MINIMUM ENTROPY OF A LOG-CONCAVE VARIABLE WITH FIXED VARIANCE

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Abstract. We show that for log-concave real random variables with fixed variance the Shannon differential entropy is minimized for an exponential random variable. We apply this result to derive upper bounds on capacities of additive noise channels with log-concave noise. We also improve constants in the reverse entropy power inequalities for log-concave random variables.

1. Introduction

For a real random variable X with density f its differential entropy is defined via the formula $h(X) = h(f) = -\int f \log f$. This definition goes back to the celebrated work of Shannon [13], but the same quantity was also considered, without the minus sign, by physicists, including Boltzmann, in the context of thermodynamics of gases. In fact, it is a classical fact going back to Boltzmann [4] that under fixed variance the entropy is maximized for a Gaussian random variable. This leads to the translation and scale invariant inequality

$$h(X) \leq \frac{1}{2}\log \mathrm{Var}(X) + \frac{1}{2}\log(2\pi e).$$

One can see that in general one cannot hope for a reverse bound, since for the density $f_\varepsilon(x) = (2\varepsilon)^{-1}\mathbf{1}_{[1,1+\varepsilon]}(|x|)$ the variance stays bounded while the entropy goes to $-\infty$ as $\varepsilon \to 0^+$. However, a reverse bound still holds if one imposes some extra assumption on X, such as log-concavity. Recall that X is said to be log-concave if its density is of the form $f = e^{-V}$, where $V : \mathbb{R} \to (-\infty, \infty]$ is convex. In [3] Bobkov and Madiman showed that indeed in this class, the inequality can be reversed, up to an absolute additive constant and it became a well-known open problem to find the optimal such bound. The sharpest inequality to date can be found in [10] where Marsiglietti and Kostina proved that if X is log-concave, then $h(X) \geq \frac{1}{2}\log \mathrm{Var}(X) + \log 2$. The goal of this article is to prove the following inequality.

Theorem 1.1. *For a log-concave random variable X we have*

$$h(X) \geq \frac{1}{2}\log \mathrm{Var}(X) + 1$$

with equality for the standard one-sided exponential random variable with density $e^{-x}\mathbf{1}_{[0,\infty)}(x)$.

Probably the most significant generalization of entropy is the so-called Rényi entropy of order $\alpha \in (0,\infty) \setminus \{1\}$, which is defined as

$$h_\alpha(X) = h_\alpha(f) = \frac{1}{1-\alpha}\log\left(\int f^\alpha(x)\mathrm{d}x\right),$$

assuming that the integral converges, see [12]. If $\alpha \to 1$ one recovers the usual Shannon differential entropy $h(f) = h_1(f) = -\int f \ln f$. Also, by taking limits one can put $h_0(f) =$

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$\log|\mathrm{Supp} f|$, where $\mathrm{Supp} f$ stand for the support of f and $h_\infty(f) = -\log\|f\|_\infty$, where $\|f\|_\infty$ is the essential supremum of f. For $p \geq q > 0$ one has

$$0 \leq h_q(f) - h_p(f) \leq \frac{\log q}{q-1} - \frac{\log p}{p-1},$$

where the fraction $\frac{\log t}{t-1}$ is interpreted as 1 for $t=1$, see [5]. Thus using this bound together with Theorem 1.1 gives the following corollary.

Corollary 1.2. *For $\alpha > 1$ and a log-concave random variable X one has*

$$h_\alpha(X) \geq \frac{1}{2}\log \mathrm{Var}(X) + \frac{\log \alpha}{\alpha - 1}.$$

Let us mention that the problem of minimizing Rényi entropy under fixed variance for symmetric log-concave random variables was solved in [8] for the case $\alpha \leq 1$ and in [2] for $\alpha > 1$. Here the worst case in the uniform random variable on an interval for $\alpha \leq \alpha^*$ and symmetric exponential random variable for $\alpha > \alpha^*$, where $\alpha^* \approx 1.241$ is the solution to the equation $\frac{\log \alpha^*}{\alpha^* - 1} = \frac{1}{2}\log 6$.

2. Applications

2.1. **Additive noise channels.** We now briefly discuss an application of our main result in the context of information theory. For more details we refer the reader to [9], where the case of symmetric log-concave random variables was discussed. Consider the memoryless transmission channel with power budged P subject to additive noise N, that is, if the input of the channel is X, then output produced by the channel is $Y = X + N$, where N is the noise independent of X. Shannon's celebrated channel coding theorem [13] asserts that the so-called *capacity* of such a channel is gives by the formula

$$C_P(N) = \sup_{X:\ \mathrm{Var}(X) \leq P} (h(X+N) - h(N)).$$

We have the following fact.

Proposition 2.1. *Let N be a random variable with finite variance and let Z be a centered Gaussian random variable with the same variance. Then*

$$C_P(Z) \leq C_P(N) \leq C_P(Z) + D(N),$$

where $D(N) = h(Z) - h(N)$ is the relative entropy of N from Gaussianity.

Our Theorem 1.1 gives

$$D(N) = h(Z) - h(N) \leq h(Z) - \frac{1}{2}\log \mathrm{Var}(N) - 1 = \frac{1}{2}\log(2\pi e) - 1 = \frac{1}{2}\log\left(\frac{2\pi}{e}\right).$$

We can therefore formulate the following corollary.

Corollary 2.2. *Let N be a log-concave noise and let Z be a centered Gaussian noise with the same variance. Then*

$$C_P(Z) \leq C_P(N) \leq C_P(Z) + \frac{1}{2}\log\left(\frac{2\pi}{e}\right).$$

It other words, using an arbitrary log-concave noise instead of the Gaussian does not increase capacity by more than $\frac{1}{2}\log\left(\frac{2\pi}{e}\right) < 0.42$ nats.

2.2. **Reverse EPI.** Recall that the entropy power of a random variable X is defined by $\mathcal{N}(X) = \frac{1}{2\pi e}\exp(2h(N))$. Note that for a Gaussian random variable one has $\mathcal{N}(Z) = \mathrm{Var}(Z)$ and in general $\mathrm{Var}(X) = \mathrm{Var}(Z)$ implies $\mathcal{N}(X) \le \mathcal{N}(Z)$. Theorem 1.1 can be rewritten in the form $\mathcal{N}(X) \ge \frac{e}{2\pi}\mathrm{Var}(X)$. The entropy power inequality of Shannon and Stam [13, 14] states that for independent random variables X, Y one has

$$\mathcal{N}(X+Y) \ge \mathcal{N}(X) + \mathcal{N}(Y).$$

It is of interest to obtain reverse bounds. Under log-concavity assumption the authors of [10] showed that if X, Y are uncorellated, then $\mathcal{N}(X+Y) \le \frac{\pi e}{2}(\mathcal{N}(X) + \mathcal{N}(Y))$. Using Theorem 1.1 we can improve this result.

Corollary 2.3. *Let X, Y be log-concave uncorrelated real random variables. Then*

$$\mathcal{N}(X+Y) \le \frac{2\pi}{e}\left(\mathcal{N}(X) + \mathcal{N}(Y)\right).$$

Indeed, one has

$$\mathcal{N}(X+Y) \le \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) \le \frac{2\pi}{e}\left(\mathcal{N}(X) + \mathcal{N}(Y)\right).$$

We can also define the Rényi entropy power via $\mathcal{N}_\alpha(X) = \exp(2h_\alpha(X))$, where we removed the normalizing constant $2\pi e$ for simplicity. Similarly as in [2], Theorem 2 from [7] together with our Theorem 1.1 gives the inequality

$$C_-(\alpha)\,\mathrm{Var}(X) \le N_\alpha(X) \le C_+(\alpha)\,\mathrm{Var}(X), \qquad \alpha > 1 \tag{1}$$

where

$$C_-(\alpha) = \alpha^{\frac{2}{\alpha-1}}, \qquad\qquad C_+(\alpha) = \frac{3\alpha-1}{\alpha-1}\left(\frac{2\alpha}{3\alpha-1}\right)^{\frac{2}{1-\alpha}} B\left(\frac{1}{2}, \frac{\alpha}{\alpha-1}\right)^2.$$

Here $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ stands for the Beta function. The right inequality does not need log-concavity. Thus, using the same computation as for the case $\alpha = 1$ we get the following corollary.

Corollary 2.4. *If X, Y are log-concave uncorrelated real random variables, then for $\alpha > 1$ one has*

$$\mathcal{N}_\alpha(X+Y) \le \frac{C_+(\alpha)}{C_-(\alpha)}\left(\mathcal{N}_\alpha(X) + \mathcal{N}_\alpha(Y)\right).$$

3. Reductions

3.1. Decreasing Rearrangement.

Definition 3.1 (Decreasing Rearrangement)**.** *For a measurable set $A \subseteq \mathbb{R}$ let $|A|$ denote its Lebesgue measure and let us define*

$$A^{\downarrow} = (0, |A|)$$

with the interpretation of $(0,0)$ as the empty set.

Definition 3.2. *For a measurable function $f : \mathbb{R} \to [0,\infty]$, define $f^{\downarrow} : (0,\infty) \to [0,\infty]$*

$$f^{\downarrow}(x) = \int_0^\infty \mathbb{1}_{\{y : f(y) > \lambda\}^{\downarrow}}(x)\, d\lambda.$$

We observe that $f^{\downarrow}$ is fully characterized by the equality $\{f > \lambda\}^{\downarrow} = \{f^{\downarrow} > \lambda\}$ and in particular, equimeasurable (with respect to the Lebesgue measure) functions possess identical decreasing rearrangements.

Proposition 3.3. *For $f : \mathbb{R} \to [0, \infty)$, its rearrangement $f^{\downarrow}$ satisfies,*

$$f^{\downarrow}(x) = \sup\{\lambda : x \in \{f > \lambda\}^{\downarrow}\} \tag{2}$$

and

$$\{f^{\downarrow} > \lambda\} = \{f > \lambda\}^{\downarrow}. \tag{3}$$

Proof. Representation (2) follows directly from the definition of $f^{\downarrow}$. To prove (3) observe that by (2) the condition $f^{\downarrow}(x) > \lambda$ is equivalent to the existence of $\lambda' > \lambda$ such that $x \in \{f > \lambda'\}^{\downarrow}$. It is therefore enough to prove the following equivalence

$$\exists_{\lambda' > \lambda} \;\; x \in \{f > \lambda'\}^{\downarrow} \quad \iff \quad x \in \{f > \lambda\}^{\downarrow}.$$

By the nestedness the implication $\implies$ is trivial. To show the other direction assume that $x \in \{f > \lambda\}^{\downarrow}$. This is equivalent to $x < |\{f > \lambda\}|$. By the continuity of Lebesgue measure

$$x < |\{f > \lambda\}| = \left| \bigcup_n \left\{ f > \lambda + \frac{1}{n} \right\} \right| = \lim_n |\{f > \lambda + 1/n\}|.$$

By taking $\lambda' = \lambda + \frac{1}{n}$ for large enough n we get $\lambda' > \lambda$ such that $x \in \{f > \lambda'\}^{\downarrow}$. □

Proposition 3.4. *For φ measurable and f non-negative,*

$$\int \varphi(f(x))dx = \int \varphi(f^{\downarrow}(x))dx.$$

Proof. This is equivalent to the statement that f pushes the Lebesgue measure dm forward to the same measure $f^{\downarrow}$ pushes the Lebesgue measure to. Thus it suffices to check

$$f_{\#}m(\lambda, \infty) = f^{\downarrow}_{\#}m(\lambda, \infty).$$

This is just $|\{f > \lambda\}| = |\{f^{\downarrow} > \lambda\}|$, which follows from the characterization $\{f^{\downarrow} > \lambda\} = \{f > \lambda\}^{\downarrow}$. □

Note that taking $\varphi(x) = -x \log x$ shows that the decreasing rearrangement $f^{\downarrow}$ preserves the entropy of a density function f.

Definition 3.5. *For a random variable X with density function f, define $X^{\downarrow}$ to be a random variable drawn from the density function $f^{\downarrow}$.*

When X has density f we will write the variance as $\mathrm{Var}(X)$ and $\mathrm{Var}(f)$ interchangeably.

Lemma 3.6. *For non-negative X, and increasing, non-negative φ,*

$$\mathbb{E}\varphi(X^{\downarrow}) \leq \mathbb{E}\varphi(X).$$

Proof. The proof follows from the elementary observation that for $a \geq 0$,

$$|(a, \infty) \cap A| \geq \left|(a, \infty) \cap A^{\downarrow}\right|. \tag{4}$$

Using the layer-cake decomposition,

$$\begin{aligned} \mathbb{E}\varphi(X) &= \int_0^\infty \int_0^\infty \left(\int_0^\infty \mathbb{1}_{\{\varphi > \lambda\}}(x) \mathbb{1}_{\{f > \sigma\}}(x) dx \right) d\lambda d\sigma \\ &= \int_0^\infty \int_0^\infty |\{\varphi > \lambda\} \cap \{f > \sigma\}| \, d\lambda d\sigma \\ &\geq \int_0^\infty \int_0^\infty \left| \{\varphi > \lambda\} \cap \{f^{\downarrow} > \sigma\} \right| d\lambda d\sigma = \mathbb{E}\varphi(X^{\downarrow}), \end{aligned}$$

where the inequality follows from (4). □

Lemma 3.7. *For probability densities f_0, f_1 and $\lambda \in [0,1]$ the density $f = \lambda f_0 + (1-\lambda) f_1$ has variance,*

$$\mathrm{Var}(f) = \lambda \mathrm{Var}(f_0) + (1-\lambda)\mathrm{Var}(f_1) + \lambda(1-\lambda)(\mu_1 - \mu_0)^2$$

where μ_i denotes the barycenter of f_i, that is $\mu_i = \int x f_i(x) dx$.

Proof.

$$\begin{aligned}
\mathrm{Var}(f) =& \int x^2(\lambda f_0(x) + (1-\lambda) f_1(x)) dx - \left(\int x(\lambda f_0(x) + (1-\lambda) f_1(x)) dx \right)^2 \\
=& \lambda \left(\int x^2 f_0(x) dx - \left(\int x f_0(x) dx \right)^2 \right) + (1-\lambda) \left(\int x^2 f_1(x) dx - \left(\int x f_1(x) dx \right)^2 \right) \\
&+ \lambda \left(\int x f_0(x) dx \right)^2 + (1-\lambda) \left(\int x f_1(x) dx \right)^2 - \left(\int x(\lambda f_0(x) + (1-\lambda) f_1(x)) dx \right)^2 \\
=& \lambda \mathrm{Var}(f_0) + (1-\lambda)\mathrm{Var}(f_1) + \lambda(1-\lambda)(\mu_1 - \mu_0)^2.
\end{aligned}$$

□

Theorem 3.8. *For X log-concave,*

$$\mathrm{Var}(X) \leq \mathrm{Var}(X^{\downarrow}).$$

Proof. We prove the result when X has a density f given by a unimodal step function by induction, that is $f = \sum_{k=0}^{n} \lambda_k \mathbb{1}_{I_k}/|I_k|$ with I_k intervals satisfying $I_{k+1} \subsetneq I_k$ and $\lambda_k > 0$. An easy limiting argument gives the result for log-concave X. When $n = 0$, X is uniform and the result is immediate. Assuming the result for $n' < n$, we proceed. The density of f can be written as $\lambda f_0 + (1-\lambda) f_1$, with $\lambda = \lambda_0$, $f_0 = \frac{\mathbb{1}_{I_0}}{|I_0|}$, and $f_1 = \sum_{k=1}^{n} \frac{\lambda_k}{1-\lambda_0} \frac{\mathbb{1}_{I_k}}{|I_k|}$. Observing that f_1 takes strictly less values than f and that by affine invariance of the inequality we may assume that $I_0 = (0,1)$, and by considering $\tilde{X} = 1 - X$, we may assume without loss of generality, that $\int x f_1(x) dx \leq \frac{1}{2}$.

By Lemma 3.7,

$$\mathrm{Var}(f) = \lambda \mathrm{Var}(f_0) + (1-\lambda)\mathrm{Var}(f_1) + \lambda(1-\lambda) \left(\frac{1}{2} - \int x f_1(x) dx \right)^2.$$

Observe that $f^{\downarrow} = \lambda f_0^{\downarrow} + (1-\lambda) f_1^{\downarrow}$. Applying Lemma 3.7 to $f^{\downarrow}$,

$$\mathrm{Var}(f^{\downarrow}) = \lambda \mathrm{Var}(f_0^{\downarrow}) + (1-\lambda)\mathrm{Var}(f_1^{\downarrow}) + \lambda(1-\lambda) \left(\frac{1}{2} - \int x f_1^{\downarrow}(x) dx \right)^2.$$

Clearly $f_0^{\downarrow} = f_0$. By induction $\mathrm{Var}(f_1) \leq \mathrm{Var}(f_1^{\downarrow})$ and by Lemma 3.6, applied to $\varphi(x) = x$, we have $\frac{1}{2} \geq \int x f_1(x) dx \geq \int x f_1^{\downarrow}(x) dx$. The result follows. □

Lemma 3.9. *For X log-concave,*

$$\frac{e^{2h(X)}}{\mathrm{Var}(X)} \geq \frac{e^{2h(X^{\downarrow})}}{\mathrm{Var}(X^{\downarrow})}$$

This follows immediately from Theorem 3.8 and Proposition 3.4 (with $\varphi(x) = -x \log x$) since decreasing rearrangement will preserve entropy and increase variance. The following result, a special case of a result from [11], will thus allow us to reduce our problem to X log-concave with monotone decreasing density.

Proposition 3.10 ([11] Proposition 7.5.8)**.** *For X log-concave, $X^{\downarrow}$ is log-concave as well.*

Proof. Note that X has a log-concave density f if and only if

$$(1-t)\{f>\lambda_1\}+t\{f>\lambda_2\}\subseteq\{f>\lambda_1^{1-t}\lambda_2^t\}. \tag{5}$$

Thus, to show that $X^{\downarrow}$ is log-concave it suffices to prove

$$(1-t)\{f>\lambda_1\}^{\downarrow}+t\{f>\lambda_2\}^{\downarrow}\subseteq\{f>\lambda_1^{1-t}\lambda_2^t\}^{\downarrow}$$

But since both sets are open intervals it suffices to prove that the right hand side has bigger volume, which follows Brunn-Minkowski inequality and the set theoretic inclusion in (5),

$$\begin{aligned}|(1-t)\{f>\lambda_1\}^{\downarrow}+t\{f>\lambda_2\}^{\downarrow}| &= |(1-t)\{f>\lambda_1\}+t\{f>\lambda_2\}| \\ &\le |\{f>\lambda_1^{1-t}\lambda_2^t\}| \\ &= |\{f>\lambda_1^{1-t}\lambda_2^t\}^{\downarrow}|.\end{aligned}$$

□

3.2. **Degrees of freedom.** Our goal is to show that in order to prove Theorem 1.1 it suffices to consider functions of the form e^{-V} where V is a two-piece affine on a finite interval. This is a standard argument appearing e.g. in [8], so we only sketch it.

Step 1. Let $\sigma^2=\mathrm{Var}(X)$. By the previous subsection, in order to prove the inequality $h(f)\ge\log(e\sigma)$ it suffices to consider non-increasing densities on $[0,\infty)$. In fact by an approximation argument we can assume that the support of f is a finite interval. Indeed if we define $f_n=e^{-V}\mathbb{1}_{(0,n)}/c_n$, where $c_n=\int_0^n e^{-V}$ is the normalizing constant, then clearly $c_n\to1$ as $n\to\infty$ and by the Lebesgue dominated convergence theorem $\mathrm{Var}(f_n)\to\mathrm{Var}(f)$. We also have

$$h(f_n)=\frac{1}{c_n}\int_0^n Ve^{-V}+\frac{\log c_n}{c_n}\int_0^n e^{-V}\to h(f)$$

by the Lebesgue dominated convergence theorem, as $|V|e^{-V}=|V|e^{-V/2}e^{-V/2}\le e^{-V/2}$ is integrable.

Step 2. We can therefore fix the interval $[0,L]$ on which the function f is defined. Let $A=A_{L,\sigma}$ denote the set of log-concave non-increasing densities supported in $[0,L]$ and having variance σ^2. We have $f(0)^2\,\mathrm{Var}(X)\le f(0)^2\mathbb{E}X^2\le2$, see [1]. To see this we follow the argument from [8]. By scaling we can assume that $f(0)=1$. If $g(x)=e^{-x}\mathbb{1}_{[0,\infty)}(x)$ then by log-concavity of f the function $f-g$ changes sign in exactly one point $a>0$ and thus

$$\mathbb{E}X^2-2=\int_0^\infty x^2(f(x)-g(x))\mathrm{d}x=\int_0^\infty(x^2-a^2)(f(x)-g(x))\mathrm{d}x\le0$$

since the integrant is non-positive. This shows that f is bounded by $2\sigma^{-2}$ and in particular $h(f)=-\int f\log f\ge-\log f(0)$ is bounded from below. This shows that the quantity $M=\inf\{h(f):f\in A_{L,\sigma}\}$ is finite. Let f_n be such that $h(f_n)\to M$. By a straightforward adaptation of Lemma 12 from [8] we get that (f_n) has a convergent subsequence $f_{n_k}\to f_0\in A$ and by the Lebesgue dominated convergence theorem $h(f_0)=M$. This shows that the infimum of $h(f)$ is attained on A.

Step 3. We now apply the theory of degrees of freedom due to Fradelizi and Guédon from [6]. We say that $f\in A$ has d degrees of freedom in there exist $\varepsilon>0$ and linearly independent functions $f_1,\ldots,f_d$ such that $f_\delta:=f+\sum_{i=1}^d\delta_if_i$ is log-concave and non-incresing for all $|\delta_i|\le\varepsilon$. Suppose f has at least 4 degrees of freedom. Then

$$\int f_\delta(x)\mathrm{d}x=\int f(x)\mathrm{d}x,\qquad\int xf_\delta(x)\mathrm{d}x=\int xf(x)\mathrm{d}x,\qquad\int x^2f_\delta(x)\mathrm{d}x=\int x^2f(x)\mathrm{d}x$$

is a system of 3 linear equations in variables $\delta_1, \dots, \delta_4$ and thus has nontrivial linear subspace of solutions. Thus there is δ such that $f_\delta, f_{-\delta} \in A$ and

$$h(f) = h(\frac{1}{2}f_\delta + \frac{1}{2}f_{-\delta}) > \frac{1}{2}h(f_\delta) + \frac{1}{2}h(f_{-\delta})$$

since entropy is a strictly convex functional. Thus f is not the extremizer of $h(f)$. This shows that extremizers have to have at most 3 degrees of freedom and Step IV of the proof of Theorem 1 from [8] shows that such functions must be piecewise log-affine with at most two pieces.

3.3. **Localization.** Alternatively, we can proceed with extreme point analysis following Fradelizi and Guedon [6]. For a fixed compact interval $K \subseteq \mathbb{R}$, and upper semi-continuous functions g_1, g_2 define $\mathcal{P}_g$ to be the space of log-concave probability measures μ supported in K such that

$$\int g_i d\mu \geq 0.$$

We will use the following special case of Fradelizi and Guedon.

Theorem 3.11 ([6] Theorem 1)**.** *Let ν be an extreme point of the convex hull of $\mathcal{P}_g$, then ν is a point mass, or ν has density e^{-V} with respect to the Lebesgue measure where on the support of ν, $V = \max\{\varphi_1, \varphi_2\}$ for φ_i affine.*

When a density f has the form e^{-V} for $V = \max\{\varphi_1, \varphi_2\}$ for φ_i affine, we will say that f is two piece log-affine.

Lemma 3.12. *For X log-concave on $\mathbb{R}$,*

$$\frac{e^{2h(X)}}{\mathrm{Var}(X)} \geq \inf_{Z \in \mathcal{K}} \frac{e^{2h(Z)}}{\mathrm{Var}(Z)},$$

where $\mathcal{K}$ is the space of compactly supported log-concave variables with monotone density on this support of the form $f = e^{-\max\{\varphi_0, \varphi_1\}}$ for φ_i affine.

Proof. Recalling the truncation argument it suffices to consider $X \sim \mu$ with density, supported on $[0, L]$ for some $L > 0$. Fix X and take

$$g_1(x) = \mathbb{E}[X] - x$$
$$g_2(x) = x^2 - \mathbb{E}[X^2].$$

For $Z \sim \nu$ an extreme point of $\mathcal{P}_g$, since Z is non-negative by $\int g_i d\nu \geq 0$, we have

$$\mathrm{Var}(Z) \geq \mathrm{Var}(X) > 0,$$

so ν is not a point mass and hence by Theorem 3.11, Z has density of the form $f = e^{-\max\{\varphi_1, \varphi_2\}}$. Since $X \sim \mu \in \mathcal{P}_g$ by definition, if we let $\mathcal{E}(\mathcal{P}_f)$ denote the extreme points of the convex hull of $\mathcal{P}_f$, by the Krein-Milman μ belongs to the closure of the convex hull of $\mathcal{E}(\mathcal{P}_f)$. Now let us show that this implies that

$$h(X) \geq \inf_{Z \sim \nu \in \mathcal{E}(\mathcal{P}_g)} h(Z). \tag{6}$$

Indeed the entropy is concave and upper semi-continuous in the weak topology on when restricted to compact sets (as can be seen from the more well known fact that the relative entropy is lower semicontinuous, and on compact sets $h(X) = h(U) - D(X||U)$ where U is the uniform distribution on the compact set), thus writing μ as $\lim_n \mu_n$ for a sequence of μ_n

that can be expressed as convex combination of extreme points, that is $\mu_n = \sum_{i=1}^{k_n} \lambda_n(i)\nu_n(i)$ for $\nu_n(i) \in \mathcal{E}(\mathcal{P}_g)$, we have

$$h(\mu) \geq \limsup_n h(\mu_n) = \limsup_n h\left(\sum_{i=1}^{k_n} \lambda_n(i)\nu_n(i)\right) \geq \limsup_n \sum_{i=1}^{k_n} \lambda_n(i) h(\nu_n(i)) \geq \inf_{\nu\in\mathcal{E}(\mathcal{P}_g)} h(\nu).$$

Since every element of $\mathcal{P}_g$ has variance no smaller than X, it follows that

$$\frac{e^{2h(X)}}{\mathrm{Var}(X)} \geq \inf_{Z\sim\nu\in\mathcal{E}(\mathcal{P}_g)} \frac{e^{2h(Z)}}{\mathrm{Var}(Z)}.$$

Consider $Z^{\downarrow}$ for $Z \in \mathcal{E}(\mathcal{P}_g)$, and applying Lemma 3.9, we have

$$\frac{e^{2h(X)}}{\mathrm{Var}(X)} \geq \inf_{Z\sim\nu\in\mathcal{E}(\mathcal{P}_g)} \frac{e^{2h(Z^{\downarrow})}}{\mathrm{Var}(Z^{\downarrow})}$$

Direct computation, shows that if Z has a two-piece log-affine density, then $Z^{\downarrow}$ does as well, completing the proof. □

4. The scheme of the computation

4.1. **Three-point inequality.** From the previous section we can assume that $f = e^{-V}$, where V is two-piece affine and non-decreasing on an interval $[0, L]$. In fact by scale invariance we can assume that we have the following parametrization of our function

$$g(t) = e^{-\frac{t}{a}}\mathbb{1}_{[-ax,0]}(t) + e^{-\frac{t}{b}}\mathbb{1}_{[0,-yb]}(t), \qquad f = \frac{g}{c}, \qquad c = \int g = a(e^x - 1) - b(e^y - 1),$$

where

$$a \geq b > 0, \qquad x \geq 0, \qquad y \leq 0.$$

Then f is a probability density and we have

$$\int xg(x)\mathrm{d}x = a^2\left(e^x(1-x) - 1\right) - b^2\left(e^y(1-y) - 1\right)$$

and

$$\int x^2 g(x)\mathrm{d}x = a^3\left(e^x\left(x^2 - 2x + 2\right) - 2\right) - b^3\left(e^y\left(y^2 - 2y + 2\right) - 2\right).$$

Also

$$-\int g\log g = a\left(e^x(1-x) - 1\right) - b(e^y(1-y) - 1).$$

We want to prove the inequality

$$-\int f\log f \geq \frac{1}{2}\log\mathrm{Var}(f) + 1.$$

This is

$$e^{-2\int f\log f} \geq e^2\,\mathrm{Var}(f).$$

In terms of g

$$e^{-2}e^{-2\int\frac{g}{c}\log\frac{g}{c}} \geq \frac{1}{c}\int x^2 g(x)\mathrm{d}x - \frac{1}{c^2}\left(\int xg(x)\mathrm{d}x\right)^2.$$

We have

$$e^{-2}e^{-2\int\frac{g}{c}\log\frac{g}{c}} = e^{-2}e^{-\frac{2}{c}\int g\log g + 2\log c} = e^{-2}c^2 e^{-\frac{2}{c}\int g\log g},$$

so after multiplying both sides by c^2 we want to prove

$$e^{-2}c^4 e^{-\frac{2}{c}\int g\log g} \geq c\int x^2 g(x)\mathrm{d}x - \left(\int xg(x)\mathrm{d}x\right)^2.$$

Observe that

$$e^{-2}e^{-\frac{2}{c}\int g\log g} = \exp\left(2\frac{a\left(e^x(1-x)-1\right)-b(e^y(1-y)-1)}{a(e^x-1)-b(e^y-1)}-2\right) = \exp\left(2\frac{-axe^x+bye^y}{a(e^x-1)-b(e^y-1)}\right).$$

Our goal is therefore to prove

$$c^4\exp\left(2\frac{-axe^x+bye^y}{a(e^x-1)-b(e^y-1)}\right) \geq c\int x^2g(x)\mathrm{d}x - \left(\int xg(x)\mathrm{d}x\right)^2.$$

Equivalently

$$\begin{aligned}&(a(e^x-1)-b(e^y-1))^4\exp\left(-2\frac{ae^xx-be^yy}{a(e^x-1)-b(e^y-1)}\right)\\ &\quad\geq (a(e^x-1)-b(e^y-1))\left(a^3\left(e^x\left(x^2-2x+2\right)-2\right)-b^3\left(e^y\left(y^2-2y+2\right)-2\right)\right)\\ &\qquad-\left(a^2\left(e^x(1-x)-1\right)-b^2\left(e^y(1-y)-1\right)\right)^2.\end{aligned}$$

The inequality is invariant under $(a,b)\to(ta,tb)$ and therefore we can assume that $b=1$ and $a\geq 1$. Let us define the function

$$\begin{aligned}G(c,x,y) &= (c(e^x-1)-(e^y-1))^4\exp\left(-2\frac{ce^xx-e^yy}{c(e^x-1)-(e^y-1)}\right)\\ &\quad-(c(e^x-1)-(e^y-1))\left(c^3\left(e^x\left(x^2-2x+2\right)-2\right)-\left(e^y\left(y^2-2y+2\right)-2\right)\right)\\ &\qquad+\left(c^2\left(e^x(1-x)-1\right)-\left(e^y(1-y)-1\right)\right)^2.\end{aligned}$$

Our goal is to prove that $G(c,x,y)\geq 0$ for $x\geq 0$, $y\leq 0$ and $c\geq 1$.

4.2. **Fifth derivative.** The first crucial observation of the proof is that $\frac{\partial^5 G}{\partial c^5}$ has a sign. We have the following lemma.

Lemma 4.1. *For a positive integer n and some real numbers A,B,C,D let $h_n(t)=(At+B)^{n-1}e^{\frac{Ct+D}{At+B}}$. Then*

$$h_n^{(n)}(t) = \frac{(BC-AD)^n}{(At+B)^{n+1}}e^{\frac{Ct+D}{At+B}}.$$

Proof. Taylor expanding the exponent and using Leibniz rule under the sum we get

$$\begin{aligned}h_n^{(n)}(t) &= \left(\sum_{k=0}^{\infty}\frac{1}{k!}(Ct+D)^k(At+B)^{n-1-k}\right)^{(n)} = \left(\sum_{k=n}^{\infty}\frac{1}{k!}(Ct+D)^k(At+B)^{n-1-k}\right)^{(n)}\\ &= \sum_{k=n}^{\infty}\frac{1}{k!}\sum_{j=0}^{n}\binom{n}{j}C^jA^{n-j}(Ct+D)^{k-j}(At+B)^{j-k-1}\frac{k!}{(k-j)!}\cdot\frac{(k-j)!}{(k-n)!}(-1)^{n-j}\\ &= \sum_{j=0}^{n}\binom{n}{j}C^jA^{n-j}(-1)^{n-j}\sum_{k=0}^{\infty}(Ct+D)^{k+n-j}(At+B)^{j-k-n-1}\\ &= \sum_{j=0}^{n}\binom{n}{j}C^jA^{n-j}(-1)^{n-j}(Ct+D)^{n-j}(At+B)^{j-n-1}e^{\frac{Ct+D}{At+B}}\\ &= e^{\frac{Ct+D}{At+B}}(At+B)^{-(n+1)}\sum_{j=0}^{n}\binom{n}{j}[C(At+B)]^j[-A(Ct+D)]^{n-j}\\ &= e^{\frac{Ct+D}{At+B}}(At+B)^{-(n+1)}[C(At+B)-A(Ct+D)]^n\\ &= e^{\frac{Ct+D}{At+B}}(At+B)^{-(n+1)}(CB-AD)^n.\end{aligned}$$

□

Applying this lemma with

$$A = e^x - 1, \quad B = -(e^y - 1), \quad C = -2xe^x, \quad D = 2ye^y$$

we get

$$\frac{\partial^5 G}{\partial c^5} = \frac{32\left(e^{x+y}(x-y) - e^x x + e^y y\right)^5 e^{-2\frac{ce^x x - e^y y}{c(e^x-1)-(e^y-1)}}}{\left(c\left(e^x-1\right)-\left(e^y-1\right)\right)^6}.$$

To prove that this is non-positive one has to show that $e^x x - e^y y - e^{x+y}(x-y) \geq 0$. For $x = 0$ we have equality, thus it is enough to show positivity of the derivative in x, namely $e^x(x+1) - e^{x+y}(x-y+1) \geq 0$ for $x \geq 0$. Equivalently $x + 1 - e^y(x-y+1) \geq 0$. For $y = 0$ we have equality and the derivative in y is $-e^y(x-y)$, which is clearly non-positive. Thus the inequality holds for $y \leq 0$.

4.3. **From $G^{(5)}$ to G.** By $G^{(j)}$ we denote the jth derivative of $G(c,x,y)$ in c. We claim that in order to show that $G(c,x,y) \geq 0$ it is enough to prove the following inequalities

$$G^{(4)}(\infty,x,y) \geq 0, \quad G^{(3)}(1,x,y) \geq 0, \quad G^{(2)}(1,x,y) \geq 0, \quad G^{(1)}(1,x,y) \geq 0, \quad G(1,x,y) \geq 0.$$

Indeed, the first inequality together with $G^{(5)}(c,x,y) \leq 0$ implies that $G^{(4)}(c,x,y) \geq 0$. This together with $G^{(3)}(1,x,y) \geq 0$ implies that $G^{(3)}(c,x,y) \geq 0$. Repeating the argument based on the above list of inequalities finishes the argument. The rest of the proof is a verification of these inequalities.

4.4. **Crucial technical bound.** The inequalities $G^{(j)}(1,x,y) \geq 0$ can always be written in the form

$$e^{2\frac{e^y y - e^x x}{e^x - e^y}} A_j(x) - B_j(x) \geq 0, \tag{7}$$

where A_j and B_j are polynomials in x, y, e^x, e^y. The problematic exponential factor makes the inequalities intractable in the current form. Therefore, one needs to bound the exponent. The following lemma provides a very tight bound on this expression.

Lemma 4.2. *For every $x, y \in \mathbb{R}$ we have*

$$\left(\left(\frac{e^2}{6} - 1\right)(x-y)^2 e^{x+y} + (e^x - e^y)^2 + e^{2+x+y}\right) e^{-2\frac{e^x x - e^y y}{e^x - e^y}} \geq 1$$

In the proof we shall need the following lemma, see [2].

Lemma 4.3. *Suppose the sequence of coefficients of the series $\sum_{n=0}^{\infty} a_n x^n$ has sign pattern $(+,-)$. Then $f(x) = 0$ for exactly one point $x > 0$.*

Proof. Let $f(x) = a_0 + a_1 x + \ldots + a_k x^k - b_{k+1} x^{k+1} + \ldots$, where a_i, b_i are non-negative. Then $f(x) = 0$ is equivalent with

$$\frac{a_0}{x^k} + \frac{a_1}{x^{k-1}} + \ldots + a_k = b_{k+1} x + b_{k+2} x^2 + \ldots$$

The left hand side is decreasing and the right hand side is increasing. □

Proof of Lemma 4.2. This inequality is invariant under $(x,y) \to (x+t, y+t)$ and therefore it is enough to consider $y = -x$ in which case we get

$$\left(\left(\frac{2e^2}{3} - 4\right)x^2 + 4\sinh^2(x) + e^2\right) e^{-2x\coth(x)} \geq 1$$

Let us define

$$f(x) = \left(\left(\frac{2e^2}{3} - 4\right)x^2 + 4\sinh^2(x) + e^2\right) e^{-2x\coth(x)}$$

It is enough to prove the inequality for $x \geq 0$ since the function is even. We will show that f' on $\mathbb{R}_+$ has sign pattern $(+,-)$. This together with $f(0) = \lim_{x\to\infty} f(x) = 1$ will finish the proof.

We have

$$e^{2x\coth(x)} f'(x) = \frac{2}{3}\left(\frac{x\left(-12x^2+2e^2\left(x^2+1\right)+e^2\cosh(2x)\right)}{\sinh^2(x)} + \frac{\left(12x^2-e^2\left(2x^2+3\right)\right)\cosh(x)}{\sinh(x)}\right).$$

We therefore have to examine the sign pattern of

$$x\left(-12x^2+2e^2\left(x^2+1\right)+e^2\cosh(2x)\right)+\left(12x^2-e^2\left(2x^2+3\right)\right)\cosh(x)\sinh(x).$$

This is

$$x\left(-12x^2+2e^2\left(x^2+1\right)+e^2\cosh(2x)\right)+\frac{1}{2}\left(12x^2-e^2\left(2x^2+3\right)\right)\sinh(2x)$$

Taking $t = 2x$ gives

$$\frac{1}{2}\left(3t^2-e^2\left(\frac{t^2}{2}+3\right)\right)\sinh(t)+\frac{1}{2}t\left(-3t^2+2e^2\left(\frac{t^2}{4}+1\right)+e^2\cosh(t)\right)$$

This is an odd function whose odd Taylor coefficient in front of t^n for $n \geq 5$ is equal to

$$-\frac{2e^2(n-2)!(3(n-1)!-n!)+\left(e^2-6\right)(n-1)!n!}{4(n-2)!(n-1)!n!} = -\frac{2e^2(3-n)+(e^2-6)n(n-1)}{4n!},$$

while the lower coefficients vanish. We have to show that the sequence

$$c_n = 2e^2(3-n)+(e^2-6)n(n-1)$$

has sign pattern $(-,+)$. Since $c_5 = -1.7751 < 0$ this follows from the fact that c_n is a convex quadratic function of n.

□

Remark. *The rest of the proof highly relies on symbolic computational software (Mathematica) in order to generate relevant formulas without making mistakes. We shall mention the Mathematica commands that we use in the crucial computations.*

In the case of the first, second and third derivative the proof has the following structure. Let

$$C(x,y) = \left(\frac{e^2}{6}-1\right)(x-y)^2e^{x+y} + (e^x-e^y)^2 + e^{2+x+y}$$

be the function from Lemma 4.2. After ensuring that $A_i(x,y) \geq 0$, we estimate the exponent using the above bound. This leads to the inequality

$$A_i(x,y) + C(x,y)B_i(x,y) \geq 0.$$

We then expand the left hand side in x and show that the coefficients $P_n(y)$ in front of $\frac{x^n}{n!}$ are non-negative, which finishes the proof. In the main body of the paper we only show the nonnegativity of $P_n(y)$ for $n \geq n_0$, where n_0 is some explicit small number. The verification of $P_n(y)$ for $n < n_0$ is a straightforward algorithmic (but sometimes a bit lengthy) exercise, we therefore move this part to the appendix.

5. Nonnegativity of $G(1,x,y)$ and $G^{(4)}(\infty,x,y)$

5.1. **The function.** We are going to show the inequality

$$G(1,x,y) = e^{-2\frac{e^x x - e^y y}{e^x - e^y}}(e^x - e^y)^4 - (e^x - e^y)^2 + e^{x+y}(x-y)^2 \geq 0.$$

The inequality is invariant under the scaling $(x,y) \to (x+t, y+t)$. This is understendable, since the case $c=1$ corresponds to only one slope, so we gain the freedom of shifting. Therefore we can assume that $x+y=0$. In this case one has to prove

$$e^{-2x\frac{e^x+e^{-x}}{e^x-e^{-x}}}\left(e^x - e^{-x}\right)^4 - (e^x - e^{-x})^2 + 4x^2 \geq 0.$$

This can be rewritten as

$$f(x) := 4e^{-2x\coth x}\frac{\sinh^4 x}{\sinh^2 x - x^2} \geq 1.$$

It is enough to consider only $x>0$ since the expression is even. The left hand side converges to 1 as $x \to \infty$ and therefore it is enough to show that the function is decreasing. We have

$$f'(x) = -\frac{8x\sinh^2(x)e^{-2x\coth(x)}\left(x^2 - 2\sinh^2(x) + x\sinh(x)\cosh(x)\right)}{\left(x^2 - \sinh^2(x)\right)^2}.$$

It is therefore enough to show the inequality

$$x^2 + x\sinh(x)\cosh(x) - 2\sinh^2(x) \geq 0.$$

This is

$$2x^2 + x\sinh(2x) - 4\sinh^2(x) \geq 0.$$

It is enough to show that the derivative is non-negative for $x \geq 0$, that is

$$4x + \sinh(2x) + 2x\cosh(2x) - 4\sinh(2x) \geq 0.$$

Therefore for $t \geq 0$ one needs to show

$$g(t) := 2t + t\cosh(t) - 3\sinh(t) \geq 0.$$

Since $g(0) = g'(0) = g''(0)$, it is enough to observe that $g'''(t) = t\sinh(t) \geq 0$.

5.2. **The fourth derivative.** Using the Mathematica command

Limit[D[G[c, x, y], c, c, c, c], c → Infinity]

and simplifying further the expression shows that

$$G^{(4)}(\infty,x,y) = 24\left(e^{-\frac{2e^x x}{e^x-1}}(e^x-1)^4 + e^x x^2 - (e^x-1)^2\right).$$

We are going to prove that this is nonnegative. We first observe that

$$(e^x-1)^2 - e^x x^2 \geq 0, \qquad x \geq 0$$

The coefficient in front of $\frac{x^n}{n!}$ is $2^n - n(n-1) - 2 > 0$ for $n \geq 4$, whereas the lower coefficients vanish. Putting the term $(e^x-1)^2 - e^x x^2$ to the right hand side and taking the logarithm shows that it is enough to prove

$$f(x) := -2\frac{e^x x}{e^x - 1} + 4\log(e^x - 1) - \log((e^x-1)^2 - e^x x^2) > 0.$$

The limit of the left hand side as $x \to \infty$ is equal to 0. Therefore it is enough to show that the function is decreasing. We have

$$(e^x-1)^2 f'(x) = -\frac{e^x x\left(2e^x\left(x^2+4\right) + e^{2x}(x-4) - x - 4\right)}{(e^x-1)^2 - e^x x^2}.$$

It is therefore enough to show that

$$h(x) := 2e^x\left(x^2+4\right) + e^{2x}(x-4) - x - 4 \geq 0.$$

First five derivatives of h at $x = 0$ vanish. We have

$$h''(x) = 2e^x\left(x^2+4x+2e^x(x-3)+6\right)$$

It is therefore enough to show that

$$g(x) := x^2 + 4x + 2e^x(x-3) + 6 \geq 0.$$

First three derivatives of g at 0 vanish, so it is enough to observe that $g'''(x) = 2e^x x \geq 0$.

6. First derivative

We have

$$G^{(1)}(1,x,y) = e^{\frac{2e^y y - 2e^x x}{e^x - e^y}} A_1(x) + B_1(x)$$

where

$$A_1(x,y) = 2\left(e^x - e^y\right)^2\left(e^{x+y}(x-y-2) - e^x(x+2) + 2e^{2x} + e^y(y+2)\right)$$

and

$$B_1(x,y) = e^{x+y}\left(3x^2 - 2x(2y+1) + y^2 + 2y + 4\right) + e^x\left(x^2+2x+4\right) - 4e^{2x} - e^y\left(y^2+2y+4\right).$$

6.1. **The factor A_1 is non-negative.** We now claim that the factor

$$A(x,y) = e^{x+y}(x-y-2) - e^x(x+2) + 2e^{2x} + e^y(y+2)$$

is non-negative and therefore $A_1(x,y) \geq 0$. We have $A(0,y) = 0$ and

$$e^{-x}\frac{\partial A}{\partial x} = e^y(x-y-1) - x + 4e^x - 3.$$

It is enough to show that this expression is non-negative. For $x = 0$ we get $1 - e^y(y+1) \geq 0$ since for $y < -1$ there is nothing to prove and for $y \in [-1,0]$ we have $e^y(y+1) \leq y+1 \leq 1$. Thus it is enough to see that

$$\frac{\partial}{\partial x}\left(e^y(x-y-1) - x + 4e^x - 3\right) = e^y - 1 + 4e^x \geq 4e^x - 1 \geq 3 > 0.$$

Using Lemma 4.2 we are left with proving

$$A_1(x,y) + C(x,y)B_1(x,y) \geq 0.$$

6.2. **Taylor expansion.** Now, the left hand side $L(x,y)$ can be expanded in x and the coefficient R[n,y] can be found using Mathematica command

$$R[n\_, y\_] := \text{SeriesCoefficient}[L[x,y], \{x, 0, n\}].$$

After manually multiplying by $n!$ (obtaining P[n,y]) and using the command

$$\text{Collect}[P[n,y], \{e^y, y, 3^n, 2^n, n\}]$$

to collect terms one gets that the coefficient in front of $\frac{x^n}{n!}$ is of the form

$$P_n(y) = a_0 + e^y(b_0 + b_1 y + b_2 y^2) + e^{2y}(c_0 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4) + e^{3y}(d_0 + d_1 y + y^2),$$

where for $n \geq 4$ we have

$$a_0(n) = 3^{n-2}(n-1)n, \qquad d_0 = 3n(n-1), \qquad d_1 = -4n,$$

$$b_0(n) = 3^{n-3}\left(\left(21-2e^2\right)n^2 + \left(2e^2-21\right)n - 108e^2\right)$$
$$+ \frac{1}{27}2^{n-5}\left(\left(9e^2-54\right)n^4 + \left(108-18e^2\right)n^3 + \left(351e^2-1242\right)n^2 + \left(1188+522e^2\right)n + 3456e^2\right)$$
$$b_1(n) = 4\left(e^2-9\right)3^{n-2}n - \frac{1}{3}\left(e^2-6\right)2^{n-3}n\left(n^2+n+14\right)$$
$$b_2(n) = \frac{1}{3}2^{n-3}\left(\left(e^2-6\right)n^2 + \left(3e^2-18\right)n + 16e^2 - 120\right) + \left(15-2e^2\right)3^{n-1}$$

and

$$c_0(n) = \left(\left(5-\frac{2e^2}{3}\right)n^2 + \left(\frac{2e^2}{3}-5\right)n - 4e^2\right)$$
$$+ \frac{1}{3}2^{n-5}\left(\left(3e^2-18\right)n^4 + \left(132-22e^2\right)n^3 + \left(133e^2-510\right)n^2 + \left(396-210e^2\right)n + 384e^2\right)$$
$$c_1(n) = \left(2-\frac{e^2}{3}\right)n^2 + \left(\frac{5e^2}{3}-10\right)n - 2e^2$$
$$+ \frac{1}{3}2^{n-3}\left(\left(30-5e^2\right)n^3 + \left(21e^2-126\right)n^2 + \left(288-80e^2\right)n + 48e^2\right)$$
$$c_2(n) = 2^n\left(\left(\frac{e^2}{2}-3\right)n^2 + \left(6-e^2\right)n + \frac{5e^2}{3} - 6\right) + \left(\left(1-\frac{e^2}{6}\right)n^2 + \left(\frac{5e^2}{6}-5\right)n - \frac{5e^2}{3} + 6\right)$$
$$c_3(n) = -\frac{1}{6}\left(e^2-6\right)\left(2^n(3n-2) - 2n + 2\right)$$
$$c_4(n) = \frac{1}{6}\left(e^2-6\right)\left(2^n-1\right).$$

6.3. **Signs of the coefficients.** We first examine the signs of the coefficients of P_n for $n \geq 4$. Clearly

$$a_0(n) > 0, \qquad d_0(n) > 0, \qquad d_1(n) < 0, \qquad c_3(n) < 0, \qquad c_4(n) > 0, \qquad b_1(n) < 0.$$

We also have

$$c_2(n) \geq 2^n\left(\frac{1}{2}n^2 - \frac{3}{2}n + 6\right) + \left(-7 + n + \frac{1}{4}n^2\right) \geq 16\left(\frac{1}{2}n^2 - \frac{3}{2}n + 6\right) + \left(-7 + n - \frac{1}{4}n^2\right)$$
$$\geq 7n^2 - 23n + 89 > 0.$$

Also

$$\begin{aligned} c_1(n) &\leq \left(-14 + \frac{5}{2}n - \frac{2}{5}n^2\right) + 2^n\left(15 - 12n + \frac{3}{2}n^2 - \frac{1}{4}n^3\right) \\ &\leq \left(-14 + \frac{5}{2}n - \frac{2}{5}n^2\right) + 16\left(15 - 12n + \frac{3}{2}n^2 - \frac{1}{4}n^3\right) \\ &= -4n^3 + \frac{118n^2}{5} - \frac{379n}{2} + 226 = \frac{1}{5}n^2\left(118 - 20n\right) - \frac{379n}{2} + 226. \end{aligned}$$

This is obviously negative for $n \geq 6$. The case $n = 4, 5$ is checked directly. We also have

$$\begin{aligned} c_0(n) &\geq \frac{7n^2}{100} + 2^n\left(\frac{n^4}{25} - \frac{8n^3}{25} + \frac{49n^2}{10} - 13n + 29\right) - \frac{2n}{25} - 30 \\ &\geq \frac{7n^2}{100} + 16\left(\frac{n^4}{25} - \frac{8n^3}{25} + \frac{49n^2}{10} - 13n + 29\right) - \frac{2n}{25} - 30 \\ &= \frac{1}{100}\left(64n^4 - 512n^3 + 7847n^2 - 20808n + 43400\right). \end{aligned}$$

We have

$$64n^4 - 512n^3 + 7847n^2 - 20808n + 43400 = 64n^3(n-8) + n(7847n - 20808) + 43400.$$

which is clearly positive for $n \geq 8$. The cases $n = 4, 5, 6, 7$ are checked directly.

We have

$$b_2(n) \geq 2^n \left(\frac{n^2}{20} + \frac{17n}{100} - \frac{2}{25} \right) + \frac{7}{100} 3^n > 0.$$

Summarizing, we get

$$c_2(n) > 0, \quad c_1(n) < 0, \quad c_0(n) > 0, \quad b_2(n) > 0.$$

The only coefficient which may not have the correct sign is $b_0(n)$. We have

$$b_0(n) \geq 2^n \left(\frac{n^4}{100} - \frac{3n^3}{100} + \frac{3n^2}{2} + 5n + 29 \right) + 3^n \left(\frac{23}{100} n(n-1) - 30 \right).$$

The first term is positive. The second term is positive for $n \geq 12$. In fact also $b_0(11) > 0$. We therefore have

$$b_0(n) > 0, \ n \geq 11, \qquad b_0(n) < 0, \ 4 \leq n \leq 10.$$

6.4. **Positivity of $P_n(y)$ for $n \geq 6$.** Note that for $n \geq 11$ all the coefficients have correct signs, which implies $P_n(y) \geq 0$.

For $n = 9, 10$ one has $P_n(y) \geq a_0(n) + e^y b_0(n) \geq a_0(n) + b_0(n) > 0$. For $n = 6, 7, 8$ we keep more terms

$$P_n(y) \geq a_0(n) + e^y b_0(n) + e^{2y} c_0(n).$$

We now have to show that for $t \in [0, 1]$ one has $a_0(n) + t b_0(n) + t^2 c_0(n) \geq 0$. It is enough to observe that the discriminant $b_0(n)^2 - 4a_0(n)c_0(n)$ is negative.

Positivity of $P_n(y)$ for $1 \leq n \leq 5$ is proved in the Appendix. We discourage the reader from checking these computations, since they are almost algorithmic. We provide a graph of these functions of readers convenience.

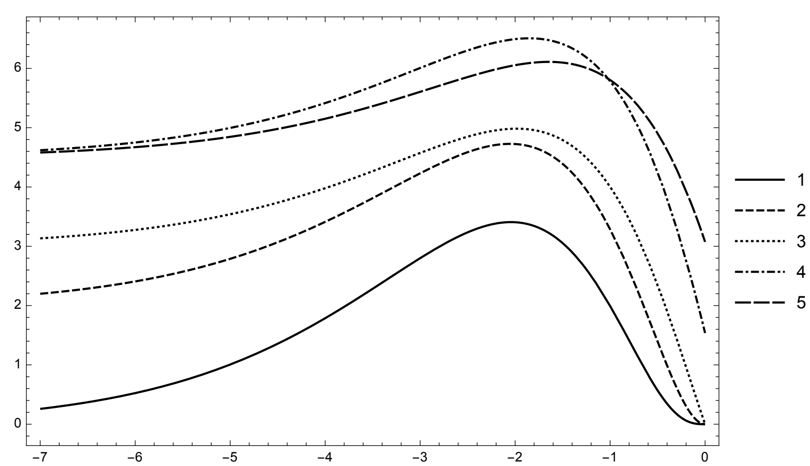

FIGURE 1. The graphs of $7P_1(y), P_2(y), \frac{1}{3!}P_3(y), \frac{1}{4!}P_4(y)$ and $\frac{1}{5!}P_5(y)$.

7. Second derivative

Using the command

$$\mathrm{Simplify}[D[G[c,x,y],c,c]/.c\to 1]$$

and further simplifying the expressions for A_2 and B_2 we find

$$G^{(2)}(1,x,y)=e^{-2\frac{e^x x-e^y y}{e^x-e^y}}A_2(x,y)+B_2(x,y),$$

where

$$\begin{aligned}A_2(x,y)=&12\left(e^x-1\right)^2\left(e^x-e^y\right)^2+12\left(e^x-1\right)\left(e^{x+y}(x-y)-e^x x+e^y y\right)\left(e^x-e^y\right)\\&+4\left(e^{x+y}(x-y)-e^x x+e^y y\right)^2\end{aligned}$$

and

$$B_2(x,y)=e^{x+y}\left(6x^2-4x(y+2)+4(y+2)\right)+2e^x\left(3x^2+4x+8\right)-12e^{2x}-4e^y(y+2)-4.$$

7.1. **The factor A_2 is non-negative.** We shall show $A_2(x,y)>0$ by showing that the first two terms sum up to something positive. This sum is of the form $12\left(e^x-1\right)\left(e^x-e^y\right)R(x,y)$, where

$$R(x,y)=e^{x+y}(x-y-1)-e^x(x+1)+e^{2x}+e^y(y+1). \tag{8}$$

We have $R(0,y)=0$ and using $e^x\ge 1+x$

$$\frac{\partial R}{\partial x}=e^x\left(e^y(x-y)+2\left(e^x-1\right)-x\right)\ge e^x\left(2\left(e^x-1\right)-x\right)\ge e^x x\ge 0.$$

7.2. **Taylor expansion.** We expand the expression

$$A_2(x,y)+C(x,y)B_2(x,y)$$

in x. The coefficient in front of $\frac{1}{n!}x^n$ is

$$P_n(y)=a_0+e^y(b_0+b_1y+b_2y^2)+e^{2y}(c_0+c_1y+c_2y^2+c_3y^3)+e^{3y}(d_0+d_1y),$$

where for $n\ge 4$ one has

$$a_0(n)=\frac{2}{3}3^n\left(n^2-3n-12\right)+2^n\left(n^2+5n+8\right)$$

and

$$\begin{aligned}d_0(n)&=6n^2-14n+8\\d_1(n)&=-4(n-1)\end{aligned}$$

$$\begin{aligned}b_0(n)=&\frac{1}{3}\left(\left(12-2e^2\right)n^2+\left(2e^2-48\right)n-12e^2-48\right)+3^{n-2}\left(\left(18-2e^2\right)n^2+\left(2e^2-6\right)n-108e^2+72\right)\\&+\frac{1}{3}2^{n-4}\left(\left(3e^2-18\right)n^4+\left(60-10e^2\right)n^3+\left(113e^2-486\right)n^2+\left(60+86e^2\right)n+768e^2+384\right)\\b_1(n)=&\frac{1}{3}2^{n-2}\left(-3\left(e^2-6\right)n^3+\left(e^2-6\right)n^2-6\left(5e^2-38\right)n+240\right)+\frac{4}{3}\left(\left(e^2-12\right)n-9\right)\\&+\frac{4}{3}3^n\left(\left(e^2-7\right)n-6\right)\\b_2(n)=&\frac{1}{3}\left(e^2-6\right)2^{n-2}\left(3n^2+5n+32\right)+2\left(6-e^2\right)3^n+\left(4-\frac{2e^2}{3}\right)\end{aligned}$$

and moreover

$$c_0(n) = \frac{1}{3}2^{n-4}\left(\left(3e^2-18\right)n^4 + \left(156-26e^2\right)n^3 + \left(145e^2-534\right)n^2 + \left(492-314e^2\right)n + 384e^2 - 768\right)$$
$$+\left(14-\frac{4e^2}{3}\right)n^2 + \left(6+\frac{4e^2}{3}\right)n - 8e^2 + 8$$
$$c_1(n) = \frac{1}{3}2^{n-1}\left(\left(12-2e^2\right)n^3 + \left(11e^2-66\right)n^2 + \left(102-29e^2\right)n + 24e^2 + 24\right)$$
$$+\left(4-\frac{2e^2}{3}\right)n^2 + \left(\frac{10e^2}{3}-12\right)n - 4e^2 - 16$$
$$c_2(n) = \frac{1}{3}2^{n-2}\left(\left(7e^2-42\right)n^2 + \left(138-23e^2\right)n + 16e^2 - 48\right) + \left(\frac{4e^2}{3}-8\right)n - \frac{4e^2}{3}$$
$$c_3(n) = \frac{1}{3}\left(6-e^2\right)2^n(n-2) - \frac{2e^2}{3} + 4$$

We have $P_1 \equiv 0$.

7.3. **Signs of the coefficients.** It is straightforward to see that

$$a_0(n) > 0, \qquad d_0(n) > 0, \qquad d_1(n) < 0, \qquad c_3(n) < 0.$$

We have

$$c_2(n) \geq 2^n\left(0.8n^2 - 3n + 5\right) + n - 10 > 16\left(0.8n^2 - 3n + 5\right) + n - 10 > 0.$$

We now claim that $c_1(n) < 0$. We have

$$c_1(n) < -0.9n^2 + 2^n\left(-0.4n^3 + 2.6n^2 - 18n + 34\right) + 13n - 45.$$

It is easy to prove that $-0.4n^3 + 2.6n^2 - 18n + 34 < 0$ for $n \geq 4$ and thus we can bound $2^n \geq 16$ and get

$$c_1(n) < -6.4n^3 + 40.7n^2 - 275n + 499 < 0.$$

Finally let us show that $c_0(n) > 0$. We have

$$c_0(n) \geq (4n^2 + 15n - 52) + 2^n\left(0.08n^4 - n^3 + 11n^2 - 39n + 43\right) > 0.$$

since both terms are positive for $n \geq 4$.

We shall now show that $b_0(n) > 0$ for $n \geq 12$. We can assume that $n \geq 21$, since the remaining cases can be checked directly. Bounding crudely

$$b_0(n) \geq -2^n n^3 + 3^n\left(0.3n^2 - 81\right) - n^2 - 12n - 46 \geq -2^n n^3 + 3^n\left(0.3n^2 - 81\right) - 3n^2$$
$$\geq -2^n n^3 + \frac{1}{9}3^n n^2 - 3n^2 \geq -2^n n^3 + \frac{1}{10}3^n n^2 = \frac{1}{10}n^2 2^n\left(\left(\frac{3}{2}\right)^n - 10n\right) > 0.$$

7.4. **Positivity of $P_n(y)$ for $n \geq 7$.** From the signs of c_i and d_i we see that

$$P_n(y) \geq a_0 + e^y(b_0 + b_1 y + b_2 y^2).$$

We are going to show that

$$a_0 + e^y(b_0 + b_1 y + b_2 y^2) \geq 0 \qquad \text{for} \quad n \geq 7. \tag{9}$$

We first assume that $8 \leq n \leq 11$, in which case we write

$$a_0 + e^y(b_0 + b_1 y + b_2 y^2) \geq a_0 - |b_0| - e^{-1}|b_1| - \frac{1}{4}e^{-2}|b_2| > 0.$$

where the last inequality can be verified directly. Now assume that $n \geq 12$. In this case $b_0(n) > 0$ and we have

$$a_0 + e^y(b_0 + b_1 y + b_2 y^2) \geq a_0 + e^y(b_1 y + b_2 y^2) \geq a_0 - e^{-1}|b_1| - \frac{1}{4}e^{-2}|b_2| > 0.$$

Observe that

$$|b_1(n)| \leq 2^n \left(n^3 + n^2 + n + 20\right) + 3^n n + 7n + 12 \leq 2^n \cdot 4n^3 + 3^n n + 2^n n^3 = 2^n \cdot 5n^3 + 3^n n$$

and

$$|b_2(n)| \leq 2^n \left(0.35n^2 + 0.6n + 4\right) + 3 \cdot 3^n + 1 \leq 2^n n^2 + 3 \cdot 3^n.$$

Therefore

$$e^{-1}|b_1| + \frac{1}{4}e^{-2}|b_2| \leq 2^n \cdot 2n^3 + 3^n n.$$

We also have

$$a_0(n) \geq \frac{2}{3}3^n(n^2 - 3n - 12) \geq 3^{n-1}n^2.$$

It is therefore enough to show that for $n \geq 12$ one has

$$3^{n-1}n^2 \geq 2^n \cdot 2n^3 + 3^n n.$$

Note that

$$3^{n-1}n^2 - 3^n n \geq \frac{1}{4}n^2 3^n \geq 2^n \cdot 2n^3,$$

since $8n < (\frac{3}{2})^n$ for $n \geq 12$.

Let us now prove (9) for $n = 7$. One has

$$1 + e^y \left(\frac{b_0}{a_0} + \frac{b_1}{a_0}y + \frac{b_2}{a_0}y^2\right) \geq 1 + e^y \left(-1 - \frac{3}{5}y - \frac{1}{10}y^2\right).$$

Since the right hand side vanishes for $y = 0$, it is enough to show that it is decreasing. This is true since the derivative is equal to $-\frac{1}{10}e^y(y+4)^2$.

We have $P_0 \equiv P_1 \equiv 0$. Positivity of $P_n(y)$ for $2 \leq n \leq 5$ is proved in the Appendix. We provide a graph of these functions of readers convenience.

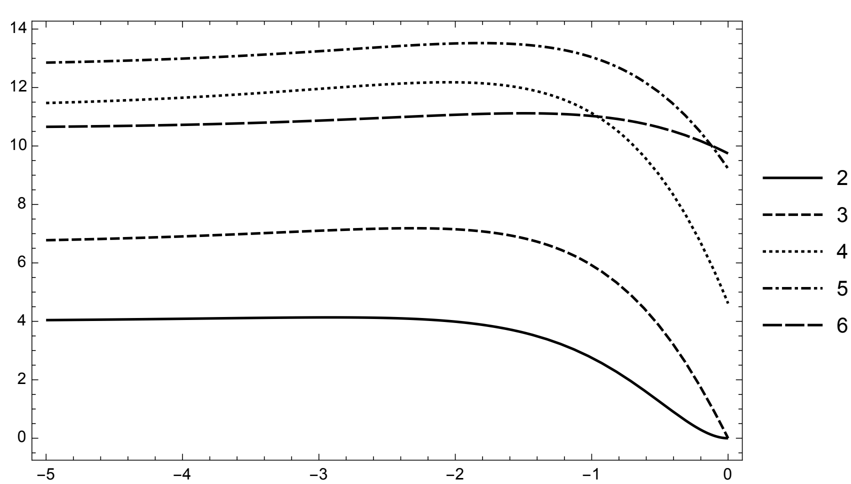

FIGURE 2. The graphs of $P_2(y)$, $\frac{1}{3!}P_3(y)$, $\frac{1}{4!}P_4(y)$, $\frac{1}{5!}P_5(y)$ and $\frac{1}{6!}P_6(y)$.

8. Third derivative

Using the command

$$\mathrm{Simplify}[D[G[c,x,y],c,c,c]/.c \to 1]$$

and further simplifying the expressions, we find that

$$G^{(3)}(1,x,y) = e^{-2\frac{e^x x - e^y y}{e^x - e^y}} \left(S(x,y) + 24\frac{(e^x-1)T(x,y)^2}{e^x - e^y} - 8\frac{T(x,y)^3}{(e^x-e^y)^2} \right) + U(x,y),$$

where

$$\begin{aligned} S(x,y) &= 12\left(e^x-1\right)^2 \left(e^{x+y}(3x-3y-2) - e^x(3x+2) + 2e^{2x} + e^y(3y+2)\right) \\ T(x,y) &= e^x x - e^y y - e^{x+y}(x-y) \\ U(x,y) &= -6\left(\left(x^2-2x+2\right)\left(-e^{x+y}\right) - e^x\left(3x^2+2x+6\right) + 4e^{2x} + 2e^y + 2\right). \end{aligned}$$

8.1. **Preparatory bound.** Our first step is to show that

$$\frac{(e^x-1)T(x,y)^2}{e^x-e^y} \geq \frac{T(x,y)^3}{(e^x-e^y)^2}.$$

This is equivalent to

$$(e^x-1)(e^x-e^y) \geq T(x,y),$$

which simplifies to

$$R(x,y) = e^{x+y}(x-y-1) - e^x(x+1) + e^{2x} + e^y(y+1) \geq 0.$$

The function R appeared in (8) and the inequality $R(x,y) \geq 0$ was already verified.

It is therefore enough to show that

$$e^{-2\frac{e^x x - e^y y}{e^x - e^y}} \left(S(x,y) + 16\frac{(e^x-1)T(x,y)^2}{e^x-e^y} \right) + U(x,y) \geq 0$$

or equivalently

$$e^{-2\frac{e^x x - e^y y}{e^x - e^y}} A_3(x,y) + B_3(x,y) \geq 0$$

with

$$A_3(x,y) = S(x,y)(e^x-e^y) + 16(e^x-1)T(x,y)^2, \qquad B_3(x,y) = U(x,y)(e^x-e^y).$$

8.2. **The factor A_3 is non-negative.** We now show that the first term in the above expression is positive, which is needed to use our standard bound on the exponent. Since $A_3(x,y) = 4(e^x-1)V(x,y)$, where

$$\begin{aligned} V(x,y) = 4&\left(e^x x - e^y y - e^{x+y}(x-y)\right)^2 \\ &+ 3\left(e^x-1\right)\left(e^x-e^y\right)\left(e^{x+y}(3x-3y-2) - e^x(3x+2) + 2e^{2x} + e^y(3y+2)\right), \end{aligned}$$

thus reduces to the inequality $V(x,y) \geq 0$. We shall expand $V(x,y)$ in x and show that the coefficients are non-negative. The coefficient in front of $\frac{x^n}{n!}$ is of the form

$$Q_n(y) = \alpha_0 + e^y(\beta_0 + \beta_1 y) + e^{2y}(\gamma_0 + \gamma_1 y + \gamma_2 y^2)$$

with

$$\alpha_0(n) = 2^n\left(n^2 + \frac{7n}{2} + 6\right) - 3^{n+1}(n+4) + 6\cdot 4^n$$

$$\beta_0(n) = 2^n\left(-2n^2+2n+24\right) - 9n + 3^n(3n-12) - 12, \qquad \beta_1(n) = -8n - 3^{n+2} + 2^n(4n+18) - 9$$

and

$$\gamma_0(n) = 2^n\left(n^2 - \frac{11n}{2} + 6\right) + 9n - 12, \qquad \gamma_1(n) = 8n + 2^n(9-4n) - 18, \qquad \gamma_2(n) = 2^{n+2} - 8,$$

whereas Q_0 and Q_1 vanish. It is straightforward to check that

$$\alpha_0(n) > 0, \qquad \beta_1(n) < 0, \qquad \gamma_0(n) > 0, \qquad \gamma_2(n) > 0.$$

It is also easy to show that $\gamma_1(n) < 0$ if and only if $n \geq 3$. We show that $\beta_0 > 0$ if and only if $n \geq 5$. For $n \geq 11$ we have

$$\beta_0(n) \geq -2 \cdot 2^n n^2 + 3 \cdot 3^n$$

where the last inequality can be proved by induction. The remaining case $2 \leq n \leq 10$ are proved directly.

Now, for $n \geq 5$ all the terms have correct sign, so $Q_n(y)$ in nonnegative. We also have

$$Q_2(y) = e^{2y}\left(8y^2 + 2y + 2\right) + e^y(-2y - 4) + 2 \geq 2e^{2y}(y+1) - 2e^y(y+2) + 2$$

It is therefore enough to prove the inequality

$$e^{2y}(y+1) - e^y(y+2) + 1 \geq 0.$$

If $y < -2$ then the second terms is positive and the first term is already smaller in absolute value than 1, so the inequality holds. If $y \in [-2, 0]$ then we shall show that the function is decreasing. The derivative is $e^y\left(-y + e^y(2y+3) - 3\right)$ so it is enough to show that $-y + e^y(2y+3) < 3$. The second derivative of the left hand side is $e^y(2y+7) > 0$, so by convexity it is enough to verify the inequality for $y = -2, 0$.

We have

$$Q_3(y) = e^{2y}\left(24y^2 - 18y + 3\right) + e^y(-36y - 24) + 21 \geq 3e^{2y} - 24e^y + 21 = 3(e^{2y} - 8e^y + 7).$$

Let $t = e^y \in [0, 1]$. We want to show $t^2 - 8t + 7 = (t-1)(t-7) > 0$, which is true. Finally

$$Q_4(y) = e^{2y}\left(56y^2 - 98y + 24\right) + e^y(-226y - 48) + 168 > -48e^y + 168 \geq -48 + 168 > 0.$$

After using the bound on the exponent, we are left with proving

$$A_3(x, y) + C(x, y)B_3(x, y) \geq 0.$$

8.3. **Taylor expansion.** We expand

$$L(x, y) := A_3(x, y) + C(x, y)B_3(x, y)$$

in x using the command

$$R[n\_, y\_] := \text{SeriesCoefficient}[\text{Expand}[L[x, y]], \{x, 0, n\}].$$

The coefficient in front of $\frac{x^n}{n!}$ is of the form

$$P_n(y) = a_0 + e^y(b_0 + b_1 y + b_2 y^2) + e^{2y}(c_0 + c_1 y + c_2 y^2) + e^{3y}(d_0 + d_1 y + d_2 y^2) + e^{4y} e_0,$$

for $n \geq 2$ (we have $P_0 = P_1 = P_2 \equiv 0$), where

$$a_0(n) = 4^n\left(\frac{9n^2}{8} - \frac{57n}{8} - 36\right) + 3^n\left(\frac{16n^2}{9} + \frac{200n}{9} + 60\right) + 2^n\left(-4n^2 - 14n - 24\right)$$
$$e_0(n) = -6n^2 + 18n - 12,$$

whereas the coefficients b_i are given by

$$b_0(n) = 2^n\left(\left(11-\frac{e^2}{2}\right)n^2+\left(\frac{e^2}{2}-29\right)n-12e^2-108\right)$$
$$+4^n\left(\left(\frac{15}{8}-\frac{e^2}{4}\right)n^2+\left(\frac{33}{8}+\frac{e^2}{4}\right)n-24e^2+36\right)$$
$$+3^n\left(\left(\frac{e^2}{27}-\frac{2}{9}\right)n^4+\left(\frac{8}{9}-\frac{4e^2}{27}\right)n^3+\left(\frac{77e^2}{27}-\frac{44}{3}\right)n^2+\left(\frac{34e^2}{27}-10\right)n+36e^2+24\right)$$
$$+(36n+48)$$
$$b_1(n) = 3^n\left(\left(\frac{4}{3}-\frac{2e^2}{9}\right)n^3+\left(\frac{2e^2}{9}-\frac{4}{3}\right)n^2+\left(\frac{104}{3}-4e^2\right)n+108\right)+2^n\left(\left(2e^2-44\right)n-108\right)$$
$$+4^n\left(\left(2e^2-12\right)n-36\right)+(32n++36)$$
$$b_2(n) = 3^n\left(\left(\frac{e^2}{3}-2\right)n^2+\left(\frac{e^2}{3}-2\right)n+6e^2-36\right)+\left(12-2e^2\right)2^n+\left(24-4e^2\right)4^n,$$

the coefficients c_i equal to

$$c_0(n) = \left(\left(2e^2-12\right)n^2+\left(-24-2e^2\right)n+12e^2+36\right)$$
$$+2^n\left(\left(\frac{9}{8}-\frac{3e^2}{16}\right)n^4+\left(\frac{7e^2}{8}-\frac{21}{4}\right)n^3+\left(\frac{235}{8}-\frac{125e^2}{16}\right)n^2+\left(\frac{115}{4}+\frac{9e^2}{8}\right)n-48e^2+72\right)$$
$$+3^n\left(\left(\frac{e^2}{81}-\frac{2}{27}\right)n^4+\left(\frac{8}{9}-\frac{4e^2}{27}\right)n^3+\left(\frac{137e^2}{81}-\frac{172}{27}\right)n^2+\left(\frac{50}{9}-\frac{50e^2}{9}\right)n+36e^2-84\right)$$
$$c_1(n) = 2^n\left(\left(\frac{3e^2}{4}-\frac{9}{2}\right)n^3+\left(\frac{15}{2}-\frac{5e^2}{4}\right)n^2+\left(\frac{17e^2}{2}-19\right)n-108\right)$$
$$+3^n\left(\left(\frac{4}{9}-\frac{2e^2}{27}\right)n^3+\left(\frac{2e^2}{3}-4\right)n^2+\left(\frac{152}{9}-\frac{124e^2}{27}\right)n+36\right)+\left(-8-4e^2\right)n+108$$
$$c_2(n) = 2^n\left(\left(\frac{9}{2}-\frac{3e^2}{4}\right)n^2+\left(\frac{3}{2}-\frac{e^2}{4}\right)n-8e^2\right)$$
$$+3^n\left(\left(\frac{e^2}{9}-\frac{2}{3}\right)n^2+\left(\frac{14}{3}-\frac{7e^2}{9}\right)n+6e^2-20\right)+\left(36+2e^2\right)$$

and finally the coefficients d_i equal to

$$d_0(n) = \left(\left(2e^2-30\right)n^2+\left(18-2e^2\right)n+12e^2-72\right)$$
$$+2^n\left(\left(\frac{3}{8}-\frac{e^2}{16}\right)n^4+\left(\frac{5e^2}{8}-\frac{15}{4}\right)n^3+\left(\frac{129}{8}-\frac{55e^2}{16}\right)n^2+\left(\frac{71e^2}{8}-\frac{123}{4}\right)n-12e^2+60\right)$$
$$d_1(n) = 2^n\left(\left(\frac{e^2}{4}-\frac{3}{2}\right)n^3+\left(\frac{21}{2}-\frac{7e^2}{4}\right)n^2+\left(\frac{7e^2}{2}-21\right)n\right)+\left(24-4e^2\right)n$$
$$d_2(n) = 2^n\left(\left(\frac{3}{2}-\frac{e^2}{4}\right)n^2+\left(\frac{5e^2}{4}-\frac{15}{2}\right)n-2e^2+12\right)+\left(2e^2-12\right)$$

8.4. **Signs of c_0, c_1, c_2 for $n \geq 4$.** Let $n \geq 4$. We show the inequalities

$$c_0(n) > 0, \qquad c_1(n) < 0, \qquad c_2(n) > 0.$$

We start with $c_0(n)$. We have

$$c_0(n) > (2n^2-39n+124)+2^n\left(-0.27n^4+n^3-29n^2+37n-283\right)$$
$$+3^n\left(0.01n^4-0.21n^3+6n^2-36n+182\right).$$

If we now assume that $n \geq 25$ then we can further estimate

$$2n^2 - 39n + 124 > 0, \qquad -0.27n^4 + n^3 - 29n^2 + 37n - 283 > -0.27n^4 - 29n^2 - 283 > -n^4$$

and

$$6n^2 - 36n + 182 > 0, \qquad 0.01n^4 - 0.21n^3 > 0.001n^4$$

thus

$$c_0(n) > -2^n n^4 + 3^n \cdot 0.001n^4 = n^4(3^n \cdot 10^{-3} - 2^n) > 0.$$

The remaining cases are checked directly.

We now show that $c_1(n) < 0$. For $n \geq 23$ we have

$$\begin{aligned} c_1(n) &< 3^n \left(-0.1n^3 + n^2 - 17n + 36\right) + 2^n \left(1.05n^3 - n^2 + 44n - 108\right) - 37n + 108 \\ &< 2^n \left(1.05n^3 + 44n\right) + 3^n \left(-0.1n^3 + n^2 + 36\right) < 2^n \cdot 2n^3 + 3^n(-0.1n^3 + 2n^2) \\ &< 2^n \cdot 2n^3 + 3^n(-0.1n^3 + 2n^2) < 2^n \cdot 2n^3 - 3^n \cdot 0.01n^3 = -n^3(3^n 10^{-3} - 2 \cdot 2^n) < 0. \end{aligned}$$

The cases $4 \leq n \leq 23$ are checked directly.

Finally, we show that $c_2(n) > 0$. We can assume $n \geq 9$ since other cases can be checked directly. We have

$$\begin{aligned} c_2(n) &> 3^n \left(0.15n^2 - 1.1n + 24\right) - 2^n \left(1.05n^2 + 0.35n + 60\right) > 3^n \cdot 0.1n^2 - 2^n \cdot 2n^2 \\ &= n^2(3^n \cdot 0.1 - 2^n \cdot 2) > 0. \end{aligned}$$

8.5. **Bounding terms containing high powers of e^y.** We are going to obtain a useful bound on $P_n(y)$ by showing that

$$e^{2y}(c_0 + c_1 y + c_2 y^2) + e^{3y}(d_0 + d_1 y + d_2 y^2) + e^{4y} e_0 \geq 0.$$

Since $c_1(n) < 0$ and $c_2(n) > 0$, it is enough to show that

$$c_0 + e^y(d_0 + d_1 y + d_2 y^2) + e^{2y} e_0 \geq 0$$

for all $n \geq 4$. Since $e^y \leq 1$, $|e^y y| < 1$ and $e^{2y} y^2 < 1$ it is enough to show that

$$c_0 - (|d_0| + |d_1| + |d_2| + |e_0|) > 0. \tag{10}$$

We have

$$|d_0| \leq 16n^2 + 4n + 17 + 2^n \left(0.1n^4 + n^3 + 10n^2 + 35n + 29\right) < 40n^2 + 2^n \cdot 2n^4 < 2^n \cdot 3n^4.$$

Moreover

$$|d_1| < 2^n \left(0.35n^3 + 2.5n^2 + 5n\right) + 6n < 2^n \cdot 2n^3 \leq \frac{1}{2} \cdot 2^n \cdot n^4$$

and

$$|d_2| < 2^n \left(0.35n^2 + 2n + 3\right) + 3 < 2^n \cdot 2n^2 < \frac{1}{8} \cdot 2^n n^4.$$

Also

$$|e_0| < 6n^2 + 18n + 12 < 12n^2 < \frac{1}{4} 2^n n^4.$$

As a result

$$|d_0| + |d_1| + |d_2| + |e_0| < 2^n \cdot 4n^4$$

We have already seen that for $n \geq 25$ one has

$$c_0 \geq n^4(3^n \cdot 10^{-3} - 2^n).$$

Therefore it is enough to check that $3^n \cdot 10^{-3} > 5 \cdot 2^n$, which is true for $n \geq 25$. For $4 \leq n \leq 24$ the inequality (10) is also valid.

8.6. **Positivity of $P_n(y)$ for $n \ge 8$.** According to the previous subsection, it is enough to prove that

$$a_0 + e^y(b_0 + b_1 y + b_2 y^2) \ge 0.$$

It turns out that for $n \ge 9$ the inequality follows from the trivial bound

$$a_0 + e^y(b_0 + b_1 y + b_2 y^2) \ge a_0 - (|b_0| + e^{-1}|b_1| + 4e^{-2}|b_2|).$$

The right hand side is positive for $9 \le n \le 25$. We have

$$a_0(n) > 3^n \left(n^2 + 22n + 60\right) + 4^n \left(1.125n^2 - 8n - 36\right) - 2^n \left(4n^2 + 14n + 24\right).$$

Moreover

$$\begin{aligned} |b_0| &\le 4^n \left(0.028n^2 + 6n + 142\right) + 2^n \left(8n^2 + 26n + 197\right) \\ &\quad + 3^n \left(0.06n^4 + 0.3n^3 + 7n^2 + n + 291\right) + 36n + 48 \\ |b_1| &\le 3^n \left(0.4n^3 + 0.4n^2 + 6n + 108\right) + 32n + 4^n(3n + 36) + 2^n(30n + 108) + 36 \\ |b_2| &\le 3^n \left(0.5n^2 + 0.5n + 9\right) + 3 \cdot 2^n + 6 \cdot 4^n. \end{aligned}$$

Putting these crude bounds together gives

$$\begin{aligned} a_0 - (|b_0| + e^{-1}|b_1| + 4e^{-2}|b_2|) &\ge a_0 - (|b_0| + |b_1| + |b_2|) \\ &\ge 4^n \left(1.097n^2 - 17n - 220\right) + 2^n \left(-12n^2 - 70n - 332\right) \\ &\quad + 3^n \left(-0.06n^4 - 0.7n^3 - 6.9n^2 + 14.5n - 348\right) - 68n - 84. \end{aligned}$$

If $n \ge 26$ then

$$1.097n^2 - 17n - 220 \ge 0.1n^2, \qquad 12n^2 + 70n + 332 < 16n^2, \qquad 68n + 84 < 2^n n^2$$

and

$$0.06n^4 + 0.7n^3 + 6.9n^2 - 14.5n + 348 < 0.1n^4.$$

Using these estimates we get a lower bound

$$4^n \cdot 0,1n^2 - 2^n \cdot 16n^2 - 3^n \cdot 0.1n^4 - 2^n n^2 = n^2(4^n \cdot 0.1 - 2^n \cdot 17 - 3^n \cdot 0.1n^2)$$

Note that

$$4^n \cdot 0.1 - 2^n \cdot 17 - 3^n \cdot 0.1n^2 \ge 4^n \cdot 0.1 - 3^n \cdot 0.2n^2 > 0.$$

It remains to prove the inequality in the case $n = 8$, in which case

$$1 + e^y \left(\frac{b_0}{a_0} + \frac{b_1}{a_0}y + \frac{b_2}{a_0}y^2\right) > 1 - e^y(1 + y + 0.12y^2).$$

The right hand side is clearly positive for, say, $y < -5$. The derivative of the right hand side is $e^y(-0.12y^2 - 1.24y - 2.)$ which on the interval $[-5, 0]$ has sign pattern $(+, -)$ and thus it is enough to check the inequality for $y = -5$ and $y = 0$.

We have $P_0 \equiv P_1 \equiv P_2 \equiv 0$. Positivity of $P_n(y)$ for $3 \le n \le 7$ is proved in the Appendix. Below we provide a graph of these functions.

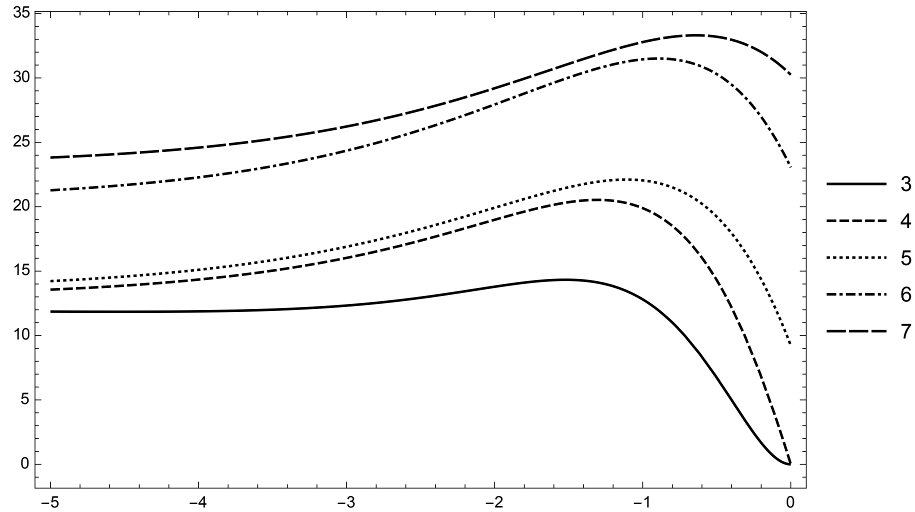

FIGURE 3. The graphs of $P_3(y)$, $\frac{2}{4!}P_4(y)$, $\frac{1}{5!}P_5(y)$, $\frac{1}{6!}P_6(y)$ and $\frac{1}{7!}P_7(y)$.

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9. Appendix

9.1. Functions $P_n(y)$ for the first derivative.

9.1.1. *Positivity of $P_1(y)$.* We have

$$\begin{aligned}6e^{-y}P_1(y) &= \left[\left(18-2e^2\right)y^2-24y-12e^2\right]\\ &\quad+e^y\left[\left(e^2-6\right)y^4+\left(12-2e^2\right)y^3+\left(8e^2-24\right)y^2+\left(48-12e^2\right)y+12e^2\right]\\ &\quad+e^{2y}\left[6y^2-24y\right].\end{aligned}$$

The first step is to forget the positive terms including $e^y y^4$ and $e^y y^3$, which leads to

$$6e^{-y}P_1(y)\geq\left[\left(18-2e^2\right)y^2-24y-12e^2\right]+e^y\left[\left(8e^2-24\right)y^2+\left(48-12e^2\right)y+12e^2\right]+e^{2y}\left[6y^2-24y\right].$$

Case 1: $y<-3$. In this case the first bracket is positive and thus the whole function is positive. *Case 2:* $y\in[-\frac{2}{5},0]$. In this case we use the bound $e^t\geq 1+t+\frac{t^2}{2}+\frac{t^3}{6}$ which gives

$$6e^{-y}P_1(y)\geq\frac{2}{3}y^3\left(\left(6+2e^2\right)y^2+\left(3e^2-36\right)y+6e^2-54\right).$$

We have

$$\left(6+2e^2\right)y^2+\left(3e^2-36\right)y+6e^2-54\leq\left(6+2e^2\right)\frac{4}{25}+\left(3e^2-36\right)\left(-\frac{2}{5}\right)+6e^2-54=-0.808033<0.$$

Case 3: $y\in[-3,-\frac{2}{5}]$. The derivative of the right hand side in the above estimate is

$$4\left(-6e^y\left(y^2-2\right)+3e^{2y}\left(y^2-3y-2\right)-e^2y+e^{y+2}(2y+1)y+9y-6\right)$$

and therefore its absolute value is bounded by

$$4\left(6\left(y^2+2\right)+3\left(y^2+3|y|+2\right)+e^2|y|+e^2(2|y|+1)|y|+9|y|+6\right)\leq 1500.$$

Therefore it is enough to show that for $y_k=-3+(-\frac{2}{5}+3)\frac{k}{50000}$ for $k=0,\dots,50000$ the values of our lower bound are greater than $\frac{1}{10}$, which is the case.

9.1.2. *Positivity of $P_2(y)$.* We have

$$\begin{aligned}P_2(y)&=e^{3y}\left(\frac{y^2}{2}-4y+3\right)+e^y\left(\left(\frac{15}{2}-\frac{5e^2}{6}\right)y^2+\left(\frac{2e^2}{3}-16\right)y-5e^2+1\right)\\ &\quad+e^{2y}\left(\left(\frac{e^2}{4}-\frac{3}{2}\right)y^4+\left(7-\frac{7e^2}{6}\right)y^3+\left(3e^2-12\right)y^2+\left(20-\frac{17e^2}{3}\right)y+5e^2-5\right)+1.\end{aligned}$$

We can again neglect positive terms

$$e^{2y}\left(\left(\frac{e^2}{4}-\frac{3}{2}\right)y^4+\left(7-\frac{7e^2}{6}\right)y^3\right)\qquad\text{and}\qquad e^y\left(\frac{15}{2}-\frac{5e^2}{6}\right)y^2\qquad\text{and}\qquad e^{3y}\frac{y^2}{2}$$

getting a bound

$$P_2(y)\geq e^{2y}\left(\left(3e^2-12\right)y^2+\left(20-\frac{17e^2}{3}\right)y+5e^2-5\right)+e^y\left(\left(\frac{2e^2}{3}-16\right)y-5e^2+1\right)+e^{3y}(3-4y)+1.$$

Case 1. $y<-4$. In this case $\left(\frac{2e^2}{3}-16\right)y-5e^2+1>0$ and the rest of the terms are also positive.

Case 2. $y \in [-\frac{1}{10}, 0]$. In this case we can assume that the bracket multiplied by e^y is negative, since otherwise there is nothing to prove. For this term we are going to use the bound $e^y \leq 1 + y + \frac{y^2}{2}$ and for the other exponents the bound $e^t \geq 1 + t + \frac{t^2}{2} + \frac{t^3}{6}$, yielding

$$P_2(y) \geq \frac{y^2}{18}\left(\left(72e^2 - 288\right)y^3 + \left(-276 - 28e^2\right)y^2 + \left(30e^2 - 57\right)y - 3e^2 + 72\right).$$

The polynomial

$$\left(72e^2 - 288\right)y^3 + \left(-276 - 28e^2\right)y^2 + \left(30e^2 - 57\right)y - 3e^2 + 72$$

clearly increases with y (checking the signs of the coefficients) at thus it is enough to show that it is positive for $y = \frac{1}{10}$, which is the case.

Case 3. $y \in [-4, -\frac{1}{10}]$. The derivative of the estimate is

$$-\frac{1}{3}e^y\left(6e^y\left(12y^2 - 8y - 5\right) + e^{y+2}\left(-18y^2 + 16y - 13\right) + 48y + 3e^{2y}(12y - 5) + e^2(13 - 2y) + 45\right)$$

and thus its absolute value can be upper bounded by

$$\frac{1}{3}\left(6\left(12y^2 + 8|y| + 5\right) + e^2\left(18y^2 + 16|y| + 13\right) + 48|y| + 3(12|y| + 5) + e^2(13 + 2|y|) + 45\right)$$

which is maximized for $y = -4$ and does not exceed 1600. Therefore it is enough to show that for $y_k = -4 + (-\frac{1}{10} + 4)\frac{k}{10^6}$ for $k = 0, \ldots, 10^6$ the values of our lower bound are greater than 0.02, which is the case.

9.1.3. *Positivity of $P_3(y)$.* We have

$$P_3(y) = 3 + e^{3y}\left(\frac{y^2}{6} - 2y + 3\right) + e^y\left(\left(\frac{59}{6} - \frac{10e^2}{9}\right)y^2 + \left(\frac{5e^2}{3} - 28\right)y - 7e^2 + 7\right)$$
$$+ e^{2y}\left(\left(\frac{7e^2}{36} - \frac{7}{6}\right)y^4 + \left(\frac{26}{3} - \frac{13e^2}{9}\right)y^3 + \left(\frac{37e^2}{9} - 20\right)y^2 + \left(28 - \frac{23e^2}{3}\right)y + 7e^2 - 13\right).$$

We can neglect all the terms having powers of y greater than 1. We get

$$P_3(y) \geq e^{3y}(-2y + 3) + e^y\left(\left(\frac{5e^2}{3} - 28\right)y - 7e^2 + 7\right) + e^{2y}\left(\left(28 - \frac{23e^2}{3}\right)y + 7e^2 - 13\right) + 3.$$

Case 1. $y < -3$. In this case the bracket multiplied by e^y is positive and there is nothing to prove.

Case 2. $y \in [-3, 0]$. We shall show that the bound is decreasing on $[-3, 0]$. The derivative is

$$-\frac{1}{3}e^y\left(84y + 3e^{2y}(6y - 7) - 6e^y(28y + 1) + e^{y+2}(46y - 19) + e^2(16 - 5y) + 63\right).$$

It is therefore enough to show that

$$84y + 3e^{2y}(6y - 7) - 6e^y(28y + 1) + e^{y+2}(46y - 19) + e^2(16 - 5y) + 63$$

is positive. The derivative of this expression is

$$12e^{2y}(3y - 2) - 6e^y(28y + 29) + e^{y+2}(46y + 27) - 5e^2 + 84$$

which in absolute value is bounded by

$$12(3|y| + 2) + 6(28|y| + 29) + e^2(46|y| + 27) + 5e^2 + 84 < 2200.$$

Therefore it is enough to show that for $y_k = -3\frac{k}{1500}$ for $k = 0, \ldots, 1500$ the values of our lower bound are greater than 0.02, which is the case.

9.1.4. *Positivity of* $P_4(y)$. We have the bound

$$e^y(-357y-894)+e^{2y}(786-569y)+108$$

and the first term is positive for $y<-2.6$. Assume $y\in[-2.6,0]$. It is enough to show that this function is decreasing on this interval. Its derivative is

$$-e^y\left(357y+e^y(1138y-1003)+1251\right)$$

and thus it is enough to show that

$$357y+e^y(1138y-1003)+1251\geq 0$$

Dividing by 357 and estimating gives the goal

$$y+e^y\left(\frac{7y}{2}-3\right)+\frac{7}{2}\geq 0 \qquad y\in[-2.6,0]$$

The derivative of this function is $\frac{1}{2}e^y(7y+1)+1$ whose absolute value can be upper bounded by $\frac{7|y|}{2}+\frac{3}{2}<10$. Therefore it is enough to show that for $y_k=-2.6\frac{k}{3000}$ for $k=0,\dots,3000$ the values of our lower bound are greater than 0.001, which is the case.

9.1.5. *Positivity of* $P_5(y)$. We have

$$P_5(y)\geq e^y(-1277y-2758)+2525e^{2y}+540$$

The first term is positive for $y<-2.2$. Let us therefore assume that $y\in[-2.2,0]$. The derivative of the estimate is

$$e^y\left(-1277y+5050e^y-4035\right)$$

So its absolute value is upper bounded by

$$1277|y|+5050+4035\leq 12000$$

Therefore it is enough to show that for $y_k=-2.2\frac{k}{300}$ for $k=0,\dots,300$ the values of our lower bound are greater than 100, which is the case.

9.2. **Functions $P_n(y)$ for the second derivative.**

9.2.1. *Positivity of* $P_2(y)$. We have

$$\begin{aligned}P_2(y)=&\frac{2}{3}e^y\left(\left(6-e^2\right)y^2-18y-6e^2-6\right)\\&+\frac{2}{3}e^{2y}\left(\left(6-e^2\right)y^3+\left(6+e^2\right)y^2+\left(24-6e^2\right)y+6e^2-6\right)+4e^{3y}(1-y)+4\end{aligned}$$

Case 1: $y<-3$. Note that

$$P_2(y)\geq\frac{2}{3}e^y\left(\left(6-e^2\right)y^2-18y-6e^2-6\right)+4\geq 4-e^y(y^2+34).$$

To prove that this is positive it is enough to observe that it is positive for $y=-3$ and then show that the expression is decreasing. Equivalently, we want to show that $e^y(y^2+34)$ is increasing. The derivative is $e^y\left(y^2+2y+34\right)=e^y\left((y+1)^2+33\right)>0$

Case 2: $y\in[-\frac{1}{2},0]$. It is enough to show that the derivative on this interval is negative. The derivative is of the form $-\frac{2}{3}e^yR(y)$, where

$$\begin{aligned}R(y)=&\left(\left(e^2-6\right)y^2+\left(6+2e^2\right)y+6e^2+24\right)+e^{2y}(18y-12)\\&+e^y\left(\left(2e^2-12\right)y^3+\left(e^2-30\right)y^2+\left(10e^2-60\right)y-6e^2-12\right).\end{aligned}$$

Using estimates $e^t \leq 1 + t + \frac{t^2}{2}$ for $t < 0$ we get the bound

$$\begin{aligned} R(y) &\geq \frac{1}{2}\left(2e^2 - 12\right) y^5 + \frac{1}{2}\left(5e^2 - 54\right) y^4 + \frac{1}{2}\left(16e^2 - 72\right) y^3 + \frac{1}{2}\left(18e^2 - 180\right) y^2 + \frac{1}{2}\left(12e^2 - 144\right) y \\ &\geq 2y^5 - 8y^4 + 24y^3 - 24y^2 - 27y. \end{aligned}$$

The positivity of this expression is equivalent with the positivity of

$$-2y^4 + 8y^3 - 24y^2 + 24y + 27.$$

Here the worst case is $y = -\frac{1}{2}$.

Case 3: $y \in [-3, -\frac{1}{2}]$. We crudely bound the derivative

$$\begin{aligned} |P_2'(y)| \leq |R(y)| &\leq \left(6 + e^2\right) y^2 + \left(\left(12 + 2e^2\right) |y|^3 + \left(30 + e^2\right) y^2 + \left(60 + 10e^2\right) |y| + 6e^2 + 12\right) \\ &\qquad + \left(6 + 2e^2\right) |y| + (18|y| + 12) + 6e^2 + 24 \\ &\leq 27y^3 + 51y^2 + 173y + 137 < 2000. \end{aligned}$$

Therefore it is enough to show that for $y_k = -3 + (-\frac{1}{2} + 3)\frac{k}{6000}$ for $k = 0, \ldots, 6000$ the values of our lower bound are greater than 1, which is the case.

9.2.2. *Positivity of $P_3(y)$.* We have

$$\begin{aligned} P_3(y) &= \frac{20}{3} + \frac{2}{9}e^y \left(\left(24 - 4e^2\right) y^2 + \left(3e^2 - 96\right) y - 24e^2 + 9\right) \\ &\quad + \frac{1}{9}e^{2y} \left(\left(30 - 5e^2\right) y^3 + \left(14e^2 - 48\right) y^2 + \left(96 - 36e^2\right) y + 48e^2 - 108\right) + \frac{2}{3}e^{3y}(5 - 2y). \end{aligned}$$

After neglecting various positive terms we get a bound

$$\begin{aligned} P_3(y) &\geq \frac{2}{9}e^y \left(\left(24 - 4e^2\right) y^2 + \left(3e^2 - 96\right) y - 24e^2 + 9\right) \\ &\quad + \frac{1}{9}e^{2y} \left(\left(96 - 36e^2\right) y + 48e^2 - 108\right) + \frac{10e^{3y}}{3} + \frac{20}{3}. \end{aligned}$$

Case 1: $y < -2.1$. Forgetting about additional positive terms we get

$$P_3(y) \geq \frac{20}{3} + \frac{2}{9}e^y \left(\left(24 - 4e^2\right) y^2 + (-24e^2 + 9)\right) \geq 6 - e^y(2y^2 + 38).$$

The derivative of $e^y(2y^2 + 38)$ is $2e^y\left(y^2 + 2y + 19\right)$ and it therefore positive. Therefore the bound is decreasing in y and so it is enough to observe that it is positive for $y = -2.1$.

Case 2: $y \in [-2.1, 0]$. The derivative of the estimate is $\frac{2}{9}e^y R(y)$ where

$$R(y) = \left(\left(24 - 4e^2\right) y^2 + \left(-48 - 5e^2\right) y - 21e^2 - 87\right) + 45e^{2y} + e^y \left(-36e^2 y + 96y + 30e^2 - 60\right).$$

It is enough to show that $R(y) < 0$. The derivative of R is

$$R'(y) = \left(48 - 8e^2\right) y - 5e^2 - 48 + 90e^{2y} + e^y(\left(96 - 36e^2\right) y - 6e^2 + 36).$$

Thus

$$|R'(y)| \leq 6 + 12|y| + 90 + 9 + 171|y| = 105 + 183|y| < 500.$$

Therefore it is enough to show that for $y_k = -2.1\frac{k}{200}$ for $k = 0, \ldots, 200$ the values of $R(y)$ are smaller than -20, which is the case.

9.2.3. *Positivity of $P_4(y)$.* We use the bound

$$P_4(y) \geq a_0 + e^y(b_0 + b_1 y + b_2 y^2) + e^{2y} c_0.$$

Therefore, bounding the coefficients crudely

$$a_0^{-1} P_5(y) \geq e^y \left(-0.2y^2 - 1.5y - 3.7\right) + 2.8e^{2y} + 1. \tag{11}$$

Assume that $y < -3$. Then

$$e^y \left(-0.2y^2 - 1.5y - 3.7\right) + 2.8e^{2y} + 1 \geq e^y \left(-0.2y^2 - 3.7\right) + 1.$$

Note that for $y < -3$ one has $e^y y^2 < \frac{1}{2}$ and thus

$$|e^y \left(-0.2y^2 - 3.7\right)| \leq 0.1 + 3.7e^{-3} < 1.$$

and therefore the inequality holds in this case.

For $y \in [-3, 0]$ we use the netting argument. The derivative of the right hand side of (11) is

$$e^y \left(-0.2y^2 - 1.5y - 3.7\right) + e^y(-0.4y - 1.5) + 5.6e^{2y}$$

and is therefore upper bounded in absolute value by

$$\left(0.2y^2 + 1.5|y| + 3.7\right) + (0.4|y| + 1.5) + 5.6 \leq 20.$$

Therefore it is enough to show that for $y_k = -3\frac{k}{2000}$ for $k = 0, \ldots, 2000$ the values of of the right hand side of (11) is greater than 0.05, which is the case.

9.2.4. *Positivity of $P_5(y)$.* We are going to use the bound

$$P_5(y) \geq a_0 + e^y(b_0 + b_1 y + b_2 y^2) + e^{2y} c_0.$$

Therefore, again bounding the coefficients crudely

$$a_0^{-1} P_5(y) \geq e^y \left(-y^2 - y - 3\right) + 2e^{2y} + 1.$$

The right hand side vanishes for $y = 0$ and therefore it is enough to show that the right hand side is decreasing. The derivative is equal to $e^y \left(4e^y - 4 - 3y - y^2\right)$. To show that $4e^y - 4 - 3y - y^2$ it is enough to show that it is increasing. The derivative is $4e^y - 3 - 2y$. The minimum of this function is achieved for $y = \log 2$ at is equal to $-1 + 2\log 2 > 0$.

9.2.5. *Positivity of $P_6(y)$.* We have

$$P_6(y) \geq a_0 + e^y(b_0 + b_2 y^2) + e^{2y} c_0.$$

Therefore, again bounding the coefficients crudely gives

$$a_0^{-1} P_6(y) \geq e^y \left(-y^2 - 2\right) + e^{2y} + 1.$$

It is enough to show that the derivative of the right hand side is negative. This derivative is equal to $2e^y \left(e^y - 1 - y - \frac{y^2}{2}\right)$ and the inequality $e^y \leq 1 + y + \frac{y^2}{2}$ is well known.

9.3. **Functions $P_n(y)$ for the third derivative.** Note that for $n = 5, 6, 7$ we have $c_0 > 0$, $c_1 < 0$, $c_2 > 0$ and $e_0 < 0$. Thus we have the bound

$$\begin{aligned} P_n(y) &> a_0 + e^y(b_0 + b_1 y + b_2 y^2) + e^{3y}(c_0 + c_1 y + c_2 y^2) + e^{3y}(d_0 + d_1 y + d_2 y^2) + e^{3y} e_0 \\ &= a_0 + e^y(b_0 + b_1 y + b_2 y^2) + e^{3y}(c_0 + d_0 + e_0 + (c_1 + d_1)y + (c_2 + d_2)y^2). \end{aligned}$$

It is also true that $c_1 + d_1 < 0$ and $c_2 + d_2 > 0$ and thus

$$P_n(y) \geq a_0 + e^y(b_0 + b_1 y + b_2 y^2) + e^{3y}(c_0 + d_0 + e_0).$$

9.3.1. *Positivity of $P_7(y)$.* We have

$$a_0^{-1}P_7(y) > 1 + e^y\left(-0.15y^2 - 1.6y - 1.6\right) + 1.8e^{3y} > 0.9 - 1.6e^y + 1.8e^{3y}$$

It is therefore enough to show that for $t \in [0,1]$ one has $0.9 - 1.6t + 1.8t^3 > 0$. By AM-GM we have

$$0.9 + 1.8t^3 = 0.45 + 0.45 + 1.8t^3 > 3t\sqrt[3]{0.45^2 \cdot 1.8} > 1.6t$$

9.3.2. *Positivity of $P_6(y)$.* We have

$$a_0^{-1}P_6(y) > 1 + e^y\left(-0.2y^2 - 2y - 2.8\right) + 2.8e^{3y} > 0.89 + e^y\left(-2y - 2.8\right) + 2.8e^{3y}$$

The right hand side is clearly positive for $y < -1.4$. Assume $y \in [-1.4, 0]$. Then the derivative of the right hand side is $e^y\left(-2y + 8.4e^{2y} - 4.8\right)$ and can be upper bounded in absolute value by 20. Therefore it is enough to show that for $y_k = -1.4\frac{k}{200}$ for $k = 0, \ldots, 200$ the values of the right hand side are greater than 0.2, which is the case.

9.3.3. *Positivity of $P_5(y)$.* We have

$$a_0^{-1}P_5(y) > 1 + e^y\left(-0.24y^2 - 3.45y - 4.68\right) + 4.3e^{3y} \tag{12}$$

We first assume that $y < -2$. In this case

$$1 + e^y\left(-0.24y^2 - 3.45y - 4.68\right) + 4.3e^{3y} > (1 - 0.24e^y y^2) + e^y(-3.45y - 4.68) > 0$$

Assume $y \in [-2, 0]$. The derivative of the right hands side of (12) is

$$e^y\left(-0.24y^2 - 3.93y + 12.9e^{2y} - 8.13\right)$$

and its absolute value can be upper bounded by 25. Therefore it is enough to show that for $y_k = -2\frac{k}{5000}$ for $k = 0, \ldots, 5000$ the values of the right hand side are greater than 0.01, which is the case.

9.3.4. *Positivity of $P_4(y)$.* Let us first assume that $y < -2.5$. Then

$$P_4(y) > 1 + e^{3y}\left(-0.13y^2 + 0.15y - 2\right) + e^y\left(-0.27y^2 - 7\right) - 0.24e^{4y}$$

and

$$\begin{aligned}&|e^{3y}\left(-0.13y^2 + 0.15y - 2\right) + e^y\left(-0.27y^2 - 7\right) - 0.24e^{4y}|\\ &\qquad \le 0.005 \cdot 0.13 + 0.002 \cdot 0.15 + 0.002 + 0.52 \cdot 0.27 + 0.58 + 0.24 \cdot 0.001 < 0.73 < 1\end{aligned}$$

Therefore the inequality holds in this case. Let $y \in [-1, 0]$. We have the bound $P_4(y) \ge g(y)$ where

$$g(y) = a_0 + e^y(b_0 + b_1 y + b_2 y^2) + e^{2y}c_0 + e^{3y}(d_0 + d_1 y + d_2 y^2) + e^{4y}e_0$$

We are going to show that $g'(y) < 0$ on $[-1, 0]$. We have

$$e^{-y}g'(y) = (b_0 + b_1 + (b_1 + 2b_2)y + b_2y^2) + e^y 2c_0 + e^{2y}(3d_0 + d_1 + (3d_1 + 2d_2)y + 3d_2y^2) + e^{3y}4e_0$$

and

$$(e^{-y}g'(y))' = b_1 + 2b_2 + 2b_2y + e^y 2c_0 + e^{2y}(6d_0 + 5d_1 + 2d_2 + (6d_1 + 10d_2)y + 6d_2y^2) + e^{3y}12e_0$$

We therefore have

$$|(e^{-y}g'(y))'| \le |b_1| + 4|b_2| + 2|c_0| + 6|d_0| + 11|d_1| + 18|d_2| + 12|e_0| < 6013$$

Therefore it is enough to show that for $y_k = -\frac{k}{100}$ for $k = 0, \ldots, 100$ the values of $e^{-y}g'(y)$ are smaller than -100, which is the case.

We are now left with $y \in [-2.5, -1]$. We shall show that $g(y) > 0$. We have

$$|g'(y)| \le |b_0| + 2|b_1| + 3|b_2| + 2|c_0| + 3|d_0| + 4|d_1| + 5|d_2| + 4|e_0| < 6113$$

Therefore it is enough to show that for $y_k = -2.5 + (-1 + 2.5)\frac{k}{200}$ for $k = 0, \ldots, 200$ the values of $e^{-y}g'(y)$ are greater than 100, which is the case.

9.3.5. *Positivity of $P_3(y)$.* Let us first assume that $y \in [-0.1, 0]$. Then it is easy to check that

$$b_0 + b_1 y + b_2 y^2 < 0, \qquad c_0 + c_1 y + c_2 y^2 > 0, \qquad d_0 + d_1 y + d_2 y^2 < 0, \qquad e_0 < 0$$

and therefore using standard bounds $e^t \le 1 + t + \frac{1}{2}t^2$ and $e^t \ge 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3$ for $t < 0$ we get

$$\begin{aligned} P_3(y) \ge{}& \frac{4}{3}\left(72 + 4e^2\right) y^5 + \left(2\left(72+4e^2\right) + 5\left(12 - 2e^2\right) + 32\right) y^4 \\ &+ \left(2\left(72+4e^2\right) + 32e^2 + 4\left(12-2e^2\right) - 28\right) y^3 + \left(2\left(12-2e^2\right) - 8e^2 + 120\right) y^2 \\ \ge{}& y^2(136y^3 + 221y^2 + 401y + 55) > y^2(0.136 - 40.1 + 55) > 0. \end{aligned}$$

Assume now that $y < -2.5$. We have

$$P_3(y) \ge e^y\left(-3y^2 - 89\right) + e^{3y}\left(-3y^2 - 41\right) + 24e^{2y}y - 12e^{4y} + 12$$

and

$$e^y\left(3y^2 + 89\right) + e^{3y}\left(3y^2 + 41\right) + 24e^{2y}|y| + 12e^{4y}$$

This expression is increasing in y for $y < -2.5$ and thus it is maximized for $y = -2.5$ in which case the value is smaller than 10.

We are now left with the case $y \in [-2.5, -0.1]$. We have

$$\begin{aligned} P_3'(y) ={}& e^{2y}\left(\left(144 + 8e^2\right)y^2 + \left(192 + 8e^2\right)y + 48e^2 - 72\right) + e^y\left(\left(12 - 2e^2\right)y^2 - 4e^2 y - 12e^2 - 24\right) \\ &+ e^{3y}\left(\left(36 - 6e^2\right)y^2 + \left(24 - 4e^2\right)y - 36e^2 + 144\right) - 48e^{4y} \end{aligned}$$

Therefore

$$|P_3'(y)| \le 144 + 8e^2 + 192 + 8e^2 + 48e^2 - 72 + |12 - 2e^2| + 4e^2 + 12e^2 + 24 + |36 - 6e^2| + |24 - 4e^2| + |-36e^2 + 144| + 48$$

This is smaller than 1070. Therefore it is enough to show that for $y_k = -2.5 + (-0.1 + 2.5)\frac{k}{7000}$ for $k = 0, \ldots, 7000$ the values of $e^{-y}g'(y)$ are greater than 0.4, which is the case.