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DEVIATION INEQUALITIES FOR CONVEX FUNCTIONS MOTIVATED BY THE TALAGRAND CONJECTURE

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Motivated by Talagrand's conjecture on regularization properties of the natural semigroup on the Boolean hypercube, and, in particular, by its continuous analogue involving regularization properties of the Ornstein–Uhlenbeck semigroup acting on integrable functions, we explore deviation inequalities for log-semiconvex functions under Gaussian measure. Bibliography: 18 *titles.*

1. Introduction

In the late eighties, Talagrand conjectured that the "convolution by a biased coin" on the hypercube $\{-1,1\}^n$ satisfies some refined hypercontractivity property. We refer to Problems 1 and 2 of [17] for precise statements. A continuous version of Talagrand's conjecture for the Ornstein–Uhlenbeck operator has recently attracted some attention [1, 6, 11]; in particular, it was resolved in [6, 11] by first proving a deviation inequality for log-semiconvex functions above their means under Gaussian measure. In this paper, we discuss a simpler approach to proving this deviation inequality for the special case of log-convex functions (which is already of interest).

Let us start by presenting the continuous version of Talagrand's conjecture and the history of its resolution. Denote by γ_n the standard Gaussian (probability) measure in dimension n with density

$$x \mapsto (2\pi)^{-n/2} \exp\left\{-\frac{|x|^2}{2}\right\}$$

(where $|x|$ denotes the standard Euclidean norm of $x \in \mathbb{R}^n$), and, for $p \geq 1$, denote by $\mathbb{L}^p(\gamma_n)$ the set of measurable functions $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $|f|^p$ is integrable with respect to γ_n. Then, given $g \in \mathbb{L}^1(\gamma_n)$, the Ornstein–Ulhenbeck semigroup is defined as

$$P_t g(x) := \int g\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) d\gamma_n(y), \qquad x \in \mathbb{R}^n,\ t \geq 0. \tag{1.1}$$

It is well known that the family $(P_t)_{t\geq 0}$ enjoys the so-called hypercontractivity property [9, 13, 14], which asserts that, for any $p > 1$, any $t > 0$, and $q \leq 1 + (p-1)e^{2t}$, $P_t g$ is more regular than g in the sense that if $g \in \mathbb{L}^p(\gamma_n)$, then $P_t g \in \mathbb{L}^q(\gamma_n)$, and, moreover,

$$\|P_t g\|_q \leq \|g\|_p.$$

However, this property is empty when one only assumes that $g \in \mathbb{L}^1(\gamma_n)$. A natural question is therefore to ask whether the semigroup has anyway some regularization effect in this case also. Given a nonnegative $g : \mathbb{R}^n \to \mathbb{R}$ with $\int g\, d\gamma_n = 1$, by Markov's inequality and the fact that $\int P_s g\, d\gamma_n = 1$, we have

$$\gamma_n(\{P_s g \geq t\}) \leq \frac{1}{t} \qquad \text{for any} \qquad t > 0.$$

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The continuous version of Talagrand's conjecture (adapted from [17, Problems 1 and 2]) states that

$$\lim_{t\to\infty} \sup_{\substack{g\geq 0,\\ \int g\, d\gamma_n=1}} t\gamma_n(\{P_s g \geq t\}) = 0$$

for $s > 0$. The most recent paper dealing with this conjecture is due to Lehec [11] who proved that for any $s > 0$ there exists a constant $\alpha_s \in (0, \infty)$ (depending only on s and not on the dimension n) such that

$$\gamma_n(\{P_s g \geq t\}) \leq \frac{\alpha_s}{t\sqrt{\log t}} \qquad \text{for any} \qquad t > 1 \tag{1.2}$$

and for any nonnegative function $g : \mathbb{R}^n \to \mathbb{R}^+$ with $\int g\, d\gamma_n = 1$, and this bound is optimal in the sense that the factor $\sqrt{\log t}$ cannot be improved. In the first paper [1] dealing with this question, Ball, Barthe, Bednorz, Oleszkiewicz, and Wolff already obtained a similar bound but with a constant α_s depending heavily on the dimension n plus some extra $\log\log t$ factor in the numerator. Later, Eldan and Lee [6] proved that the above bound holds with a constant α_s independent on n but again with the extra $\log\log t$ factor in the numerator. Finally, the conjecture was fully proved by Lehec removing the $\log\log t$ factor [11] and giving an explicit bound on α_s, namely, that $\alpha_s := \alpha \max(1, \frac{1}{2s})$ for some numerical constant α.

In both Eldan–Lee and Lehec's papers, the two key ingredients are the following:
(1) for any $s > 0$, the Ornstein–Uhlenbeck semigroup satisfies the inequality

$$\mathrm{Hess}\,(\log P_s g) \geq -\frac{1}{2s}\mathrm{Id}$$

for all nonnegative functions $g \in \mathbb{L}^1(\gamma_n)$, where Hess denotes the Hessian matrix and Id is the identity matrix of $\mathbb{R}^n$. This is a somehow standard property easily proved thanks to the kernel representation (1.1);
(2) for any positive function g with $\mathrm{Hess}\,(\log g) \geq -\beta\mathrm{Id}$, for some $\beta > 0$, and $\int g\, d\gamma_n = 1$,

$$\gamma_n(\{g \geq t\}) \leq \frac{C_\beta}{t\sqrt{\log t}} \qquad \text{for any} \qquad t > 1,$$

with $C_\beta = \alpha \max(1, \beta)$.

It will be more convenient to deal with $g = e^f$ in the sequel; thus, now we move to this setting. The last inequality can be reformulated as follows: For any $f : \mathbb{R}^n \to \mathbb{R}$ with $\int e^f d\gamma_n = 1$ and $\mathrm{Hess}\,(f) \geq -\beta\mathrm{Id}$,

$$\gamma_n\left(\{f \geq t\}\right) \leq C_\beta \frac{e^{-t}}{\sqrt{t}} \qquad \text{for any} \qquad t > 0. \tag{1.3}$$

We now describe the two main contributions of this note (which were independently obtained by Ramon van Handel). First, as a warm up, we give in Sec. 2 a short proof of (1.3) in dimension 1. The main argument of this proof is that due to the semiconvexity of f, the condition $(2\pi)^{-1/2} \int e^{f-\frac{1}{2}|x|^2}\, d\gamma = 1$ implies a *pointwise* comparison between f and the function $|x|^2/2$, which then can be turned into a tail comparison.

Then, in dimension n, we give in Sec. 3 a sharp version of the upper bound (1.3) for convex functions. Our main result is as follows.

Theorem 1.4. *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function such that $\int e^f\, d\gamma_n = 1$; then*

$$\gamma_n(f \geq t) \leq \overline{\Phi}(\sqrt{2t}) \qquad \textit{for any} \qquad t \geq 0, \tag{1.5}$$

where $\overline{\Phi}(t) = \frac{1}{\sqrt{2\pi}} \int\limits_t^{+\infty} e^{-u^2/2}\, du$, $t \in \mathbb{R}$.

Let us make a few comments on this result. First, using the following classical bound (which is asymptotically optimal):

$$\overline{\Phi}(s) = \frac{1}{\sqrt{2\pi}} \int\limits_s^{\infty} e^{-x^2/2}\, dx \le \frac{1}{\sqrt{2\pi}} \int\limits_s^{\infty} \frac{x}{s} e^{-x^2/2}\, dx = \frac{e^{-s^2/2}}{\sqrt{2\pi} s} \text{ for any } s > 0, \tag{1.6}$$

one immediately recovers (1.3) with the constant $C_0' = 1/(2\sqrt{\pi})$. Furthermore, the bound (1.9) is sharp. Indeed, for a given value of $t \ge 0$, inequality (1.9) becomes an equality for the function

$$f_t(x) = \sqrt{2t} x_1 - t, \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Finally, since the Ornstein–Uhlenbeck semigroup preserves log-convexity (this follows from the fact that any positive combination of log-convex functions remains log-convex, see, e.g., [12, p. 649]), Theorem 1.4 immediately implies the following corollary.

Corollary 1.7. *Let g be a log-convex function such that $\int g\, d\gamma_n = 1$; then for any $s \ge 0$,*

$$\gamma_n(P_s g \ge t) \le \overline{\Phi}(\sqrt{2 \log(t)}) \qquad \textit{for any} \qquad t \ge 1.$$

In the special case where g is log-convex, Corollary 1.7 is a sharp improvement of Lehec's result (1.2). Note that for log-convex g, the constant α_s can be taken independent of s unlike in (1.2), but this already followed from Lehec's inequality (1.3) combined with the preservation of log-convexity by the Ornstein–Uhlenbeck semigroup.

Another consequence of Theorem 1.4 is that a deviation inequality for structured functions also follows for other measures that can be obtained by "nice" pushforwards of Gaussian measure. Indeed, observe that for any coordinatewise nondecreasing, convex function f on $\mathbb{R}^n$ and any convex functions $g_1, \ldots, g_n : \mathbb{R}^N \to \mathbb{R}$, the composition $f(g_1(x), \ldots, g_n(x))$ is convex on $\mathbb{R}^N$. Hence, we immediately have the following corollary.

Corollary 1.8. *For a standard Gaussian random vector Z in $\mathbb{R}^N$, let the probability measure μ on $\mathbb{R}^n$ be the joint distribution of $(g_1(Z), \ldots, g_n(Z))$. Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a coordinatewise nondecreasing, convex function such that $\int\limits_{\mathbb{R}^n} e^f\, d\mu = 1$. Then*

$$\mu(f \ge t) \le \overline{\Phi}(\sqrt{2t}) \qquad \textit{for any} \qquad t \ge 0. \tag{1.9}$$

For example, consider the exponential distribution, whose density is e^{-x} on $\mathbb{R}_+ = (0, \infty)$ and which can be realized as $\frac{Z_1^2 + Z_2^2}{2}$ with Z_1 and Z_2 i.i.d. standard Gaussian. Clearly, a product of exponential distributions on the line is an instance covered by Corollary 1.8, since we can take $N = 2n$ and $g_i(x) = \frac{x_i^2 + x_{i+1}^2}{2}$. More generally, Corollary 1.8 applies to a product of χ^2 distributions with arbitrary degrees of freedom, and also to some cases with correlation (consider, for example, $N = 3, g_1(x) = \frac{x_1^2 + x_2^2}{2}$, and $g_2(x) = \frac{x_2^2 + x_3^2}{2}$). The proof of Theorem 1.4 is given in Sec. 3. It relies on the Ehrhard inequality which we recall now: According to [5, Theorem 3.2], if $A, B \subset \mathbb{R}^n$ are two convex sets, then

$$\Phi^{-1}(\gamma_n(\lambda A + (1 - \lambda) B)) \ge \lambda \Phi^{-1}(\gamma_n(A)) + (1 - \lambda) \Phi^{-1}(\gamma_n(B)) \text{ for any } \lambda \in [0, 1], \tag{1.10}$$

where $\lambda A + (1 - \lambda) B := \{\lambda a + (1 - \lambda) b : a \in A, b \in B\}$ denotes the usual Minkowski sum and Φ^{-1} is the inverse of the cumulative distribution function Φ of γ_1:

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{t} e^{-u^2/2}\, du, \qquad t \in \mathbb{R}. \tag{1.11}$$

After Ehrhard's pioneer work, inequality (1.10) was shown to be true if only one set is assumed to be convex by Latała [10] and finally was extended to arbitrary measurable sets by Borell [4]. See also [2, 18] and references therein for recent developments on this inequality. Inequality (1.10) (for arbitrary sets A and B) is a very strong statement in the hierarchy of Gaussian geometric and functional inequalities. For instance, it gives back the celebrated Gaussian isoperimetric result of Sudakov–Tsirelson [16] and Borell [3]. Another elegant consequence of (1.10) due to Kwapień is that if f is a convex function on $\mathbb{R}^n$ that is integrable with respect to γ_n, then the median of f is always less than or equal to the mean of f under γ_n. The key ingredient in Kwapień's proof is the observation that the function

$$\alpha(t) = \Phi^{-1}(\gamma_n(f \leq t)), \qquad t \in \mathbb{R},$$

is *concave* over $\mathbb{R}$; this observation (already made in Ehrhard's original paper) also plays a key role in our proof of Theorem 1.4.

After the completion of this work, we learned that Paouris and Valettas [15] developed in a recent paper similar ideas to derive from (1.10) deviation inequalities for convex functions *under* their mean.

In Sec. 4, we give a second proof of Theorem 1.4 and also discuss (following an observation by R. van Handel) the difficulty of its extension to the log-semiconvex case.

Acknowledgment. The results of this note were independently obtained by Ramon van Handel a few months before us, as we learnt after a version of this note was circulated. Although he chose not to publish them, these observations should be considered as due to him. We are also grateful to him for numerous comments on earlier drafts of this note.

2. The continuous Talagrand conjecture in dimension 1

In the next lemma, we take advantage of the semiconvexity property $\mathrm{Hess}\,(f) \geq -\beta \mathrm{Id}$ to derive information on f. More precisely, we may compare f to $x \mapsto |x|^2/2$. The result holds in any dimension, and we give two proofs for completeness.

Lemma 2.1. *Let* $f\colon \mathbb{R}^n \to \mathbb{R}$ *and* $\beta \geq 0$ *be such that* $\int e^f \, d\gamma_n = 1$*,* f *is smooth, and* $\mathrm{Hess}\,(f) \geq -\beta \mathrm{Id}$*. Then*

$$f(x) \leq \frac{n}{2}\log(1+\beta) + \frac{1}{2}|x|^2 \qquad \textit{for any} \qquad x \in \mathbb{R}^n.$$

First proof of Lemma 2.1. Let $h(x) = f(x) + \frac{\beta}{2}|x|^2$. By the assumption on f, the function h is convex on $\mathbb{R}^n$; hence,

$$h(x) = \sup_{t \in \mathbb{R}^n} \{\langle x, t\rangle - h^*(t)\} \qquad \text{for any} \qquad x \in \mathbb{R}^n,$$

where

$$h^*(t) := \sup_{x \in \mathbb{R}^n} \{\langle t, x\rangle - h(x)\}, \qquad t \in \mathbb{R}^n,$$

is the Legendre transform of h. Now,

$$\begin{aligned} 1 = \int e^f \, d\gamma_n = \int \exp\left\{h(x) - \frac{\beta}{2}|x|^2\right\} d\gamma_n(x) \\ \geq (2\pi)^{-n/2} e^{-h^*(t)} \int \exp\left\{\langle x, t\rangle - \frac{1+\beta}{2}|x|^2\right\} dx \\ = (1+\beta)^{-n/2} \exp\left\{-h^*(t) + \frac{1}{2(1+\beta)}|t|^2\right\} \end{aligned}$$

for all $t \in \mathbb{R}^n$. Therefore,

$$h^*(t) \geq -\frac{n}{2}\log(1+\beta) + \frac{1}{2(1+\beta)}|t|^2$$

for all $t \in \mathbb{R}^n$. In its turn,

$$\begin{aligned} h(x) &= \sup_t \{\langle x, t\rangle - h^*(t)\} \leq \frac{n}{2}\log(1+\beta) + \sup_t \left\{\langle x, t\rangle - \frac{1}{2(1+\beta)}|t|^2\right\} \\ &= \frac{1}{2}\left(n\log(1+\beta) + (1+\beta)|x|^2\right), \end{aligned}$$

which leads to the desired conclusion. □

Second proof of Lemma 2.1. Define $\tilde{h}(x) = h(x) + \frac{\beta}{2}|x|^2$, $x \in \mathbb{R}^n$, and let $\gamma_{n,\beta}$ be the Gaussian measure $\mathcal{N}(0, \frac{1}{1+\beta}I)$; then

$$1 = \int e^{h(x)}\, d\gamma_n(x) = (1+\beta)^{-n/2}\int e^{\tilde{h}(x)}\, d\gamma_{n,\beta}(x).$$

For all $a \in \mathbb{R}^n$, the change of variable formula implies that

$$1 = (1+\beta)^{-n/2}e^{-\frac{(1+\beta)}{2}|a|^2}\int e^{\tilde{h}(y+a)-(1+\beta)y\cdot a}\, d\gamma_{n,\beta}(dy).$$

The function $y \mapsto \tilde{h}(y+a) - (1+\beta)y\cdot a$ is convex and the function $x \mapsto e^x$ is convex and increasing; thus, the function $y \mapsto \exp\left(\tilde{h}(y+a) - (1+\beta)y\cdot a\right)$ is also convex. Applying the Jensen inequality, we see that

$$\begin{aligned} 1 &\geq (1+\beta)^{-n/2}e^{-\frac{(1+\beta)}{2}|a|^2}\exp\left(\tilde{h}\left(a + \int y\, d\gamma_{n,\beta}(y)\right) - (1+\beta)\int y\cdot a\, d\gamma_{n,\beta}(y)\right) \\ &= e^{-\frac{(1+\beta)}{2}|a|^2 + \tilde{h}(a)}; \end{aligned}$$

hence, $h(a) \leq |a|^2/2 + \frac{n}{2}\log(1+\beta)$. □

Remark 2.2. *The case of* $\beta = 0$ *in Lemma* 2.1 (*i.e., of convex functions* f*, which is the essential case*) *is contained in Graczyk et al.* [7, Lemma 3.7] (*curiously, it does not appear in the published version* [8] *of the paper*)*, and, in fact, was proved in the more general setting of subharmonic functions. The second proof given above is theirs and works for the more general setting. Also note that neither proof requires the smoothness of* f*, which however is sufficient for our purposes.*

In principle, one would hope to already get some deviation bound from the above lemma. More precisely, given f as in Lemma 2.1, we have the inequality

$$\gamma_n(\{f \geq t\}) \leq \gamma_n\left(\left\{|x|^2 \geq 2t - n\log(1+\beta)\right\}\right)$$

thanks to Lemma 2.1, and we are left with a tail estimate for a χ^2 distribution with n degrees of freedom. In dimension $n = 1$, the tail of the χ^2 distribution behaves like $e^{-t}/\sqrt{t}$. Therefore, the above simple argument already gives back estimate (1.3) and thus, provides a quick proof of the continuous Talagrand's conjecture for $n = 1$, moreover, with a clean dependence on β, as detailed below.

Theorem 2.3. *If a smooth function* $f\colon \mathbb{R} \to \mathbb{R}$ *and* $\beta \geq 0$ *are such that* $\int e^f\, d\gamma_1 = 1$ *and* $f'' \geq -\beta$ *pointwise, then*

$$\gamma_1(\{f \geq t\}) \leq \frac{1+\beta}{\sqrt{2}}\frac{e^{-t}}{\sqrt{t}} \qquad \text{for any} \qquad t \geq 1.$$

Proof. Assume first that $t \geq (1+\beta)\log(1+\beta)/(2\beta)$. Using inequality (1.6), we deduce from Lemma 2.1 that

$$\gamma_1(\{f \geq t\}) \leq \gamma_1\left(\left\{|x| \geq \sqrt{2t - \log(1+\beta)}\right\}\right) \leq 2(2\pi)^{-1/2} \frac{\exp\left\{-t + \frac{1}{2}\log(1+\beta)\right\}}{\sqrt{2t - \log(1+\beta)}}$$

$$= \sqrt{\frac{1+\beta}{\pi}} \frac{e^{-t}}{\sqrt{t}} \frac{1}{\sqrt{1 - (\log(1+\beta)/(2t))}} \leq \sqrt{\frac{1+\beta}{\pi}} \frac{e^{-t}}{\sqrt{t}} \frac{1}{\sqrt{1 - (\beta/(1+\beta))}} = \frac{1+\beta}{\sqrt{\pi}} \frac{e^{-t}}{\sqrt{t}}.$$

Now assume that $t \leq (1+\beta)\log(1+\beta)/(2\beta)$. By Markov's inequality,

$$\gamma_1(\{f > t\}) \leq e^{-t} \leq \sqrt{(1+\beta)\log(1+\beta)/(2\beta)} \frac{e^{-t}}{\sqrt{t}} \leq \frac{\sqrt{1+\beta}}{\sqrt{2}} \frac{e^{-t}}{\sqrt{t}} \leq \frac{1+\beta}{\sqrt{2}} \frac{e^{-t}}{\sqrt{t}},$$

where, in the third inequality, we have used that $\log(1+\beta) \leq \beta$. The result follows. □

Unfortunately, this naive approach of using the pointwise bound from Lemma 2.1 is specific to dimension 1 since in higher dimension, the tail of the $\mathcal{X}^2$ distribution does not have the correct behavior. It should be noticed that Ball et al. [1] also have a quick direct proof of the Talagrand conjecture for $n = 1$ which also uses a similar tail comparison with the $\mathcal{X}^2$ distribution, and it is also noticed that such a tail is not of the correct order for $n \geq 2$.

3. The deviation inequality for log-convex functions

Throughout this section, $f : \mathbb{R}^n \to \mathbb{R}$ is a *convex* function satisfying $\int e^f \, d\gamma_n = 1$, where γ_n is the standard Gaussian measure on $\mathbb{R}^n$. Given $s \in \mathbb{R}$, let

$$A_s := \{f \leq s\}$$

and

$$\varphi(s) := \Phi^{-1}(\gamma_n(A_s)),$$

where Φ^{-1} is the inverse of the Gaussian cumulative distribution function Φ defined by (1.11).

The key ingredient in the proof of Theorem 1.4 is the concavity of the function φ, which, as we see in the proof of the next lemma, is a direct consequence of Ehrhard's inequality (1.10).

Lemma 3.1. *Let f and φ be defined as above. Then φ is concave, nondecreasing, $\lim_{s\to\infty} \varphi(s) = +\infty$, and $\lim_{s\to-\infty} \varphi(s) = -\infty$.*

The concavity of φ was first observed by Ehrhard in [5]. Below we recall the proof for the reader's convenience.

Proof. The fact that φ is nondecreasing and satisfies $\lim_{s\to\infty} \varphi(s) = +\infty$ and $\lim_{s\to-\infty} \varphi(s) = -\infty$ is a direct and obvious consequence of the definition. Now we prove that φ is concave using Ehrhard's inequality. Given $\lambda \in [0,1]$ and $s_1, s_2 \in \mathbb{R}$, we have, by the convexity of f, the inclusion

$$A_{\lambda s_1 + (1-\lambda)s_2} \supset \lambda A_{s_1} + (1-\lambda)A_{s_2}.$$

Hence, by the monotonicity of Φ^{-1},

$$\varphi(\lambda s_1 + (1-\lambda)s_2) \geq \Phi^{-1}(\gamma_n(\lambda A_{s_1} + (1-\lambda)A_{s_2})).$$

Then, Ehrhard's inequality (1.10) implies that

$$\begin{aligned}\Phi^{-1}(\gamma_n(\lambda A_{s_1} + (1-\lambda)A_{s_2})) &\geq \lambda\Phi^{-1}(\gamma_n(A_{s_1})) + (1-\lambda)\Phi^{-1}(\gamma_n(A_{s_2})) \\ &= \lambda\varphi(s_1) + (1-\lambda)\varphi(s_2),\end{aligned}$$

from which the concavity of φ follows. □

Proof of Theorem 1.4. Let f and φ be defined as above. Then it is enough to show that

$$\varphi(u) \geq \sqrt{2u} \qquad \text{for any} \qquad u \geq 0.$$

Since $-\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is convex by Lemma 3.1 and lower-semicontinuous, the Fenchel–Moreau theorem applies and guarantees that

$$-\varphi(u) = \sup_{t \in \mathbb{R}} \{ut - \psi(t)\} \qquad \text{for any} \qquad u \in \mathbb{R},$$

where

$$\psi(t) = (-\varphi)^*(t) := \sup_{u \in \mathbb{R}} \{ut + \varphi(u)\}$$

is the Fenchel–Legendre transform of $-\varphi$. We also observe that, since $\lim_{u\to\infty} \varphi(u) = +\infty$, necessarily $\psi(t) = +\infty$ for all $t > 0$, so that

$$\varphi(u) = -\sup_{t \leq 0} \{ut - \psi(t)\} = \inf_{t \leq 0} \{-ut + \psi(t)\}.$$

Now observe that

$$1 = \int e^f \, d\gamma_n = \int_{-\infty}^{\infty} e^u \gamma_n(f \geq u) \, du = \int_{-\infty}^{\infty} e^u (1 - \Phi(\varphi(u)) \, du = \int_{-\infty}^{\infty} e^u \overline{\Phi}(\varphi(u)) \, du,$$

where we recall that $\overline{\Phi} = 1 - \Phi$. Using integration by parts and the fact that $\overline{\Phi}$ is decreasing, we conclude that

$$1 = \int_{-\infty}^{\infty} e^u \overline{\Phi}(\varphi(u)) \, du \geq \int_{-\infty}^{\infty} e^u \overline{\Phi}(-ut + \psi(t)) \, du = (-t) e^{\frac{\psi(t)}{t}} \int_{-\infty}^{+\infty} e^{\frac{-v}{t}} \overline{\Phi}(v) \, dv$$

$$= e^{\frac{\psi(t)}{t}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-v}{t}} e^{-v^2/2} \, dv = \exp\left\{ \frac{\psi(t)}{t} + \frac{1}{2t^2} \right\}$$

for all $t \leq 0$. Therefore,

$$\psi(t) \geq -\frac{1}{2t}$$

for all $t \leq 0$. In its turn,

$$\varphi(u) = \inf_{t \leq 0} \{-ut + \psi(t)\} \geq \inf_{t \leq 0} \left\{ -ut - \frac{1}{2t} \right\} = \sqrt{2u},$$

as expected. □

4. Revisiting the deviation inequality, with a discussion of the semi-convex case

Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a function such that $\int e^f \, d\gamma_n = 1$. Define μ_f as the distribution of f under γ_n, i.e.,

$$\mu_f(A) := \gamma_n(\{x \in \mathbb{R}^n : f(x) \in A\}) \ \text{ for any } \ \text{Borel } A \subset \mathbb{R}.$$

Consider the monotone rearrangement transport map T_f sending γ_1 to μ_f. It is defined by

$$T_f(u) = F_f^{-1} \circ \Phi(u) \ \text{ for any } \ u \in \mathbb{R},$$

where $F_f(t) = \mu_f((-\infty, t])$, $t \in \mathbb{R}$, denotes the cumulative distribution function of μ_f and

$$F_f^{-1}(s) = \inf\{t : F_f(t) \geq s\}, \quad s \in (0, 1),$$

is its generalized inverse.

The following proposition yields a slightly different proof of Theorem 1.4.

Proposition 4.1. *With the notation above, if T_f is κ-semiconvex for some $\kappa \ge 0$, i.e.,*

$$T_f((1-t)x+ty) \le (1-t)T_f(x) + tT_f(y) + \frac{\kappa}{2}t(1-t)|x-y|^2$$
$$\text{for any} \quad x,y \in \mathbb{R} \quad \text{and} \quad t\in[0,1],$$

then

$$\gamma_n(\{f>u\}) \le \overline{\Phi}\left(\sqrt{2u-\log(1+\kappa)}\right) \quad \text{for any} \quad u \ge \frac{1}{2}\log(1+\kappa).$$

Proof. The κ-semiconvexity condition is equivalent to the convexity of the function $x \mapsto T_f(x) + \kappa\frac{x^2}{2}$. Now we observe that

$$1 = \int e^f \, d\gamma_n = \int e^y \, d\mu_f(y) = \int e^{T_f(x)} \, d\gamma_1(x).$$

Applying Lemma 2.1 to the function T_f in dimension 1, one concludes that

$$T_f(x) \le \frac{1}{2}x^2 + \frac{1}{2}\log(1+\kappa) \quad \text{for any} \quad x \in \mathbb{R}.$$

This is equivalent to

$$\Phi(x) \le F_f\left(\frac{1}{2}x^2 + \frac{1}{2}\log(1+\kappa)\right);$$

thus,

$$F_f(u) \ge \Phi\left(\sqrt{2u-\log(1+\kappa)}\right) \qquad \text{for any} \qquad u \ge \frac{1}{2}\log(1+\kappa),$$

or, in other words,

$$\gamma_n(\{f>u\}) \le \overline{\Phi}\left(\sqrt{2u-\log(1+\kappa)}\right) \qquad \text{for any} \qquad u \ge \frac{1}{2}\log(1+\kappa). \qquad \square$$

Second proof of Theorem 1.4. Assume that $f:\mathbb{R}^n \to \mathbb{R}$ is convex and such that $\int e^f\,d\gamma_n = 1$. Then, according to Lemma 3.1, the function $\Phi^{-1}\circ F_f = T_f^{-1}$ is concave. Being also nondecreasing, its inverse T_f is convex. Applying Proposition 4.1 with $\kappa = 0$ completes the proof. $\square$

In view of Proposition 4.1, a natural conjecture would be the following.

Conjecture. *There exists a function $\kappa:[0,\infty)\to[0,\infty)$ such that if $f:\mathbb{R}^n\to\mathbb{R}$ is a smooth function with* $\operatorname{Hess} f \ge -\beta\operatorname{Id}$*, $\beta \ge 0$, then the map T_f is $\kappa(\beta)$-semiconvex on $\mathbb{R}$.*

If this conjecture were true, then one would recover completely the Eldan–Lee–Lehec result (1.3). In addition to the convex case, let us observe that the conjecture is obviously true in dimension 1 for *nondecreasing* functions f. Indeed, f is clearly a transport map between γ_1 and μ_f. Being nondecreasing, f is necessarily a monotone rearrangement map, i.e., $f = T_f$. Since f is κ-semiconvex, then so is T_f.

Unfortunately, this probably too naive conjecture turns out to be false in general. As explained to us by R. van Handel, the presence of local minimizers for f breaks down the semi-convexity of T_f. Let us illustrate this in dimension 1. Consider a function $f:\mathbb{R}\to\mathbb{R}$ of class $\mathcal{C}^1$ such that $f'(x)$ vanishes only at a finite number of points and such that there is some point $x_o \in \mathbb{R}$ and an $\eta > 0$ such that $f'(x_o) = 0$, $f'(x) < 0$ on $[x_o-\eta, x_o[$, and $f'(x_o) > 0$ on $]x_o, x_o+\eta]$. Denoting $t_o = f(x_o)$, we assume that $\inf_{\mathbb{R}} f < t_o$, i.e., f only presents a local minimizer at x_o. Let us further assume that there are some $\alpha_o, \beta_o > 0$ and some positive integer N such that, for all $t_o - \alpha_o \le t < t_o$,

$$\operatorname{Card}\{x\in\mathbb{R} : f(x) = t\} \le N$$

and $|f'(x)| \geq \beta_o$ for all x such that $t_o - \alpha_o \leq f(x) < t_o$.

Claim. There is no $\lambda \geq 0$ for which the map $T := T_f$ is λ-semiconvex.

It is not difficult to exhibit semiconvex functions f enjoying the assumptions above, which disclaims the conjecture.

Proof of the claim. First let us remark that if T were λ-semiconvex for some $\lambda \geq 0$, then the map $x \mapsto T(x) + \frac{\lambda}{2}x^2$ would be convex, and so it would admit finite left and right derivatives everywhere. Moreover, for a convex function, the left derivative at some point is always less than or equal to the right derivative at this same point. Thus, the λ-semiconvexity of T would, in particular, imply that

$$T'_-(x) \leq T'_+(x) \qquad \text{for any} \qquad x \in \mathbb{R}.$$

We are going to show that $T'_-(u_o) > T'_+(u_o)$ for some $u_o \in \mathbb{R}$, which will prove the claim. Since, denoting $F := F_f$,

$$T'_\pm(u) = \frac{\varphi(u)}{F'_\pm \circ T(u)}$$

at every point $u \in \mathbb{R}$ where the derivative exists, one concludes that it is enough to show that

$$F'_-(t_o) < F'_+(t_o)$$

to have the desired inequality at $u_o = T^{-1}(t_o)$. Note that $|T^{-1}(t_o)| < \infty$ because

$$\mu_f((t_o, +\infty)) = \gamma_1((T^{-1}(t_o), +\infty)) > 0$$

and

$$\mu_f((\infty, t_o)) = \gamma_1((-\infty, T^{-1}(t_o))) > 0,$$

as easily follows from our assumptions.

According to the one-dimensional general change of variable formula, the probability measure μ_f admits the following density:

$$h(t) = \sum_{x \in \{f=t\}} \frac{\varphi(x)}{|f'(x)|}, \qquad t \in \mathbb{R},$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in \mathbb{R}$. Define $\varepsilon_o = \max_{[x_o-\eta, x_o+\eta]} f - t_o > 0$; then, for $h < \varepsilon_o$,

$$F(t_o + h) - F(t_o) = \int_{t_o}^{t_o+h} h(t)\, dt \geq h \frac{m}{M(h)},$$

where

$$m = \inf_{[x_o-\eta, x_o+\eta]} \varphi$$

and

$$M(h) = \sup \left\{ |f'(x)| : x \in [x_o - \eta, x_o + \eta], f(x) \in [t_o, t_o + h] \right\}.$$

It is easily seen that $M(h) \to 0$ as h tends to 0^+, which implies that $F'_+(t_o) = +\infty$. Now let us consider the left derivative. Let us note that one can assume without loss of generality that the left derivative exists at t_o, since otherwise, the function T is clearly not semiconvex. For any $h > 0$,

$$F(t_o) - F(t_o - h) = \int_{t_o-h}^{t_o} h(t)\, dt \leq h \frac{N}{\sqrt{2\pi}\beta_o} ;$$

thus, $F'_-(t_o) < +\infty$, which completes the proof of the claim. □

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