{u(T)|T'|} \geq 1$. Then,*

$$g \prec_\mu f.$$

A more general version of the above lemma will be stated in Section 2.5.

As a simple but useful application, let us restate the conclusion of the above result in the spirit of Nazarov and Podkorytov's lemma (take $u \equiv 1$ and $\varphi(x) = x^{s/s_o}$ in Lemma 1.5, details are left to the reader).

Lemma 1.6. *Let $f, g : \mathbb{R}^n \to [0, \infty)$ be measurable such that there exists $T : \mathbb{R}^n \to \mathbb{R}^n$ and $s_o > 0$ such that T is a change of variables from g^{s_o} and f^{s_o}, with $|T'(x)| \leq 1$ and $\int g^{s_o}(x)dx = \int f^{s_o}(x)dx < \infty$. Then $\int g^s(x)dx \leq \int f^s(x)dx$ for all $s \geq s_0$.*

As a second application of Lemma 1.5, with the help of the celebrated Cafarelli's contaction Theorem, we will prove that any strongly log-concave density majorizes the standard Gaussian density (see Section 2), a result that is very natural and, to the best of our knowledge, was not known. Related to information theory, as a corollary, we will deduce that the Gaussian distribution maximizes, among all strongly log-concave distributions, the Rényi and Tsallis entropies of all orders. The Rényi entropy [27] unifies the Shannon, min, Hartley, and collision entropies, and has been long used in information theory, see the survey [30] for more background. The Tsallis entropy [29] is an alternative generalization of the Shannon entropy proven useful in statistical mechanics in the last few decades.

To further demonstrate the efficacy of Lemma 1.5 (and in practice Lemma 1.6), in Section 3.3 we give simple proofs of some integral inequalities previously derived through the method of Nazarov and Podkorytov. We also use Lemma 1.6 to derive the following main result.

Theorem 1.7. *For $p \geq 2$, and $2 \leq n \in \mathbb{N}$,*

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{\sin(n\pi x)}{n \sin \pi x} \right|^p dx < \sqrt{\frac{2}{p\ (n^2 - 1)}}$$

We note that using the method of Nazarov-Podkorytov, Theorem 1.7 was obtained for $n \geq N$ for a fixed $N \in \mathbb{N}$ in an unpublished work [19]. As a corollary of Theorem 1.7, we obtain sharp upper-bounds on the cardinality of $A_k := \{z \in \mathbb{Z}^n : z_i \in [0, n_i], z_1 + \cdots + z_n = k\}$. The continuous version of this problem, upper-bounds on the volume of $\tilde{A}_\lambda := \{x \in \mathbb{R}^n : x_i \in [0, n_i], x_1 + \cdots + x_n = \lambda\}$ is equivalent to upper-bounds on slices of the cube, and it is in this sense that we consider Theorem 1.7 to be a discrete analog of

Ball's integral inequality. Moreover, it will be shown by letting $n \to \infty$ in Theorem 1.7 one recovers Ball's integral inequality.

As a final remark, we mention that some of our results can be extended to functions f and g leaving on different measure spaces $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$. For this we will need to restrict to convex functions vanishing at 0 and to introduce the appropriate corresponding definition of majorization (see Section 2.5).

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2. Majorization and transport

This section collects some aspects of majorization related to Nazarov and Podkorytov's lemma. We first recall some basic properties of majorization. Then we prove Lemma 1.3. In the next sub-sections we deal with the transport approach of majorization, and, together with Cafarelli's contraction Theorem, with strongly log-concave distributions. Finally, in the last sub-section we extend some of our results to functions f and g leaving on different measure spaces.

All along the section $(\mathcal{X}, \mathcal{A}, \mu)$ denotes a measure space and g and f non-negative measurable functions $g, f : \mathcal{X} \to \mathbb{R}_+$. Their respective distribution functions G and F denote $G(\lambda) = \mu\{g > \lambda\}$ and $F(\lambda) = \mu\{f > \lambda\}$.

2.1. Basics on majorization

For more background on majorization see [20]. The following theorem is a reformulation in terms of a single signed measure, of well known, equivalent, useful descriptions of majorization, see for instance [8] for a classical reference.

Theorem 2.1. *For a signed measure σ on $[0, \infty)$ such that $\int_0^\infty L(x) d\sigma = 0$ for any affine function*[4] *$L(x) = ax + b$, the following are equivalent;*

(1)

$$\sigma \succ 0$$

(2) *For $t \geq 0$,*

$$\int_t^\infty \sigma(\lambda, \infty) d\lambda \geq 0.$$

[4] We assume tacitly that the integrals are well defined, which will only be true if $\int L d|\sigma| < \infty$.

(3) *For* $t \geq 0$,

$$\int_0^\infty [x-t]^+ d\sigma(x) \geq 0.$$

Note that $\int_0^\infty L(x)d\sigma(x) = 0$ is a necessary condition for $\sigma \succ 0$ since L and $-L$ are both convex functions.

Proof. Note that the finite assumptions on σ justify the change of order of integration,

$$\int_t^\infty \sigma[\lambda,\infty)d\lambda = \int_0^\infty \left(\int_0^\infty \mathbb{1}_{\{x \geq \lambda \geq t\}} d\lambda \right) d\sigma(x) = \int_0^\infty [x-t]_+ d\sigma(x),$$

so that (2) $\iff$ (3). To prove (3) $\Rightarrow$ (1) the Taylor series expansion of a smooth convex function φ as

$$\varphi(x) = \varphi(0) + x\varphi'(0) + \int_0^\infty [x-t]^+ \varphi''(t)dt,$$

we have

$$\int_0^\infty \varphi(x)d\sigma(x) = \int_0^\infty (\varphi(0) + x\varphi'(0))\, d\sigma(x) + \int_0^\infty \left(\int_0^\infty [x-t]^+ d\sigma(x) \right) \varphi''(t)dt \geq 0.$$

By approximation the result follows for general convex φ. This completes the proof since (1) $\Rightarrow$ (3) is immediate. $\square$

2.2. Nazarov and Podkorytov's lemma as a consequence of majorization

We start by proving Lemma 1.3.

Proof of Lemma 1.3. The signed measure $\sigma = f\#\mu - g\#\mu$, satisfies

$$\int_0^\infty \psi d\sigma = \int_{\mathcal{X}} \psi(f) - \psi(g) d\mu$$

for measurable ψ for which either side of the equality is well defined. Hence, it follows that $\int_0^\infty 1 d\sigma = \int_0^\infty x d\sigma = 0$, and hence $\int_0^\infty L d\sigma = 0$ for any $L(x) = ax + b$. Moreover, $\sigma(t,\infty) = F(t) - G(t)$. Thus it follows from the assumptions on $F - G$ that $\Phi(t) := \int_t^\infty \sigma(\lambda,\infty)d\lambda$ is non-decreasing for $t \leq \lambda_0$ and non-increasing for $t \geq \lambda_0$, and since

$\lim_{t\to\infty}\Phi(t) = \Phi(0) = \int(g-f)d\mu = 0$ we have $\Phi(t) \geq 0$. Hence, $0 \prec \sigma = g\#\mu - f\#\mu$ by item (2) of Theorem 2.1. $\square$

As an immediate consequence of the majorization property, we re-prove Nazarov and Podkorytov's lemma in the following form.

Corollary 2.2 *(Nazarov-Podkorytov [24], Karlin-Novikoff [16]). Let g and f be two non-negative measurable functions on $(\mathcal{X}, \mathcal{A}, \mu)$. Let G and F be their distribution functions. Assume that $\int f^{s_o} - g^{s_o} d\mu = 0$ for some $s_o > 0$. Assume also that at some point λ_o, $F(\lambda) \leq G(\lambda)$ for all $\lambda \in (0, \lambda_o)$ and $F(\lambda) \geq G(\lambda)$ for all $\lambda > \lambda_o$. Then*

$$\int (f^s - g^s) d\mu \geq 0$$

for all $s \geq s_0$.

Proof. Writing, $G_{s_0}(t) = \mu\{g^{s_0} > t\}$, and $F_{s_0}(t) = \nu\{f^{s_0} > t\}$ then $F_{s_0}(t) = F(t^{\frac{1}{s_0}})$ and $G_{s_0}(t) = G(t^{\frac{1}{s_0}})$, so that F_{s_0} and G_{s_0} have a single crossing at $\lambda_0^{\frac{1}{s_0}}$ and satisfy the hypothesis of Lemma 1.3, and we have $g^{s_0} \prec_\mu f^{s_0}$. Consequently, by Theorem 2.1 for φ convex we have

$$\int (\varphi(f^{s_0}) - \varphi(g^{s_0})) d\mu \geq 0. \tag{2}$$

For $s \geq s_0$ taking $\varphi(x) = x^{\frac{s}{s_0}}$ gives the result. $\square$

2.3. Majorization via transport

This section is dedicated to the proof of Lemma 1.5.

Before moving to the proof, let us make some comments on the existence and regularity of the map T.

The change of variable assumption guarantees the existence of a map T that transports gu to fv: $T\#gu = fv$. Set $\tilde{\mu} = g\mu / \int g d\mu$ and $\tilde{\nu} = f\nu / \int f d\nu$ to turn $g\mu$ and $f\nu$ into probability measures on $\mathbb{R}^n$. We stress that T can be defined arbitrarily on any set of Lebesgue measure 0. Indeed, if $T = T_o$ for almost all $x \in \mathbb{R}^n$, $\int\int h(T) d\tilde{\mu} = \int_{\{T=T_o\}} \int h(T) d\tilde{\mu} = \int_{\{T=T_o\}} \int h(T_o) d\tilde{\mu} = \int\int h(T_o) d\tilde{\mu}$.

Brenier's theorem [3] asserts that, as soon as $\mathcal{T}_2(\tilde{\mu}, \tilde{\nu}) < \infty$ (that is a very mild assumption), where $\mathcal{T}_2$ is the optimal transport associated to the quadratic cost (see [32]), the transport map T exists and is the gradient of a convex function $\Phi\colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ with $\tilde{\mu}(\{\Phi < \infty\}) = 1$. Furthermore, T' exists and satisfies the Monge-Ampère equation in the Aleksandrov sense. We refer the reader to [31] for more details.

The invertibility of the map T is not guaranteed in general in the transport theory, even for the Brenier map. However, since $f\nu$ and $g\mu$ are absolutely continuous with respect to the Lebesgue measure, $T = \nabla\Phi$, with Φ convex, has an inverse and in fact $T^{-1} = \nabla\Phi^*$ where Φ^* is the Legendre transform of Φ and $T^{-1}\#(f\nu) = g\mu$. In particular, in our setting, the Brenier map $T = \nabla\Phi$ is always a change of variables.

In dimension 1 the situation is simpler since $T = F^{-1} \circ G$, with F and G the distribution functions of $f\nu$ and $g\mu$, is increasing, $T^{-1} = G^{-1} \circ F$ is always well-defined.

Proof of Lemma 1.5. Let $t \geq 0$. Since $\varphi\colon x \mapsto [x-t]^+$ is a convex function vanishing at 0, $\varphi(sx) \leq s\varphi(x)$ for any $s \in (0,1)$ and $x \geq 0$. Therefore

$$\begin{aligned}
\int [g(x)-t]^+ d\mu &= \int_{\{u\neq 0\}} \left[\frac{u(T(x))f(T(x))|T'(x)|}{u(x)} - t\right]^+ u(x)dx \\
&\leq \int [f(T(x))-t]^+ u(T(x))|T'(x)|dx \\
&= \int [f(x)-t]^+ d\mu.
\end{aligned}$$

Since f and g are integrable, the above inequality is equivalent to

$$\int_0^\infty [x-t]^+ d\sigma \geq 0$$

for $\sigma = f\#\mu - g\#\mu$. The expected result follows from Theorem 2.1 Item (3). □

2.4. Strongly log-concave distributions and majorization

We prove here that any strongly log-concave density majorizes the Gaussian density and that the Gaussian density maximizes the Rényi and Tsallis entropies among all strongly log-concave densities. To that aim, we need first to recall the definition of strongly log-concave densities and Cafarelli's contraction Theorem.

We denote by γ_n the standard Gaussian measure on $\mathbb{R}^n$, with density $g(x) := \frac{d\gamma_n}{dx} = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-|x|^2/2}$.

Definition 2.3. A probability density function $f : \mathbb{R}^n \to [0,\infty)$ is strongly log-concave when $\nu(dx) = f(x)dx$ is log-concave with respect to γ_n. In other words, there exists a convex function V such that $\nu(dx) = e^{-V(x)}\gamma_n(dx)$.

Theorem 2.4 *(Caffarelli [5,6]). Let ν, with density f with respect to the Lebesgue measure, be strongly log-concave. Then there exists a 1-Lipschitz change of variables T from g to f such that $T = \nabla\varphi$, for φ convex.*

The existence of a connection between majorization and Caffarellli's contraction theorem seems to go back to Hargé [14] (see [12,11] for more recent results in this direction).

Here we may put together our transport approach of the majorization in Lemma 1.5 and the latter theorem to get the following natural statement.

Corollary 2.5. *If f is a strongly log-concave density function, then $g \prec_{dx} f$ where g is the standard Gaussian density defined above.*

Proof. By Theorem 2.4, there exists a 1-Lipschitz change of variables $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T = \nabla\varphi$, for a convex $\varphi \colon \mathbb{R}^n \to \mathbb{R}$ such that $T\#\gamma_n = \nu$ (with $\nu(dx) = f(x)dx$). As the Hessian of a convex function $T'(x)$ is symmetric and positive definite, thus it has non-negative eigenvalues. Since T is Lipschitz, its eigenvalues are all bounded by 1, and hence $|T'(x)| \leq 1$ for all x. Applying Lemma 1.5 leads to the desired conclusion. □

Remark 2.6. For comparison let us mention that Hargé proved in [14] that $0 \prec \gamma_n - \nu$ (under the assumption that $\int x d\nu = 0$), which amounts to saying that, for all convex function φ it holds $\int \varphi d\gamma_n \geq \int \varphi d\nu$, while our conclusion $g \prec_{dx} f$ reads $\int \varphi(g)dx \leq \int \varphi(f)dx$ for all φ convex.

As a corollary, we will prove that the Gaussian distribution maximizes the Rényi and Tsallis entropies of all orders (among the set of all strongly log-concave densities). Let us recall some definition.

Definition 2.7 *(Tsallis [29]/Rényi [27] Entropy).* For a probability density function $f : \mathbb{R}^n \to \mathbb{R}_+$, and $q \in (0,1) \cup (1,\infty)$ we denote the q-Tsallis entropy by

$$S_q(f) := \frac{\int f^q(x)dx - 1}{1-q}.$$

For $q = 1$, $S_1(f) := h(f) = -\int f(x)\log f(x)dx$ is the Shannon entropy. Via continuous extension, one can define $S_0(f) = |\{f > 0\}| - 1$ and $S_\infty(f) = 0$ when $\|f\|_\infty \leq 1$ and $S_\infty(f) = -\infty$ otherwise.

We denote the q-Rényi entropy,

$$h_q(f) := \frac{\log \int f^q(x)dx}{1-q},$$

and define $h_1(f) := h(f)$, $h_0(f) = |\{f > 0\}|$, $h_\infty(f) = \|g\|_\infty$.

Observe that $S_q(f) = \Psi_q(h_q(f))$ where

$$\Psi_q(x) := \frac{\exp[(1-q)x] - 1}{1-q}, \qquad x \in \mathbb{R} \tag{3}$$

is a strictly increasing function.

We are in position to state our corollary.

Corollary 2.8. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a strongly log-concave probability density, then for $q \in [0,\infty]$*

$$h_q(f) \leq h_q(g)$$

and

$$S_q(f) \leq S_q(g)$$

where g is the Gaussian density defined above.

Proof. In light of the one-to-one relationship between h_q and S_q given in (3), it suffices to prove the result for the Rényi entropy. When $q = 0$ there is nothing to prove since $h_q(g) = \infty$. For $q \in (0,1)$, the function $\varphi(x) = -x^q$ is convex and hence by Corollary 2.5, $g \prec_{dx} f$, and applying Theorem 2.1,

$$\int \varphi(g(x))dx \leq \int \varphi(f(x))dx,$$

which gives $\int f^q \leq \int g^q$, or $h_q(f) \leq h_q(g)$. When $q = 1$, the convex function $\varphi(x) = x \log x$, gives the result for the Shannon entropy. Again, when $q > 1$, take $\varphi(x) = x^q$. For $q = \infty$, observe that f strongly log-concave is more than sufficient to give $\lim_{q\to\infty} h_q(f) = h_\infty(f)$ from which the result follows, ending the proof of the Corollary. $\square$

As a final remark note that if X is a random variable and $Y = T(X)$ for $|T'(x)| \geq 1$, then if Y has density f, X has density $g(x) = f(T(x))|T'(x)|$ and $h_q(T(X)) \geq h_q(X)$, and similarly for S_q.

2.5. Extensions to different measure spaces

This section collects some generalization of the previous results. In particular, we will extend Lemma 1.5 to functions f and g leaving on different spaces and relax the condition $\operatorname{inf\,ess} \frac{u}{v(T)|T'|} \geq 1$.

Here $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ denote measure spaces and g and f non-negative measurable functions $g : \mathcal{X} \to \mathbb{R}_+$ and $f : \mathcal{Y} \to \mathbb{R}_+$. Their respective distribution functions G and F denote $G(\lambda) = \mu\{g > \lambda\}$ and $F(\lambda) = \nu\{f > \lambda\}$.

The measure spaces will not be assumed to have the same measure ($\mu(\mathcal{X})$ and $\nu(\mathcal{Y})$ need not be equal). This will not guarantee anymore that $\int_0^\infty L(x)d\sigma = 0$ for any affine function. In particular, the assumption $\int_0^\infty d\sigma = 0$ will not be satisfied leading us to consider only convex functions vanishing at 0 and therefore a modified definition of majorization. Similarly to Theorem 2.1, the following holds (the proof being similar, we left it to the reader).

Theorem 2.9. *For a signed measure σ on $[0,\infty)$ such that $\int_0^\infty x d\sigma = 0$, the following are equivalent;*

(i) *For φ convex with $\varphi(0) = 0$,*

$$\int_0^\infty \varphi(x) d\sigma(x) \geq 0.$$

(ii) *For $t \geq 0$,*

$$\int_t^\infty \sigma(\lambda,\infty) d\lambda \geq 0.$$

(iii) *For $t \geq 0$,*

$$\int_0^\infty [x-t]^+ d\sigma(x) \geq 0.$$

We will write $0 \prec_0 \sigma$ when any of the above are satisfied (and accordingly $g\#\nu \prec_0 f\#\mu$).

In [24] Nazarov and Podkorytov's lemma (Lemma 1.2) is stated in the case that $\mu = \nu$. However their proof can be easily adapted in the case that ν differs from μ and f and g are integrable. In such a setting, the following counterpart of Lemma 1.3 and Corollary 2.2 hold. Their proof are left to the reader.

Lemma 2.10 *([16]). Let g and f be two non-negative measurable functions on $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ respectively. Let G and F be their distribution functions. Assume that $\int g d\mu = \int f d\mu < \infty$. Assume also that at some point λ_o the difference $F - G$ changes sign from $-$ to $+$, i.e., $F(\lambda) \leq G(\lambda)$ for all $\lambda \in (0, \lambda_o)$ and $F(\lambda) \geq G(\lambda)$ for all $\lambda > \lambda_o$. Then*

$$g\#\mu \prec_0 f\#\nu,$$

and consequently $\int_{\mathcal{X}} \varphi(g) d\mu \leq \varphi(f)) d\nu$ when φ is convex and $\varphi(0) = 0$.

Corollary 2.11 *(Nazarov-Podkorytov [24], Karlin-Novikoff [16]). Let g and f be two non-negative measurable functions on $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ respectively. Let G and F be their distribution functions. Assume that $\int f^{s_o} d\nu = \int g^{s_o} d\mu < \infty$ for some $s_o > 0$. Assume also that at some point λ_o, $F(\lambda) \leq G(\lambda)$ for all $\lambda \in (0, \lambda_o)$ and $F(\lambda) \geq G(\lambda)$ for all $\lambda > \lambda_o$. Then*

$$\int g^s d\mu \leq \int f^s d\nu$$

for all $s \geq s_0$.

Next we turn to a generalization of Lemma 1.5.

Lemma 2.12. *Let* μ *and* ν *be measures on* $\mathbb{R}^n$ *(not necessarily finite), such that* $\mu(dx) = u(x)dx$ *and* $\nu(dx) = v(x)dx$, $f, g : \mathbb{R}^n \to \mathbb{R}_+$ *be non-negative and measurable such that* $\int g d\mu < \infty$, *and assume that there exists a change of variables* $T : \mathbb{R}^n \to \mathbb{R}^n$ *from* gu *to* fv. *Then for all convex functions* $\varphi : \mathbb{R}_+ \to \mathbb{R}$, *with* $\varphi(0) = 0$,

$$\int \varphi(g) d\mu \geq A \int \varphi(f/A) d\nu$$

if

$$A := \sup\operatorname{ess} \frac{u}{v(T)|T'|} \in (0, \infty).$$

Further if $A \leq 1$, $f\#\nu \prec_0 g\#\mu$.
Also,

$$\int \varphi(g) d\mu \leq A' \int \varphi(f/A') d\nu$$

if

$$A' := \inf\operatorname{ess} \frac{u}{v(T)|T'|} \in (0, \infty),$$

in which case $g\#\mu \prec_0 f\#\nu$ *if* $A' \geq 1$.

Remark 2.13. In the application we will use the lemma only when $A' \geq 1$ (*i.e.* Lemma 1.5), which specifying to $\varphi(x) = |x|^s$ leads to the family of inequalities

$$\int g^s d\mu \leq \int f^s d\nu, \qquad s \geq 1.$$

Note however that the lemma could potentially lead to a much stronger result in the situation where $A' > 1$. Indeed, for $\varphi(x) = |x|^s$ the second conclusion of the lemma reads

$$\int g^s d\mu \leq {A'}^{-(s-1)} \int f^s d\nu$$

i.e. there is an extra exponentially small factor.

Remark 2.14. Observe that, if one looks for integral comparison, one can separate variables in, say, the conclusion $\int \varphi(g)d\mu \geq A \int \varphi(f/A)d\nu$, when $\varphi \geq 0$, by considering $w(x) := \sup_{u>0} \varphi(ux)/\varphi(u)$. Indeed, by definition of w it holds

$$\int \varphi(g)d\mu \geq A \int \varphi(f/A)d\nu \geq \frac{A}{w(A)} \int \varphi(f)d\nu.$$

Also, for $\varphi \geq 0$ satisfying the following so-called Δ_2-condition (see *e.g.* [26]) $\varphi(2x) \leq K\varphi(x)$, for all $x > 0$ and some $K \geq 2$, it holds for $A > 1$,

$$A\varphi\left(\frac{f}{A}\right) = K^{\log_K(A)}\varphi\left(\frac{f}{A}\right) \geq K^{\lfloor \log_K(A) \rfloor}\varphi\left(\frac{f}{A}\right) \geq \varphi\left(\frac{2^{\lfloor \log_K(A) \rfloor}}{A} f\right)$$

(where the floor signs denotes the entire part and $\log_K$ the logarithm in base K). As a conclusion we get that $\frac{2^{\lfloor \log_K(A) \rfloor}}{A} f\#\mu \prec_0 g\#\nu$. For the conclusion of the lemma involving A', one needs to consider instead the ∇_2-condition: $\varphi(x) \geq \frac{1}{2\ell}\varphi(\ell x)$, for all $x > 0$ and some $\ell > 1$ [26].

Proof of Lemma 2.12. Define $\mu_0 = \mu/A$, then $G_0(\lambda) := \mu_0(g > \lambda) = \mu(g > \lambda)/A = G(\lambda)/A$. Further, define $\tilde{f} := f/A$, $\tilde{F}(\lambda) = \nu(\tilde{f} > \lambda) = \nu(f > \lambda A) = F(\lambda A)$, then

$$\begin{aligned}
\int [g-t]^+ d\mu &= \int\limits_{\{u>0\}} \left[\frac{f(T)v(T)|T'|}{u} - t\right]^+ d\mu \\
&\geq \int\limits_{\{u>0\}} [f(T(x)) - tA]^+ \, v(T(x))|T'(x)|dx \\
&= \int [f - tA]^+ d\nu
\end{aligned}$$

where the inequality follows from the fact that $\left[\frac{x}{y} - t\right]_+ \geq [x - t\bar{y}]_+/y$ when $y \leq \bar{y}$ holds for $x, y, t \geq 0$, and the first equality follows a change of variables after observing that $[f(T) - tA]^+ \, v(T)|T'| > 0$ implies $u > 0$. Observe that since $\int [f - tA]^+ d\nu = A \int_t^\infty \tilde{F}(\lambda)d\lambda$ the above inequality can be re-written as

$$\int\limits_t^\infty G_0(\lambda)d\lambda \geq \int\limits_t^\infty \tilde{F}(\lambda)d\lambda.$$

Thus by Theorem 2.1,

$$\int \varphi(g)d\mu_0 \geq \int \varphi(\tilde{f})d\nu$$

and the result follows. If $A \leq 1$, then since $\varphi(0) = 0$, $\varphi(tx) \geq t\varphi(x)$ for $t \geq 1$ and hence

$$\int A\varphi(f/A)d\nu \geq \int \varphi(f)d\nu,$$

and the majorization follows from Theorem 2.9.

The argument for A' is similar and left to the reader. □

Remark 2.15. In short, Lemma 2.12 demonstrates that *transportation implies majorization.* More explicitly, consider the case $u \equiv v \equiv 1$. If there exists a change of variable $T : \mathbb{R}^n \to \mathbb{R}^n$ from g to f (therefore such that $g(x) = f(T(x))|T'|$ for almost all x), and an $\varepsilon > 0$ such that $|T'(x)| \geq \varepsilon$ holds almost surely, then $\tilde{T}(x) = T(x)/\varepsilon^{\frac{1}{n}}$ is a change of variable from $\tilde{g}(x) := g(x)/\varepsilon$ to $\tilde{f}(x) = f(x\varepsilon^{\frac{1}{n}})$ (almost surely $\tilde{g}(x) = \tilde{f}(\tilde{T}(x))|\tilde{T}'(x)|$), with $|\tilde{T}'(x)| \geq 1$. Thus $\tilde{f}\#\nu \prec_0 \tilde{g}\#\mu$.

In the following corollary we apply this observation to the convex function $\varphi(x) = x^s$.

Corollary 2.16. *For $f, g : \mathbb{R}^n \to \mathbb{R}_+$ integrable such that there exists a change of variables T from g to f with $\mathcal{T} := \inf_x |T'(x)| > 0$, then*

$$\int f^s(x)dx \leq \mathcal{T}^{1-s} \int g^s(x)dx, \qquad s \geq 1.$$

Remark 2.17. Written in terms of L_p norms, the conclusion of the corollary becomes

$$\|f\|_s \leq \mathcal{T}^{-\frac{1}{s'}} \|g\|_s$$

Note that the statement is given for $s_o = 1$ (we assumed $\int f(x)dx = \int g(x)dx$), where s_o is as in Lemma 1.6. Similar conclusion could be stated for any $s_o > 0$.

Proof. For $\tilde{g}(x) = g(x)/\mathcal{T}$, $\tilde{f}(x) = f(x\mathcal{T}^{\frac{1}{n}})$, and $\tilde{T}(x) = T(x)/\mathcal{T}^{\frac{1}{n}}$, by $g(x) = f(T(x))|T'(x)|$, it holds $\tilde{g}(x) = \tilde{f}(\tilde{T}(x))|\tilde{T}'(x)|$ and $|\tilde{T}'(x)| \geq 1$. Thus, by Lemma 2.12, $\tilde{f}\#d\nu \prec_0 \tilde{g}\#\mu$ and applying $\int \varphi(\bar{f}) \leq \int \varphi(\bar{g})$ to the function $\varphi(x) = x^s$, we get $\int \tilde{f}^s(x)dx \leq \int \tilde{g}^s(x)dx$. This leads to the desired conclusion. □

We end this section with an alternative direct proof of Corollary 2.16, based on transport arguments.

Lemma 2.18. *Let μ and ν be measures on $\mathbb{R}^n$ (not necessarily finite), such that $\mu(dx) = u(x)dx$ and $\nu(dx) = v(x)dx$, $f, g : \mathbb{R}^n \to \mathbb{R}_+$ be non-negative and measurable such that $\int f(x)v(x) < \infty$, and assume that there exists a change of variables $T : \mathbb{R}^n \to \mathbb{R}^n$ from gu to fv. Then for all $s \geq 1$,*

$$\int f^s d\nu \leq A^{s-1} \int g^s d\mu \qquad \text{if} \quad A := \sup \frac{u}{v(T)|T'|} \in (0, \infty).$$

Also, if T is invertible

$$\int g^s d\mu \le \frac{1}{A'^{s-1}} \int f^s d\nu \qquad \textit{if} \quad A' := \inf \frac{u}{v(T)|T'|} \in (0,\infty).$$

Remark 2.19. As earlier mentioned, if T is the Brenier map, then it is invertible in our setting and therefore the second part of the Lemma applies.

Proof. Since T is a change of variables from ug to fv, for any measurable function h, $\int h(T) g d\mu = \int h f d\nu$. Applying this to $h = f^{s-1}$ it follows that

$$\begin{aligned}\int f^s d\nu &= \int f^{s-1} f d\nu \\ &= \int f^{s-1}(T) g d\mu \\ &\le \left(\int f^s(T) d\mu\right)^{\frac{s-1}{s}} \left(\int g^s d\mu\right)^{\frac{1}{s}},\end{aligned}$$

where the last inequality follows from Hölder's inequality applied with $\frac{s-1}{s} + \frac{1}{s} = 1$.

By definition of A and changing variables it holds $\int f^s(T) d\mu \le A \int f^s(T) v(T)|T'| = A \int f^s d\nu$ from which the expected result follows.

The part with A' is similar and left to the reader. □

3. Ball's integral inequality and beyond

The aim of this section is to prove Theorem 1.7. As a warm up, and to show the efficiency of the transport approach of Lemma 1.6 (and Lemma 2.12), we may first reprove Ball's integral inequality and a 2-dimensional analog due to Oleskiewicz and Pełczyński [25]. Both proofs are very short and elementary.

3.1. Ball's integral inequality

Recall that Ball's integral inequality asserts that

$$\int_{-\infty}^{\infty} g(x)^s dx < \int_{-\infty}^{\infty} f(x)^s dx, \qquad s > 1$$

with

$$f(x) := e^{-\pi x^2} \qquad \text{and} \qquad g(x) := \left(\frac{\sin(\pi x)}{\pi x}\right)^2, \qquad x \in \mathbb{R}.$$

Ball's original proof is based on series expansion. As already mentioned, Nazarov and Podkorytov gave a very elegant and simple alternative proof of the latter, using

Lemma 1.2. In this section, we present yet another proof, very elementary, based on Lemma 1.6.

Since $\int f(x)dx = \int g(x)dx = 1$, f and g are probability densities on the line. In that case, the transport map T that pushes forward the probability measure with density f onto that of density g is increasing and given by $T := F^{-1} \circ G$ with

$$F(x) := \int_{-\infty}^{x} f(t)dt, \qquad G(x) := \int_{-\infty}^{x} g(t)dt, \qquad x \in \mathbb{R}.$$

The transport map T is a change of variables from g to f. Note that it is one-to-one increasing on $\mathbb{R}$ and that it satisfies by construction the Monge-Ampère Equation $g = f(T)T'$.

Now Ball's integral inequality will follow from Lemma 1.6 if we can prove that $T' \leq 1$, which is the aim of the next lemma.

Lemma 3.1. *For all $x \in \mathbb{R}$, $T'(x) \leq 1$.*

Proof. Observe that, $T' \leq 1$ on $\mathbb{R}$ is equivalent[5] to saying that $G' \circ G^{-1} \leq F' \circ F^{-1}$, and so to $g \circ G^{-1} \leq f \circ F^{-1}$ on $(0, 1)$. Since f and g are even, $g \circ G^{-1}$ and $f \circ F^{-1}$ are symmetric about $1/2$. Therefore one needs to prove the inequality on $(1/2, 1)$ only. The density f being decreasing on $\mathbb{R}_+$ with inverse $f^{-1}(y) = \sqrt{\frac{1}{\pi} \log\left(\frac{1}{y}\right)}$ and F being increasing, the inequality $g \circ G^{-1} \leq f \circ F^{-1}$ on $(1/2, 1)$ is in turn equivalent to $G \leq F \circ f^{-1} \circ g$ on $(0, \infty)$. This can be recast as

$$\int_{-\infty}^{x} \left(\frac{\sin(\pi u)}{\pi u}\right)^2 du \leq \int_{-\infty}^{\sqrt{\frac{2}{\pi} \log\left(\left|\frac{\pi x}{\sin(\pi x)}\right|\right)}} e^{-\pi u^2} du, \tag{4}$$

for $x > 0$. For $x \in (0, 1)$ we mimic an argument borowed from [24]. Using the infinite product representation of the sinus, for $x \in (0, 1)$, on one hand one has

$$\begin{aligned} \left(\frac{\sin(\pi x)}{\pi x}\right)^2 &= \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right)^2 \\ &\leq \prod_{k=1}^{\infty} e^{-\frac{2x^2}{k^2}} \\ &= e^{-\frac{\pi^2 x^2}{3}} \end{aligned}$$

[5] As a curiosity, in other contexts, the expression $F' \circ F^{-1}$ appears to be the isoperimetric profile associated to the probability measure with density f, and similarly for g. Therefore, the lemma asks for a comparison between two isoperimetric profiles.

$$\leq e^{-\pi x^2}.$$

This implies $\sqrt{\frac{2}{\pi}\log\left(\left|\frac{\pi x}{\sin(\pi x)}\right|\right)} \geq x$ for any $x \in (0,1)$ and therefore Inequality (4) holds for any $x \in [0,1]$. For $x > 1$, we reformulate (4) as

$$\int\limits_{\sqrt{\frac{2}{\pi}\log\left(\left|\frac{\pi x}{\sin(\pi x)}\right|\right)}}^{\infty} e^{-\pi u^2} du \leq \int\limits_{x}^{\infty} \left(\frac{\sin(\pi u)}{\pi u}\right)^2 du, \qquad x > 0, \tag{5}$$

and observe that for $y = \sqrt{\frac{2}{\pi}\log\left(\left|\frac{\pi x}{\sin(\pi x)}\right|\right)}$, it holds

$$\begin{aligned}
\int\limits_{y}^{\infty} e^{-\pi u^2} du &\leq \int\limits_{y}^{\infty} \frac{2\pi u}{2\pi y} e^{-\pi u^2} du \\
&= \frac{e^{-\pi y^2}}{2\pi y} \\
&= \frac{\left(\frac{\sin(\pi x)}{\pi x}\right)^2}{2\sqrt{2\pi \log\left(\left|\frac{\pi x}{\sin(\pi x)}\right|\right)}}.
\end{aligned}$$

Meanwhile for an integer $k \geq 2$,

$$\begin{aligned}
\int\limits_{k}^{k+1} \left(\frac{\sin(\pi u)}{\pi u}\right)^2 du &\geq \frac{1}{(k+1)^2\pi^2} \int\limits_{k}^{k+1} \sin^2(\pi u) du \\
&= \frac{1}{2\pi^2}\frac{1}{(k+1)^2}.
\end{aligned}$$

For comparison, $\int_k^{k+1} \frac{1}{u^2} du = \frac{1}{k(k+1)}$ so that, for $k \geq 2$

$$\begin{aligned}
\int\limits_{k}^{k+1} \left(\frac{\sin(\pi u)}{\pi u}\right)^2 du &\geq \frac{k}{2\pi^2(k+1)} \int\limits_{k}^{k+1} \frac{1}{u^2} du \\
&\geq \frac{1}{3\pi^2} \int\limits_{k}^{k+1} \frac{1}{u^2} du.
\end{aligned}$$

Put $\lceil x \rceil = \inf_{k\in\mathbb{Z}}\{k : k \geq x\}$ for the ceiling part of x that is greater or equal to 2, since $x > 1$. We have

$$\int_x^\infty \left(\frac{\sin(\pi u)}{\pi u}\right)^2 du \geq \frac{1}{3\pi^2}\int_{\lceil x\rceil}^\infty \frac{1}{u^2}du$$
$$= \frac{1}{3\pi^2\lceil x\rceil}.$$

We claim that

$$\frac{(\sin(\pi x))^2/\left(\pi x\right)^2}{2\sqrt{2\pi\log\left(\left|\frac{\pi x}{\sin(\pi x)}\right|\right)}} \leq \frac{1}{3\pi^2\lceil x\rceil}, \qquad \forall x>1$$

from which (5) for $x > 1$ immediately follows. Therefore, to complete the proof of the lemma, it only remains to prove the claim. We proceed to further reductions. Squaring and exploiting the fact that, for $x > 1$, $\frac{\lceil x\rceil}{x} \leq 2$ and $|\sin(\pi x)| \leq 1$, the claim will in fact be a consequence of

$$9\pi \leq y\log y, \qquad y > 20$$

where we changed variable ($y = \left|\frac{\pi x}{\sin(\pi x)}\right|^2$ whose minimum is achieved on the interval $[\pi, 2\pi]$ and is greater than 20). The claim immediately follows and this ends the proof of the lemma. □

Remark 3.2. In [17, Proposition 7] the authors prove, using Nazarov and Podkorytov's lemma, the following inequality which is a refined version of Ball's integral inequality

$$\int_{-\infty}^\infty \left|\frac{\sin(\pi u)}{\pi u}\right|^{2s} du \leq \sqrt{\frac{3}{\pi}}\int_{-\infty}^\infty e^{-\pi s u^2}du, \qquad s \geq \frac{9}{8}.$$

The refinement is coming from the factor $\sqrt{3/\pi} < 1$. We mention that Lemma 1.6 can also be applied to prove such an inequality with the exact same line of argument as above.

3.2. Oleszkiewicz-Pełczyński: a 2-dimensional Ball's type integral inequality

For $v \geq 1$, let $j_v(s) = 2^v\Gamma(v+1)J_v(s)/s^v$ where $J_v(s) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\Gamma(m+v+1)}\left(\frac{s}{2}\right)^{2m+v}$ is the Bessel function of order ν. For $p \geq 2$ and $m \geq 2$ integer, consider the following integral inequality

$$\int_0^\infty |j_{\frac{m}{2}}(s)|^p s^{m-1}ds \leq \int_0^\infty \left(e^{-\frac{s^2}{2m+4}}\right)^p s^{m-1}ds. \tag{6}$$

The case $m = 1$, as observed in [25, Remark 4.3], reduces to Ball's integral inequality since $J_{1/2}(t) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin t}{t}$ (see [33, Page 54 Inequality (3)]) and therefore $j_{1/2}(s) = \frac{\sin s}{s}$.

The case $m = 2$, treated below with the help of our transport Lemma 2.12, was originally proved in [25] through careful and direct analysis, using detailed expansions and approximations. Yet another proof can be found in [10] which relied on Nazarov-Podkorytov's lemma (to be complete, H. Dirksen mentions the existence of an unpublished note by König that inspired him and that uses Nazarov-Podkorytov's lemma).

We refer the reader to Remark 3.5 below for more comments on Inequality (6) for $m \geq 3$ and related inequalities.

For now, we fix $m = 2$. Set $s = p/2$, $f(x) = e^{-x^2/4}$ and $g(x) = \left(\frac{2J_1(x)}{x}\right)^2$, $x > 0$. Set μ for the measure on $(0, \infty)$ with density x with respect to the Lebesgue measure. Then the inequality we want to prove takes the form

$$\int_0^\infty g^s d\mu \leq \int_0^\infty f^s d\mu, \qquad s \geq 1.$$

Observe that f and g have same mass: $\int f d\mu = \int g d\mu = 2$. Let $T := F^{-1} \circ G$ with, for $x > 0$,

$$F(x) := \int_0^x f(t)\mu(dt) = 2(1 - e^{-x^2/4}), \qquad G(x) := \int_0^x g(t)\mu(dt) = 2 - 2(J_1^2(x) + J_0^2(x)) \tag{7}$$

(see Lemma 3.4 below for the computation of G). By construction, T is the (increasing) pushforward of $g\mu$ onto $f\mu$, $T\#(g\mu) = f\mu$ and satisfies the Monge-Ampère equation $g(x)x = f(T(x))T(x)|T'(x)|$. Therefore, by Lemma 2.12 (with $u(x) = v(x) = x$), the desired inequality would follow if we can prove that $T(x)T'(x) \leq x$. This is achieved in the next lemma.

Lemma 3.3. *For all $x > 0$ it holds $T(x)T'(x) \leq x$.*

Proof. We distinguish between two cases.

• For $x \in (0, 2)$ we prove first that $g(x) \leq f(x)$. Expanding, this is equivalent to proving that (note that $J_1 \geq 0$ on $[0, 2]$)

$$xe^{-x^2/8} - 2J_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \left(\frac{1}{2^{n-1}} - \frac{1}{(n+1)!}\right) \left(\frac{x}{2}\right)^{2n+1} \geq 0, \qquad 0 \leq x \leq 2.$$

We set, for $n \geq 0$, $u_n := \frac{1}{n!}\left(\frac{1}{2^{n-1}} - \frac{1}{(n+1)!}\right)\left(\frac{x}{2}\right)^{2n+1}$ so that $xe^{-x^2/8} - 2J_1(x) = \sum_{n=0}^{\infty}(-1)^n u_n$ is an alternating series ($u_n > 0$ for $n \geq 0$). Now for $x \in [0, 2]$, it is easy to check that $(u_n)_n$ is decreasing. In particular the alternating series has the sign of its first term u_0, which is positive, proving the claim.

Now $g \leq f$ on $[0, 2]$ implies that $G \leq F$ and therefore that $T(x) \leq x$ on $[0, 2]$. In particular, the claimed inequality $T(x)T'(x) \leq x$ would be a consequence of $T' \leq 1$, or $g \leq f(T)$. Since $g \leq f$ this is implied by $f(T) \geq f$, which holds since f is decreasing and $T(x) \leq x$.

• For $x \geq 2$, since $T'(x)T(x)/x = g(x)/f(T(x))$ by the Monge-Ampère Equation, the thesis $T(x)T'(x) \leq x$ is equivalent to saying that $g(x) \leq f(T(x))$, $x \geq 2$. Using the explicit expressions of F and G given in (7), this amounts to proving that

$$g(x) \leq J_1^2(x) + J_0^2(x), \qquad x \geq 2.$$

This trivially holds, since for $x \geq 2$, $g(x) = \frac{4}{x^2} J_1^2(x) \leq J_1^2(x)$. □

Lemma 3.4. *For all $x \geq 0$, it holds*

$$\int_0^x g(t)tdt = 2 - 2(J_1^2(x) + J_0^2(x)).$$

Proof. Observe that $J_1 = -J_0'$ and that J_0 is satisfying the following equation $J_0''(t) + J_0(t) = J_0'(t)/t$. Therefore

$$\frac{J_1^2(t)}{t} = \frac{J_0'(t)^2}{t} = -J_0''(t)J_0'(t) - J_0'(t)J_0(t)$$

from which the result follows by integration. □

Remark 3.5. We comment on Inequality (6). For $m \geq 3$, observe that $\int_0^\infty e^{-y^2/2} y^{m-1} dy = 2^{\frac{m-2}{2}} \Gamma(m/2)$. Therefore, after change of variable

$$\int_0^\infty \left(e^{-\frac{s^2}{2m+4}}\right)^p s^{m-1} ds = \left(\frac{m+2}{p}\right)^{\frac{m}{2}} \int_0^\infty e^{-\frac{x^2}{2}} x^{m-1} dx = \frac{1}{2}\Gamma(m/2)\left(\frac{2m+4}{p}\right)^{\frac{m}{2}}.$$

In turn, Inequality (6) can be recast as

$$\int_0^\infty |j_{\frac{m}{2}}(s)|^p s^{m-1} ds \leq \frac{1}{2}\Gamma(m/2)\left(\frac{2m+4}{p}\right)^{\frac{m}{2}}.$$

Such an inequality was proved by Brzezinski, [4, Lemma 3.5], for $m \geq 3$ integer and $p > p_o > 2$ for some well defined p_o. His proof relies on Nazarov-Podkorytov's lemma.

Let us mention two other inequalities of similar type. In [25, Inequality (10)] the authors mention the following one

$$\int_0^\infty |j_{\frac{m}{2}}(s)|^p s^{m-1} ds \leq \left(\int_0^\infty |j_{\frac{m}{2}}(s)|^2 s^{m-1} ds\right) \frac{2^{m/2}}{p^{m/2}}, \qquad p \geq 2.$$

They suggest that this might hold iff $1 \leq m \leq 2$. They also mention that "König noticed that it is false for $m = 3, 4, \ldots$"...

On the other hand, Dirksen [10, Theorem 4] proved the following inequality

$$\int_0^\infty |j_{\frac{m}{2}}(s)|^p s^{m-1} ds \leq \sqrt{\frac{\pi(m+2)}{2p}}, \qquad p \geq 2, \quad m \geq 2 \text{ integer}.$$

It should be clear that the latter is different from (6). In fact, it is equivalent to saying that

$$\int_0^\infty |j_{\frac{m}{2}}(s)|^p s^{m-1} ds \leq \int_0^\infty \left(e^{-\frac{x^2}{m+2}}\right)^p dx, \qquad p \geq 2, \quad m \geq 2 \text{ integer}.$$

The difference between the latter and (6) is coming from the absence of the factor s^{m-1} in the integral in the right hand side. Dirksen's proof uses Nazarov-Podkorytov's lemma (for $m = 2$ and $m \geq 5$), the case $m = 3, 4$ uses the techniques of [25]).

The above inequalities are related to convex geometry (slices of cylinders, volume estimates etc.). Their proofs are quite involved and we hope that the techniques developed in Lemma 2.12 could help for smoother approaches.

3.3. Discrete analog of Ball's integral inequality: proof of Theorem 1.7

We recast the statement of Theorem 1.7, which can be considered a discrete analog of Ball's integral inequality, as L^p-norm comparison. That is, for $p \geq 2$, and $2 \leq n \in \mathbb{N}$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\sin(n\pi x)}{n \sin \pi x}\right|^p dx < \int_{-\infty}^{\infty} \left(e^{-\pi(n^2-1)x^2/2}\right)^p dx = \sqrt{\frac{2}{p(n^2-1)}}. \tag{8}$$

The proof of the theorem uses Lemma 1.6. Before moving to it, let us explain why (8) is stronger than Ball's integral inequality and give a Corollary on discrete slicing.

Using the substitution $u = nx$, and $|\sin x| \leq |x|$, it holds

$$\int_{-1/2}^{1/2} \left|\frac{\sin(n\pi x)}{n \sin(\pi x)}\right|^p dx = \int_{-n/2}^{n/2} \left|\frac{\sin(\pi x)}{n \sin(\pi x/n)}\right|^p \frac{dx}{n} \geq \int_{-n/2}^{n/2} \left|\frac{\sin(\pi x)}{\pi x}\right|^p \frac{dx}{n}$$

Therefore, (8) implies that

$$\sqrt{\frac{n^2-1}{n^2}} \int_{-n/2}^{n/2} \left|\frac{\sin(\pi x)}{\pi x}\right|^p dx \leq \sqrt{\frac{2}{p}} = \int_{-\infty}^{\infty} e^{-p\pi x^2/2} dx,$$

which yields Ball's inequality with $n \to \infty$.

Corollary 3.6 *(Discrete slicing). For $k_i \in \mathbb{Z}$ and $1 \le l_i \in \mathbb{Z}$, the rectangular subset of $\mathbb{Z}^n$, $L = [\![k_1, k_1 + l_1 - 1]\!] \times \cdots \times [\![k_n, k_n + l_n - 1]\!]$ satisfies*

$$\#\left\{z \in L : \sum_{i=1}^{n} z_i = k\right\} < \sqrt{2}\frac{\prod_{i=1}^{n} l_i}{\sqrt{\sum_{j=1}^{n}(l_j^2 - 1)}}$$

for any $k \in \mathbb{Z}$.

Proof. Without loss of generality, let $k_i = 1$. We will split the argument in two cases, for the first, suppose that there exists $l_{i'}$ such that

$$\sum_{j=1}^{n}(l_j^2 - 1) < 2(l_{i'}^2 - 1)$$

In this case,

$$\sqrt{2}\frac{\prod_{i=1}^{n} l_i}{\sqrt{\sum_{j=1}^{n}(l_j^2 - 1)}} > \frac{\prod_{j=1}^{n} l_j}{\sqrt{l_{i'}^2 - 1}} > \prod_{j \neq i'} l_j,$$

which is clearly larger than $\#\{z \in L : \sum_{i=1}^{n} z_i = k\}$, since for $m = \{m_j\}_{j \neq i'}$, $\{z \in L : z_j = m_j \text{ for } j \neq i', \sum_l z_l = k\}$ has at most one element.

Now we assume $\sum_{j=1}^{n}(l_j^2 - 1) \geq 2(l_{j_o}^2 - 1)$ holds for all j_o. Take X_j to be independent and uniformly distributed on $[\![1, l_j]\!]$ and set $X = X_1 + \cdots + X_n$. Then, using the triangle inequality and then Hölder's inequality with exponents p_j, $j = 1, \ldots, n$, satisfying $\sum \frac{1}{p_j} = 1$, we get

$$\begin{aligned}
\frac{\#\left\{z \in L : \sum_{j=1}^{n} z_j = k\right\}}{\prod_{j=1}^{n} l_j} &= \mathbb{P}(X = k) \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbb{E}e^{2i\pi t(X-k)} dt \\
&\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |\mathbb{E}e^{2i\pi t(X-k)}| dt \\
&\leq \prod_{j=1}^{n}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\mathbb{E}e^{2i\pi t X_j}|^{p_j}\right)^{\frac{1}{p_j}}. \qquad (9)
\end{aligned}$$

Observing that $|\mathbb{E}e^{2i\pi tX_j}| = \left|\frac{\sin(l_j\pi t)}{l_j \sin \pi t}\right|$, we set $p_j = \frac{\sum_{i=1}^n (l_i^2-1)}{l_j^2-1} \geq 2$ and apply (8) to (9) to obtain

$$\mathbb{P}(X=k) < \prod_{j=1}^n \left(\sqrt{\frac{2}{p_j(l_j^2-1)}}\right)^{\frac{1}{p_j}} = \sqrt{\frac{2}{\sum_{i=1}^n (l_i^2-1)}},$$

and the result follows. □

We note that equality can be obtained in following limit. Take $l_1 = l_2 = m$ and fixing $l_3, \dots, l_n = 1$, then

$$\#\{z \in L : \sum_{i=1}^n z_i = m+n-3\} = m$$

while

$$\frac{\prod_{j=1}^n l_j}{\sum_{i=1}^n (l_i^2-1)} = \frac{m^2}{\sqrt{2(m^2-1)}}.$$

Thus, the constant in Corollary 3.6 cannot be improved, as

$$\lim_{m\to\infty} \frac{\#\left\{z \in L : \sum_{i=1}^n z_i = k\right\}}{\frac{\prod_{i=1}^n l_i}{\sqrt{\sum_{j=1}^n (l_j^2-1)}}} = \sqrt{2}.$$

We now turn to the proof of Theorem 1.7.

Proof of Theorem 1.7. The proof of Theorem 1.7 relies on Lemma 1.6. Observe that the two relevant functions in (8) do not have equal mass[6]:

$$\int_{-1/2}^{1/2} \left(\frac{\sin(n\pi x)}{n\sin(\pi x)}\right)^2 dx = \frac{1}{n} \qquad \text{and} \qquad \int_{-\infty}^{\infty} e^{-\pi(n^2-1)x^2} = \frac{1}{\sqrt{n^2-1}}. \tag{10}$$

Therefore, we define for $x \in [0, \infty)$,

$$g(x) = \mathbb{1}_{[0,\frac{1}{2}]}(x)\left(\frac{\sin(n\pi x)}{n\sin(\pi x)}\right)^2 \qquad \text{and} \qquad f(x) = \mathbb{1}_{[0,A]}e^{-\pi(n^2-1)x^2},$$

where A is determined by the equation

[6] To see the first equality take a uniform random variable X on $\{0, 1, \dots, n-1\}$ and set $\varphi_X(t) = \mathbb{E}e^{2\pi iX}$. By Plancherel, $\frac{1}{n} = \sum_{k=0}^{n-1} P(X=k)^2 = \int_0^1 |\varphi_X(t)|^2 dt = \int_{-1/2}^{1/2} |\varphi_X(t)|^2 dt$ by periodicity, while $\varphi_X(t) = \frac{1}{n}\sum_{k=0}^{n-1} \left(e^{2\pi it}\right)^k = e^{\pi it(n-1)} \frac{\sin(\pi nt)}{n\sin(\pi t)}$.

$$\int_0^A e^{-\pi(n^2-1)x^2}dx = \frac{1}{2n}.$$

Note that $\int_0^\infty e^{-\pi(n^2-1)x^2}dx = \frac{1}{2\sqrt{n^2-1}} > \frac{1}{2n}$ so that A is well defined. Further since A is finite, upon completing the proof, we arrive at the strict inequality. By construction $\int_0^\infty g(x)dx = \int_0^\infty f(x)dx = 1/(2n)$.

Define $G : [0, 1/2] \to [0, \frac{1}{2n}]$ as $G(x) = \int_0^x g(t)dt$ and $F : [0, A] \to [0, \frac{1}{2n}]$ by $F(x) = \int_0^x f(t)dt$. Put $T = F^{-1} \circ G$ that is, by construction, the pushforward of the measure with density g onto that of density f, which satisfies the Monge-Ampère equation $g = f(T)T'$ (observe that T is increasing). Therefore, by Lemma 1.6, the Theorem will follow if we can prove that $T' \leq 1$ which holds by Lemma 3.7. This achieves the proof of the Theorem. $\square$

Lemma 3.7. *For all $x \in [0, 1/2]$, $T'(x) \leq 1$.*

As a technical preparation, we observe that,

$$\sin a \geq \frac{a}{b}\sin(b), \qquad 0 < a \leq b \leq \pi/2, \tag{11}$$

a direct consequence of the fact that $x \mapsto \frac{\sin(x)}{x}$ is decreasing on $[0, \pi/2]$.

Proof of Lemma 3.7. We need to prove that $g(x) \leq f(F^{-1}(G(x)))$ for any $x \in [0, 1/2]$. Put

$$I := \{x \in [0, 1/2] : g(x) > f(A)\}.$$

Then, for any $x \in [0, 1/2] \setminus I$, it holds $g(x) \leq f(A) \leq f(F^{-1}(G(x)))$. Therefore we only need to prove the desired inequality $g(x) \leq f(F^{-1}(G(x)))$ for $x \in I$.

For $x \in I$, composing by f^{-1}, that is decreasing, and then by F, that is increasing, $g(x) \leq f(F^{-1}(G(x)))$ is equivalent to $F(f^{-1}(g((x))) \geq G(x)$. Hence, we need to prove that

$$\int_0^x g(t)dt \leq \int_0^{f^{-1}(g(x))} f(t)dt, \tag{12}$$

for all $x \in I$. To that aim, we need to distinguish between different regimes and to proceed to further successive reductions.

We first prove the inequality for $x \in [0, 1/n] \cap I$. For this we need only to prove $f(x) \geq g(x)$, as this will give $f^{-1}(g(x)) \geq x$ since f is decreasing, and hence

$$\int_0^x g(t)dt \leq \int_0^x f(t)dt \leq \int_0^{f^{-1}(g(x))} f(t)dt.$$

The inequality $g(x) \leq f(x)$ on $(0, 1/n)$ is a direct consequence of the fact that, on $(0, 1/n)$, $\frac{\sin(n\pi x)}{n\sin(\pi x)} \leq e^{-(n^2-1)\pi^2 x^2/6}$ that we now prove. By the product expansion for sin,

$$\frac{\sin(n\pi x)}{n\sin(\pi x)} = \frac{\frac{\sin(n\pi x)}{n\pi x}}{\frac{\sin(\pi x)}{\pi x}} = \prod_{k=1}^{\infty} \frac{1 - \left(\frac{nx}{k}\right)^2}{1 - \left(\frac{x}{k}\right)^2},$$

while the identity $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, gives

$$e^{-(n^2-1)\pi^2 x^2/6} = \prod_{k=1}^{\infty} e^{-\frac{(n^2-1)x^2}{k^2}}.$$

Comparing term-wise, using $e^{-\frac{(n^2-1)x^2}{k^2}} \geq 1 - (n^2-1)\frac{x^2}{k^2}$, and writing $y = x^2/k^2$ it suffices to prove

$$(1 - (n^2 - 1)y)(1 - y) \geq 1 - n^2 y \tag{13}$$

for $y \in (0, 1/n^2)$. But this is equivalent to $(n^2 - 1)y^2 \geq 0$ so the claim follows.

Note that the above argument shows that (12) holds for $n = 2$ for all $x \in I$. We therefore deal in the sequel with $n \geq 3$. One key ingredient is the following lemma whose proof is postponed to the end of the section.

Lemma 3.8. *For $n \geq 3$, and $x \in [1/2 - 1/n, 1/2] \cap (1/n, 1/2]$ it holds*

$$f(A) \geq g(x).$$

This lemma guarantees that, for $n = 3$, $I \cap (1/3, 1/2] = \emptyset$ and therefore (12) is proved for $n = 3$ and we can assume that $n \geq 4$. The lemma also guarantees that, for any $n \geq 4$, $I \cap [1/2 - 1/n, 1/2] = \emptyset$. Therefore, it only remains to prove (12) for $n \geq 4$ in the intermediate regime $[1/n, 1/2 - 1/n] \cap I$. Observe that $[1/n, 1/2 - 1/n] = \{\frac{1}{4}\}$ for $n = 4$ so that (12) holds for $n = 4$ also, by continuity and we are left with the regime $[1/n, 1/2 - 1/n] \cap I$ for $n \geq 5$.

Assume that $n \geq 5$ and consider the regime $[1/n, 1/2 - 1/n] \cap I$. Since by construction

$$\int_0^{1/2} g(t)dt = \int_0^{A} f(t)dt = \frac{1}{2n}$$

Inequality (12) in the studied regime is equivalent to the tail inequality

$$\int_{f^{-1}(g(x))}^{A} f(t)dt \leq \int_x^{1/2} g(t)dt, \qquad x \in [1/n, 1/2 - 1/n]. \tag{14}$$

Observe that

$$\begin{aligned}\int_{f^{-1}(g(x))}^{A} f(t)dt &= \int_{f^{-1}(g(x))}^{A} \frac{2\pi(n^2-1)t}{2\pi(n^2-1)t} e^{-\pi(n^2-1)t^2} dt \\ &\leq \frac{1}{2\pi(n^2-1)f^{-1}(g(x))} \int_{f^{-1}(g(x))}^{A} 2\pi(n^2-1)te^{-\pi(n^2-1)t^2} dt \\ &= \frac{g(x)-f(A)}{2\pi(n^2-1)f^{-1}(g(x))} \\ &\leq \frac{g(x)}{2\pi(n^2-1)f^{-1}(g(x))}.\end{aligned}$$

Therefore, using the explicit expression for $f^{-1}(y) = \sqrt{\log(1/y)/(\pi(n^2-1))}$, (14) would follow from

$$\frac{g(x)}{2\sqrt{\pi(n^2-1)\log(1/g(x))}} \leq \int_{x}^{1/2} g(t)dt. \tag{15}$$

In order to bound from below $1/g(x)$, we need to distinguish between $n = 5$ and $n \geq 6$.

Let us first consider the case $n \geq 6$. We claim that, for $x \in [1/n, 1/2]$, $\frac{1}{g(x)} \geq 14$ (with the convention that $1/g(x) = \infty$ when $g(x) = 0$). Now fix $\theta \in (0, 1/2)$. If $\frac{1}{n} \leq x \leq \frac{1+\theta}{n}$, we have for $n \geq 6$

$$\begin{aligned}\frac{1}{g(x)} &= \frac{n^2\sin(\pi x)^2}{\sin(n\pi x)^2} \\ &\geq n^2\frac{\sin(\pi/n)^2}{\sin(\theta\pi)^2} && \text{(since } x \mapsto \sin^2(x) \text{ is increasing on } (0,\pi/2)\cup(\pi,3\pi/2)) \\ &\geq \frac{36\sin(\pi/6)^2}{\sin(\theta\pi)^2} && \text{(by (11) with } a = \frac{\pi}{n} \text{ and } b = \frac{\pi}{6}) \\ &= \frac{9}{\sin(\theta\pi)^2}\end{aligned}$$

Similarly, for $x \in [(1+\theta)/n, 1/2]$, it holds

$$\frac{1}{g(x)} = \frac{n^2\sin(\pi x)^2}{\sin(n\pi x)^2} \geq n^2\sin^2\left(\frac{(1+\theta)\pi}{n}\right) \geq 36\sin^2\left(\frac{(1+\theta)\pi}{6}\right).$$

Therefore,

$$\frac{1}{g(x)} \geq 9 \max_{\theta \in (0,1/2)} \min\left(\frac{1}{\sin(\theta\pi)^2}, 4\sin^2\left(\frac{(1+\theta)\pi}{6}\right)\right).$$

For $\theta = 0.295$, we obtain $\min\left(\frac{1}{\sin(\theta\pi)^2}, 4\sin^2\left(\frac{(1+\theta)\pi}{6}\right)\right) = \frac{1}{\sin(\theta\pi)^2} \simeq 1.56$ from which we deduce that $1/g(x) \geq 14$ as announced, proving the claim.

It follows that, for any $n \geq 6$,

$$2\sqrt{\pi(n^2-1)\log(1/g(x))} = 2\sqrt{\pi\log(1/g(x))}\sqrt{\frac{n^2-1}{n^2}}n \geq 2\sqrt{\pi\log(14)}\sqrt{\frac{35}{36}}n \geq 5.67n.$$

In turn (14) would follow from

$$g(x) \leq 5.67n \int_x^{1/2} g(t)dt.$$

Our aim is now to bound from below the right hand side of the latter. Using (11) and a change of variables, it holds

$$\begin{aligned}
\int_x^{1/2} g(t)dt &= \frac{1}{n^2\sin(\pi x)^2}\int_x^{1/2}\left(\frac{\sin(\pi x)}{\sin(\pi t)}\right)^2 \sin(n\pi t)^2 dt \\
&\geq \frac{1}{n^2\sin(\pi x)^2}\int_x^{1/2}\left(\frac{x}{t}\right)^2 \sin(n\pi t)^2 dt \\
&= \frac{\pi x^2}{n\sin(\pi x)^2}\int_{n\pi x}^{n\pi/2}\left(\frac{\sin s}{s}\right)^2 ds.
\end{aligned}$$

Therefore, the inequality $g(x) \leq 5.67n\int_x^{1/2} g(t)dt$ would be a consequence of

$$\sin(n\pi x)^2 \leq 5.67n^2\pi x^2 \int_{n\pi x}^{n\pi/2}\left(\frac{\sin s}{s}\right)^2 ds.$$

Set $y := n\pi x$. We need to prove that

$$\pi\sin(y)^2 \leq 5.67y^2 \int_y^{n\pi/2}\left(\frac{\sin s}{s}\right)^2 ds$$

holds for all $y \in [\pi, \frac{n\pi}{2} - \pi]$. This would be a consequence of

$$\pi \sin(y)^2 \leq 5.67 y^2 \int_y^{y+\pi} \left(\frac{\sin s}{s}\right)^2 ds, \qquad y \geq \pi.$$

Now, observe that

$$\int_y^{y+\pi} \left(\frac{\sin s}{s}\right)^2 ds \geq \frac{1}{(y+\pi)^2} \int_y^{y+\pi} \sin^2 s \, ds = \frac{\pi}{2(y+\pi)^2}.$$

Therefore, it suffices to prove that, for any $y \geq \pi$, it holds $2(y+\pi)^2 \sin^2 y \leq 5.67 y^2$, which is a consequence of $(y+\pi)|\sin y| \leq 1.68 y$ proved in Lemma 3.9 below.

As an intermediate conclusion, we established (12) for all n except $n = 5$ and we are left with proving (14) only in the regime $[1/5, 1/2 - 1/5] = [1/5, 3/10]$. Our starting point is Inequality (15). As for the case $n \geq 6$, we need to bound from below $1/g(x)$ and $\int_x^{1/2} g(t)dt$. Using similar arguments, we have for $\theta \in (0, 1/2)$ and $x \in [1/5, (1+\theta)/5]$,

$$\frac{1}{g(x)} = \frac{25 \sin(\pi x)^2}{\sin(5\pi x)^2} \geq \frac{25 \sin(\pi/5)^2}{\sin(\theta\pi)^2}$$

and for $x \in [(1+\theta)/5, 3/10]$,

$$\frac{1}{g(x)} = \frac{25 \sin(\pi x)^2}{\sin(5\pi x)^2} \geq 25 \sin^2\left(\frac{(1+\theta)\pi}{5}\right).$$

Therefore,

$$\frac{1}{g(x)} \geq 25 \max_{\theta \in (0,1/2)} \min\left(\frac{\sin(\pi/5)^2}{\sin(\theta\pi)^2}, \sin^2\left(\frac{(1+\theta)\pi}{5}\right)\right).$$

For $\theta = 0.299$, the above minimum equals $\frac{\sin(\pi/5)^2}{\sin(\theta\pi)^2}$ from which we deduce that $1/g(x) \geq 13.25$.

It follows that

$$\frac{g(x)}{2\sqrt{\pi(n^2-1)\log(1/g(x))}} \leq \frac{\sin(5\pi x)^2}{\sin(\pi x)^2} \frac{1}{25 \times 2\sqrt{24\pi \log(13.25)}} \leq \frac{\sin(5\pi x)^2}{25 \times 27.9 \sin(\pi x)^2}.$$

On the other hand,

$$\int_x^{1/2} g(t)dt \geq \frac{1}{25} \int_x^{x+\frac{1}{5}} \sin(5\pi t)^2 dt = \frac{1}{250}.$$

Therefore, (15) would be a consequence of $\sin(5\pi x)^2 \leq 2.79 \sin(\pi x)^2$, $x \in [1/5, 3/10]$. Setting $y = \pi x$ and taking the root (note that $5y \in [\pi, 3\pi/2]$), it is enough to prove

$$-\sin(5y) \leq 1.67 \sin y, \qquad y \in [\pi/5, 3\pi/10].$$

As a last reduction, we set $z = \sin(y)$ and use that $\sin(5y) = 16z^5 - 20z^3 + 5z$ so that the latter can be recast as $16z^4 - 20z^2 + 6.67 \geq 0$ for all $z \in [\sin(\pi/5), \sin(3\pi/10)]$. But the second order polynomial $16X^2 - 20X + 6.67$ is always positive since its discriminant is negative, ending the proof of the lemma provided we can prove Lemma 3.8 and Lemma 3.9. □

In the proof of Lemma 3.7 we used the following lemma.

Lemma 3.9. *For all $y \geq \pi$, it holds*

$$(y + \pi)|\sin y| \leq 1.68y. \tag{16}$$

Proof. For $y \geq 3\pi/2$, we have

$$\frac{y + \pi}{y} \leq \frac{5}{3} \leq 1.68,$$

from which (16) follows. Therefore, it remains to prove (16) for $\pi \leq y \leq 3\pi/2$. Let

$$H(y) := 1.68y - (y + \pi)|\sin y| = 1.68y + (y + \pi)\sin y, \qquad y \in (\pi, 3\pi/2).$$

The successive derivatives are

$$H'(y) = 1.68 + \sin y + (y + \pi)\cos y, \qquad H''(y) = 2\cos y - (y + \pi)\sin y$$

and

$$H'''(y) = -3\sin y - (y + \pi)\cos y.$$

Notice that, for $y \in (\pi, 3\pi/2)$, $\cos y, \sin y \leq 0$ so that $H''' > 0$ and H'' is increasing. Since $H''(\pi) = -2$ and $H''(3\pi/2) = 5\pi/2$, H'' changes sign from $-$ to $+$ at a unique point y_o. Therefore H' is decreasing on (π, y_o) and increasing on $(y_o, 3\pi/2)$. Since $H'(\pi) = -0.32$ and $H'(3\pi/2) = 0.68$ we can conclude that H' changes sign from $-$ to $+$ at a point $y_1 > y_o$ and that H has a unique minimum at y_1. Therefore, the thesis will follow if we can prove that $H(y_1) \geq 0$.

Now by construction $H'(y_1) = 0$. Hence

$$1.68 \sin y_1 + \sin^2 y_1 + (y_1 + \pi)\sin y_1 \cos y_1 = 0$$

from which we deduce that

$$(y_1 + \pi)\sin y_1 = -\frac{1}{\cos y_1}\left(1.68 \sin y_1 + \sin^2 y_1\right)$$

(note that $\pi < y_1 < 3\pi/2$ so that $\cos y_1 \neq 0$). In turn

$$H(y_1) = \frac{1}{\cos y_1}\left(1.68 y_1 \cos y_1 - 1.68 \sin y_1 - \sin^2 y_1\right)$$

Since $\cos y_1 < 0$, $H(y_1) \geq 0$ amounts to proving that $G(y) := 1.68y\cos y - 1.68\sin y - \sin^2 y$, $y \in (\pi, 3\pi/2)$, satisfies $G(y_1) \leq 0$.

Since $G'(y) = -\sin y(1.68y + 2\cos y)$, and since $1.68y + 2\cos y \geq 0$ on $(\pi, 3\pi/2)$ (a consequence of the fact that $y \mapsto 1.68y+2\cos y$ is increasing on $(\pi, 3\pi/2)$), G is increasing. Therefore, to prove that $G(y_1) \leq 0$, it is enough to find $y^* \geq y_1$ with $G(y^*) \leq 0$.

To that aim, take $y^* := 4.6244$. Observe that $H'(y^*) \simeq 0.0014$ so that, since H' changes sign from $-$ to $+$ and $H'(y_1) = 0$, necessarily $y^* \geq y_1$. Since $G(y^*) \simeq -0.0015$ the lemma is proved. □

Next we prove Lemma 3.8.

Proof of Lemma 3.8. We distinguish different cases.

- $n \geq 6$. For $x \in [1/2 - 1/n, 1/2]$, $g(x) = \frac{\sin^2(n\pi x)}{n^2 \sin^2(\pi x)} \leq \frac{1}{n^2 \sin^2(\pi(1/2-1/n))} = \frac{1}{n^2 \cos^2(\pi/n)}$. Thus by Lemma 3.10 below it suffices to prove

$$\frac{1}{6}\frac{1}{n^2\cos^2(\pi/n)} + \frac{1}{2}\left(\frac{1}{n^2\cos^2(\pi/n)}\right)^{\frac{4}{3}} \leq \frac{1}{2n^2},$$

  or, rearranging,

$$\frac{1}{6} + \frac{1}{2}\frac{1}{(n\cos(\pi/n))^{\frac{2}{3}}} \leq \frac{1}{2}\cos^2(\pi/n).$$

  Note that the left hand side is decreasing in n, while the right hand side is increasing in n, so to prove the result for all $n \geq 6$, one needs only check the $n = 6$ case, which can be evaluated exactly,

$$\frac{1}{6} + \frac{1}{2}\frac{1}{(6\cos(\pi/6))^{\frac{2}{3}}} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \leq \frac{3}{8} = \frac{1}{2}\cos^2(\pi/6).$$

- $n = 3$. When $n = 3$, a derivative computation shows that the maximum of $g(x)$ on $[1/3, 1/2]$ occurs when $x = 1/2$, with $g(1/2) = \frac{1}{9}$. Thus to finish this case we need only prove $f(A) \leq \frac{1}{9}$ or by Lemma 3.10

$$\frac{1}{6\times 9} + \frac{1}{2\times 9^{4/3}} < 0.046 < 0.55 < \frac{1}{2\times 3^2}.$$

- $n = 4$. In this case $g'(x) = -\frac{1}{2}\pi\sin(4\pi x)\left(3\cos(2\pi x) + 2\right)$ shows that g takes its maximum on $[1/4, 1/2]$ at $x_0 = \arccos(-2/3)/(2\pi)$. Applying half and double angle formulas for sin and $\sin(\arccos(x)) = \sqrt{1-x^2}$ yields $g(x_0) = \frac{2}{27}$. Similarly to prove that $f(A) \leq g(x)$ on $(1/4, 1/2)$ by Lemma 3.10 we need to check

$$\frac{2}{6 \times 27} + \frac{2^{4/3}}{2 \times 27^{4/3}} \leq \frac{1}{2 \times 4^2}.$$

This can be checked by hand or by numerical approximation where $\frac{2}{6\times 27} + \frac{2^{4/3}}{2\times 27^{4/3}} < 0.028$ and $\frac{1}{2\times 4^2} = 0.03125$.

- $n = 5$. In this case, the maximum of g on $[2/5, 1/2]$ on occurs at $x_0 = 1/2$, where $g(1/2) = 1/25$. On $[1/5, 2/5]$, g takes its maximum at $x_0 = \frac{2}{\pi} \arctan\left(\sqrt{\frac{11-4\sqrt{6}}{5}}\right)$, with $g(x_0) = \frac{1}{16}$. Thus by Lemma 3.10 we need only check

$$\frac{1}{\sqrt{\pi} 16 \sqrt{\log(16)}} \left(1 - \frac{1}{2\log(16)} + \frac{3}{4\log(16)^2}\right) \leq \frac{1}{2 \times 5^2}$$

The result follows since the left hand side approximately equals 0.019 and the right hand side equals 0.02.

This achieves the proof. □

Lemma 3.10. *Given $I \subset [0, 1/2]$, set $g^* := \sup_{x\in I} g(x)$. Then, to prove $f(A) \geq g(x)$ on I, it suffices to prove*

$$\frac{g^*}{6} + \frac{g^{*4/3}}{2} \leq \frac{1}{2n^2} \tag{17}$$

provided $g^* \leq e^{-1/4}$, or to prove

$$\frac{g^*}{\sqrt{\pi}\sqrt{\log(1/g^*)}} \left(1 - \frac{1}{2\log(1/g^*)} + \frac{3}{4\log(1/g^*)^2}\right) \leq \frac{1}{2n^2}. \tag{18}$$

Proof. The definition of A, $\frac{1}{2n} = \int_0^A e^{-\pi(n^2-1)x^2} dx$, can be written through change of variables $y = \sqrt{\pi(n^2-1)}x$ as

$$\mathrm{Erf}(\sqrt{\pi(n^2-1)}A) = \sqrt{1 - \frac{1}{n^2}}$$

where $\mathrm{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ is the error function. We denote its inverse function Erf^{-1} and can give an explicit expression for A,

$$A = \frac{\mathrm{Erf}^{-1}\left(\sqrt{1-\frac{1}{n^2}}\right)}{\sqrt{\pi(n^2-1)}}.$$

Thus, the relation

$$f(A) = e^{-\pi(n^2-1)A^2} = e^{-\operatorname{Erf}^{-1}\left(\sqrt{1-\frac{1}{n^2}}\right)^2} \geq g(x),$$

can be rearranged as

$$\operatorname{Erfc}\left(\sqrt{\log\frac{1}{g(x)}}\right) \leq 1 - \sqrt{1-\frac{1}{n^2}}.$$

Since

$$\frac{1}{2n^2} \leq 1 - \sqrt{1-\frac{1}{n^2}}$$

it is enough to prove that

$$\operatorname{Erfc}\left(\sqrt{\log\frac{1}{g(x)}}\right) \leq \frac{1}{2n^2}.$$

The first claim follows from the following upper bound for the Erfc function which is popular in Engineering (see *e.g.* [7]) valid for $x \geq 1/2$,

$$\operatorname{Erfc}(x) \leq \frac{e^{-x^2}}{6} + \frac{e^{-4x^2/3}}{2}.$$

For the second claim, we use successive integration by parts to get that, for $x > 0$,

$$\begin{aligned}
\frac{\sqrt{\pi}}{2}\operatorname{Erfc}(x) &= \int_x^\infty e^{-t^2}dt = \int_x^\infty \frac{-2te^{-t^2}}{-2t}dt = \frac{e^{-x^2}}{2x} - \frac{1}{2}\int_x^\infty \frac{e^{-t^2}}{t^2}dt \\
&= \frac{e^{-x^2}}{2x} - \frac{1}{4}\frac{e^{-x^2}}{x^3} + \frac{3}{4}\int_x^\infty \frac{e^{-t^2}}{t^4}dt \\
&= \frac{e^{-x^2}}{2x} - \frac{1}{4}\frac{e^{-x^2}}{x^3} + \frac{3}{8}\frac{e^{-x^2}}{x^5} - \frac{15}{8}\int_x^\infty \frac{e^{-t^2}}{t^6}dt \\
&\leq \frac{e^{-x^2}}{2x} - \frac{1}{4}\frac{e^{-x^2}}{x^3} + \frac{3}{8}\frac{e^{-x^2}}{x^5}
\end{aligned}$$

from which the second claim follows. $\square$

Data availability

No data was used for the research described in the article.

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