Spectral gaps for spin systems: some non-convex phase examples

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Abstract

We prove that a convex phase may be perturbed into a non-convex phase preserving the spectral gap properties of the unbounded spin system with nearest neighbour interaction associated to this potential. The proof is based on HELFFER's method that reduces the spectral properties of the unbounded spin system to some uniform spectral gap of the one-dimensional phase. We then make use of HARDY's criterion for POINCARÉ inequalities on the real line to construct our examples.

1 Introduction

The purpose of this work is to establish some perturbation results for spectral gaps to produce some examples of unbounded spin systems with nearest neighbour interaction associated to non-convex phases satisfying a spectral gap inequality uniformly in finite subsets of the lattice and boundary conditions. These examples thus show that the recent results by YOSHIDA (see [Yos99]), HELFFER [Hel99a], BODINEAU-HELFFER [BH99a, BH99b] on spectral gaps and logarithmic SOBOLEV inequalities can actually hold for families of phases that go beyond the usual convexity at infinity.

To introduce to the results of this paper, let us first describe, following [Hel99a], the spin systems we will investigate. Consider the measure $\exp(-\Phi_{\Lambda,\omega}(X))dX$, where $\Phi_{\Lambda,\omega}$ is a function associated to a finite subset Λ in $\mathbb{Z}^d$ (for $d \in \mathbb{N}^*$) and to some $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ which defines the boundary conditions. The function $\Phi_{\Lambda,\omega}$ has the form, for $X = X^\Lambda \in \mathbb{R}^{|\Lambda|}$ (where $|\Lambda|$ is the cardinal of Λ) :

$$\Phi_{\Lambda,\omega}(X) = \sum_{i\in\Lambda} \psi(x_i) + J \sum_{\{i,j\}\cap\Lambda\neq\emptyset,\, i\sim j} V(z_i - z_j)$$

where

- $X = (x_i)_{i\in\Lambda}$, $z_i = \begin{cases} x_i & \text{if } i \in \Lambda \\ \omega_i & \text{if } i \notin \Lambda \end{cases}$.

- ψ and V are real-valued functions, respectively called *phase* and *potential of the interactions* between sites. We assume that V satisfy

$$\|V''\|_\infty < \infty. \tag{1}$$

- $i \sim j$ means that j and i are neighbours in $\mathbb{Z}^d$.

- J is a positive real parameter (the coupling constant).

Assume that there exists $J_0 > 0$ such that for any J in $[0, J_0]$, any finite subset Λ of $\mathbb{Z}^d$ and ω in $\mathbb{R}^{\mathbb{Z}^d}$, the integral of $\exp(-\Phi_{\Lambda,\omega})$ on $\mathbb{R}^\Lambda$ is finite. In this case, define the probability measure $\mu_{\Phi_{\Lambda,\omega}}$ as:

$$d\mu_{\Phi_{\Lambda,\omega}}(X) = \frac{1}{Z_{\Phi_{\Lambda,\omega}}}\exp(-\Phi_{\Lambda,\omega}(X))dX, \tag{2}$$

where $Z_{\Phi_{\Lambda,\omega}} = \int \exp(-\Phi_{\Lambda,\omega}(X))dX$.

This model is described in [BH99a] (see also [Hel99a]). The particular case where $\psi(x) = ax^4 - bx^2$ $(a, b > 0)$ and $V(x) = x^2$ is considered by YOSHIDA (see [Yos99]).

We will investigate spectral gaps and decays of correlations of the family of probability measures $\mu_{\Phi_{\Lambda,\omega}}$ uniformly over Λ and ω. More precisely, we want to find two constants C and C' such that, for any Λ finite subset of $\mathbb{Z}^d$, $\omega \in \mathbb{Z}^d$ and any smooth functions f and F, G we have:

$$\mathbf{E}_{\mu_{\Phi_{\Lambda,\omega}}}(f^2) - \mathbf{E}_{\mu_{\Phi_{\Lambda,\omega}}}(f)^2 \leqslant C \int \|\nabla f\|^2 d\mu_{\Phi_{\Lambda,\omega}},$$

$$\begin{aligned} \mathbf{E}_{\mu_{\Phi_{\Lambda,\omega}}}(F,G) &\stackrel{\text{def.}}{=} \mathbf{E}_{\mu_{\Phi_{\Lambda,\omega}}}\left(\left(F - \mathbf{E}_{\mu_{\Phi_{\Lambda,\omega}}}(F)\right)\left(G - \mathbf{E}_{\mu_{\Phi_{\Lambda,\omega}}}(G)\right)\right) \\ &\leqslant C' \exp(-\mathrm{d}(S_F, S_G)) \left(\int \|\nabla F\|^2 d\mu_{\Phi_{\Lambda,\omega}}\right)^{1/2} \left(\int \|\nabla G\|^2 d\mu_{\Phi_{\Lambda,\omega}}\right)^{1/2}, \end{aligned} \tag{3}$$

where $\|\nabla f\|^2 = \sum_{i\in\Lambda}(\partial_i f)^2$, $\mathbf{E}_{\mu_{\Phi_{\Lambda,\omega}}}(f) = \int f d\mu_{\Phi_{\Lambda,\omega}}$, S_F is the support of F in $\mathbb{Z}^d$ and d is the distance between subsets of the lattice $\mathbb{Z}^d$.

It has been shown in [Hel99a], [BH99a] and [Yos99] that whenever the phase ψ is convex at infinity and J_o is small enough, then the measures $\left(\mu_{\Phi_{\Lambda,\omega}}\right)_{\Lambda,\omega}$ satisfy such a uniform spectral gap and decay of correlations (inequality (3)).

As announced, the aim of this paper is to present examples of non-convex phases such that the uniform spectral gap still hold. The main tool of the construction is the method developed by HELFFER which is presented in the next section. It will reduce to the study of some uniform spectral gap property of the phase ψ in dimension one that we investigate by means of HARDY's criterion for POINCARÉ inequalities in dimension one. This is the subject of Section 3.

In the last section, we prove our main result:

Theorem 1.1 *Let φ be a strictly uniformly convex function (i.e. for all $x \in \mathbb{R}$, $\varphi''(x) \geqslant a > 0$), let g be a bounded function ($\|g\|_\infty < \infty$), and let h be a perturbation function satisfying $S = \int_{\mathbb{R}}(e^{|h|} - 1) < \infty$. Then, the measure $\mu_{\Phi_{\Lambda,\omega}}$ defined in (2) with $\psi = \varphi + g + h$ satisfies, uniformly in Λ and ω, a spectral gap inequality and a decay of correlations for every J small enough.*

The simple criterion on h will easily produce examples for which ψ is not convex at infinity, and not even bounded above and below by two power type functions as will be showm at the end of the section 4.

2 HELFFER's method for spectral gap inequality

Let us first recall the definition of the spectral gap or POINCARÉ inequality for a measure on $\mathbb{R}^n$ and for the set of measures $\left(\mu_{\Phi_{\Lambda,\omega}}\right)_{\Lambda,\omega}$.

Definition 2.1 (Spectral gap inequality) *Let assume that $d\mu_\psi = exp(-\psi(X))dX$ is a probability measure on $\mathbb{R}^n$, where ψ is a real-valued function. The measure μ_ψ satisfies a* spectral gap inequality *if there exists a positive constant C_ψ such that for any smooth enough function $f : \mathbb{R}^n \to \mathbb{R}$,*

$$\mathbf{Var}_{\mu_\psi}(f) = \mathbf{E}_{\mu_\psi}(f^2) - \mathbf{E}_{\mu_\psi}(f)^2 \leqslant C_\psi \int \|\nabla f\|^2 d\mu_\psi \tag{4}$$

where $\|\nabla f\|^2 = \sum_{i=1}^n (\partial_i f)^2$ and $\mathbf{E}_{\mu_\psi}(f) = \int f d\mu_\psi$.

The constant C_ψ is the spectral gap constant associated to either the measure μ_ψ or the function ψ.

We say that the set of measures $\left(\mu_{\Phi_{\Lambda,\omega}}\right)_{\Lambda,\omega}$ defined by (2) satisfies a uniform spectral gap inequality if each measure $\mu_{\Phi_{\Lambda,\omega}}$ satisfies a spectral gap inequality with a constant $C = C_{\Phi_{\Lambda,\omega}}$ independent of Λ in $\mathbb{Z}^d$ and ω in $\mathbb{R}^{\mathbb{Z}^d}$.

Helffer proved recently spectral gap inequalities for the preceding spin systems using a criterion on the Witten Laplacian [Hel99a]. The main feature of the approach is that it reduces to some uniform spectral gap for the phase ψ that we investigate in the next section by Hardy's inequalities. For the sake of completness, we present this criterion that we however reformulate via the Bakry-Emery $\mathbf{\Gamma_2}$ operator (see [BE85], [Bak94] and [Led92] with more simple semigroup tools).

Theorem 2.2 *Let assume that $d\mu_\psi(X) = exp(-\psi(X))dX$ is a probability measure on $\mathbb{R}^n$, where ψ is a C^2 real-valued function. Let $\mathbf{L} = \Delta - (\nabla\psi.\nabla)$ denote the infinitesimal diffusion generator with invariant measure μ_ψ. Then the spectral gap inequality is equivalent to the inequality:*

$$\int (-f)\mathbf{L}f d\mu_\psi \leqslant C_\psi \int (\mathbf{L}f)^2 d\mu_\psi, \tag{5}$$

holding for any smooth function f.

We briefly recall the proof.

Proof

◀ Assume that the measure μ_ψ satisfies (5). Denote by $(\mathbf{P}_t)_{t\geqslant 0}$ the semigroup with generator $\mathbf{L}$. For any smooth function f, we have $\mathbf{P}_0 f = f$ and $\mathbf{P}_\infty = \int f d\mu_\psi$, so that

$$\mathbf{Var}_{\mu_\psi}(f) = -2 \int\int_0^\infty \mathbf{P}_t f \; \mathbf{L}\mathbf{P}_t f \; dt d\mu_\psi = -2 \int_0^\infty \int \mathbf{P}_t f \; \mathbf{L}\mathbf{P}_t f \; d\mu_\psi dt.$$

Let $F(t) = -\int \mathbf{P}_t f \; \mathbf{L}\mathbf{P}_t f d\mu_\psi$, $t \geqslant 0$. Integration by parts shows that

$$F'(t) = -2 \int (\mathbf{L}\mathbf{P}_t f)^2 d\mu_\psi.$$

Then, by (5), $F'(t) \leqslant -(2/C_\psi)F(t)$ so that $F(t) \leqslant e^{-\frac{2}{C_\psi}t} F(0)$, for every $t \geqslant 0$. Hence,

$$\mathbf{Var}_{\mu_\psi}(f) \leqslant C_\psi \int \|\nabla f\|^2 d\mu_\psi.$$

On the other hand, by invariance and the Cauchy-Schwarz inequality,

$$\int -f\mathbf{L}f d\mu_\psi = \int (f - \mathbf{E}_{\mu_\psi}(f))(-\mathbf{L}f) d\mu_\psi \leqslant \mathbf{Var}_{\mu_\psi}(f)^{1/2} \left(\int (\mathbf{L}f)^2 d\mu_\psi\right)^{1/2}.$$

Therefore inequality (5) follows from the spectral gap inequality with the same constant C_ψ. The proof of the theorem is complete. ▶

Applying Theorem 2.2 shows that a spectral gap inequality for the set of measures $(\mu_{\Phi\Lambda,\omega})$ amounts to establish (5) uniformly in Λ and ω. Now, for a given smooth function f, integration by parts allows us to write that

$$\begin{aligned}
\int (\mathbf{L}f)^2 d\mu_{\Phi_{\Lambda,\omega}} &= \int \left(\sum_{i,j\in\Lambda} (\partial_{i,j} f)^2 + \sum_{i,j\in\Lambda} \partial_i f \left(\partial_{i,j}\Phi_{\Lambda,\omega}\right) \partial_j f \right) d\mu_{\Phi_{\Lambda,\omega}} \\
&\geqslant \int \sum_{i\in\Lambda} \left((\partial_{i,i} f)^2 + (\partial_i f)^2 \{\psi''(x_i) + \sum_{j\in N(i)} JV''(x_i - z_j)\} \right) d\mu_{\Phi_{\Lambda,\omega}} \\
&+ \int \sum_{i,j\in\Lambda, i\sim j} -JV''(x_i - x_j)\partial_i f \; \partial_j f \, d\mu_{\Phi_{\Lambda,\omega}},
\end{aligned} \tag{6}$$

where $N(i) = \{j \in \mathbb{Z}^d \; ; \; j \sim i\}$. For any i in Λ, $|N(i)| = 2d$, so

$$\sum_{i,j\in\Lambda, i\sim j} \partial_i f \; \partial_j f \leqslant 2d \sum_{i\in\Lambda} (\partial_i f)^2.$$

The condition (1) on V'' easily shows that

$$\sum_{i,j\in\Lambda, i\sim j} -JV''(x_i - x_j)\partial_i f \partial_j f \geqslant -2d\|V''\|_\infty J \sum_{i\in\Lambda} (\partial_i f)^2. \tag{7}$$

Therefore, (6) holds as soon as

$$\begin{aligned}
\int (\mathbf{L}f)^2 d\mu_{\Phi_{\Lambda,\omega}} &\geqslant \sum_{i\in\Lambda} \int \left((\partial_{i,i} f)^2 + (\partial_i f)^2 (\psi''(x_i) + \sum_{j\in N(i)} JV''(x_i - z_j)) \right) d\mu_{\Phi_{\Lambda,\omega}} \\
&+ \sum_{i\in\Lambda} -2dJ\|V''\|_\infty \int (\partial_i f)^2 d\mu_{\Phi_{\Lambda,\omega}}.
\end{aligned} \tag{8}$$

For each i in Λ, denote by $\mu^{(i)}_{\Phi_{\Lambda,\omega}}$ the conditional measure on $\mu_{\Phi_{\Lambda,\omega}}$ given $\{(x_j), j \in \Lambda, j \neq i\}$. Therefore $\mu^{(i)}_{\Phi_{\Lambda,\omega}}$ is a measure on the real line and

$$\mu^{(i)}_{\Phi_{\Lambda,\omega}}(dx_i) = \frac{\exp(-\psi(x_i) - \sum_{j\in N(i)} JV''(x_i - z_j))}{Z^{(i)}_{\Phi_{\Lambda,\omega}}} dx_i,$$

where $Z^{(i)}_{\Phi_{\Lambda,\omega}} = \int \exp(-\psi(x_i) - \sum_{j\in N(i)} JV''(x_i - z_j)) dx_i$. The measure $\mu^{(i)}_{\Phi_{\Lambda,\omega}}$ depends of the variables $(x_j)_{j\neq i}$ and $\omega \in \mathbb{Z}^d$.

Suppose now that all the measures $\mu^{(i)}_{\Phi_{\Lambda,\omega}}$, $i \in \Lambda$, satisfy a spectral gap inequality with a constant C_{USG} (USG as uniform spectral gap) independent from the variables $(\omega, (x_j)_{j\in\Lambda, j\neq i})$ and from the site i.

The equivalence provided by Theorem 2.2 then indicates that for every $i \in \Lambda$,

$$\int \left((\partial_{i,i} f)^2 + (\partial_i f)^2 (\psi''(x_i) + \sum_{j\in N(i)} JV''(x_i - z_j)) \right) d\mu^{(i)}_{\Phi_{\Lambda,\omega}} \geqslant \frac{1}{C_{USG}} \int (\partial_i f)^2 d\mu^{(i)}_{\Phi_{\Lambda,\omega}}.$$

After integration all the $(x_j)_{j\neq i}$ and summation on $i\in\Lambda$ we find:

$$\int\sum_{i\in\Lambda}\left((\partial_{i,i}f)^2+(\partial_i f)^2(\psi''(x_i)+\sum_{j\in N(i)}JV''(x_i-z_j))\right)d\mu_{\Phi_{\Lambda,\omega}}\geqslant\frac{1}{C_{USG}}\int\sum_{i\in\Lambda}(\partial_i f)^2d\mu_{\Phi_{\Lambda,\omega}}. \tag{9}$$

It then follows from (8) and (9) that (6) became

$$\int(\mathbf{L}f)^2d\mu_{\Phi_{\Lambda,\omega}}\geqslant\left(\frac{1}{C_{USG}}-2dJ\|V''\|_\infty\right)\int\|\nabla f\|^2d\mu_{\Phi_{\Lambda,\omega}}.$$

As a consequence, the measure $\mu_{\Phi_{\Lambda,\omega}}$ satisfies the spectral gap inequality defined by 2.1 with a constant, independant of $\Lambda\subset\mathbb{Z}^d$ and $\omega\in\mathbb{R}^{\mathbb{Z}^d}$, equal to $C_{USG}/(1-C_{USG}2dJ\|V''\|_\infty)$ as soon as $1-C_{USG}2dJ\|V''\|_\infty>0$.

The main point in this argument is the uniformity of the spectral gap inequalities for the measures $\mu^{(i)}_{\Phi_{\Lambda,\omega}}$ on $\mathbb{R}$. We emphasize this property with the following definition.

Definition 2.3 (Uniform spectral gap inequality (USG)) *Let define* $\overline{\theta}=(\theta_i)_{i\in N(0)}$ *and* $N(0)=\{j\in\mathbb{Z}^d\ ;\ j\sim 0\}$. *Denote by* $\mu_{\psi_{\overline{\theta}}}$ *the probability measure on* $\mathbb{R}$ *defined by*

$$d\mu_{\psi_{\overline{\theta}}}(x)=\frac{1}{Z_{\psi_{\overline{\theta}}}}exp\left(-\psi_{\overline{\theta}}(x)\right)dx, \tag{10}$$

where $\psi_{\overline{\theta}}(x)=\psi(x)+\sum_{i\in N(0)}JV(x-\theta_i)$ *and* $Z_{\overline{\theta}}=\int exp\left(-\psi_{\overline{\theta}}(x)\right)dx$. *We say that the phase* ψ *satisfies a uniform spectral gap inequality (USG) if the measure* $\mu_{\psi_{\overline{\theta}}}$ *satisfies a spectral gap inequality with a constant* C_{USG} *independent of* $\overline{\theta}$ *in* $\mathbb{R}^{|N(0)|}$.

At the light of this definition 2.3 and the preceding argument, we may state Helffer's result in the following way (see [Led99]).

Theorem 2.4 *Let* ψ *be a real-valued function on* $\mathbb{R}$ *such that* ψ *satisfies the condition (USG) of the definition 2.3. Then there exists* J_0 *such that for every* $J\in[0,J_0]$, *the set of measures* $(\mu_{\Phi_{\Lambda,\omega}})_{\Lambda,\omega}$ *satisfies spectral gap inequality with*

$$C_{\Phi_{\Lambda,\omega}}\leqslant\frac{C_{USG}}{1-C_{USG}2dJ\|V''\|_\infty}.$$

The right hand side of the inequality does not depend of $\Lambda\subset\mathbb{Z}^d$, *and* $\omega\in\mathbb{R}^{\mathbb{Z}^d}$.

With the same method, we also get the corresponding decay of correlations as in [Hel99a].

Theorem 2.5 *Let* ψ *be a real-valued function on* $\mathbb{R}$ *such that* ψ *satisfies the condition (USG) of the definition 2.3. Then there exists* J_0 *and a constant* C' *such that for every* $J\in[0,J_0]$, *the set of measures* $(\mu_{\Phi_{\Lambda,\omega}})_{\Lambda,\omega}$ *satisfies:*

$$\mathbf{E}_{\mu_{\Phi_{\Lambda,\omega}}}(F,G)\leqslant C'\exp(-d(S_F,S_G))\left(\int\|\nabla F\|^2d\mu_{\Phi_{\Lambda,\omega}}\right)^{1/2}\left(\int\|\nabla G\|^2d\mu_{\Phi_{\Lambda,\omega}}\right)^{1/2},$$

for any smooth functions F,G, Λ *and* ω. *The constant* C' *only depends on* C_{USG},J,d *and* $\|V''\|_\infty$.

Remark 2.6 In the particular case where $V(x) = x^2$, (10) can be reparametrized in terms of a single one-dimensional parameter θ, and (USG) amounts to a uniform spectral gap for the set of measures (μ_{ψ_θ}) defined by

$$d\mu_{\psi_\theta}(x) = \frac{1}{Z_{\psi_\theta}} \exp(-\psi_\theta(x))dx. \tag{11}$$

where $\psi_\theta(x) = \psi(x) + \theta x$.

A natural question is thus to ask when the condition (USG) is satisfied? Simple arguments show that (USG) holds when ψ is strictly convex at infinity, that is $\psi = \varphi_a + g$ with $\varphi_a''(x) \geqslant a$ for some $a > 0$ and $\|g\|_\infty < \infty$ (see [Yos99] and [Aa99]). This is the classical phase behaviour investigated in the recent contributions on spectral gap and logarithmic Sobolev inequalities for unbounded spin systems by Zegarlinski [Zeg96], Yoshida [Yos99], Helffer [Hel99a] and Bodineau-Helffer [BH99a, BH99b].

That (USG) for a given phase ψ may be quite restrictive is shown by the example of $\psi(x) = |x|^s$ with some s in $[1, 2[$ that is known to satisfy a spectral gap inequality (see [Aa99]) but that does not satisfy a uniform spectral gap in the sense of the preceding definition as shown by the next proposition.

Proposition 2.7 *Let ψ be a real-valued function on $\mathbb{R}$ such that*

$$\psi'(\mathbb{R}) = \mathbb{R} \ \textit{and} \ \lim_{|x|\to\infty} \psi''(x) = 0.$$

Then the phase $\psi_\theta(x) = \psi(x) + \theta x$, define on (11), could satisfy for any θ in $\mathbb{R}$ a spectral gap inequality but the phase ψ does not satisfy the condition (USG).

Remark 2.8 The function $\psi(x) = |x|^s$ satisfies the hypotheses of Proposition 2.7 and for any θ there is a spectral gap inequality. However by the last proposition, it does not satisfy the condition (USG). This case is treated by Helffer in the Exercise 6.3.4 in [Hel99b].

Proof

◀ We assume that the measure μ_{ψ_θ} satisfies the condition (USG). By Theorem 2.2, there exists a constant $C_{USG} > 0$ such that, for any $\theta \in \mathbb{R}$ and any smooth function f with compact support, we have:

$$\int f'^2 d\mu_{\psi_\theta} \leqslant C_{USG} \int \left(f''^2 + f'^2 \psi''\right) d\mu_{\psi_\theta}.$$

The hypotheses on ψ insure the existence of α_θ such that $\psi'(\alpha_\theta) + \theta = 0$, and $\lim_{|\theta|\to\infty} |\alpha_\theta| = \infty$. Then for any y by Taylor's formula on $\mathbb{R}$, there exists $u_{y,\theta} \in]0, 1[$ such that

$$\psi(y + \alpha_\theta) + \theta y = \psi(\alpha_\theta) + \frac{y^2}{2}\psi''(\alpha_\theta + u_{y,\theta} y).$$

By a change of variables,

$$\int f'(y)^2 e^{-\frac{y^2}{2}\psi''(\alpha_\theta + u_{y,\theta}y)} dy \leqslant C_{USG} \int \left(f''(y)^2 + f'(y)^2\psi''(y + \alpha_\theta)\right) e^{-\frac{y^2}{2}\psi''(\alpha_\theta + u_{y,\theta}y)} dy.$$

By the dominated convergence theorem as $\theta \to \infty$,

$$\int f'^2 dx \leqslant C_{USG} \int f''^2 dx.$$

However, there is no spectral gap inequality for the LEBESGUE measure on $\mathbb{R}$. Therefore the phase ψ does not satisfy the condition (USG). ▶

Proposition 2.7 deals in particular with the case $|x|^s$ for $1 < s < 2$. On the other hand, one can easily see that $\psi_\theta(x) = |x|^2 + \theta x$ satisfies a spectral gap inequality with constant independant of θ (and so a uniform spectral gap). To complete this setting, our next result deals with the behaviour of the spectal gap constant of ψ_θ (define in 11) with $\psi(x) = |x|^s$ and $s > 2$. More precisly, we have (we omit the proof):

Proposition 2.9 *Let ψ be a real-valued function on $\mathbb{R}$ such that*

$$\lim_{|x|\to\infty} \psi''(x) = \infty.$$

Then the measure μ_{ψ_θ} define in (11) satisfy for any θ a spactral gap inequality and the constant C_{ψ_θ} satisfy:

$$\lim_{|\theta|\to\infty} C_{\psi_\theta} = 0.$$

Theorem 2.4 reduces the question of spectral gap for families $(\mu_{\Lambda,\omega})$ to the (USG) property of a real-valued phase ψ. We now investigate this condition by means of HARDY type inequalities.

3 HARDY type inequalities and applications

This section introduces the notion of HARDY type inequalities which will be our basic tool to prove our main result (Theorem 4.1 below).

Let μ and ν be two probability measures on $\mathbb{R}^+$. We assume ν to be absolutely continuous with respect to LEBESGUE's measure on $\mathbb{R}^+$, and its density $d\nu/dx$ to be strictly positive. In 1972, MUCKENHOUPT generalized results of HARDY and TOMASELLI (see [Tom69]), TALENTI (see [Tal69]) and ARTOLA on some specific functional inequalities, now called HARDY type inequalities. They were namely interested in controlling the best constant A such that

$$\int_0^\infty \left(\int_x^\infty f(t)\, dt \right)^2 d\mu(x) \leqslant A \int_0^\infty f^2(x)\, d\nu(x) \tag{12}$$

for all continuous functions for which the preceding integrals are well-defined. HARDY gave a result for this inequality with $d\mu(x) = x^{2b}dx$ and $d\nu(x) = x^{2b+2}dx$. TOMASELLI, TALENTI and ARTOLA proved the inequality for general measure $d\mu(x) = U^2(x)dx$ and $d\nu(x) = V^2(x)dx$. One of MUCKENHOUPT's main results [Muc72] is summarized in the next statement.

Theorem 3.1 *The constant A defined by* (12) *is finite if and only if*

$$B \stackrel{def.}{=} \sup_{x\geqslant 0} \int_x^\infty d\mu(x) \int_0^x \left(\frac{d\nu}{dx} \right)^{-1} dx < \infty ,$$

and in this case,

$$B \leqslant A \leqslant 4B .$$

In [Mic99], Miclo establishes a link between the Hardy type inequalities and the spectral gap inequalities. While he was concerned with the case of probability measures on $\mathbb{Z}$, we briefly present here the corresponding results on the real line.

Let us first remark that if $F(x) = \int_0^x f(t)\,dt$ and $\mu = \nu$ (that we will suppose from now), then the previous inequality (12) becomes

$$\int_0^\infty (F(x) - F(0))^2\, d\mu(x) \leqslant A \int_0^\infty F'^2(x)\, d\mu(x) \; .$$

This latter inequality is rather close to a spectral gap inequality for the probability measure μ. More precisely, it is well known that for every $m \in \mathbb{R}$,

$$\mathbf{Var}_\mu(F) \leqslant \int_{\mathbb{R}} (F(x) - F(m))^2\, d\mu(x) \; .$$

In order to apply Theorem 3.1, we need to cut the real line into two parts (to reduce $\mathbb{R}$ to $\mathbb{R}^+$). We denote by A_m^+ and A_m^- the best constants satisfying respectively

$$\int_m^\infty (F(x) - F(m))^2\, d\mu(x) \leqslant A_m^+ \int_m^\infty F'^2(x)\, d\mu(x)$$

and

$$\int_{-\infty}^m (F(x) - F(m))^2\, d\mu(x) \leqslant A_m^- \int_{-\infty}^m F'^2(x)\, d\mu(x) \; ,$$

for all $\mathcal{C}^1$ functions F. By a simple change of variables, they are controlled by

$$B_m^+ \leqslant A_m^+ \leqslant 4B_m^+ \quad \text{and} \quad B_m^- \leqslant A_m^- \leqslant 4B_m^- \; ,$$

where

$$B_m^+ \stackrel{\text{def.}}{=} \sup_{x \geqslant m} \int_x^\infty d\mu(x) \int_m^x \left(\frac{d\mu}{dx}\right)^{-1} dx \quad \text{and} \quad B_m^- \stackrel{\text{def.}}{=} \sup_{x \leqslant m} \int_{-\infty}^x d\mu(x) \int_x^m \left(\frac{d\mu}{dx}\right)^{-1} dx \; . \tag{13}$$

We may summarize these observations in the following proposition.

Proposition 3.2 *For every $m \in \mathbb{R}$, the best constant C in the spectral gap inequality for μ is such that*

$$C \leqslant 4(B_m^+ \vee B_m^-) \; .$$

While the latter bound holds for every $m \in \mathbb{R}$, we get a more precise control if m is a median for μ.

Proposition 3.3 *Let m be a median of μ. Then μ satisfies a spectral gap inequality if and only if $B_m^+ \vee B_m^-$ is finite, and in this case, the best constant C in the spectral gap inequality for μ is such that*

$$\frac{1}{2}(B_m^+ \vee B_m^-) \leqslant C \leqslant 4(B_m^+ \vee B_m^-) \; .$$

Proof

◀ We need only prove the lower bound. Assume that $B_m^+ \vee B_m^-$ is finite and that $B_m^+ \vee B_m^- = B_m^+$. For any $\epsilon > 0$, we can find f such that

$$\int_m^\infty \left(\int_m^x f(t)\, dt\right)^2 d\mu(x) \geqslant (A_m^+ - \epsilon) \int_m^\infty f^2(x)\, d\mu(x) \; .$$

Without loss of generality, we can assume that f is non negative. Now, define

$$F(x) = \begin{cases} 0 & \text{if } x \geqslant m \;, \\ \int_m^x f(t)\,dt & \text{if } x \leqslant m \;. \end{cases}$$

As m is a median of μ, $\mu(F=0) \geqslant \mu(x \leqslant m) \geqslant \frac{1}{2}$. From the CAUCHY-SCHWARZ inequality, we get therefore that

$$\mu(F)^2 \leqslant \mu(F^2)\mu(F>0) \leqslant \frac{1}{2}\mu(F^2) \;.$$

Hence

$$\begin{aligned} \mathbf{Var}_\mu(F) = \mu(F^2) - \mu(F)^2 &\geqslant \frac{1}{2}\mu(F^2) \\ &\geqslant \frac{1}{2}(A_m^+ - \epsilon)\int_m^\infty F'^2(x)\,d\mu(x) \\ &\geqslant \frac{1}{2}(B_m^+ - \epsilon)\int_{-\infty}^\infty F'^2(x)\,d\mu(x) \\ &\geqslant \frac{1}{2}(B_m^+ - \epsilon)\frac{1}{C}\mathbf{Var}_\mu(F) \;, \end{aligned}$$

from which the result follows since $\epsilon > 0$ is arbitrary.

Now, assume that $B_m^+ = \infty$. Following step by step the previous argument with f_n such that

$$\int_m^\infty \left(\int_m^x f_n(t)\,dt\right)^2 d\mu(x) \geqslant n\int_m^\infty f_n^2(x)\,d\mu(x)$$

for n large enough, we conclude similarly that $C = \infty$. The proof of the proposition is complete. ▶

In the following, we describe with the preceding results some perturbation properties. Consider a function $h : \mathbb{R} \to \mathbb{R}$. Given a phase φ, we modify the probability measure $d\mu_\varphi(x) = Z_\varphi^{-1}\exp(-\varphi(x))dx$ as

$$d\mu_{\varphi+h}(x) = Z_{\varphi+h}^{-1}\exp(-\varphi(x) - h(x))dx \;. \tag{14}$$

We assume that $Z_{\varphi+h}$ is finite. h can be considered as a perturbation function.

The following result gives conditions on φ and h so that $\mu_{\varphi+h}$ satisfies a spectral gap inequality. It also gives an upper bound on the spectral gap constant.

Theorem 3.4 *Assume that there exist $m \in \mathbb{R}$ and a constant K independent of x such that for all $x \geqslant m$,*

$$\int_x^\infty e^{-\varphi(t)}\,dt \leqslant Ke^{-\varphi(x)} \quad \text{and} \quad \int_m^x e^{\varphi(t)}\,dt \leqslant Ke^{\varphi(x)} \;, \tag{15}$$

and the corresponding inequalities for $x \leqslant m$. Assume furthermore that φ is increasing on $[m,\infty)$ and decreasing on $(-\infty, m]$. Assume also that

$$S \overset{def.}{=} \int_{\mathbb{R}} (e^{|h|} - 1) < \infty \;. \tag{16}$$

Then, the probability measure $\mu_{\varphi+h}$ defined by (14) satisfies the spectral gap inequality. Furthermore, the best constant $C_{\varphi+h}$ appearing in the spectral gap inequality is controled by

$$C_{\varphi+h} \leqslant 4(K^2 + 2KS + S^2) \;.$$

Proof

◀ From Proposition 3.2, we see that $C_{\varphi+h} \leqslant 4(B_m^+(\varphi+h) \vee B_m^-(\varphi+h))$. By symmetry, we may reduce to the control of

$$B_m^+(\varphi+h) \stackrel{\text{def.}}{=} \sup_{x \geqslant m} \int_m^x e^{\varphi(t)+h(t)}\, dt \int_x^\infty e^{-\varphi(t)-h(t)}\, dt\ .$$

Now, given that $x \geqslant m$, we may write

$$\int_m^x e^{\varphi(t)+h(t)}\, dt \int_x^\infty e^{-\varphi(t)-h(t)}\, dt \leqslant \left(\int_m^x e^{\varphi} + \int_m^x e^{\varphi} \left| e^h - 1 \right| \right) \left(\int_x^\infty e^{-\varphi} + \int_x^\infty e^{-\varphi} \left| e^{-h} - 1 \right| \right)\ .$$

We develop the right hand side using (15) and monotonicity of φ to get

$$\begin{aligned} \int_m^x e^{\varphi(t)+h(t)}\, dt \int_x^\infty e^{-\varphi(t)-h(t)}\, dt \quad &\leqslant \quad \left(K e^{\varphi(x)} \right) \left(K e^{-\varphi(x)} \right) + \left(K e^{\varphi(x)} \right) \left(e^{-\varphi(x)} \int_x^\infty \left| e^{-h(t)} - 1 \right|\, dt \right) \\ &\quad + \left(e^{\varphi(x)} \int_m^x \left| e^{h(t)} - 1 \right|\, dt \right) \left(K e^{-\varphi(x)} \right) \\ &\quad + \left(e^{\varphi(x)} \int_m^x \left| e^{h(t)} - 1 \right|\, dt \right) \left(e^{-\varphi(x)} \int_x^\infty \left| e^{-h(t)} - 1 \right|\, dt \right)\ . \end{aligned}$$

Now, note that $|e^h - 1| \leqslant e^{|h|} - 1$. It follows that for all $x \geqslant m$

$$\int_m^x e^{\varphi(t)+h(t)}\, dt \int_x^\infty e^{-\varphi(t)-h(t)}\, dt \leqslant K^2 + 2KS + S^2\ .$$

Together with the corresponding result for $B_m^-(\varphi+h)$, this completes the proof. ▶

Remark 3.5 Turning back the proof above, we can replace K^2 in the upper bound of $C_{\varphi+h}$ by $B_m^+(\varphi) \vee B_m^-(\varphi)$ (this quantity is finite by (15)). Moreover, if m is a median of μ_φ, keeping all the hypotheses of Theorem 3.4 and applying Proposition 3.3, we get that $B_m^+(\varphi) \vee B_m^-(\varphi) \leqslant 2C_\varphi \leqslant 8B_m^+(\varphi) \vee B_m^-(\varphi)$ so that

$$C_{\varphi+h} \leqslant 8C_\varphi + 4(2KS + S^2)\ .$$

This inequality clearly describes the contribution of the perturbation in the spectral gap constant.

4 A new class of measures which satisfies a spectral gap inequality

As already mentioned in Section 1 and 2, the keystone of HELFFER's method is the (USG) condition (see Definition 2.3). It allows us to reduce the initial problem on unbounded spin systems to a simple problem on the real line. We will then be able to apply Theorem 3.4.

In this section, we first present and establish the main result of this paper. We then discuss why the new class of phases $\psi = \varphi + g + h$ is strictly bigger than the usual class of phases convex at infinity.

Theorem 4.1 *Let φ be a strictly uniformly convex function (i.e. for all $x \in \mathbb{R}$, $\varphi''(x) \geqslant a > 0$), let g be a bounded function ($\|g\|_\infty < \infty$), and let h be a perturbation function satisfying $S = \int_{\mathbb{R}} (e^{|h|} - 1) < \infty$. Then, there exists J_0 such that for all $J \in [0, J_0]$, the set of measures $(\mu_{\Phi_{\Lambda,\omega}})$ defined in (2) with $\psi = \varphi + g + h$ satisfies a spectral gap inequality uniformly in Λ and ω, with*

$$C_{\Phi_{\Lambda,\omega}} \leqslant \frac{C_{USG} e^{4\|g\|_\infty}}{1 - C_{USG} 2dJ \|V''\|_\infty}\ ,$$

where $C_{USG} \leqslant 4(K^2+2KS+S^2)$ and $K \leqslant 1+4/a$.

This set of measures satisfies also a decay of correlations, uniformly in Λ and ω, with constant C' depending only on C_{USG}, J, d, $\|V''\|_\infty$ and $\|g\|_\infty$, that is, for any smooth functions F, G,

$$\mathbf{E}_{\mu_{\Phi_{\Lambda,\omega}}}(F,G) \leqslant C' \exp(-d(S_F,S_G)) \left(\int \|\nabla F\|^2 d\mu_{\Phi_{\Lambda,\omega}}\right)^{1/2} \left(\int \|\nabla G\|^2 d\mu_{\Phi_{\Lambda,\omega}}\right)^{1/2} .$$

The proof of this theorem requires two technical lemmata. Note that from the hypothesis on φ, it is obvious that φ has a unique minimum $m \in \mathbb{R}$ and that φ is increasing on $[m,\infty)$ and decreasing on $(-\infty, m]$.

Lemma 4.2 *Let φ be as defined in Theorem 4.1. Then there exists a constant $K \leqslant 1+2/a$ such that for all $x \geqslant m$,*

$$\int_m^x e^{\varphi(t)}\, dt \leqslant K e^{\varphi(x)} .$$

Proof

◀ By convexity, for all $x \geqslant m+1$,

$$\varphi'(x) = \varphi'(x) - \varphi'(m) \geqslant a(x-m) \geqslant a . \tag{17}$$

For $x \geqslant m+1$, integration by parts yields that

$$\begin{aligned}
\int_{m+1}^x e^{\varphi(t)}\, dt &= \frac{e^{\varphi(x)}}{\varphi'(x)} - \frac{e^{\varphi(m+1)}}{\varphi'(m+1)} + \int_{m+1}^x \frac{\varphi''(t)}{\varphi'^2} e^{\varphi(t)}\, dt \\
&\leqslant \frac{e^{\varphi(x)}}{\varphi'(x)} + e^{\varphi(x)} \int_{m+1}^x \frac{\varphi''(t)}{\varphi'^2}\, dt , \\
&\leqslant \frac{e^{\varphi(x)}}{\varphi'(x)} + \frac{1}{\varphi'(m+1)} e^{\varphi(x)} .
\end{aligned}$$

So, by (17),

$$\int_{m+1}^x e^{\varphi(t)}\, dt \leqslant \frac{2}{a} e^{\varphi(x)} .$$

Finally, for all $x \geqslant m+1$, we have

$$\begin{aligned}
\int_m^x e^{\varphi(t)}\, dt &= \int_m^{m+1} e^{\varphi(t)}\, dt + \int_{m+1}^x e^{\varphi(t)}\, dt \\
&\leqslant e^{\varphi(x)} + \frac{2}{a} e^{\varphi(x)} \\
&\leqslant \left(1 + \frac{2}{a}\right) e^{\varphi(x)} ,
\end{aligned}$$

namely, the expected result in this case with $K = 1+2/a$. Now, for $m \leqslant x \leqslant m+1$, write

$$\int_m^x e^{\varphi(t)}\, dt \leqslant e^{\varphi(x)} \leqslant K e^{\varphi(x)} ,$$

from which the proof follows. ▶

In a similar way, we can prove the following lemma (we omit the proof).

Lemma 4.3 *Let φ be as defined in Theorem 4.1. Then there exists a constant $K \leqslant 1 + 1/a$ such that for all $x \geqslant m$,*

$$\int_x^\infty e^{-\varphi(t)}\, dt \leqslant K e^{-\varphi(x)} .$$

Note that it exists two corresponding lemmata for $x \leqslant m$.

We now prove Theorem 4.1.

Proof

◀ As announced, we use the reduction to the (USG) property provided by Theorem 2.4. With the notation of the first two sections and in particular Definition 2.3, let us define

$$\varphi_{\overline{\theta}}(x) = \varphi(x) + \sum_{i \in N(0)} JV(x - \theta_i) .$$

For all $x \in \mathbb{R}$ and J small enough, $\varphi''_{\overline{\theta}}(x) \geqslant a - 2dJ\|V''\|_\infty \geqslant a/2$. It follows from lemmata 4.2 and 4.3 that the hypothesis (15) holds for $\varphi_{\overline{\theta}}$, with $K \leqslant 1 + 4/a$. We can thus apply Theorem 3.4 to $\varphi_{\overline{\theta}}$ and h to get

$$C_{USG} \leqslant 4(K^2 + 2KS + S^2) .$$

The preceding constant is independent of $\overline{\theta} = (\theta_i)_{i \in N(0)}$, so that (USG) is satisfied. In this way we may apply Theorem 2.4. The set of measures $(\mu_{\Phi_{\Lambda,\omega}})$ will satisfy a spectral gap inequality with a constant equal to

$$\frac{C_{USG}}{1 - C_{USG} 2dJ\|V''\|_\infty} .$$

Finally, it is well known that adding a bounded function g gives no more than $e^{4\|g\|_\infty}$ in the control of the constant (see [Aa99]).

On the other hand, applying Theorem 2.5 instead of Theorem 2.4 gives the result on the decay of correlations. Theorem 4.1 is established. ▶

We now present examples of functions $\psi = \varphi + g + h$ defined in Theorem 4.1 that are not convex at infinity. To do so, we give an example where we add a special perturbation function h to φ (where φ is as in Theorem 4.1). Indeed, let $h(x) \stackrel{\text{def.}}{=} \sum_{-\infty}^{\infty} h_i(x)$ where $h_i : \mathbb{R} \to \mathbb{R}$ $(i \in \mathbb{Z})$ is a piecewise linear continuous function defined from its derivative by

$$h'_i(x) = \begin{cases} 0 & \text{if } x \notin [u_i, u_i + \alpha_i] , \\ -\beta_i & \text{if } x = u_i + \frac{\alpha_i}{4} , \\ \beta_i & \text{if } x = u_i + \frac{3\alpha_i}{4} , \end{cases}$$

where (β_i) is a sequence of non negative numbers and (u_i) and (α_i) are two sequences of real numbers such that $u_i < u_i + \alpha_i < u_{i+1}$. Moreover we assume that $[u_i, u_i + \alpha_i]$ is the support of h_i. One can see h_i as a well of depth $\alpha_i\beta_i/4$ and of width α_i.

To be outside the classical class of convex at infinity functions, it is enough to show that the depth of the well increases faster than φ. Write $L_i = \|h_i\|_\infty - (\varphi(u_i + \alpha_i/2) - \varphi(u_i))$. We have the following obvious sufficient condition (we omit the proof).

Proposition 4.4 *Let φ be as in Theorem 4.1 and h as above. Then, if*

$$\lim_{i \to \infty} L_i = \infty ,$$

then $\varphi + h$ is not convex at infinity.

It is now easy to choose the sequences $(\alpha_i)_{i\in\mathbb{Z}}$ and $(\beta_i)_{i\in\mathbb{Z}}$ such that

$$\lim_{i\to\infty} L_i = \infty$$

and at the same time,

$$\int_{\mathbb{R}} e^{|h|} - 1 \leqslant \sum_{i\in\mathbb{Z}} \alpha_i e^{\|h_i\|_\infty} < \infty \ ,$$

since we are free to choose α_i as we want. For example, one can chose $\varphi(x) = x^2/2$ and h as above, even, with $u_i = i$, $\beta_i = 4i^3 e^i$ and $\alpha_i = e^{-i}/i^2$ for all $i \in \mathbb{N}^*$. It's well known that $d\mu_\varphi(x) = Z^{-1}e^{-x^2/2}dx$ satisfies a spectral gap inequality with constant $C_\varphi = 2$. On the other hand, it follows from an obvious calculus that $L_i = i - o(1/i)$ and $S \leqslant 2\sum_{i\in\mathbb{N}^*} 1/i^2$. Proposition 4.4 holds and so $\varphi + h$ is not convex at infinity. Moreover the hypotheses of Theorem 4.1 are satisfied.

We may note that Theorem 4.1 is quite general since the condition on the perturbation h ($\int_{\mathbb{R}}(e^{|h|} - 1) < \infty$) is rather weak. Until now, the results were based on convexity conditions (following semi-group methods) whereas we just need integrability conditions here.

In this note, we proved a perturbation theorem (Theorem 3.4) for POINCARÉ inequalities using HARDY's criterion. One may try to obtain the same kind of result for logarithmic SOBOLEV inequalities using the corresponding criterion by BOBKOV and GÖTZE (see [BG99]). In an other direction, one can try to prove logarithmic SOBOLEV inequalities for unbounded spin systems with nearest neighbor interaction associated to non convex phases introduced in this note. But here, HELFFER's method is no more avaible because there is no result like Theorem 2.2 (see for example [Led99, Aa99]). It is an open problem to find its analogue for logarithmic SOBOLEV inequalities. One may also try to use a more classical approache via the decay of correlations (see [Zeg96, Yos99, BH99a]). In this direction, HELFFER (see [Hel99b] Remark 8.5.2) remarks that under the decay of correlations (see Theorem 2.5) and an hypothesis relative to the existence for each n of a uniform logarithmic SOBOLEV constant C_n (uniform on the boundary conditions, in J and in the functions f with support "nearly" includes in a box of size n), one can prove the logarithmic SOBOLEV inequality for unbounded spin systems. Actually, taking back the proof given by HELFFER, we can see that we only need the existence of the uniform logarithmic SOBOLEV constant in dimension one. Thus, by this note, as the logarithmic SOBOLEV inequality implies the spectral gap inequality and the decay of correlations, we have the following interesting result: under a uniform logarithmic SOBOLEV inequality on the line, the unbounded spin systems satisfy a logarithmic SOBOLEV inequality uniformly on the boundary conditions, on the box Λ, for J small enough. By this way, we reduce the problem from $\mathbb{R}^n$ to a more simple problem on the line. However, at this point, we have not been able to prove a uniform logarithmic SOBOLEV inequality in this context.

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