A NEW CHARACTERIZATION OF TALAGRAND'S TRANSPORT-ENTROPY INEQUALITIES AND APPLICATIONS

NATHAEL GOZLAN, CYRIL ROBERTO, PAUL-MARIE SAMSON

A . We show that Talagrand's transport inequality is equivalent to a restricted logarithmic Sobolev inequality. This result clarifies the links between these two important functional inequalities. As an application, we give the first proof of the fact that Talagrand's inequality is stable under bounded perturbations.

1. I

Talagrand's transport inequality and the logarithmic Sobolev inequality are known to share important features: they both hold for the Gaussian measure in any dimension, they enjoy the tensorization property and they imply Gaussian concentration results. We refer to [30, 18, 1, 15] for surveys about these notions. Otto and Villani [25] proved that the logarithmic Sobolev inequality implies, in full generality, Talagrand's transport inequality (see also [5]) and under a curvature condition, that the converse also holds (see also [14]). However, since the work by Cattiaux and Guillin [8], it is known that the two inequalities are not equivalent, in general.

In this paper, we prove that Talagrand's transport inequality is actually equivalent to some restricted form of the logarithmic Sobolev inequality. Our strategy easily generalizes to other transport inequalities. As a byproduct we obtain an elementary and direct proof of the fact that transport inequalities can be perturbed by bounded functions.

In order to present our main results, we need some definitions and notation.

1.1. **Definitions and notation.** In all what follows, $c : \mathbb{R}^k \to \mathbb{R}^+$ is a differentiable function such that $c(0) = \nabla c(0) = 0$. Let μ and ν be two probability measures on $\mathbb{R}^k$; the *optimal transport cost* between ν and μ (with respect to the cost function c) is defined by

$$\mathcal{T}_c(\nu,\mu) := \inf_{\pi} \left\{ \iint c(x-y)\, d\pi(x,y) \right\},$$

where the infimum is taken over all the probability measures π on $\mathbb{R}^k \times \mathbb{R}^k$ with marginals ν and μ. Optimal transport costs are used in a wide class of problems, in statistics, probability and PDE theory, see [30]. Here we shall focus on the following transport inequality.

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Definition 1.1 (Transportation-cost inequality $\mathbf{T}_c(C)$)**.** *A probability measure μ on $\mathbb{R}^k$ satisfies $\mathbf{T}_c(C)$, with $C > 0$, if*

$$(\mathbf{T}_c(C)) \qquad \mathcal{T}_c(\nu,\mu) \le CH(\nu|\mu), \qquad \forall \nu \in \mathcal{P}(\mathbb{R}^k),$$

where

$$H(\nu|\mu) = \begin{cases} \int \log \frac{d\nu}{d\mu}\, d\nu & \text{if } \nu \ll \mu \\ +\infty & \text{otherwise} \end{cases}$$

is the relative entropy of ν with respect to μ and $\mathcal{P}(\mathbb{R}^k)$ is the set of all probability measures on $\mathbb{R}^k$.

The inequality $\mathbf{T}_c(C)$ implies concentration results as shown by Marton [20], see also [6], [18], and [15] for a full introduction to this notion.

The quadratic cost $c(x) = |x|^2/2$ (where $|\cdot|$ stands for the Euclidean norm) plays a special role. In this case, we write $\mathbf{T}_2(C)$ and say that Talagrand's transport, or the quadratic transport, inequality is satisfied. Talagrand proved in [29], among other results, that the standard Gaussian measure satisfies $\mathbf{T}_2(1)$ in all dimensions. In turn, Inequality $\mathbf{T}_2(C)$ implies dimension free Gaussian concentration results. Recently, the first author showed that the converse is also true, namely that a dimension free Gaussian concentration result implies $\mathbf{T}_2(C)$ [14].

Now, we introduce the notion of restricted logarithmic Sobolev inequalities. To that purpose, we need first to define K-semi-convex functions.

Definition 1.2 (K-semi-convex function)**.** *A function $f : \mathbb{R}^k \to \mathbb{R}$ is K-semi-convex ($K \in \mathbb{R}$) for the cost function c if for all $\lambda \in [0,1]$, and all $x, y \in \mathbb{R}^k$*

$$(1.3) \quad f(\lambda x+(1-\lambda)y) \le \lambda f(x)+(1-\lambda)f(y)+\lambda Kc((1-\lambda)(y-x))+(1-\lambda)Kc(\lambda(y-x)).$$

As shown in Proposition 5.1 below, for differentiable functions, (1.3) is equivalent to the condition

$$f(y) \ge f(x) + \nabla f(x)\cdot(y-x) - Kc(y-x), \qquad \forall x, y \in \mathbb{R}^k.$$

The reader might see the semi-convexity as an answer to the question: how far is the function f from being convex? The quadratic case $c(x) = \frac{1}{2}|x|^2$ is particularly enlightening since a function f is K-semi-convex if and only if $x \mapsto f(x) + \frac{K}{2}|x|^2$ is convex. Note that the semi-convexity can be related to the notion of convexity-defect, see *e.g.* [3] and references therein where it is largely discussed and used. Note also that our definition differs from others, such as [30][Definition 10.10] or [10][Lemma 3 in Chapter 3, page 130].

Dealing only with semi-convex functions leads to the following definition.

Definition 1.4 (Restricted (modified) logarithmic Sobolev inequality)**.** *A probability measure μ on $\mathbb{R}^k$ verifies* the restricted logarithmic Sobolev inequality *with constant $C > 0$, in short* **rLSI**(C)*, if for all $0 \le K < \frac{1}{C}$ and all K-semi-convex $f : \mathbb{R}^k \to \mathbb{R}$,*

$$(\mathbf{rLSI}(C)) \qquad \mathrm{Ent}_\mu\left(e^f\right) \le \frac{2C}{(1-KC)^2}\int |\nabla f|^2 e^f\, d\mu,$$

where $\mathrm{Ent}_\mu(g) := \int g \log g \, d\mu - \int g \, d\mu \log \int g \, d\mu$. *More generally, a probability measure* μ *on* $\mathbb{R}^k$ *verifies the* restricted modified logarithmic Sobolev inequality *with constant* $C > 0$ *for the cost* c, *in short* **rMLSI**(c, C), *if for all* $K \geq 0$, $\eta > 0$ *with* $\eta + K < 1/C$ *and all* K*-semi-convex* $f : \mathbb{R}^k \to \mathbb{R}$ *for the cost* c,

$$(\mathbf{rMLSI}(c, C)) \qquad \mathrm{Ent}_\mu\left(e^f\right) \leq \frac{\eta}{1 - C(\eta + K)} \int c^*\left(\frac{\nabla f}{\eta}\right) e^f \, d\mu,$$

where $c^*(u) := \sup_{h \in \mathbb{R}^k} \{u \cdot h - c(h)\}$ *and* $u \cdot h$ *is the usual scalar product in* $\mathbb{R}^k$.

Note that **rMLSI**(c, C) reduces to **rLSI**(C) for $c(x) = c^*(x) = \frac{1}{2}|x|^2$, optimizing over η.

Without the restriction on the set of K-semi-convex functions, the first inequality corresponds to the usual logarithmic Sobolev inequality introduced by Gross [16] (see also [27]). For the second one (without the restriction) we recognize the modified logarithmic Sobolev inequalities introduced first by Bobkov and Ledoux [7], with $c^*(t) = 2|t|^2/(1-\gamma)$ for $|t| \leq \gamma$ and $c^*(t) = +\infty$ otherwise, $t \in \mathbb{R}$, in order to recover the celebrated result by Talagrand [28] on the concentration phenomenon for products of exponential measures. Gentil, Guillin and Miclo [11] established modified logarithmic Sobolev inequalities for products of the probability measures $d\nu_p(t) = e^{-|t|^p}/Z_p$, $t \in \mathbb{R}$ and $p \in (1, 2)$, with $c^*(t)$ that compares to $\max(t^2, |t|^q)$ where $q = p/(p-1) \in (2, \infty)$ is the dual exponent of p. In a subsequent paper [12] they generalized their results to a large class of measures with tails between exponential and Gaussian (see also [4] and [13]). In [11] the authors also prove that the modified logarithmic Sobolev inequality (without the restriction, and with $c^*(t)$ that compares to $\max(t^2, |t|^q)$) implies the corresponding transport inequality $\mathbf{T}_c(C)$.

Our results below show that the functional inequalities **rMLSI**$(c, \cdot)$ and $\mathbf{T}_c(\cdot)$ are equivalent (up to universal factors in the constants). To give a more complete description of this equivalence, let us consider yet another type of logarithmic Sobolev inequalities that we call inf-convolution logarithmic Sobolev inequality.

Definition 1.5 (Inf-convolution logarithmic Sobolev inequality)**.** *A probability measure* μ *on* $\mathbb{R}^k$ *verifies* the inf-convolution logarithmic Sobolev inequality *with constant* $C > 0$, *in short* **ICLSI**(c, C), *if for all* $\lambda \in (0, 1/C)$ *and all* $f : \mathbb{R}^k \to \mathbb{R}$,

$$(\mathbf{ICLSI}(c, C)) \qquad \mathrm{Ent}_\mu(e^f) \leq \frac{1}{1 - \lambda C} \int \left(f - Q^\lambda f\right) e^f \, d\mu,$$

where $Q^\lambda f : \mathbb{R}^k \to \mathbb{R}$ *denotes the infimum-convolution of* f:

$$Q^\lambda f(x) = \inf_{y \in R^k} \{f(y) + \lambda c(x - y)\}.$$

1.2. **Main results.** Our first main result is the following.

Theorem 1.6. *Let* $\alpha : \mathbb{R} \to \mathbb{R}^+$ *be a convex symmetric function of class* C^1 *such that* $\alpha(0) = \alpha'(0) = 0$, α' *is concave on* $\mathbb{R}^+$. *Define* $c(x) = \sum_{i=1}^k \alpha(x_i)$ *and let* μ *be a probability measure on* $\mathbb{R}^k$. *The following propositions are equivalent:*

(1) There exists $C_1 > 0$ *such that* μ *verifies the inequality* $\mathbf{T}_c(C_1)$.

(2) There exists $C_2 > 0$ such that μ verifies the inequality **ICLSI**(c, C_2).
(3) There exists $C_3 > 0$ such that μ verifies the inequality **rMLSI**(c, C_3).

The constants C_1, C_2 and C_3 are related in the following way:

$$(1) \Rightarrow (2) \Rightarrow (3) \quad \textit{with} \quad C_1 = C_2 = C_3$$
$$(3) \Rightarrow (1) \quad \textit{with} \quad C_1 = 8C_3.$$

The typical example of function α satisfying the setting of Theorem 1.6 is a smooth version of $\alpha(x) = \min(x^2, x^p)$, with $p \in [1, 2]$.

The first part (1) $\Rightarrow$ (2) $\Rightarrow$ (3) actually holds in a more general setting (see Theorem 2.1), it is proven in Section 2. Moreover the inequality **ICLSI**(c, C) has a meaning even if $\mathbb{R}^k$ is replaced by an abstract metric space X. The proof of the second part (3) $\Rightarrow$ (1) is given in Section 3. It uses the Hamilton-Jacobi approach of [5] based on explicit computations on the sup-convolution semi-group (Hopf-Lax formula). An alternative proof of (3) $\Rightarrow$ (1), with a worst constant, is given in the subsequent Section 4 in the particular case of the quadratic cost $c(x) = |x|^2/2$. We believe that such an approach may lead to further developments in the future and so that it is worth mentioning it.

In order to keep the arguments as clean as possible and to go straight to the proofs, we decided to collect most of results on semi-convex functions, and most of the technical lemmas, in an independent section (Section 5).

Finally, we present some extensions and comments in Section 6. We first give an extension of our main Theorem 2.1 to Riemannian manifolds verifying a certain curvature condition (see Theorem 6.11). Then, in Section 6.2, we show that other types of logarithmic Sobolev inequalities can be derived from transport inequalities (see Theorem 6.13). The last Section 6.3 is a discussion on the links between Poincaré inequality and (restricted) modified logarithmic Sobolev inequality.

Let us end this introduction with an important application of Theorem 1.6. It is well known that many functional inequalities of Sobolev type are stable under bounded perturbations. The first perturbation property of this type was established by Holley and Stroock in [17] for the logarithmic Sobolev inequality:

Theorem 1.7 (Holley-Stroock)**.** *Let μ be a probability measure verifying the logarithmic Sobolev inequality with a constant $C > 0$ (***LSI**(C) *for short):*

$$\mathrm{Ent}_\mu(f^2) \leq C \int |\nabla f|^2 \, d\mu, \qquad \forall f.$$

Let φ be a bounded function; then the probability measure $d\tilde{\mu} = \frac{1}{Z} e^{\varphi} \, d\mu$ verifies **LSI** *with the constant $\tilde{C} = e^{\mathrm{Osc}(\varphi)} C$, where the oscillation of φ is defined by*

$$\mathrm{Osc}(\varphi) = \sup \varphi - \inf \varphi.$$

A longstanding open question was to establish such a property for transport inequalities. We have even learned from C. Villani that this question was one of the initial motivations behind the celebrated work [25]. The representation furnished by Theorem 1.6 is the key that enables us to give the first bounded perturbation property for transport inequalities. The following corollary is our second main result.

Corollary 1.8. *Let α be a convex symmetric function of class C^1 such that $\alpha(0) = \alpha'(0) = 0$, α' is concave on $\mathbb{R}^+$. Let $c(x) = \sum_{i=1}^k \alpha(x_i)$ and μ be a probability measure on $\mathbb{R}^k$. Assume that μ verifies $\mathbf{T}_c(C)$. Let $\varphi : \mathbb{R}^k \to \mathbb{R}$ be bounded and define $d\tilde{\mu}(x) = \frac{1}{Z} e^{\varphi(x)}\, d\mu(x)$, where Z is the normalization constant. Then $\tilde{\mu}$ verifies $\mathbf{T}_c\left(8Ce^{\mathrm{Osc}(\varphi)}\right)$ where $\mathrm{Osc}(\varphi) = \sup\varphi - \inf\varphi$.*

Proof. The proof below is a straightforward adaptation of the original proof of Theorem 1.7. Using the following representation of the entropy

$$\mathrm{Ent}_\mu(g) = \inf_{t>0}\left\{\int\left(g\log\left(\frac{g}{t}\right) - g + t\right)d\mu\right\}$$

with $g = e^f$, we see that (since $g\log\left(\frac{g}{t}\right) - g + t \geq 0$)

$$\mathrm{Ent}_{\tilde{\mu}}(g) \leq \frac{e^{\sup\varphi}}{Z}\,\mathrm{Ent}_\mu(g).$$

From the first part of Theorem 1.6 it follows that for all $K \geq 0$, $\eta > 0$, with $\eta + K < 1/C$ and all K-semi-convex functions f for the cost c,

$$\begin{aligned}\mathrm{Ent}_{\tilde{\mu}}\left(e^f\right) &\leq \frac{e^{\sup\varphi}}{Z}\frac{\eta}{1 - C(\eta + K)}\int c^*\left(\frac{\nabla f}{\eta}\right)e^f\, d\mu \\ &\leq \frac{\eta e^{\mathrm{Osc}(\varphi)}}{1 - C(\eta + K)}\int c^*\left(\frac{\nabla f}{\eta}\right)e^f\, d\tilde{\mu}.\end{aligned}$$

Let $u = e^{\mathrm{Osc}(\varphi)}$ and $c_u(x) := uc(x/u)$, $x \in \mathbb{R}^k$. Let f be a K-semi-convex function for the cost c_u. Since $u \geq 1$ the convexity of α yields $c_u(x) \leq c(x)$, $x \in \mathbb{R}^k$. Hence f is a K-semi-convex function for the cost c. Observing that $c_u^*(x) = uc^*(x)$, $x \in \mathbb{R}^k$, from the above inequality, it follows that $\tilde{\mu}$ verifies the inequality $\mathbf{rMLSI}(c_u, C)$. Then, the second part of Theorem 1.6 implies that $\tilde{\mu}$ verifies $\mathbf{T}_{c_u}(8C)$. From Point (i) of the technical Lemma 5.6, one has $uc(x/u) \geq c(x)/u$ for $u \geq 1$, $x \in \mathbb{R}^k$. This inequality completes the proof. □

Remark 1.9. *After the preparation of this work, we have learned from E. Milman that he has obtained in* [23] *new perturbation results for various functional inequalities on a Riemannian manifold equipped with a probability measure μ absolutely continuous with respect to the volume element. His results also cover transport inequalities but are only true under an additional curvature assumption. To be more precise, suppose that μ verifies say $\mathbf{T}_2(C)$ and consider another probability measure of the form $d\tilde{\mu}(x) = e^{-V(x)}\, dx$ such that*

$$\mathrm{Ric} + \mathrm{Hess}\, V \geq -\kappa,$$

for some $\kappa \geq 0$. Then if $C > \frac{\kappa}{2}$ and if μ and $\tilde{\mu}$ are close in some sense to each other, then $\tilde{\mu}$ verifies $\mathbf{T}_2(\tilde{C})$ for some $\tilde{C}$ depending only on C, κ and on the "distance" between μ and $\tilde{\mu}$. Actually, the curvature assumption above makes possible to go beyond the classical Holley-Stroock property and to work with measures $\tilde{\mu}$ which are more serious perturbations of μ. Proofs of these results are based on the remarkable equivalence between concentration and isoperimetric inequalities under curvature bounded from below, discovered by E. Miman in [22].

2. F S

In this section we prove the first part (1) $\Rightarrow$ (2) $\Rightarrow$ (3) of Theorem 1.6. As mentioned in the introduction, this implication holds in a more general setting as we explain now.

Let X denote a Polish space equipped with the Borel σ-algebra. Then the optimal transport cost between two probability measures μ and ν on X, with cost $\mathrm{c} : X \times X \to \mathbb{R}^+$ is

$$\mathcal{T}_{\mathrm{c}}(\nu, \mu) := \inf_{\pi} \iint \mathrm{c}(x, y)\, d\pi(x, y),$$

where the infimum is taken over all probability measures π on $X \times X$ with marginals ν and μ. Assume c is symmetric so that $\mathcal{T}_{\mathrm{c}}(\nu, \mu) = \mathcal{T}_{\mathrm{c}}(\mu, \nu)$. The transport inequality $\mathbf{T}_{\mathrm{c}}(C)$ is defined accordingly as in Definition 1.1. For $f : X \to \mathbb{R}$ and $\lambda > 0$, the inf-convolution $Q^\lambda f : X \to \mathbb{R}$ is given by

$$Q^\lambda f(x) = \inf_{y \in X} \{ f(y) + \lambda \mathrm{c}(x, y) \}.$$

The first part of Theorem 1.6 will be a consequence of the following general result.

Theorem 2.1. *Let $\mathrm{c} : X \times X \to \mathbb{R}^+$ be a symmetric continuous function. Let μ be a probability measure on X satisfying $\mathbf{T}_{\mathrm{c}}(C)$ for some $C > 0$. Then for all functions $f : X \to \mathbb{R}$ and all $\lambda \in (0, 1/C)$, it holds*

$$\mathrm{Ent}_\mu \left(e^f \right) \leq \frac{1}{1 - \lambda C} \int \left(f - Q^\lambda f \right) e^f \, d\mu.$$

Assume moreover that $\mathrm{c}(x, y) = c(x - y)$, $x, y \in \mathbb{R}^k$, where $c : \mathbb{R}^k \to \mathbb{R}^+$ is a differentiable function such that $c(0) = \nabla c(0) = 0$. Then μ verifies the inequality **rMLSI**(c, C).

Proof of Theorem 2.1. Fix $f : X \to \mathbb{R}$, $\lambda \in (0, 1/C)$ and define $d\nu_f = \frac{e^f}{\int e^f \, d\mu} \, d\mu$. One has

$$\begin{aligned} H(\nu_f | \mu) &= \int \log \left(\frac{e^f}{\int e^f \, d\mu} \right) \frac{e^f}{\int e^f \, d\mu} \, d\mu = \int f \, d\nu_f - \log \int e^f \, d\mu \\ &\leq \int f \, d\nu_f - \int f \, d\mu, \end{aligned}$$

where the last inequality comes from Jensen inequality. Consequently, if π is a probability measure on $X \times X$ with marginals ν_f and μ

$$H(\nu_f | \mu) \leq \iint (f(x) - f(y))\, d\pi(x, y).$$

It follows from the definition of the inf-convolution function that $f(x) - f(y) \leq f(x) - Q^\lambda f(x) + \lambda \mathrm{c}(x, y)$, for all $x, y \in X$. Hence,

$$H(\nu_f | \mu) \leq \iint \left(f(x) - Q^\lambda f(x) \right) d\pi(x, y) + \lambda \iint \mathrm{c}(x, y)\, d\pi(x, y),$$

and optimizing over all π with marginals ν_f and μ

$$H(\nu_f|\mu) = \int \left(f - Q^\lambda f\right) d\nu_f + \lambda \mathcal{T}_c(\nu_f, \mu)$$
$$\leq \frac{1}{\int e^f \, d\mu} \, d\mu \int \left(f - Q^\lambda f\right) e^f \, d\mu + \lambda C H(\nu_f|\mu).$$

The first part of Theorem 2.1 follows by noticing that $\left(\int e^f d\mu\right) H(\nu_f|\mu) = \mathrm{Ent}_\mu\left(e^f\right)$. Then the proof of Theorem 2.1 is completed by applying Lemma 2.2 below. □

Lemma 2.2. *Let $c : \mathbb{R}^k \to \mathbb{R}^+$ be a differentiable function such that $c(0) = \nabla c(0) = 0$ and define $c^*(x) = \sup_y\{x \cdot y - c(y)\} \in \mathbb{R} \cup \{+\infty\}$, $x \in \mathbb{R}^k$. Then, for any K-semi-convex differentiable function $f : \mathbb{R}^k \to \mathbb{R}$ for the cost c, it holds*

$$f(x) - Q^{K+\eta} f(x) \leq \eta c^*\left(-\frac{\nabla f(x)}{\eta}\right), \qquad \forall x \in \mathbb{R}^k, \ \forall \eta > 0.$$

Proof. Fix a K-semi-convex differentiable function $f : \mathbb{R}^k \to \mathbb{R}$. Also fix $x \in \mathbb{R}^k$ and $\eta > 0$. By Proposition 5.1 and the Young inequality $X \cdot Y \leq \eta c^*\left(\frac{X}{\eta}\right) + \eta c(Y)$ we have

$$f(x) - f(y) - Kc(y - x) \leq -\nabla f(x) \cdot (y - x) \leq \eta c^*\left(-\frac{\nabla f(x)}{\eta}\right) + \eta c(y - x).$$

Hence, for any $y \in \mathbb{R}^k$,

$$f(x) - f(y) - (K + \eta)c(y - x) \leq \eta c^*\left(-\frac{\nabla f(x)}{\eta}\right).$$

This yields the expected result. □

3. F S - I: H -J

In this section we prove the second part (3) ⇒ (1) of Theorem 1.6. The proof is based on the approach of Bobkov, Gentil and Ledoux [5], using the Hamilton-Jacobi equation. We will use the following notation: given a convex function $\alpha : \mathbb{R} \to \mathbb{R}^+$ with $\alpha(u) \neq 0$ for $u \neq 0$, we define

$$\omega_\alpha(x) = \sup_{u>0} \frac{\alpha(ux)}{\alpha(u)}, \qquad \forall x \in \mathbb{R}. \tag{3.1}$$

Proof of (3) ⇒ (1) *of Theorem 1.6.* Let $f : \mathbb{R}^k \to \mathbb{R}$ be a bounded continuous function. For $x \in \mathbb{R}^k$ and $t \in (0, 1)$, define

$$P_t f(x) = \sup_{y \in \mathbb{R}^k} \left\{ f(y) - tc\left(\frac{x - y}{t}\right)\right\}.$$

It is well known that the latter sup-convolution semi-group is a solution to the following Hamilton-Jacobi equation (see *e.g.* [10])

$$\begin{cases} \partial_t u_t(x) &= c^*\left(-\nabla u_t(x)\right) \\ u_0 &= f \end{cases}$$

Hence, defining $Z(t) = \int e^{\ell(t)P_{1-t}f}\,d\mu$, where ℓ is a smooth non-negative function on $\mathbb{R}^+$ with $\ell(0) = 0$ that will be chosen later, one gets

$$\begin{aligned} Z'(t) &= \int \left(\ell'(t)P_{1-t}f + \ell(t)\frac{\partial}{\partial t}P_{1-t}f\right) e^{\ell(t)P_{1-t}f}\,d\mu \\ &= \int \ell'(t)P_{1-t}f\, e^{\ell(t)P_{1-t}f}\,d\mu - \ell(t)\int c^*\left(\nabla P_{1-t}f\right) e^{\ell(t)P_{1-t}f}\,d\mu. \end{aligned}$$

On the other hand,

$$\mathrm{Ent}_\mu\left(e^{\ell(t)P_{1-t}f}\right) = \ell(t)\int P_{1-t}f\, e^{\ell(t)P_{1-t}f}\,d\mu - Z(t)\log Z(t).$$

Therefore provided $\ell'(t) \neq 0$,

$$\begin{aligned} \mathrm{Ent}_\mu\left(e^{\ell(t)P_{1-t}f}\right) &= \frac{\ell(t)}{\ell'(t)}Z'(t) - Z(t)\log Z(t) \\ &\quad + \frac{\ell(t)^2}{\ell'(t)}\int c^*\left(\nabla P_{1-t}f\right) e^{\ell(t)P_{1-t}f}\,d\mu. \end{aligned} \tag{3.2}$$

By Lemma 5.5 (with $A = \ell(t)(1-t)$ and $B = 1-t$), the function $g = \ell(t)P_{1-t}f$ is $K(t)$ semi-convex for the cost function $c(x) = \sum_{i=1}^k \alpha(x_i)$, $x \in \mathbb{R}^k$, where $K(t) = 4\ell(t)(1-t)\omega_\alpha\left(\frac{1}{2(1-t)}\right)$. Hence we can apply the restricted logarithmic Sobolev inequality to get that for any $\eta > 0$, any $t \in (0,1)$ such that $K(t) + \eta < 1/C_3$,[1]

$$\begin{aligned} \mathrm{Ent}_\mu\left(e^{\ell(t)P_{1-t}f}\right) &\leq \frac{\eta}{1-(K(t)+\eta)C_3}\int c^*\left(\frac{\ell(t)\nabla P_{1-t}f}{\eta}\right) e^{\ell(t)P_{1-t}f}\,d\mu \\ &\leq \frac{\eta\omega_{\alpha^*}\left(\frac{\ell(t)}{\eta}\right)}{1-(K(t)+\eta)C_3}\int c^*\left(\nabla P_{1-t}f\right) e^{\ell(t)P_{1-t}f}\,d\mu. \end{aligned}$$

Combining this bound with (3.2) leads to

$$\begin{aligned} &\frac{\ell(t)}{\ell'(t)}Z'(t) - Z(t)\log Z(t) \leq \\ &\qquad \left(\frac{\eta\omega_{\alpha^*}\left(\frac{\ell(t)}{\eta}\right)}{1-(K(t)+\eta)C_3} - \frac{\ell(t)^2}{\ell'(t)}\right)\int c^*\left(\nabla P_{1-t}f\right) e^{\ell(t)P_{1-t}f}\,d\mu. \end{aligned}$$

Our aim is to choose the various parameters so that to have the right hand side of the latter inequality non-positive. We will make sure to choose ℓ so that $\ell(t)/\eta < 1$; then by Lemma 5.6 below $K(t) \leq \ell(t)/(1-t)$ and $\omega_{\alpha^*}\left(\frac{\ell(t)}{\eta}\right) \leq \frac{\ell^2(t)}{\eta^2}$. Setting $\nu = 1 - C_3\eta$, one has $0 < \nu < 1$,

$$C_3(K(t)+\eta) \leq (1-\nu)\left(\frac{\ell(t)}{\eta(1-t)} + 1\right), \tag{3.3}$$

and

$$\left(\frac{\eta\omega_{\alpha^*}\left(\frac{\ell(t)}{\eta}\right)}{1-(K(t)+\eta)C_3} - \frac{\ell(t)^2}{\ell'(t)}\right) \leq \ell^2(t)\left(\frac{1}{\eta\nu - (1-\nu)\frac{\ell(t)}{1-t}} - \frac{1}{\ell'(t)}\right). \tag{3.4}$$

[1]Note that this condition is not empty since $K(0) = 0$.

We choose $\ell(t) = \eta\Big((1-t)^{1-v} - (1-t)\Big), t \in (0,1)$, so that $\ell(0) = 0$ and the right hand side of (3.4) is equal to zero. Furthermore $\ell'(t) = \eta\Big(1 - \frac{1-v}{(1-t)^v}\Big) \geq 0, \forall t \in [0, 1-(1-v)^{1/v}]$. As assumed earlier, $\ell(t)$ is non-negative and $\ell(t)/\eta < 1$ on $(0,1)$. Let us observe that

$$\left[\frac{\log Z(t)}{\ell(t)}\right]' = \frac{\ell'(t)}{Z(t)\ell^2(t)}\left[\frac{\ell(t)}{\ell'(t)}Z'(t) - Z(t)\log Z(t)\right].$$

Let $T = T(v) := 1-(1-v)^{1/v}$, since $\ell'(t) > 0$ on $(0, T(v))$, the above inequalities imply that on that interval $\left[\frac{\log Z(t)}{\ell(t)}\right]' \leq 0$ provided $C_3(K(t)+\eta) < 1$. By (3.3), this is indeed satisfied for $t \in [0, T(v)]$. This gives that the function $t \mapsto \frac{\log Z_t}{\ell(t)}$ is non-increasing on $(0, T]$. Hence, we have

$$\int e^{\ell(T)P_T f}\, d\mu = Z_T \leq \exp\left(\ell(T)\lim_{t\to 0}\frac{\log Z_t}{\ell(t)}\right) = e^{\ell(T)\int P_1 f\, d\mu}.$$

In other words, since $P_T f \geq f$, then for all bounded continuous functions $g = \ell(T)f$,

$$\int e^g\, d\mu \leq e^{\int \widetilde{P}g\, d\mu},$$

with

$$\widetilde{P}g(x) = \sup_{y\in\mathbb{R}^k}\{g(y) - \ell(T)c(x-y)\}.$$

According to the Bobkov and Götze sup-convolution characterization of transport inequalities (which for the reader's convenience we quote below as Theorem 3.5), this implies that μ verifies $\mathbf{T}_c(1/\ell(T))$. One has $\ell(T) = \eta v(1-v)^{(1/v)-1}$ and $C_3\ell(T) = v(1-v)^{1/v}$. Hence μ verifies $\mathbf{T}_c(K)$ with

$$K = \frac{C_3}{\sup_{v\in(0,1)} v(1-v)^{1/v}} \leq 7,7C_3.$$

The proof of (3) $\Rightarrow$ (1) is complete.

□

Theorem 3.5 ([6])**.** *Let μ be a probability measure on $\mathbb{R}^k$, $\lambda > 0$ and c defined as in Theorem 1.6. Then, the following two statements are equivalent:*
(*i*) *μ satisfies $\mathbf{T}_c(1/\lambda)$;*
(*ii*) *for any bounded function $f : \mathbb{R}^k \to \mathbb{R}$ it holds*

$$\int e^f\, d\mu \leq \exp\left\{\int \sup_{y\in\mathbb{R}^k}\{f(y) - \lambda c(x-y)\}\right\} d\mu.$$

Note that Theorem 3.5 holds in much more general setting, see [30].

4. F S $\mathbf{T}_2$ - II:

In this section we give an alternative proof of the second part (3) $\Rightarrow$ (1) of Theorem 1.6. The final result will lead to a worst constant, so we will present our approach only in the particular case of the quadratic cost function $c(x) = \frac{1}{2}|x|^2$.

More precisely we will prove that **rLSI**$(C) \Rightarrow \mathbf{T}_2(9C)$ (leading, for the quadratic cost, to the implication (3) $\Rightarrow$ (1) of Theorem 1.6 with $C_1 = 9C_3$). We believe that this alternative approach may lead to other results in the future and so that it is worth mentioning it.

The strategy is based on the following recent characterization of Gaussian dimension free concentration by the first author.

Theorem 4.1 ([14])**.** *A probability measure μ on $\mathbb{R}^k$ verifies the inequality $\mathbf{T}_2(C/2)$ if and only if there are some $r_o \geq 0$ and $b > 0$ such that for all positive integer n and all subset A of $\left(\mathbb{R}^k\right)^n$ with $\mu^n(A) \geq 1/2$, the following inequality holds*

$$\mu^n(A + rB_2) \geq 1 - be^{-(r-r_o)^2/C}, \qquad \forall r \geq r_o,$$

where B_2 is the Euclidean unit ball of $(\mathbb{R}^k)^n$.

So, in order to get that **rLSI**$(C) \Rightarrow \mathbf{T}_2(9C)$ it is enough to prove that the dimension free Gaussian concentration inequality holds with $-(r-r_o)^2/(18C)$ in the exponential.

First let us observe that the restricted logarithmic Sobolev inequality tensorizes.

Proposition 4.2. *If a probability measure μ on $\mathbb{R}^k$ verifies* **rLSI**(C) *for some $C > 0$, then for all positive integer n the probability μ^n verifies* **rLSI**(C)*.*

Proof. If $f : \left(\mathbb{R}^k\right)^n \to \mathbb{R}$ is K-semi-convex, then for all $i \in \{1, \dots, n\}$ and all $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in \mathbb{R}^k$ the function $f_i : \mathbb{R}^k \to \mathbb{R}$ defined by $f_i(x) = f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n)$ is K-semi-convex. According to the classical additive property of the entropy functional (see *e.g.* [1, Chapter 1]),

$$\mathrm{Ent}_{\mu^n}(e^f) \leq \int \sum_{i=1}^n \mathrm{Ent}_\mu(e^{f_i})\, d\mu^n.$$

Applying to each f_i the restricted logarithmic Sobolev inequality completes the proof. □

The next proposition uses the classical Herbst argument (see e.g [18]).

Proposition 4.3. *If μ verifies the restricted logarithmic Sobolev inequality* **rLSI**(C) *then for all $f : \mathbb{R}^k \to \mathbb{R}$ which is 1-Lipschitz with respect to the Euclidean norm and K-semi-convex with $K \geq 0$ one has*

$$\int e^{\lambda(f(x) - \int f\, d\mu)}\, d\mu(x) \leq \exp\left(\frac{2\lambda^2 C}{1 - \lambda KC}\right), \qquad \forall \lambda \in (0, 1/(CK))\,.$$

Proof. Let us denote $H(\lambda) = \int e^{\lambda f}\, d\mu$, for all $\lambda \geq 0$. The function λf is λK semi-convex, so if $0 \leq \lambda < 1/(CK)$, one can apply the inequality **rLSI**(C) to the function λf. Doing so yields the inequality

$$\begin{aligned} \lambda H'(\lambda) - H(\lambda)\log H(\lambda) = \mathrm{Ent}_\mu\left(e^{\lambda f}\right) &\leq \frac{2C\lambda^2}{(1-\lambda KC)^2} \int |\nabla f|^2 e^{\lambda f}\, d\mu \\ &\leq \frac{2C\lambda^2}{(1-\lambda KC)^2} H(\lambda), \end{aligned}$$

where the last inequality comes from the fact that f is 1-Lipschitz. Consequently, for all $0 \le \lambda < 1/(CK)$,

$$\frac{d}{d\lambda}\left(\frac{\log H(\lambda)}{\lambda}\right) \le \frac{2C}{(1-\lambda KC)^2}.$$

Observing that $\log H(\lambda)/\lambda \to \int f\, d\mu$ when $\lambda \to 0$ and integrating the differential inequality above gives the result. □

Now let us show how to approach a given 1-Lipschitz function by a 1-Lipschitz and K-semi-convex function.

Proposition 4.4. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a 1-Lipschitz function. Define*

$$P_t f(x) = \sup_{y\in\mathbb{R}^k}\left\{ f(y) - \frac{1}{2t}|x-y|^2 \right\}, \qquad \forall x \in \mathbb{R}^k,\ \forall t > 0.$$

Then,

(i) For all $t > 0$, $P_t f$ is 1-Lipschitz.
(ii) For all $t > 0$, $P_t f$ is $1/t$-semi-convex.
(iii) For all $t > 0$ and all $x \in \mathbb{R}^k$, $f(x) \le P_t f(x) \le f(x) + \frac{t}{2}$.

Proof. (*i*) Write $P_t f(x) = \sup_{z\in\mathbb{R}^k}\left\{ f(x-z) - \frac{1}{2t}|z|^2 \right\}$. For all $z \in \mathbb{R}^k$, the function $x \mapsto f(x-z) - \frac{1}{2t}|z|^2$ is 1-Lipschitz. So $P_t f$ is 1-Lipschitz as a supremum of 1-Lipschitz functions.

(*ii*) Expanding $|x-y|^2$ yields $P_t f(x) = \sup_{y\in\mathbb{R}^k}\left\{ f(y) - \frac{1}{2t}|y|^2 + \frac{1}{t}x\cdot y \right\} - \frac{1}{2t}|x|^2$. Since a supremum of affine functions is convex, one concludes that $x \mapsto P_t f(x) + \frac{|x|^2}{2t}$ is convex, which means that $P_t f$ is $1/t$-semi-convex.

(*iii*) The inequality $P_t f(x) \ge f(x)$ is immediate. Since f is 1-Lipschitz,

$$\begin{aligned} P_t f(x) - f(x) &= \sup_{y\in\mathbb{R}^k}\left\{ f(y) - f(x) - \frac{1}{2t}|x-y|^2 \right\} \\ &\le \sup_{y\in\mathbb{R}^k}\left\{ |y-x| - \frac{1}{2t}|x-y|^2 \right\} \\ &= \sup_{r\ge 0}\left\{ r - \frac{r^2}{2t} \right\} = \frac{t}{2}. \end{aligned}$$

□

We are now ready to complete the proof.

Proof of **rLSI**$(C) \Rightarrow \mathbf{T}_2(9C)$. Let $n \ge 1$. Consider a 1-Lipschitz function g on $\left(\mathbb{R}^k\right)^n$ and define $P_t g(x) = \sup_{y\in(\mathbb{R}^k)^n}\left\{ g(y) - \frac{1}{2t}|x-y|^2 \right\}$, $t > 0$. Thanks to Proposition 4.4, the function $P_t g$ is 1-Lipschitz and $1/t$-semi-convex, so according to Propositions 4.2 and 4.3, for all $0 \le \lambda < t/C$, one has

$$\int e^{\lambda(P_t g(x) - \int P_t g\, d\mu^n)}\, d\mu^n(x) \le \exp\left(\frac{2\lambda^2 C}{1-\frac{\lambda C}{t}}\right).$$

Moreover, according to point (*iii*) of Proposition 4.4, $P_t g(x) - \int P_t g \, d\mu^n \geq g(x) - \int g \, d\mu^n - \frac{t}{2}$, for all $x \in \left(\mathbb{R}^k\right)^n$. Plugging this in the inequality above gives

$$\int e^{\lambda(g(x)-\int g\, d\mu^n)} \, d\mu^n(x) \leq \exp\left(\frac{\lambda t}{2} + \frac{2\lambda^2 C}{1-\frac{\lambda C}{t}}\right).$$

For a given $\lambda \geq 0$, this inequality holds as soon as $t > C\lambda$. Define $\varphi(t) = \frac{\lambda t}{2} + \frac{2\lambda^2 C}{1-\frac{\lambda C}{t}}$, $t > 0$. It is easy to check that φ attains its minimum value at $t_{\min} = 3C\lambda$ (which is greater than $C\lambda$) and that $\varphi(t_{\min}) = 9C\lambda^2/2$. Consequently, we arrive at the following upper bound on the Laplace transform of g:

$$\int e^{\lambda(g(x)-\int g\, d\mu^n)} \, d\mu^n(x) \leq e^{9C\lambda^2/2}, \qquad \forall \lambda \geq 0.$$

From this we deduce that every 1-Lipschitz function g verifies the following deviation inequality around its mean

$$\mu^n(g \geq \int g \, d\mu^n + r) \leq e^{-r^2/(18C)}, \qquad \forall r \geq 0.$$

Let r_o be any number such that $e^{-r_o^2/(18C)} < 1/2$, then denoting by $m(g)$ any median of g, we get $\int g \, d\mu^n + r_o \geq m(g)$. Applying this inequality to $-g$, we conclude that $|m(g) - \int g \, d\mu^n| \leq r_o$. So the following deviation inequality around the median holds

$$\mu^n(g \geq m(g) + r) \leq e^{-(r-r_o)^2/(18C)}, \qquad \forall r \geq r_o.$$

Take $A \subset (\mathbb{R}^k)^n$ with $\mu^n(A) \geq 1/2$, and define $g_A(x) = d_2(x, A)$ where d_2 is the usual Euclidean distance. Since 0 is a median of g_A, the preceding inequality applied to g_A reads

$$\mu^n(A + rB_2) \geq 1 - e^{-(r-r_o)^2/(18C)}, \qquad \forall r \geq r_o.$$

According to Theorem 4.1, this Gaussian dimension free concentration property implies $\mathbf{T}_2(9C)$.

□

5. S

In this section we collect some useful results on semi-convex functions.

In the case of differentiable functions, it is easy to rephrase the definition of semi-convexity, in the following way.

Proposition 5.1. *Let $c : \mathbb{R}^k \to \mathbb{R}^+$ be a differentiable function with $c(0) = \nabla c(0) = 0$. Then, a differentiable function $f : \mathbb{R}^k \to \mathbb{R}$ is K-semi-convex for the cost function c if and only if*

$$f(y) \geq f(x) + \nabla f(x) \cdot (y - x) - Kc(y - x), \qquad \forall x, y \in \mathbb{R}^k. \tag{5.2}$$

Proof. Suppose that f is K-semi-convex; according to the definition, for all $x, y \in \mathbb{R}^k$ and $\lambda \in [0, 1]$, the following holds

$$f(y) \geq f(x) + \frac{f(\lambda x + (1-\lambda)y) - f(x)}{1-\lambda} - K\frac{\lambda}{1-\lambda} c((1-\lambda)(x-y)) - Kc(\lambda(y-x)).$$

Letting $\lambda \to 1$ and using $c(0) = \nabla c(0) = 0$ one obtains (5.2). Let us prove the converse; according to (5.2),

$$f(x) \geq f(\lambda x + (1-\lambda)y) - (1-\lambda)\nabla f(\lambda x + (1-\lambda)y)\cdot(y-x) + Kc((1-\lambda)(y-x)),$$

and

$$f(y) \geq f(\lambda x + (1-\lambda)y) + \lambda\nabla f(\lambda x + (1-\lambda)y)\cdot(y-x) + Kc(\lambda(y-x)).$$

This gives immediately (1.3). □

Lemma 5.3. *If $\alpha : \mathbb{R} \to \mathbb{R}^+$ is a convex symmetric function of class C^1 such that $\alpha(0) = \alpha'(0) = 0$ and α' is concave on $\mathbb{R}^+$, then the following inequality holds*

$$\alpha(u+v) \leq \alpha(u) + v\alpha'(u) + 4\alpha(v/2), \qquad \forall u, v \in \mathbb{R}. \tag{5.4}$$

In particular, the function $-c(x) = -\sum_{i=1}^k \alpha(x_i)$, $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$, is 4-semi-convex for the cost $x \mapsto c(x/2)$.

Note that (5.4) is an equality for $\alpha(t) = t^2$.

Proof. Since $\alpha(v) = \alpha(-v)$, it is enough to prove the inequality (5.4) for $u \leq 0$ and $v \in \mathbb{R}$. Let us consider the function $G(w) := \alpha(u+w) - \alpha(u) - w\alpha'(u)$. For $w \geq 0$, using the concavity of α' on $\mathbb{R}^+$, either $u + w \geq 0$ and one has

$$G'(w) = \alpha'(u+w) - \alpha'(u) = \alpha'(u+w) + \alpha'(-u) \leq 2\alpha'(w/2),$$

or $u + w \leq 0$ and one has

$$G'(w) = \alpha'(-u) - \alpha'(-u-w) \leq \alpha'(w) \leq 2\alpha'(w/2),$$

since $w \geq 0$ and

$$\frac{\alpha'(w/2) - \alpha'(0)}{w/2} \geq \frac{\alpha'(w) - \alpha'(0)}{w} \geq \frac{\alpha'(w) - \alpha'(-u-w)}{2w+u} \geq \frac{\alpha'(-u) - \alpha'(-u-w)}{w}.$$

Similarly, if $w \leq 0$, from the convexity of α' on $\mathbb{R}^-$, $G'(w) \geq \alpha'(w) \geq 2\alpha'(w/2)$. The proof is complete integrating the above inequalities between 0 and v either for $v \geq 0$ or for $v \leq 0$.

The second part of the lemma is immediate. □

The next lemma gives some conditions on α under which the sup-convolution semi-group P_t transforms functions into semi-convex. Let us recall that ω_α is defined by

$$\omega_\alpha(x) = \sup_{u>0} \frac{\alpha(ux)}{\alpha(u)}, \qquad \forall x \in \mathbb{R}.$$

Lemma 5.5. *Let $\alpha : \mathbb{R} \to \mathbb{R}^+$ be a convex symmetric function of class C^1 such that $\alpha(0) = \alpha'(0) = 0$ and α' is concave on $\mathbb{R}^+$. Let $f : \mathbb{R}^k \to \mathbb{R}$, $A, B > 0$ and define $g(x) = \sup_{y\in\mathbb{R}^k}\{f(y) - Ac((y-x)/B)\}$ with $c(x) = \sum_{i=1}^k \alpha(x_i)$, $x \in \mathbb{R}^k$. Then g is $4A\omega_\alpha\left(\frac{1}{2B}\right)$-semi-convex for the cost function c.*

Proof. By Lemma 5.3, the function $-c$ is 4-semi-convex with the cost function $x \mapsto c(x/2)$. Consequently, for all $y \in \mathbb{R}^k$, the function $x \mapsto f(y) - Ac((y-x)/B)$ is 4-semi-convex with the cost function $x \mapsto Ac(x/(2B))$. From the definition (1.3), we observe that a supremum of K-semi-convex functions remains K-semi-convex. Consequently, by definition of ω_α, we finally get

$$\begin{aligned} g(y) &\geq g(x) + \nabla g(x) \cdot (y-x) - 4Ac\left(\frac{y-x}{2B}\right) \\ &\geq g(x) + \nabla g(x) \cdot (y-x) - 4A\omega_\alpha\left(\frac{1}{2B}\right) c\,(y-x)\,. \end{aligned}$$

□

Lemma 5.6. *Let α be a convex symmetric function of class C^1 such that $\alpha(0) = \alpha'(0) = 0$, α' is concave on $\mathbb{R}^+$. Denote by α^* the conjugate of α. Then,*
(i) for any $u \in (0,1)$, $x \in \mathbb{R}$, $\alpha(x/u) \leq \alpha(x)/u^2$.
(ii) for any $u \in (0,1)$, $\omega_\alpha(1/u) \leq 1/u^2$.
(iii) for any $u \in (0,1)$, $\omega_{\alpha^*}(u) \leq u^2$.

Proof. Point (*i*). Let $x \geq 0$, by concavity of α' on $\mathbb{R}^+$, $\alpha'(x) \geq u\alpha'(x/u) + (1-u)\alpha'(0) = u\alpha'(x/u)$. The result follows for $x \geq 0$ by integrating between 0 and x and then for $x \leq 0$ by symmetry. Point (*ii*) is a direct consequence of point (i).

Point (*iii*). Observing that $(\alpha^*)' = (\alpha')^{-1}$, it follows that $(\alpha^*)'$ is convex on $\mathbb{R}^+$ and $(\alpha^*)'(0) = \alpha^*(0) = 0$. Then the proof is similar to the proof of point (ii). □

6. F

In this final section we state some remarks and extensions on the topic of this paper.

6.1. **Extension to Riemannian manifolds.** Otto-Villani theorem holds true on general Riemannian manifolds [25]. Furthermore, efforts have been made recently to extend the Otto-Villani theorem to spaces with poorer structure such as length spaces [19, 2] or general metric spaces [14]. This section is an attempt to extend our main result to spaces other than Euclidean spaces. We will focus our attention on the inequality $\mathbf{T}_2$ on a Riemannian manifold.

In all what follows, X will be a complete and connected Riemannian manifold equipped with its geodesic distance d:

$$d(x,y) = \inf\left\{\int_0^1 |\dot\gamma_t|\,dt;\, \gamma \in C^1([0,1],X), \gamma_0 = x, \gamma_1 = y\right\}, \qquad \forall x,y \in X. \tag{6.1}$$

A minimizing path γ in (6.1) is called a minimal geodesic from x to y; in general it is not unique. It is always possible to consider that minimal geodesics are parametrized in such a way that

$$d(\gamma_s,\gamma_t) = |s-t|d(x,y), \qquad \forall s,t \in [0,1],$$

and this convention will be in force in all the sequel.

A function $f : X \to \mathbb{R}$ will be said K-semi-convex, $K \geq 0$ if for all $x, y \in X$ and all minimal geodesic γ between x and y, the following inequality holds

$$f(\gamma_t) \leq (1-t)f(x) + tf(y) + t(1-t)\frac{K}{2}d^2(x,y), \qquad \forall t \in [0,1].$$

When f is of class C^1 this is equivalent to the following condition:

$$f(y) \geq f(x) + \langle \nabla f(x), \dot{\gamma}_0 \rangle - \frac{K}{2}d^2(x,y), \qquad \forall x, y \in X \tag{6.2}$$

for all minimal geodesic γ from x to y (see e.g. [30, Proposition 16.2]). If f is semi-convex, then it is locally Lipschitz [30]. According to Rademacher's theorem (see e.g [30, Theorem 10.8]), f is thus almost everywhere differentiable. So the inequality (6.2) holds for almost all $x \in X$ and for all $y \in X$. A function f will be said K-semi-concave if $-f$ is K-semi-convex.

Lemma 6.3. *If f is K-semi-convex, then for almost all $x \in X$, the inequality*

$$f(y) \geq f(x) - |\nabla f|(x)d(x,y) - \frac{K}{2}d^2(x,y),$$

holds for all $y \in X$.

Proof. Since the geodesic is constant speed, $|\dot{\gamma}_0| = d(x,y)$. Applying Cauchy Schwarz inequality in (6.2) yields the desired inequality. □

With this inequality at hand, the proofs of Lemma 2.2 generalizes at once, and we get the following half part of our main result.

Proposition 6.4. *Suppose that an absolutely continuous probability measure μ on X verifies the inequality $\mathbf{T}_2(C)$, then it verifies the following restricted logarithmic Sobolev inequality: for all $0 \leq K < \frac{1}{C}$ and all K-semi-convex $f : X \to \mathbb{R}$,*

$$\mathrm{Ent}_\mu\left(e^f\right) \leq \frac{2C}{(1-KC)^2}\int |\nabla f|^2 e^f \, d\mu.$$

The generalization of the second half part of our main result is more delicate. We have seen two proofs of the fact that the restricted logarithmic Sobolev inequality implies $\mathbf{T}_2$: one based on the Hamilton-Jacobi equation and the other based on dimension free concentration. The common point of these two approaches is that we have used in both cases the property that the sup-convolution operator $f \mapsto P_t f$ transforms functions into semi-convex functions (see Proposition 4.4 and Lemma 5.5). Let us see how this property can be extended to Riemannian manifolds.

Proposition 6.5. *Suppose that there is some constant $S \geq 1$, such that the inequality*

$$d^2(\gamma_t, y) \geq (1-t)d^2(x,y) + td^2(z,y) - t(1-t)S^2 d^2(x,z), \qquad \forall t \in [0,1], \tag{6.6}$$

holds for all $x, y, z \in X$, where γ is a minimal geodesic joining x to z. This amounts to say that for all $y \in X$, the function $x \mapsto d^2(x,y)$ is $2S^2$-semi-concave.

Then for all $f : X \to \mathbb{R}$, the function

$$x \mapsto P_t f(x) = \sup_{y \in X}\left\{ f(y) - \frac{1}{2S^2 t}d^2(x,y)\right\}, \tag{6.7}$$

is $1/t$*-semi-convex.*

Proof. Under the assumption made on d^2, for all $y \in X$, the function $x \mapsto f(y) - \frac{1}{2S^2t}d^2(x,y)$ is $1/t$-semi-convex. Since a supremum of $1/t$ semi-convex functions is $1/t$-semi-convex, this ends the proof. □

Let us make some remarks on condition (6.6). This condition was first introduced by Ohta in [24] and Savare in [26] in their studies of gradient flows in the Wasserstein space over non-smooth metric spaces. The condition (6.6) is related to the Alexandrov curvature of geodesic spaces which generalizes the notion of sectional curvature in Riemannian geometry.

The first point is a classical consequence of Toponogov's theorem [9]. The second point in the following proposition is due to Ohta [24, Lemma 3.3].

Proposition 6.8. *Let X be a complete and connected Riemannian manifold.*

(1) The condition (6.6) *holds with $S = 1$ if and only if the sectional curvature of X is greater than or equal to* 0 *everywhere.*

(2) Suppose that the sectional curvature is greater than or equal to κ, where $\kappa \leq 0$, then for all $x, y, z \in X$ and every geodesic γ joining x to z, one has

$$d^2(\gamma_t, y) \geq (1-t)d^2(x,y) + td^2(z,y) - \left(1 + \kappa^2 \sup_{s\in[0,1]} d^2(\gamma_s, y)\right)(1-t)td^2(x,z). \tag{6.9}$$

In particular, if (X,d) is bounded, then (6.6) *holds with*

$$S = (1 + \kappa^2 \mathrm{diam}(X)^2)^{1/2}.$$

In particular, the case of the Euclidean space, studied in the preceding sections, corresponds to the case where the sectional curvature vanishes everywhere.

Now, let us have a look to Hamilton-Jacobi equation. The following theorem comes from [30, Proposition 22.16 and Theorem 22.46].

Theorem 6.10. *Let f be a bounded and continuous function on X, the function $(t,x) \mapsto P_t f(x)$ defined by* (6.7) *verifies the following: for all $t > 0$ and $x \in X$,*

$$\lim_{h\to 0^+} \frac{P_{t+h}f(x) - P_t f(x)}{h} = \frac{S^2 |\nabla^-(-P_t f)|^2(x)}{2},$$

where the metric sub-gradient $|\nabla^- g|$ of a function g is defined by

$$|\nabla^- g|(x) = \limsup_{y\to x} \frac{[g(y) - g(x)]_-}{d(y,x)}, \qquad \forall x \in X.$$

Under the condition (6.6), $x \mapsto P_t f(x)$ is semi-convex, and so differentiable almost everywhere, so for all t and almost all $x \in X$,

$$\lim_{h\to 0^+} \frac{P_{t+h}f(x) - P_t f(x)}{h} = \frac{S^2 |\nabla P_t f|^2(x)}{2}.$$

Theorem 6.11. *Suppose that the Riemannian manifold X verifies condition* (6.6) *for some $S \geq 1$; if an absolutely continuous probability measure μ on X verifies the following restricted logarithmic Sobolev inequality: for all $0 \leq K < \frac{1}{C}$ and all K-semi-convex $f : X \to \mathbb{R}$,*

$$\mathrm{Ent}_\mu\left(e^f\right) \leq \frac{2C}{(1-KC)^2} \int |\nabla f|^2 e^f \, d\mu,$$

then it verifies $\mathbf{T}_2(8CS^4)$.

Proof. To adapt the proof of Theorem 1.6 (3)$\Rightarrow$ (1), a first technical problem is to deal with right derivatives. We refer to the proof of [30, Theorem 22.17], where this difficulty has been circumvented. In the sequel, we will consider that $\frac{\partial P_t}{\partial t}(x) = \frac{S^2}{2}|\nabla P_t f|^2(x)$ for all $t > 0$, and almost all $x \in X$.

Take a bounded continuous function $f : X \to \mathbb{R}$ and define $Z(t) = \int e^{\ell(t)P_{1-t}f} \, d\mu$, where ℓ is a smooth non-negative function on $\mathbb{R}^+$ with $\ell(0) = 0$ that will be chosen later. Reasoning as in the proof of Theorem 2.1, one gets, provided $\ell'(t) \neq 0$,

$$\mathrm{Ent}_\mu\left(e^{\ell(t)P_{1-t}f}\right) = \frac{\ell(t)}{\ell'(t)} Z'(t) - Z(t)\log Z(t) + \frac{S^2\ell(t)^2}{2\ell'(t)} \int |\nabla P_{1-t}f|^2 e^{\ell(t)P_{1-t}f} \, d\mu. \tag{6.12}$$

According to Proposition 6.5, we see that $\ell(t)P_{1-t}$ is $\frac{\ell(t)}{1-t}$ semi-convex. Applying the restricted logarithmic Sobolev inequality we get that for any $\eta > 0$, any $t \in (0,1)$ such that $\frac{\ell(t)}{1-t} + \eta < 1/C$

$$\mathrm{Ent}_\mu\left(e^{\ell(t)P_{1-t}f}\right) \leq \frac{\eta\ell^2(t)}{2\left(1-\left(\frac{\ell(t)}{1-t}+\eta\right)C\right)} \int |\nabla P_{1-t}f|^2 e^{\ell(t)P_{1-t}f} \, d\mu.$$

Combining this bound with the one above leads to

$$\frac{\ell(t)}{\ell'(t)} Z'(t) - Z(t)\log Z(t) \leq \left(\frac{\ell^2(t)}{2\eta(1-(\frac{\ell(t)}{1-t}+\eta)C)} - \frac{S^2\ell(t)^2}{2\ell'(t)}\right) \int |\nabla P_{1-t}f|^2 e^{\ell(t)P_{1-t}f} \, d\mu.$$

For a given $\eta \in (0, 1/(S^2C))$, consider the function

$$\ell(t) = \frac{\eta S^2(1-\eta C)}{1-\eta S^2 C}\left((1-t)^{C\eta S^2} - (1-t)\right), \qquad \forall t \in [0,1).$$

This function is the unique solution on $[0,1)$ of the differential equation

$$\ell'(t) = \eta S^2\left(1-\left(\frac{\ell(t)}{1-t}+\eta\right)C\right), \qquad \ell(0) = 0.$$

It is easy to check that $\ell(t) \geq 0$, for all $t \in [0,1]$ and $\ell'(t) > 0$, for all $t < T = 1 - \left(C\eta S^2\right)^{\frac{-1}{C\eta S^2-1}}$. Moreover, if $t < T$, then the condition $\frac{\ell(t)}{1-t} + \eta < \frac{1}{C}$ is also

fulfilled. We conclude that the function $t \mapsto \frac{\log Z_t}{\ell(t)}$ is non-increasing on $(0, T]$ and reasoning exactly as in the proof of Theorem 1.6, we get easily that

$$\int e^g \, d\mu \leq e^{\int \widetilde{P} g \, d\mu},$$

with

$$\widetilde{P} g(x) = \sup_{y \in X} \left\{ g(y) - \frac{\ell(T)}{2S^2} d^2(x, y) \right\}.$$

According to Theorem 3.5, this implies that μ verifies the inequality $\mathbf{T}_2(S^2/(\ell(T))$. After some computations, we get

$$\frac{S^2}{\ell(T)} = \frac{S^2 C}{1 - \eta C} (C\eta S^2)^{\frac{1}{C\eta S^2 - 1}}.$$

So, letting $v = 1 - \eta C S^2 \in (1 - S^2, 1)$, and optimizing over v we obtain that μ verifies $\mathbf{T}_2$ with the constant $\tilde{C}$

$$\begin{aligned} \tilde{C} &= \frac{CS^2}{\sup_{v \in (1-S^2,1)} (1 - (1-v)/S^2)(1-v)^{1/v}} \\ &\leq \frac{CS^4}{\sup_{v \in (0,1)} v(1-v)^{1/v}} \leq 7,7 CS^4. \end{aligned}$$

□

To conclude this section let us say that the proof presented in Section 4 can also be adapted to the Riemannian framework. Essentially, all we have to do is to adapt the first point of Proposition 4.4: the fact that $P_t f$ is 1-Lipschitz when f is 1-Lipschitz. A proof of this can be found in the proof of [2, Theorem 2.5 (iv)].

6.2. **From transport inequalities to other logarithmic Sobolev type inequalities.** Following the ideas of Theorem 2.1, we may simply recover other types of logarithmic Sobolev inequalities. These new forms of inequalities should be of interest for further developments. Let X denote a Polish space equipped with the Borel σ-algebra. Given Borel functions $\mathrm{c} : X \times X \to \mathbb{R}$, and $f : X \to \mathbb{R}$, define for $\lambda > 0$, $x \in X$,

$$P^\lambda f(x) = \sup_{y \in X} \{ f(y) - \lambda \mathrm{c}(x, y) \}.$$

By definition, one says that a function $f : X \to \mathbb{R}$ is K-semi-concave for the cost c if $-f$ is K-semi-convex for the cost c.

Theorem 6.13. *Let* $\mathrm{c} : X \times X \to \mathbb{R}^+$ *be a symmetric Borel function. Let* μ *be a probability measure on* X *satisfying* $\mathbf{T}_\mathrm{c}(C)$ *for some* $C > 0$*. Then for all* $\lambda \in (0, 1/C)$*, and all function* $f : X \to \mathbb{R}$*,*

$$\mathrm{Ent}_\mu \left(e^f \right) \leq \frac{1}{1 - \lambda C} \int \left(P^\lambda f - f \right) d\mu \int e^f \, d\mu. \tag{6.14}$$

Assume moreover that $\mathrm{c}(x,y) = c(x-y)$, $x,y \in \mathbb{R}^k$, *where* $c : \mathbb{R}^k \to \mathbb{R}^+$ *is a differentiable symmetric function with* $c(0) = \nabla c(0) = 0$. *Then for all* $K \geq 0, \eta > 0$ *with* $\eta + K < 1/C$ *and all* K*-semi-concave differentiable function* $f : \mathbb{R}^k \to \mathbb{R}$,

$$\mathrm{Ent}_\mu\left(e^f\right) \leq \frac{\eta}{1 - C(\eta + K)} \int c^*\left(\frac{\nabla f}{\eta}\right) d\mu \int e^f \, d\mu. \tag{6.15}$$

Proof of Theorem 6.13. Following the proof of Theorem 2.1, one has for every probability measure π with marginals ν_f and μ,

$$H(\nu_f|\mu) \leq \iint (f(x) - f(y))\, d\pi(x,y).$$

From the definition of the sup-convolution function $P^\lambda f$, one has

$$H(\nu_f|\mu) \leq \iint \left(P^\lambda f(y) - f(y)\right) d\pi(x,y) + \lambda \iint c(y,x)\, d\pi(x,y).$$

Optimizing over all probability measure π and since μ satisfies $\mathbf{T}_c(C)$, this yields

$$H(\nu_f|\mu) \leq \int \left(P^\lambda f(y) - f(y)\right) d\mu + \lambda C H(\nu_f|\mu).$$

This is exactly the inequality (6.14). Now, if $\mathrm{c}(x,y) = c(x-y)$, $x,y \in \mathbb{R}^k$, and $f : \mathbb{R}^k \to \mathbb{R}$ is a K-semi-concave differentiable function, then by Lemma 2.2 one has: for all $\eta > 0$,

$$P^{K+\eta} f - f \leq \eta c^*\left(\frac{\nabla f}{\eta}\right).$$

The restricted modified logarithmic Sobolev inequalities (6.15) then follows. □

6.3. **On Poincaré inequalities.** Let $c : \mathbb{R}^k \to \mathbb{R}$ be a differentiable function such that $c(0) = \nabla c(0) = 0$, with invertible Hessian at point 0 and such that $\mathrm{Hess}_{|0} c > 0$ (as symmetric matrices). As for the logarithmic Sobolev inequalities, it is known that a linearized version of the transport inequality $\mathbf{T}_c(C)$ is Poincaré inequality (see [21, 25, 5]).

Naturally, **rMLSI**(c,C) or **ICLSI**(c,C) also provide Poincaré inequality by using basic ideas given in [21] (see also [5]). Namely, starting from **ICLSI**(c,C), we apply it with εf, where $f : \mathbb{R}^k \to \mathbb{R}$ is a smooth function with compact support. The infimum $\inf_{y\in\mathbb{R}^k} \{\varepsilon f(y) + \lambda c(x-y)\}$ is attained at some y_ε such that $\varepsilon \nabla f(y_\varepsilon) = \lambda \nabla c(x - y_\varepsilon)$. Since for $h \in \mathbb{R}^k$, $\nabla c^*(\nabla c)(h) = h$, one has

$$x - y_\varepsilon = \nabla c^*\left(\frac{\varepsilon \nabla f(y_\varepsilon)}{\lambda}\right) = \frac{\varepsilon}{\lambda} \mathrm{Hess}_{|0} c^*(\nabla f(x)) + o(\varepsilon).$$

Therefore, since $\mathrm{Hess}_{|\nabla c(h)} c^*(\mathrm{Hess}_{|h} c(u)) = u$ and after some computations, we get the following Taylor expansion

$$\begin{aligned} Q^\lambda(\varepsilon f)(x) &= \varepsilon f(y_\varepsilon) + \lambda c(x - y_\varepsilon) \\ &= \varepsilon f(x) - \frac{\varepsilon^2}{2\lambda} \nabla f(x).\mathrm{Hess}_{|0} c^*(\nabla f(x)) + o\left(\varepsilon^2\right) \end{aligned}$$

It is a classical fact that

$$\mathrm{Ent}_\mu\left(e^{\varepsilon f}\right) = \frac{\varepsilon^2}{2}\,\mathrm{Var}_\mu(f) + o\left(\varepsilon^2\right).$$

Finally, as $\varepsilon \to 0$, **ICLSI**(c, C) implies: for every $\lambda \in (0, 1/C)$,

$$\mathrm{Var}_\mu(f) \le \frac{1}{\lambda(1-\lambda C)} \int \nabla f . \mathrm{Hess}_{|0} c^*(\nabla f)\, d\mu.$$

Optimizing over all λ yields the following Poincaré inequality for the metric induced by $\mathrm{Hess}_{|0} c^*$

$$\mathrm{Var}_\mu(f) \le 4C \int \nabla f . \mathrm{Hess}_{|0} c^*(\nabla f)\, d\mu.$$

Denoting by $\|\cdot\|$ the usual operator norm, one also has a Poincaré inequality with respect to the usual Euclidean metric

$$\mathrm{Var}_\mu(f) \le 4C \|\mathrm{Hess}_{|0} c^*\| \int |\nabla f|^2\, d\mu.$$

From the infimum-convolution characterization of transport inequality $\mathbf{T}_c(C)$ (see Theorem 3.5), a similar proof gives the same Poincaré inequality with the constant C instead of $4C$ (see [21]).

Conversely, Bobkov and Ledoux [7, Theorem 3.1] obtained that Poincaré inequality implies a modified logarithmic Sobolev inequality. Let $\alpha_{2,1} : \mathbb{R} \to \mathbb{R}^+$ and $c_{2,1} : \mathbb{R}^k \to \mathbb{R}^+$ be the cost function defined by

$$\alpha_{2,1}(h) = \min\left(\frac{1}{2}h^2, |h| - \frac{1}{2}\right), \qquad \forall h \in \mathbb{R},$$

and $c_{2,1}(x) = \sum_{i=1}^k \alpha_{2,1}(x_i)$, $x \in \mathbb{R}^k$. One has $\alpha^*_{2,1}(h) = h^2/2$ if $|h| \le 1$ and $\alpha^*_{2,1}(h) = +\infty$ otherwise. Bobkov-Ledoux's result is the following.

Theorem 6.16 ([7])**.** *Let μ be a probability measure on $\mathbb{R}^k$ satisfying the Poincaré inequality:*

(**P**(C)) $$\mathrm{Var}_\mu(f) \le C \int |\nabla f|^2\, d\mu,$$

for every smooth function f on $\mathbb{R}^k$. Then the following modified logarithmic Sobolev inequality holds (in short **BLI**(C)*): for all $\kappa < 2/\sqrt{C}$ and every smooth function f,*

(**BLI**(C)) $$\mathrm{Ent}_\mu\left(e^f\right) \le C\kappa^2 K(\kappa, C) \int \alpha^*_{2,1}\left(\frac{\nabla f}{\kappa}\right) e^f\, d\mu,$$

where $K(\kappa, C) = \left(\frac{2+\kappa\sqrt{C}}{2-\kappa\sqrt{C}}\right)^2 e^{\kappa\sqrt{5C}}$.

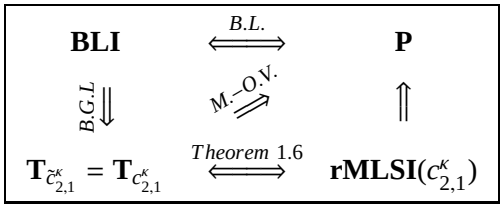
Applying **BLI**(C) to εf, as $\varepsilon \to 0$, **BLI**(C) yields **P**$(CK(\kappa, C))$ but also **P**(C) since $K(\kappa, C) \to 1$ as $\kappa \to 0$. Theorem 6.16 therefore indicates that **P**(C) and **BLI**(C) are exactly equivalent. Thanks to the Hamilton-Jacobi approach, Bobkov

Gentil and Ledoux [5] obtained that **BLI**(C) implies $\mathbf{T}_{\tilde{c}^{\kappa}_{2,1}}(C)$ for all $\kappa < 2/\sqrt{C}$ where

$$\tilde{c}^{\kappa}_{2,1}(x) = \kappa^2 C^2 K(\kappa, C)\, \alpha_{2,1}\left(\frac{|x|}{\kappa C K(\kappa, C)}\right), \qquad \forall x \in \mathbb{R}^k. \tag{6.17}$$

By linearization and optimization over κ, $\mathbf{T}_{\tilde{c}^{\kappa}_{2,1}}(C)$ implies $\mathbf{P}(C)$, and therefore **BLI**(C) is also equivalent to $\mathbf{T}_{\tilde{c}^{\kappa}_{2,1}}(C)$ for all $\kappa < 2/\sqrt{C}$.

Let $c^{\kappa}_{2,1}$ denote the cost function defined similarly as $\tilde{c}^{\kappa}_{2,1}$ replacing $\alpha_{2,1}(|\,.\,|)$ by $c_{2,1}$ in (6.17). One has $\tilde{c}^{\kappa}_{2,1} \le c^{\kappa}_{2,1}$ (this is a consequence of the subadditivity of the concave function $h \to \alpha_{2,1}(\sqrt{h})$). Therefore $\mathbf{T}_{c^{\kappa}_{2,1}}(C)$ implies $\mathbf{T}_{\tilde{c}^{\kappa}_{2,1}}(C)$. Consider now the case of dimension 1, $k = 1$, so that $c^{\kappa}_{2,1} = \tilde{c}^{\kappa}_{2,1}$. Theorem 1.6 indicates that $\mathbf{T}_{c^{\kappa}_{2,1}}$ is equivalent, up to constant, to **rMLSI**$(c^{\kappa}_{2,1})$. Actually **rMLSI**$(c^{\kappa}_{2,1})$ can be interpreted as **BLI** restricted to a class of semi-convex function for the cost $c^{\kappa}_{2,1}$. However, from the discussions above, **rMLSI**$(c^{\kappa}_{2,1})$ and **BLI** are equivalent up to constant. It would be interesting to directly recover **BLI** from **rMLSI**$(c^{\kappa}_{2,1})$ or from $\mathbf{T}_{c^{\kappa}_{2,1}}$. The known results can be summarized by the following diagram for $k = 1$:



where:

B.L. : Bobkov, Ledoux [7];
B.G.L. : Bobkov, Gentil, Ledoux [5];
M. : Maurey [21];
O.V. : Otto, Villani [25].

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U ´P E M V ´ - L 'A M ´ A ´
(UMR CNRS 8050), 5 D , 77454 M V ´ C 2, F

E-mail address: nathael.gozlan@univ-mlv.fr, cyril.roberto@univ-mlv.fr, paul-marie.samson@univ-mlv.fr