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Quantitative isoperimetric inequalities for log-convex probability measures on the line ☆

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ABSTRACT

The purpose of this paper is to analyze the isoperimetric inequality for symmetric log-convex probability measures on the line. Using geometric arguments we first reprove that extremal sets in the isoperimetric inequality are intervals or complement of intervals (a result due to Bobkov and Houdré). Then we give a quantitative form of the isoperimetric inequality, leading to a somehow anomalous behavior. Indeed, it could be that a set is very close to be optimal, in the sense that the isoperimetric inequality is almost an equality, but at the same time is very far (in the sense of the symmetric difference between sets) from any extremal sets! From the results on sets we derive quantitative functional inequalities of weak Cheeger type.

Contents

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1. Introduction

The isoperimetric problem in metric probability spaces is a very rich and extensive theory, with many applications in probability, analysis and geometry, as for example concentration of measure, phenomena in high dimension [28], rearrangement, PDEs [30], etc. See *e.g.* [27,2,36,25] for overview papers and monographs.

The isoperimetric inequality for the Gaussian measure in dimension 1 (the result holds in any dimension [11,39]) reads

$$P(E) \geq I\big(\gamma(E)\big) \quad \text{for all Borel sets } E \subset \mathbb{R},$$

where $P(E)$ is the perimeter of E (see below for a precise definition), $\gamma(E) = \frac{1}{\sqrt{2\pi}} \int_E e^{-x^2/2} dx$ is the Gaussian measure of the set E and $I = \varphi \circ \Phi^{-1}$ is the *isoperimetric profile* (here φ stands for the density of γ and Φ for its cumulative distribution function). Equality cases are given by half-lines (half-spaces in dimension higher than 1 [19,12]). Very recently Cianchi, Fusco, Maggi and Pratelli [16] solved the harder question about the almost equality cases (see [32,33,20] for further developments). If $\bar{E}$ is an extremal set in the above isoperimetric inequality, defining *the deficit* as

$$\delta(E) := P(E) - P(\bar{E}),$$

the authors proved the following quantitative isoperimetric inequality

$$\delta(E) \geq C\big(\gamma(E)\big)\lambda(E)\sqrt{\log \frac{1}{\lambda(E)}}, \tag{1.1}$$

where $\lambda(E) := \inf_{\substack{H \text{ half line:} \\ \gamma(H)=\gamma(E)}} \gamma(E \Delta H)$, Δ stands for the symmetric difference between sets and C is a constant that depends on the measure $\gamma(E)$ of the set. The quantity $\lambda(E)$ is called the *asymmetry* of E: it encodes, in the sense of the symmetric difference, how far the set is from the extremal sets in the isoperimetric inequality. One of the main issues here is to find the sharp dependence in δ. Observe that (1.1) relates two different "distances" from a set E to the extremal sets in the isoperimetric inequality.

Using a geometric argument in the spirit of [16] de Castro in [17] is able to identify all extremal sets in the isoperimetric inequality that have also a fixed asymmetry. More precisely, he proves that among sets of given measure *and* given asymmetry, intervals or complements of intervals (depending on the range) have minimal perimeter. Furthermore, he deals more generally with any log-concave probability measure and not only the Gaussian measure.

In the present paper, our aim is to analyze quantitative isoperimetric inequalities for the class of log-convex probability measures on the line. Assume for simplicity that μ is a symmetric absolutely continuous (with respect to the Lebesgue measure) probability measure on the line, with density f. Then μ is said to be *log-convex* if $\log f$ is convex on $(-\infty, 0]$. This class of probability measures includes for example distributions $dm_\alpha(x) = \frac{\alpha dx}{2(1+|x|)^{1+\alpha}}$, $\alpha > 0$ (such distributions[1] are related to the well-known κ-concave probability measures [10] whose corresponding isoperimetric problem is studied in [7]). The isoperimetric problem for

[1] Sometimes (mistakenly) referred to as "generalized Cauchy distributions" in the literature [3,13,36].

log-convex probability measures in dimension 1 is fully solved by Bobkov and Houdré [9] (see also [38]). In higher dimension, for product of log-convex probability measures, extremal sets are not known. However, some estimates on the isoperimetric profile (dimension dependent as it must be [40]) are given in [13] (see also [37,3]) with links with the concentration of measure.

Using a geometric argument of the type of Cianchi et al. [16], we shall first re-prove the result by Bobkov and Houdré [9] on the extremal sets in the isoperimetric inequality (see Section 3). Then, we will obtain a quantitative isoperimetric inequality (in the form of (1.1), see Section 4) which appears to be surprising, due to the presence of different shapes in the extremal sets when the measure of the set is precisely 1/2. Indeed, it could be that a set has very small deficit (δ above) but large asymmetry (see Section 4.2). This is one of our main results. We emphasize that log-convex probability measures are the first examples of measures, to the best of our knowledge, displaying such an anomalous property.

Contrary to the case of the log-concave probability measures, there is not a unique description of extremal sets with given measure and given asymmetry. We shall illustrate this with two explicit examples (see Section 4.1). However, under few additional assumptions on the density f of the measure, we will give a unified description of extremal sets with given measure and given asymmetry. From our estimates on sets we finally derive quantitative functional inequalities of weak Cheeger type in some specific cases (see Section 5).

There is an important activity on the questions of quantitative inequalities. To give a complete overview of the literature would be out of reach. Let us mention only few very recent works somehow related to the present paper. In [22] the authors deal with the isoperimetric problem for radially symmetric log-convex probability measures (finding extremal sets in $\mathbb{R}^n$). In [23] the authors deal with quantitative Brunn–Minkowski inequality (which is related to the isoperimetric problem in Euclidean space), while functional counter parts can be found in [15,18] on Sobolev inequalities, and in [26,8] on log-Sobolev inequalities for the Gaussian measure.

We observe that, for the clarity of the exposition, we shall postpone some (technical) proofs to Appendices A–C.

2. Log-convex measures: definition and first properties

In this section we introduce the notion of log-convex probability measures on the line, we give some examples and prove few basic properties.

Throughout the paper, we assume that μ is an absolutely continuous probability measure (with respect to the Lebesgue measure) on $\mathbb{R}$ with density f. Set $F(x) = \mu((-\infty, x])$, $x \in \mathbb{R}$, for its distribution function and let

$$a = \inf\{x \in \mathbb{R} : F(x) > 0\} \quad \text{and} \quad b = \sup\{x \in \mathbb{R} : F(x) < 1\}.$$

In general $-\infty \le a < b \le +\infty$. In analogy with the family of log-concave probability measures, we define the family of log-convex probability measures.

Definition 2.1 *(Log-convex measure).* Let μ be an absolutely continuous probability measure on $\mathbb{R}$ with density f. Then, μ is said to be *log-convex* (respectively *strictly log-convex*) if there exists $x_0 \in (a, b)$ such that $\log f$ is convex (respectively strictly convex) on (a, x_0) and on (x_0, b).

The family of log-convex measures includes distributions

$$dm_\alpha(x) = \frac{\alpha}{2(1+|x|)^{1+\alpha}} dx \tag{2.1}$$

where $\alpha > 0$ is a parameter (notice that such distributions are strictly log-convex). It also includes the two-sided exponential measure

$$d\mu_1(x) = \frac{e^{-|x|}}{2} dx \tag{2.2}$$

(which is not strictly log-convex) and more generally any probability measures of the form

$$d\mu_\Phi(x) = Z_\Phi^{-1} e^{-\Phi(x)} dx \tag{2.3}$$

with Φ *e.g.* even and concave on $(0, +\infty)$. The family of log-convex measures intersects (but is different from) the family of κ-concave probability measures introduced by Borell [10] (in particular it does not contain the usual Cauchy distribution whose density is $f(x) = 1/(\pi[1 + x^2])$).

Observe that F is strictly increasing on (a, b) and if one sets

$$J(t) = f\big(F^{-1}(t)\big) \quad 0 < t < 1, \tag{2.4}$$

then $\lim_{t\to 0} J(t) = \lim_{t\to 1} J(t) = 0$, and the map $t \mapsto J(t)$ is increasing on $(0, F^{-1}(x_0))$ and decreasing on $(F^{-1}(x_0), 1)$. Without any further mention, in the rest of the paper we will extend J up to 0 and 1, setting $J(0) = J(1) = 0$. For example, for the two-sided exponential measure (2.2), one can easily check that $J(t) = \min(t, 1 - t)$, that for $m\alpha$ defined in (2.1), $J_\alpha(t) = \alpha 2^{\frac{1}{\alpha}} \min(t, 1 - t)^{1+\frac{1}{\alpha}}$ (see *e.g.* [13]) and that for (2.3), under mild assumption on Φ (see [13, Proposition 5.21] for a precise statement), $J_\Phi(t) \sim t\Phi'(\Phi^{-1}(\log \frac{1}{t}))$, as t goes to 0.

The following characterization holds.

Proposition 2.2. *Let μ be an absolutely continuous probability measure on $\mathbb{R}$ with density f and distribution function F. Set $a := \inf\{x \in \mathbb{R} : F(x) > 0\}$ and $b := \sup\{x \in \mathbb{R} : F(x) < 1\}$. Assume that F is strictly increasing on (a, b) and denote by $F^{-1}: (0, 1) \to (a, b)$ the inverse of F. Then, the following properties are equivalent:*

- *(i) μ is log-convex (resp. strictly log-convex);*
- *(ii) f is continuous and positive on (a, b) and $J = f \circ F^{-1}$ is convex (resp. strictly convex) on $(0, F^{-1}(x_0))$ and on $(F^{-1}(x_0), 1)$.*

The proof (that we omit) is analogous to the case of log-concave measures (see [6, Proposition A.1]).

For simplicity (mainly to avoid unnecessary technicalities), we will restrict ourself to the study of a sub-class of log-convex probability measures.

In what follows, we will only consider log-convex probability measures symmetric with respect to the origin[2] (*i.e.* $x_0 = 0$ and J is symmetric with respect to $\frac{1}{2}$), such that $a = \inf\{x \in \mathbb{R} : F(x) > 0\} = -\infty$ and $b = \sup\{x \in \mathbb{R} : F(x) < 1\} = +\infty$.

Definition 2.3 *(The set $\mathcal{F}$).* We set $\mathcal{F}$ for the set of all strictly log-convex probability measures μ on $\mathbb{R}$, symmetric with respect to the origin, satisfying

$$\inf\big\{x \in \mathbb{R} : F(x) > 0\big\} = -\infty \quad \text{and} \quad \sup\big\{x \in \mathbb{R} : F(x) < 1\big\} = +\infty.$$

The following lemma holds.

Lemma 2.4. *Let $\mu \in \mathcal{F}$. Then $t \mapsto \frac{J(t)}{t}$ is a strictly increasing function on $[0, \frac{1}{2}]$.*

[2] Observe that the choice of the origin is not restrictive since the measure $\mu(\cdot + \alpha)$, with $\alpha \in \mathbb{R}$, shares the same isoperimetric properties as the measure μ.

3. Isoperimetric inequality

In this section we recover known isoperimetric inequalities for log-convex measures on the line [9], using geometric arguments (see [16,17]) that will allow us to prove quantitative estimates.

In order to give the definition of perimeter, we first recall that the *essential boundary* of a measurable set E is $\partial^M E = \mathbb{R}\backslash(E_0 \cup E_1)$, where

$$E_i = \left\{ x \in \mathbb{R} : \lim_{\rho \to 0} \frac{|E \cap B_\rho(x)|}{|B_\rho(x)|} = i \right\} \quad \text{for } i = 0, 1,$$

$B_\rho(x) = (x - \rho, x + \rho)$ and $|A|$ is the Lebesgue measure of the set A.

Given an absolutely continuous probability measure μ on the line with density f, the μ-perimeter of a Borel set E is defined as

$$P_\mu(E) = \int_{\partial^M E} f(x) \quad d\mathcal{H}^0(x),$$

where $\mathcal{H}^0(x)$ denotes the 0-dimensional Hausdorff measure in $\mathbb{R}$ (*i.e.* the counting measure, see [21, Theorem 2, p. 63]) and $\partial^M E$ is the essential boundary of E (see *e.g.* [1]). In most occurrences we will write for simplicity P for P_μ.

In what follows we will use the following property.

Lemma 3.1. *Let E be a set of finite perimeter. Then there exists a countable set H such that, up a set of measure zero, $E = \bigcup_{h \in H}(a_h, b_h)$, where $-\infty \le a_h < b_h \le +\infty$ and $\operatorname{dist}(E \setminus (a_h, b_h), (a_h, b_h)) > 0$ for all $h \in H$.*

For a proof we refer to *e.g.* [1, Proposition 3.52].

Warning. We note that in our framework we can indifferently consider open, closed and semi-open intervals, since they have all the same measure and perimeter. In particular, in many results below, we shall give equality cases only with open intervals (or union of open intervals) but of course the corresponding closed intervals and semi-open ones are also equality cases.

Bobkov and Houdré proved the following very general statement.

Theorem 3.2. *(See [9], Corollary 13.10.) Let $d\mu_\Phi(x) = Z_\Phi^{-1} e^{-\Phi(x)} dx$ be a probability measure, with $\Phi: \mathbb{R} \to \mathbb{R}$ even, and Z_Φ the normalization constant. Then the extremal sets in the isoperimetric inequality can be found among half-lines, symmetric segments and their complements.*

Our goal is to state a more precise result about extremal sets for symmetric strictly log-convex probability measures. As already mentioned, we will make use of a geometric argument that we present now.

3.1. The shifting property

Following [16] (for Gauss measures, see also [17] for log-concave measure on the line), we prove in this section a “shifting property” for intervals and complement of intervals.

We start with a definition of shifted intervals.

Definition 3.3 *(Right/left shifted interval).* Let (a, b) be an interval of $\mathbb{R}$ with $-\infty < a < b < +\infty$. Then,

- any interval (a', b') such that $a < a' < b' \leq +\infty$ and $\mu((a, b)) = \mu((a', b'))$ is said to be a *right-shifted interval* of (a, b);
- any interval (a', b') such that $-\infty \leq a' < b' < b$ and $\mu((a, b)) = \mu((a', b'))$ is said to be a *left-shifted interval* of (a, b).

The next proposition is one of our key ingredient. It encodes the fact that, depending on the measure of the interval and on its position on the line, the μ-perimeter decreases for left/right shifted intervals.

Proposition 3.4 *(Shifting property for intervals). Let $\mu \in \mathcal{F}$ (see Definition 2.3).*

(1) Let (a, b) be an interval of measure $\mu((a, b)) < \frac{1}{2}$.
(1a) If $a \geq 0$ (resp. $b \leq 0$), then

$$P\big((a, b)\big) > P\big((a', b')\big)$$

for any right-shifted (resp. left-shifted) interval of (a, b).
(1b) If $a < 0$, $b > 0$ and $a + b \geq 0$ (resp. $a + b \leq 0$), then

$$P\big((a, b)\big) > P\big((a', b')\big)$$

for any left-shifted (resp. right-shifted) interval of (a, b) with $a' + b' \geq 0$ (resp. $a' + b' \leq 0$).
(2) Let (a, b) be an interval of measure $\mu((a, b)) \geq \frac{1}{2}$. If $a + b \geq 0$ (resp. $a + b \leq 0$), then

$$P\big((a, b)\big) > P\big((a', b')\big)$$

for any left-shifted (resp. right-shifted) interval of (a, b) with $a' + b' \geq 0$ (resp. $a' + b' \leq 0$).

Remark 3.5. Without the strict log-convexity assumption, the results above still hold but no more with strict inequalities.

Proof. Let (a, b) be an interval of measure $p \in (0, 1)$, with $-\infty \leq a < b \leq +\infty$. Its perimeter is

$$P\big((a, b)\big) = f(a) + f(b) = J\big(F(a)\big) + J\big(F(b)\big).$$

The value $p \in (0, 1)$ being fixed, necessarily $b = F^{-1}(p + F(a))$. Denoting $a = F^{-1}(t)$, we may study the function

$$P\big((a, b)\big) = J(t) + J(p + t) := \psi_p(t)$$

as a function of $t \in [0, 1 - p]$. The expected results follow at once from Lemma 3.6 below. $\square$

Lemma 3.6. *Let $\mu \in \mathcal{F}$. For $p \in (0, 1)$ and $t \in [0, 1 - p]$, set $\psi_p(t) = J(t) + J(p + t)$. Then,*

i) ψ_p is symmetric about $\frac{1-p}{2}$,
ii) if $p \geq 1/2$, ψ_p is strictly convex on $[0, 1 - p]$, decreasing on $(0, \frac{1-p}{2})$, increasing on $(\frac{1-p}{2}, 1 - p)$, $\frac{1-p}{2}$ is a minimum and $0, 1 - p$ are maxima.

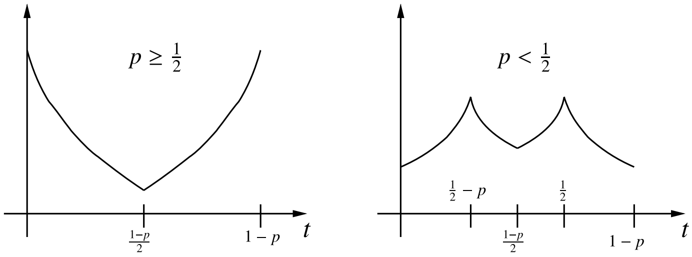

Fig. 1. The shape of the function $t \mapsto \psi_p(t)$ for $p \geq 1/2$ and $p < 1/2$.

iii) if $p < 1/2$, ψ_p is strictly convex on $(0, \frac{1}{2}-p)$, on $(\frac{1}{2}-p, \frac{1}{2})$ and on $(\frac{1}{2}, 1-p)$, it is increasing on $(0, \frac{1}{2}-p)$, decreasing on $(\frac{1}{2}-p, \frac{1-p}{2})$, increasing on $(\frac{1-p}{2}, \frac{1}{2})$ and decreasing on $(\frac{1}{2}, 1-p)$. Moreover $\frac{1}{2}-p, \frac{1}{2}$ are maxima and $\frac{1-p}{2}, 0, 1-p$ are minima. (See Fig. 1.)

Proof. The proof is elementary and left to the reader. □

We end this section with a converse of Proposition 3.4, namely that the shifting property of Proposition 3.4 (to be really precise only a weaker form is needed) implies that μ is log-convex.

Proposition 3.7. *Let μ be a probability measure on the line, symmetric with respect to a point (say the origin for simplicity) with density f. Assume that f is continuous, positive on $(-\alpha, \alpha)$ for some $\alpha \in (0, \infty]$ and such that the following shifting properties hold:*

1) *if (a, b) is an interval with $\mu((a, b)) < \frac{1}{2}$, $a < 0$, $b > 0$ and $a + b \geq 0$ (resp. $a + b \leq 0$) then*

$$P\big((a, b)\big) \geq P\big((a', b')\big)$$

for any left-shifted (resp. right-shifted) of (a, b) with $a' + b' \geq 0$ (resp. $a' + b' \leq 0$);

2) *if (a, b) is an interval with $\mu((a, b)) \geq \frac{1}{2}$ and $a + b \geq 0$ (resp. $a + b \leq 0$) then*

$$P\big((a, b)\big) \geq P\big((a', b')\big)$$

for any left-shifted (resp. right-shifted) of (a, b) with $a' + b' \geq 0$ (resp. $a' + b' \leq 0$).

Then μ is log-convex.

Proof. By Proposition 2.2, continuity and symmetry of $J(t)$ we only have to prove that, for all $t \in (0, \frac{1}{2})$ and all d so that $t \pm d \in (0, \frac{1}{2})$, it holds

$$J(t) \geq \frac{1}{2}\big(J(t-d) + J(t+d)\big). \tag{3.1}$$

Fix $t \in (0, \frac{1}{2})$ and d such that $t \pm d \in (0, \frac{1}{2})$ and set $a = F^{-1}(t)$, $b = F^{-1}(1-t)$, $a' = F^{-1}(t+d)$ and $b' = F^{-1}(1-t+d)$. With these notations in hand, we observe that, by symmetry of J,

$$P\big((a, b)\big) = J(t) + J(1-t) = 2J(t)$$

and

$$P\big((a', b')\big) = J(t+d) + J(1-t+d) = J(t+d) + J(t-d).$$

Then (3.1) precisely means that $P((a, b)) \geq P((a', b'))$ which is guaranteed by the shifting property assumption. This ends the proof. □

3.2. Isoperimetric problem for intervals and complement of intervals

The geometric tool given in the previous section will allow us to answer the following warm-up isoperimetric problem: among all intervals (and then among all complement of intervals) of given measure, which one(s) has(have) minimal perimeter? The answer for intervals is stated in the next corollary: depending on the measure of the interval, the interval with minimal perimeter has to be found at infinity (half-line), or centered around the origin.

We need a preliminary result.

Lemma 3.8. *Let $\mu \in \mathcal{F}$. Then, there exists a unique $p_0 \in (0, \frac{1}{2})$ satisfying $J(1-p_0) = 2J((1-p_0)/2)$ and such that $J(1-p) < 2J((1-p)/2)$ for $p \in [0, p_0)$ and $J(1-p) > 2J((1-p)/2)$ for $p \in (p_0, 1/2]$, where $J(t)$ is defined in (2.4).*

Remark 3.9. In general p_0 is known only implicitly. However, in the case of the measure m_α defined in (2.1), one easily sees that $p_0 = \frac{1}{1+2^{1/(1+\alpha)}}$.

Proof. Let us consider the auxiliary function $g(p) = J(1-p) - 2J(\frac{1-p}{2}) = J(1-p) - 2J(\frac{1+p}{2})$ for $p \in [0, \frac{1}{2}]$. We observe that g is continuous, increasing and $g(0) = -2J_\mu(\frac{1}{2}) < 0$. Moreover, Lemma 2.4 guarantees that $g(\frac{1}{2}) = \frac{1}{2}(\frac{J(1/2)}{1/2} - \frac{J(1/4)}{1/4}) > 0$. Hence the result. □

Introduce the following notation.

$$\alpha_p = -F^{-1}\left(\frac{1-p}{2}\right), \qquad \sigma_p = -F^{-1}(p), \quad p \in [0,1]. \tag{3.2}$$

We are now in position to state the corollary.

Corollary 3.10 *(Extremal sets in the isoperimetric problem for intervals). Let $\mu \in \mathcal{F}$ and p_0 defined in Lemma 3.8. Let us fix a, b with $-\infty \le a < b \le +\infty$ and set $p = \mu((a,b))$. Then,*

i) if $p > p_0$,

$$P\big((a,b)\big) \ge P\big((-\alpha_p, \alpha_p)\big), \tag{3.3}$$

with equality iff $(a,b) = (-\alpha_p, \alpha_p)$;

ii) if $p < p_0$,

$$P\big((a,b)\big) \ge P\big((-\infty, -\sigma_p)\big) \quad \big(= P\big((\sigma_p, +\infty)\big)\big), \tag{3.4}$$

with equality iff $(a,b) = (-\infty, -\sigma_p)$ or $(a,b) = (\sigma_p, +\infty)$;

iii) if $p = p_0$,

$$P\big((a,b)\big) \ge P\big((-\alpha_p, \alpha_p)\big) = P\big((-\infty, -\sigma_p)\big) = P\big((\sigma_p, +\infty)\big)$$

with equality iff (a,b) equals $(-\alpha_p, \alpha_p)$, $(-\infty, -\sigma_p)$ or $(\sigma_p, +\infty)$.

Proof. If $p \ge 1/2$, the result of point (i) immediately follows from Proposition 3.4 point (2).

Hence, we need to deal with $0 \le p < \frac{1}{2}$. As in the proof of Proposition 3.4, we have

$$P\big((a,b)\big) = J\big(F(a)\big) + J\big(p + F(a)\big) = \psi_p\big(F(a)\big),$$

where ψ_p has been defined in Lemma 3.6. Thanks to Lemma 3.6, we need to compare the two minima of ψ_p, namely $\psi_p(1-p) = J(1-p) = P((-\infty, -\sigma_p)) = P((\sigma_p, +\infty))$ and $\psi_p(\frac{1-p}{2}) = 2J(\frac{1-p}{2}) = P((-\alpha_p, \alpha_p))$.

By definition of p_0 (see Lemma 3.8), we conclude that $a \mapsto \psi_p(F(a))$ has a unique global minimum at $a = F^{-1}(\frac{1-p}{2})$ if $0 < p < p_0$ and has two minima at $a = F^{-1}(0)$ and $a = F^{-1}(1-p)$ if $p_0 < p < \frac{1}{2}$. This ends the proof of the inequalities. Equality cases follow at once from the strict monotonicity of ψ_p. □

Remark 3.11. Corollary 3.10 implies that

$$P\big((a,b)\big) \geq \min\left\{2J\left(\frac{1-p}{2}\right), J(p)\right\}, \quad p \in [0,1].$$

Remark 3.12 *(Isoperimetric problem for complements of an interval).* Observing that intervals and complement of intervals have the same perimeter, *i.e.* that $P((-\infty, a) \cup (b, +\infty)) = P((a, b))$ one can easily solve, using Corollary 3.10, the isoperimetric problem among sets of prescribed measure, that are complements of an interval. More precisely, one obtains the following (details are left to the reader): Fix a, b with $-\infty \leq a < b \leq +\infty$ and set $p = \mu((-\infty, a) \cup (b, +\infty))$. Then (using notations of Corollary 3.10),

i) if $p > 1 - p_0$,

$$P\big((-\infty, a) \cup (b, +\infty)\big) \geq P\big((-\infty, \sigma_{1-p})\big) \quad \big(= P\big((-\sigma_{1-p}, +\infty)\big)\big) \tag{3.5}$$

with equality iff $(-\infty, a) \cup (b, +\infty) = (-\infty, \sigma_{1-p})$ or $(-\infty, a) \cup (b, +\infty) = (-\sigma_{1-p}, \infty)$;

ii) if $p < 1 - p_0$,

$$P\big((-\infty, a) \cup (b, +\infty)\big) \geq P\big((-\infty, -\alpha_{1-p}) \cup (\alpha_{1-p}, +\infty)\big) \tag{3.6}$$

with equality iff $(-\infty, a) \cup (b, +\infty) = (-\infty, -\alpha_{1-p}) \cup (\alpha_{1-p}, +\infty)$;

iii) if $p = 1 - p_0$,

$$\begin{aligned} P\big((-\infty, a) \cup (b, +\infty)\big) &\geq P\big((-\infty, -\alpha_{1-p}) \cup (\alpha_{1-p}, +\infty)\big) \\ &= P\big((-\infty, \sigma_{1-p})\big) = P\big((-\sigma_{1-p}, +\infty)\big) \end{aligned}$$

with equality iff $(-\infty, a) \cup (b, +\infty)$ equals $(-\infty, \sigma_{1-p})$, $(-\infty, \sigma_{1-p})$ or the set $(-\infty, -\alpha_{1-p}) \cup (\alpha_{1-p}, +\infty)$.

3.3. Isoperimetric inequality for strictly log-convex probability measures

From the results of the previous sections, we can now solve the isoperimetric problem for strictly log-convex probability measures.

In what follows we need to recall the definition of α_p and σ_p given in (3.2). For simplicity, we set also $\beta_p = \alpha_{1-p}$, for $p \in [0, 1]$.

Theorem 3.13 *(Isoperimetry for strictly log-convex probability measures). Let $\mu \in \mathcal{F}$ and E be a Borel set of $\mathbb{R}$ with measure $\mu(E) = p$. Then*

i) if $p < 1/2$,

$$P(E) \geq P\big((-\infty, -\beta_p) \cup (\beta_p, +\infty)\big), \tag{3.7}$$

with equality iff $E = (-\infty, -\beta_p) \cup (\beta_p, +\infty)$;

ii) if $p > 1/2$,

$$P(E) \geq P\big((-\alpha_p, \alpha_p)\big) \tag{3.8}$$

with equality iff $E = (-\alpha_p, \alpha_p)$;

iii) if $p = \frac{1}{2}$

$$P(E) \geq P\big((-\infty, -\beta_p) \cup (\beta_p, +\infty)\big) = P\big((-\alpha_p, \alpha_p)\big) \tag{3.9}$$

with equality iff E *equals* $(-\infty, -\beta_p) \cup (\beta_p, +\infty)$ *or* $(-\alpha_p, \alpha_p)$.

Remark 3.14. The measure 1/2 can be seen as an *isoperimetric threshold*, in the sense that extremal sets move from complement of symmetric intervals (when $p < 1/2$) to symmetric intervals (when $p > 1/2$).

Proof. Let E be a Borel set with measure $p = \mu(E)$. Without loss of generality we can assume that E has finite perimeter.

Step 1. Assume that $E = (a_1, b_1) \cup (a_2, b_2)$ with $a_1 < b_1 < a_2 < b_2$ and $a_1 > 0$. Then, consider the right shifted interval (a_1', b_1') of (a_1, b_1), with $b_1' = a_2$ so that, thanks to Proposition 3.4,

$$P(E) \geq P\big((a_1', b_1') \cup (a_2, b_2)\big) \geq P\big((a_1', b_2)\big).$$

In conclusion, we get that, given two disconnected intervals contained in $(0, \infty)$, the perimeter decreases by moving (and gluing) the left most interval toward the right most one. Clearly, the same property holds for $a_1 < b_1 < a_2 < b_2 < 0$ by symmetry.

Step 2. Let E be any set of finite μ-perimeter. From Lemma 3.1 there exists a countable set H such that, up to a set of measure zero, $E = \bigcup_{h \in H} (a_h, b_h)$, where $-\infty \leq a_h < b_h \leq +\infty$ and $\operatorname{dist}(E \setminus (a_h, b_h), (a_h, b_h)) > 0$ for all $h \in H$. Without loss of generality we can assume that the set of measure zero is the empty set. Then, iterating the arguments used in step 1 (for two intervals), we obtain that, either

$$P(E) \geq P\big((-\infty, -a) \cup (b, +\infty)\big)$$

if $0 \notin E$, or, if $0 \in (a_{h_o}, b_{h_o})$ for some $h_o \in H$,

$$P(E) \geq P\big((-\infty, -\bar{a}) \cup (a_{h_o}, b_{h_o}) \cup (\bar{b}, +\infty)\big),$$

where $a, \bar{a}, b, \bar{b} \in (0, \infty]$ and $(-\infty, -a) \cup (b, +\infty)$ and $(-\infty, -\bar{a}) \cup (a_{h_o}, b_{h_o}) \cup (\bar{b}, +\infty)$ are sets with measure $p = \mu(E)$. Here and below we use the convention that $(\infty, \infty) = (-\infty, -\infty) = \emptyset$. In the second case (*i.e.* when $0 \in E$), we continue the reduction by considering the complementary set

$$\big[(-\infty, -\bar{a}) \cup (a_{h_o}, b_{h_o}) \cup (\bar{b}, +\infty)\big]^c = [-\bar{a}, a_{h_o}] \cup [b_{h_o}, \bar{b}],$$

which has the same measure and same perimeter as $\widehat{E} := (-\bar{a}, a_{h_o}) \cup (b_{h_o}, \bar{b})$ that we will deal with. By construction and since μ is symmetric, necessarily $\mu((-\bar{a}, a_{h_o})) < 1/2$ and $\mu((b_{h_o}, \bar{b})) < 1/2$. Hence, thanks to the shifting property of Proposition 3.4 (point (1)), $P((-\bar{a}, a_{h_o})) \geq P((-\infty, -\alpha))$ and $P((b_{h_o}, \bar{b})) \geq P((\beta, \infty))$, where $\alpha, \beta \in (0, \infty]$ are such that $\mu((-\infty, -\alpha)) = \mu((-\bar{a}, a_{h_o}))$ and $\mu((\beta, \infty)) = \mu((b_{h_o}, \bar{b}))$. Going back to the complementary set, we end up with the following bound

$$P(E) \geq P(\widehat{E}) \geq P\big((-\infty, -\alpha) \cup (\beta, \infty)\big) = P\big((-\alpha, \beta)\big)$$

with $\mu((-\alpha, \beta)) = p$.

As a summary, after few reductions, we obtained the following two cases: either

$$(I) \quad P(E) \geq P\big((-\infty, -a) \cup (b, +\infty)\big) \quad \text{if } 0 \notin E$$

or

$$(II) \quad P(E) \geq P\big((-\alpha, \beta)\big) \quad \text{if } 0 \in E,$$

where $\mu((-\infty, -a) \cup (b, +\infty)) = \mu((-\alpha, \beta)) = \mu(E) = p$ and $a, \alpha, b, \beta \in (0, \infty]$.

Now assume that $p \in (0, 1/2]$. We distinguish between cases (I) and (II).

Case (I). Applying Remark 3.12 point *ii*) (observe that, since $p \leq 1/2$, necessarily $p < 1 - p_0$, where p_0 is defined in Lemma 3.8), the perimeter decreases if we consider the symmetric set $(-\infty, -\alpha_{1-p}) \cup (\alpha_{1-p}, +\infty)$, unless $E = (-\infty, -\alpha_{1-p}) \cup (\alpha_{1-p}, +\infty)$, where we recall that $\alpha_{1-p} = -F^{-1}(p/2) = \beta_p$.

As a conclusion, in case (I), $P(E) \geq P((-\infty, -\beta_p) \cup (\beta_p, +\infty))$.

Case (II). Corollary 3.10 (point *ii*)) guarantees that

$$P(E) \geq P\big((-\alpha, \beta)\big) \geq P\big((-\infty, -\sigma_p)\big) = J(p)$$

(where $\sigma_p = -F^{-1}(p)$). Lemma 2.4 implies that, for $p \in [0, \frac{1}{2}]$, $J(p) \geq 2J(\frac{p}{2}) = P((-\infty, -\beta_p) \cup (\beta_p, +\infty))$. The inequality of point *i*) follows. Keeping track of the equality cases in the various steps above leads to the desired result of points *i*) and *iii*).

Finally, point *ii*) is an easy consequence of point *i*) considering the complementary set. □

Remark 3.15. Notice that in Theorem 3.13, if μ is not strictly log-convex then the uniqueness in the equality cases no longer holds (see next Section 3.4).

Also, observe that Theorem 3.13 gives the following explicit expression of the isoperimetric profile, recovering [9] for strictly log-convex probability measures,

$$I(p) = 2J\left(\frac{1}{2}\min(p, 1-p)\right), \quad p \in [0, 1].$$

3.4. Example of the two-sided exponential measure

In this subsection, we briefly deal with an example of non-strictly log-convex probability measure, the two-side exponential measure defined in (2.2). It is a symmetric probability measure, log-convex *and* log-concave, with $J(t) = \min(t, 1-t)$.

The perimeter $P_{\mu_1}((a, b))$ of an interval (a, b) of fixed measure p can be explicitly computed. If $p \geq \frac{1}{2}$ then $P_{\mu_1}((a, b)) = 1 - p$. In particular, we stress that all intervals of measure bigger than $1/2$ have the same perimeter. If $p < \frac{1}{2}$ we have

$$P_{\mu_1}\big((a, b)\big) = P_p(a) = \begin{cases} 2F(a) + p & \text{if } -\infty \leq a \leq F^{-1}(\frac{1}{2} - p), \\ 1 - p & \text{if } F^{-1}(\frac{1}{2} - p) \leq a \leq 0, \\ 2 - 2F(a) - p & \text{if } 0 \leq a \leq F^{-1}(1 - p). \end{cases}$$

Therefore among the intervals (a, b) of measure $p < \frac{1}{2}$ the half-lines have minimal perimeter. Moreover the shifting property for interval is the following:

Proposition 3.16. *If (a,b) is an interval of measure such that $\mu_1((a,b)) < \frac{1}{2}$ with $b \le 0$ (resp. $a \ge 0$), then*

$$P_{\mu_1}\big((a,b)\big) > P_{\mu_1}\big((a',b')\big)$$

for any left-shifted (resp. right-shifted) interval of (a,b).

Otherwise

$$P_{\mu_1}\big((a,b)\big) = P_{\mu_1}\big((a',b')\big)$$

for any right-shifted or left-shifted interval of (a,b).

Arguing as in Theorem 3.13 we obtain that for a fixed measure $p < \frac{1}{2}$ any complement of an interval and the half-lines are sets with minimal perimeter. For $p \ge \frac{1}{2}$ any interval and half-lines have minimal perimeter.

4. Quantitative isoperimetric inequality

In this section, following [16], we introduce and study a notion of asymmetry, which quantify the "distance" between any measurable set E and the family of extremal sets in the isoperimetric problem. Then we state a preliminary result on the sets that have minimal perimeter and given measure and asymmetry.

We define the asymmetry $\lambda(E)$ of a set E of measure $p = \mu(E)$ as

$$\lambda(E) = \begin{cases} \mu(E \triangle (-\infty,-\beta_p) \cup (\beta_p,+\infty)) & \text{if } p < \frac{1}{2} \\ \mu(E \triangle (-\alpha_p,\alpha_p)) & \text{if } p > \frac{1}{2} \\ \min\{\mu(E \triangle (-\infty,-\beta_p) \cup (\beta_p,+\infty)), \mu(E \triangle (-\alpha_p,\alpha_p))\} & \text{if } p = \frac{1}{2}, \end{cases} \tag{4.1}$$

where $\beta_p = \alpha_{1-p} = -F^{-1}(p/2)$, $p \in [0,1]$ are defined in the previous section and $\triangle$ stands for the symmetric difference between sets.

Remark 4.1. To help the reader in many computations throughout all this section, we observe that the set $E = (-\beta_a, -\beta_b)$, with $0 \le a \le b$, has perimeter $P(E) = J(b/2) + J(a/2)$ and measure $\mu(E) = (b-a)/2$.

The next lemma summarizes some basic properties on the asymmetry $\lambda(E)$.

Lemma 4.2. *Let $\mu \in \mathcal{F}$ and E, F be two sets with finite μ-perimeter. Then,*

i) $E \triangle F = E^c \triangle F^c$,
ii) $\lambda(E) = \lambda(E^c)$,
iii) $0 \le \lambda(E) \le 2\min(\mu(E), 1-\mu(E))$,

where E^c denotes the complement of E.

Proof. The assertions are easy and left to the reader. □

4.1. A preliminary reduction and application

The next result is a first reduction to find the sets with minimal perimeter and given measure and asymmetry. We first consider the case $0 < \mu(E) \le \frac{1}{2}$. The other case $\frac{1}{2} \le \mu(E) < 1$ can be obtained using complementary sets and Lemma 4.2. We may use the following notation: given a set E, $-E = \{-x,\ x \in E\}$ denotes its symmetric with respect to the origin.

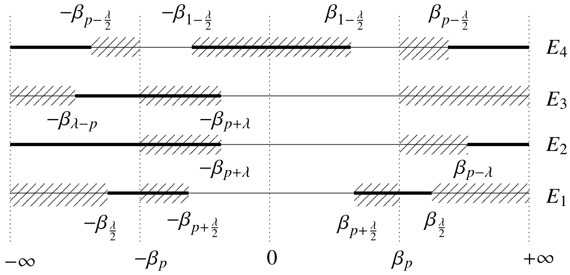

Fig. 2. The sets $E_1, \dots, E_4$ (bold lines). Hachured parts (of measure λ) correspond to the symmetric difference between the set E_i and the optimal set $(-\infty, -\beta_p) \cup (\beta_p, \infty)$. Observe that there is no universal order between the extremal points of the intervals defining $E_1, \dots, E_4$, except that $\beta_{\lambda-p} \geq \beta_{\lambda/2}$, and $\beta_{p-\lambda} \geq \beta_{p-\frac{\lambda}{2}}$. Also, depending on the value of p and λ, it could be that $0 \in E_3$.

We will show, in Proposition 4.3 below, that the minimal sets among all sets of given measure p and given asymmetry λ have to be found among the following sets and their symmetric (see Fig. 2):

$$E_1 = (-\beta_{\frac{\lambda}{2}}, -\beta_{p+\frac{\lambda}{2}}) \cup (\beta_{p+\frac{\lambda}{2}}, \beta_{\frac{\lambda}{2}}), \qquad \text{if } 0 \leq \lambda \leq 2p, \tag{4.2}$$

$$E_2 = (-\infty, -\beta_{p+\lambda}) \cup (\beta_{p-\lambda}, +\infty) \text{ and } -E_2, \qquad \text{if } 0 \leq \lambda \leq p, \tag{4.3}$$

$$E_3 = (-\beta_{\lambda-p}, -\beta_{\lambda+p}) \text{ and } -E_3, \qquad \text{if } p \leq \lambda \leq 2p, \tag{4.4}$$

and, if $0 \leq \lambda \leq 2p$,

$$E_4 = (-\infty, -\beta_{p-\frac{\lambda}{2}}) \cup (-\beta_{1-\frac{\lambda}{2}}, \beta_{1-\frac{\lambda}{2}}) \cup (\beta_{p-\frac{\lambda}{2}}, +\infty). \tag{4.5}$$

Observe that E_2 and $-E_2$ are not defined when $\lambda > p$ and that E_3 and $-E_3$ are not defined when $\lambda < p$.

Proposition 4.3. *Let $\mu \in \mathcal{F}$ and E be a Borel set with measure $\mu(E) = p \in (0, \frac{1}{2}]$ and asymmetry $\lambda(E) = \lambda \in [0, 2p]$. Then,*

$$P(E) \geq \begin{cases} \min_{i=1,2,4} P(E_i) & \text{if } 0 \leq \lambda \leq p \\ \min_{i=1,3,4} P(E_i) & \text{if } p \leq \lambda \leq 2p. \end{cases} \tag{4.6}$$

Moreover equality holds if and only if $E \in \{E_1, E_2, E_3, E_4, -E_2, -E_3\}$.

If $0 \notin E$, then

$$P(E) \geq \begin{cases} \min_{i=1,2} P(E_i) & \text{if } 0 \leq \lambda \leq p \\ \min_{i=1,3} P(E_i) & \text{if } p \leq \lambda \leq \min(1-p, 2p) \\ P(E_1) & \text{if } 1-p < \lambda \leq 2p \end{cases}$$

with equality if and only if $E \in \{E_1, E_2, E_3, -E_2, -E_3\}$.

Remark 4.4. The second part of the above proposition (together with Proposition 4.5 below) will be used in Section 5 where we will only consider sets that do not contain the origin.

The proof of Proposition 4.3 can be found in Appendix A.

At this point it is not possible to conclude which one of the sets E_i, $i = 1, \dots, 4$, has minimal perimeter on the range $p \in [0, 1/2]$, $\lambda \in [0, 2p]$ for the whole class of probability measures $\mathcal{F}$. Indeed, depending on the choice of $\mu \in \mathcal{F}$, one can exhibit very different behaviors.

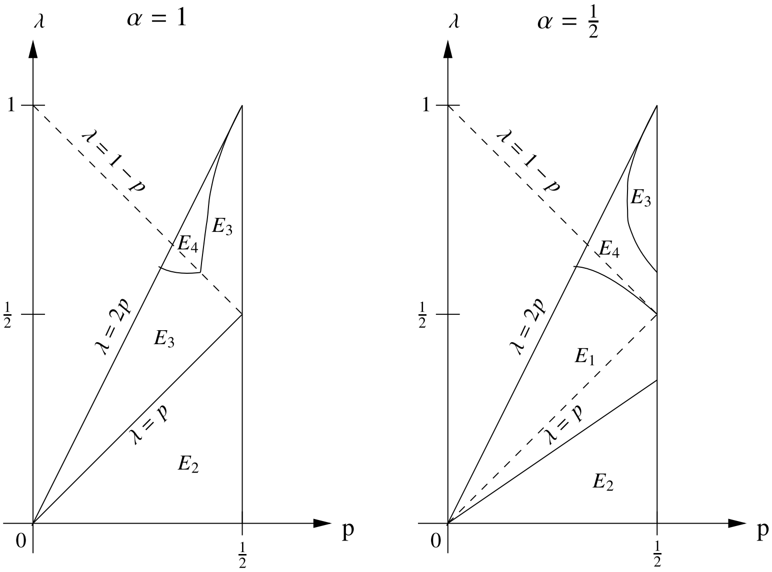

Fig. 3. The region $p \in [0, \frac{1}{2}]$, $0 \le \lambda \le 2p$ with the different areas where the set E_i has minimal perimeter among E_1, E_2, E_3 and E_4. The picture on the left corresponds to the distribution m_α with $\alpha = 1$. Here, the E_4-domain is delimited by two curves of equation $\lambda = -1 - p + \sqrt{3 + 2p + p^2}$ (bottom, that intersect the line $\lambda = 2p$ at $p_1 = \frac{\sqrt{5}-1}{4}$) and $-1 + 2p + \lambda(1-p) - \frac{\lambda^2}{2} = 0$ (top). The two curves intersect on the line of equation $\lambda = 1 - p$ at $p_2 = \sqrt{2} - 1$. The picture on the right corresponds to $\alpha = \frac{1}{2}$. The E_2-domain and E_1-domain are delimited by a straight line of equation $\lambda = \frac{4\sqrt{3}p}{3\sqrt{3}+\sqrt{19}}$. The two curves delimiting the E_3, E_4 region are degree 3 polynomials in p, λ.

To illustrate this phenomenon, let us deal with two specific distributions (2.1), with parameter $\alpha = 1$ and $\alpha = 1/2$:

$$dm_1(x) = \frac{1}{2(1+|x|)^2}dx \quad \text{and} \quad dm_{1/2}(x) = \frac{1}{4(1+|x|)^{3/2}}dx.$$

Recall that $J_\alpha(t) = \alpha 2^{\frac{1}{\alpha}} \min(t, 1-t)^{1+\frac{1}{\alpha}}$ so that

$$J_1(t) = 2\min(t, 1-t)^2 \quad \text{and} \quad J_{1/2}(t) = 2\min(t, 1-t)^3.$$

Since functions J are explicit, one can compute the various perimeter $P(E_1)$, $P(E_2)$, $P(E_3)$, $P(E_4)$ and compare them. It is simple (but very tedious, details are left to the reader) to obtain Fig. 3 that depicts, for $p \in [0, 1/2]$, the different regions with minimal perimeter (note that the region $p \in [1/2, 1]$ can be obtained by symmetry about $p = 1/2$, using Lemma 4.2).

If m_1 and $m_{1/2}$ have very different behaviors, under additional assumptions, one can however prove that a large sub-class of $\mathcal{F}$ behaves like m_1 (*i.e.* have the same type of picture than the left one in Fig. 3). This is stated in the next proposition.

Proposition 4.5. *Let $\mu \in \mathcal{F}$ and assume in addition that $J \in \mathcal{C}^1(0, \frac{1}{2})$, J' is concave on $(0, \frac{1}{2})$ and $J'(0^+) = 0$. Let us fix $p \in [0, \frac{1}{2}]$ and $\lambda \in [0, 1]$, and let us define E_2, E_3 and E_4 as in (4.3), (4.4) and (4.5) respectively.*

Then, there exist $p_1 \in (0, \frac{1}{3})$, $p_2 \in (\frac{1}{3}, 12)$, a function $\lambda_0 : [p_1, p_2] \to [0, 1]$ satisfying $\lambda_0(p_1) = 2p_1$ and $\lambda_0(p_2) = 1 - p_2$, and a $\mathcal{C}^1$-increasing function $p_0 : [1-p_2, 1] \to [0, \frac{1}{2}]$ satisfying $p_0(1-p_2) = p_2$ and $p_0'(1) = \frac{1}{2}$, such that for any Borel set E with measure p and asymmetry λ it holds

$$P(E) \ge \begin{cases} P(E_2) & \text{if } 0 \le \lambda \le p \\ P(E_4) & \text{if } p \in [p_1, p_2] \text{ and } \lambda \in [\lambda_0(p), \min(2p, 1-p)] \\ P(E_4) & \text{if } \lambda \in [1-p_2, 1] \text{ and } p \in [\frac{\lambda}{2}, p_0(\lambda)] \\ P(E_3) & \text{otherwise.} \end{cases}$$

Moreover, if $0 \notin E$, then

$$P(E) \geq \begin{cases} P(E_2) & \text{if } 0 \leq \lambda \leq p \\ P(E_3) & \text{if } p \leq \lambda \leq \min(1-p, 2p) \\ P(E_1) & \text{if } 1-p < \lambda \leq 2p. \end{cases} \tag{4.7}$$

Remark 4.6. Observe that m_1 satisfies the assumption of the proposition and more generally any distribution m_α with $\alpha \geq 1$. Also, Fig. 3 is an illustration of the result of the proposition (with $p_1 = (\sqrt{5}-1)/4$, $p_2 = \sqrt{2}-1$, $\lambda_0(p) = -1-p+\sqrt{3+2p+p^2}$ and $p_0(\lambda) = (1-\lambda+\frac{\lambda^2}{2})/(2-\lambda)$).

The property $p_0'(1) = 1/2$ means that the curve has $\lambda = 2p$ as tangent in $(1/2, 1)$.

On the other hand, the assumptions on J guarantee that J' is sub-linear, *i.e.* $J'(a+b) \leq J'(a) + J'(b)$ for all $a, b \in [0, 1/2]$, and that $t \mapsto \frac{J(t)}{t^2}$ is decreasing on $(0, 1/2)$. We will make use of these properties repeatedly.

For the clarity of the exposition, the proof of Proposition 4.5 is postponed to Appendix B.

4.2. Estimates on the deficit

In this section we prove a quantitative estimate on the deficit.

The deficit of a set E is defined as

$$\delta(E) = \begin{cases} P(E) - P((-\infty, -\beta_p) \cup (\beta_p, +\infty)) & \text{if } \mu(E) \leq \frac{1}{2} \\ P(E) - P((-\alpha_p, \alpha_p)) & \text{if } \mu(E) \geq \frac{1}{2}. \end{cases} \tag{4.8}$$

In words, the deficit measures (in the sense of the perimeter) how far the set is from the optimal set in the isoperimetric inequality.

Recall that the convex function $J: (0, 1/2) \to [0, \infty)$ is said to satisfy the ∇_2-condition if there exists $\varepsilon > 0$ such that, for all $x \in (0, 1/2)$, it holds $J(x) \geq (2+\varepsilon)J(x/2)$ (see [35]).

What follows is one of our main theorems.

Theorem 4.7. *Let $\mu \in \mathcal{F}$ and assume that $J \in C^2(0, \frac{1}{2})$. Assume furthermore that $M(p) := \inf_{t\in[p/2,1/2]} J''(t) > 0$ for all $p \in (0, 1/2]$. Fix $p \in [0, 1/2]$ and $\lambda \in [0, 2p]$. Then, there exist $c = c(p) > 0$ and $c' > 0$ such that the following hold:*
(i) for any Borel set E of measure p and asymmetry λ, it holds

$$\delta(E) \geq c\big[(1-\lambda)^2 + (1-2p)\big]\lambda^2; \tag{4.9}$$

(ii) if in addition J satisfies the ∇_2-condition with $\varepsilon \in (0, 1)$, J' is concave on $(0, 1/2)$ and $J'(0^+) = 0$, then for any Borel set $E \not\ni 0$ of measure p and asymmetry λ, it holds

$$\delta(E) \geq c'\lambda^2. \tag{4.10}$$

Moreover, one can choose $c' = \varepsilon J''(1/2^-)/32$ and

$$c = \frac{1}{32}\min\left(8J'\left(\frac{p}{2}\right), M(p), 16J'\left(\frac{1}{6}\right), 8\left[J\left(\frac{1}{2}\right) - 2J\left(\frac{1}{4}\right)\right], 4M\left(\frac{J(\frac{1}{2}) - 2J(\frac{1}{4})}{J'(1/2^-)}\right)\right). \tag{4.11}$$

The proof of Theorem 4.7 is technical and can be found in Appendix C. Let us comment on the result.

(a) We stress that the constant c' in the right hand side of (4.10) does not depend on p. This part of the theorem will be useful in the next section. Moreover, the quantity $J(\frac{1}{2}) - 2J(\frac{1}{4})$ in (4.11) is positive thanks to Lemma 2.4.

(b) Using Lemma 4.2, the above result extends at once to the whole region $p \in [0,1]$: given $p \in [0,1]$ and $\lambda \in [0, 2\min(p, 1-p)]$, there exists positive constant $c'' = c''(p)$ such that for any Borel set E of measure p and asymmetry λ, it holds

$$\delta(E) \geq c''\big[(1-\lambda)^2 + \big(1 - 2\min(p, 1-p)\big)\big]\lambda^2.$$

(c) The assumptions $J \in C^2(0, \frac{1}{2})$ and $M(p) := \inf_{t \in [p/2, 1/2]} J''(t) > 0$ for all $p \in (0, 1/2]$ are technical. The result would certainly hold under weaker assumptions. Also, the constant c is clearly not optimal.

(d) It is possible to construct an example for which λ is close to 1 and δ is small. More precisely, given $\varepsilon, \eta \in (0, 1/2)$, consider the set defined in (4.4) with $p = \frac{1}{2} - \eta$ and $\lambda = 1 - \varepsilon$: $E = E(\varepsilon) = (-\beta_{\frac{1}{2}+\eta-\varepsilon}, -\beta_{\frac{1}{2}+\eta+\varepsilon})$. Assuming that J is twice differentiable, an expansion for ε, η small (recall that J is symmetric about $1/2$) leads to

$$\begin{aligned}
\delta(E) &= P(E) - P\big((-\infty, -\beta_p) \cup (\beta_p, +\infty)\big) \\
&= J\left(\frac{1}{4} + \frac{1}{2}(\eta + \varepsilon)\right) + J\left(\frac{1}{4} + \frac{1}{2}(\eta - \varepsilon)\right) - 2J\left(\frac{1}{4} - \frac{1}{2}\eta\right) \\
&= 2J'\left(\frac{1}{4}\right)\eta + \frac{1}{4}J''\left(\frac{1}{4}\right)\varepsilon^2 + o\big(\eta^2\big) + o\big(\varepsilon^2\big) \approx (1-\lambda)^2 + (1-2p)
\end{aligned}$$

where $a \approx b$ means that $\frac{a}{c} \leq b \leq ca$ for some constant c. Hence, for ε and η small, the deficit $\delta(E)$ is small while the asymmetry $\lambda(E)$ is close to 1. This anomalous phenomenon is coming from the fact that, at $p = 1/2$, extremal sets have two different shapes. Indeed, in the above example, E is closer to the set $(-\alpha_p, \alpha_p)$ than to the isoperimetric set $(-\infty, -\beta_p) \cup (\beta_p, +\infty)$!

(e) Observe that, according to the above example, the pre-factor $(1-\lambda)^2 + (1-2p)$, in (4.9), is necessary. Finally, we stress that the behavior λ^2 in (4.9) and in (4.10) is optimal.

5. Functional forms

As it is well known, isoperimetric inequalities have often equivalent functional forms, see *e.g.* [31,14]. In this section, using the results of the preceding sections, we shall derive some weak embedding properties, and also some weak Cheeger inequalities, in quantitative forms.

We need first to introduce some notations, and in particular the notion of rearrangement of a function with respect to a probability measure. We refer to [5] for more on this topic.

Let Ω be a measurable set and $u : \Omega \subset \mathbb{R} \to \mathbb{R}_+$ be a measurable function. The *level sets* of u are the sets

$$E_h^u := \big\{x \in \Omega : u(x) > h\big\}, \quad h \in \mathbb{R}_+.$$

Then, we define the *distribution function* of u as $\mu_u(h) = \mu(E_h^u)$ for every $h \geq 0$. The mapping $h \mapsto \mu_u(h)$ is non-negative, decreasing and right continuous on $[0, +\infty[$. Moreover μ_u has a jump at h if and only if $\mu_u(\{x \in \Omega : u(x) = h\}) \neq 0$. The *decreasing rearrangement* u^* of u is the generalized inverse of $\mu_u(h)$, namely

$$u^*(s) = \sup\big\{h \geq 0 : \mu_u(h) > s\big\} \quad \text{for } s \in \big(0, \mu(\Omega)\big).$$

Now, let E be a measurable subset of $\mathbb{R}$. We denote by $E^\#$ the complement of the (unique) symmetric interval such that $\mu(E^\#) = \mu(E)$, that is $E^\# = (-\infty, F^{-1}(\frac{\mu(E)}{2})) \cup (-F^{-1}(\frac{\mu(E)}{2}), +\infty)$. Finally, the *rearrangement* of the function u with respect to μ is the function

$$u^\#(x) = u^\#(-x) = u^*\big(2F(x)\big) \quad \text{for } -\infty < x < F^{-1}\left(\frac{\mu(\Omega)}{2}\right)$$

defined on $\Omega^\#$ (symmetric with respect to the origin). The rearrangement is non-increasing on $(-\infty, 0) \cap \Omega^\#$ and non-decreasing on $(0, \infty) \cap \Omega^\#$ and for every $h \geq 0$

$$E_h^{u^\#} = \big(E_h^u\big)^\# = \left(-\infty, F^{-1}\left(\frac{\mu(E_h^{u^\#})}{2}\right)\right) \cup \left(-F^{-1}\left(\frac{\mu(E_h^{u^\#})}{2}\right), +\infty\right).$$

The idea behind such a construction is that the level sets of $u^\#$ are precisely the extremal sets in the isoperimetric inequality related to μ. Hence, in view of Theorem 3.13, when $\mu \in \mathcal{F}$, the definition of the rearrangement above will be useful only for functions u satisfying $\mu(\operatorname{supp} u) \leq \frac{1}{2}$. Indeed, for function with $\mu(\operatorname{supp} u) > \frac{1}{2}$, one would have to consider on one hand sets $E^\#$ that are complement of symmetric intervals (for sets of measure less than $1/2$), and on the other hand symmetric intervals for sets of measure greater than $1/2$, which is impossible: there is no construction of rearrangement compatible with those two shapes. Observe that by construction one can easily check that the rearrangement is a homogeneous mapping. More precisely, if $v = \lambda u$, with $\lambda > 0$, then, $v^\# = \lambda u^\#$.

Finally, recall that m is a μ-median of u if $\mu(\{u \geq m\}) \geq 1/2$ and $\mu(\{u \leq m\}) \geq 1/2$.

We are now in position to state our embedding results.

5.1. Embedding inequality

The following result is a consequence of isoperimetric inequality and can be obtained in a classical way (see *e.g.* [7, Corollary 8.2], [29]).

Proposition 5.1. *Let μ be a probability measure, on the line, satisfying $P(E) \geq I(\mu(E))$ for all Borel set $E \subset \mathbb{R}$, for some isoperimetric profile I. Then, for any non-negative smooth function u on $\mathbb{R}$ with μ-median zero, it holds*

$$\sup_{h \geq 0}\big\{hI\big(\mu(u > h)\big)\big\} = \sup_{0<t<1/2}\big\{u^*(t)I(t)\big\} \leq \int_{\mathbb{R}} |u'| \, d\mu. \tag{5.1}$$

We shall use Theorem 3.13 to give some explicit examples of application in the setting of log-convex probability measures.

Recall that, for any measurable function u (on $\mathbb{R}$) and any $p \geq 1$, the weighted Lorentz pseudo-norm is defined by

$$\|u\|_{\mathbb{L}^{p,\infty}(\mu)} := \sup_{t>0}\big\{t\mu\big(|u| > t\big)^{1/p}\big\}.$$

Now, since the two-sided exponential measure μ_1 satisfies the isoperimetric inequality with isoperimetric profile $I(t) = \min(t, 1-t)$ (see Remark 3.15), Inequality (5.1) implies

$$\|u\|_{L^{1,\infty}(\mu_1)} \leq \int_{\mathbb{R}} |u'| \, d\mu_1$$

for any smooth positive function u with μ_1-median zero.

For the distribution (2.1) with parameter $\alpha > 0$, whose isoperimetric profile is $I(t) = \alpha \min(t, 1-t)^{1+\frac{1}{\alpha}}$ (see Remark 3.15), we have (see also [7])

$$\|u\|_{L^{\frac{\alpha}{\alpha+1},\infty}(m_\alpha)} \leq \frac{1}{\alpha} \int_{\mathbb{R}} |u'| dm_\alpha$$

for any smooth positive function u with m_α-median zero.

Finally the probability measure μ_Φ defined in (2.3) with $\Phi(x) = |x|^\alpha$ and $0 < \alpha < 1$ has an isoperimetric profile comparable to $\min(t, 1-t)(\log \frac{1}{\min(t,1-t)})^{\frac{\alpha-1}{\alpha}}$ (see [13, Proposition 3.22]). Hence, Inequality (5.1) implies

$$\|u\|_{L^{1,\infty}(\log L)^{1-\frac{1}{\alpha}}(\mu_\Phi)} \leq c \int_{\mathbb{R}} |u'| d\mu_\Phi$$

for some constant $c > 0$ and any smooth positive function u on $\mathbb{R}$ with μ_Φ-median zero (we refer to [34] for a definition of the Orlicz–Lorentz spaces $L^{p,\infty}(\log L)^\beta(\mu)$ for $p > 1$ and $\beta \in \mathbb{R}$).

5.2. Weak inequality of Cheeger type

Isoperimetric inequalities imply some weak Cheeger inequalities and for some measure (see [7]) they are equivalent. In this subsection we investigate on the consequences of the isoperimetric inequality and the quantitative isoperimetric inequality in terms of Cheeger type inequalities.

A probability measure μ, on the line, is said to satisfy a *weak Cheeger inequality* if there exists some non-increasing function $\beta : (0, \infty) \to [0, \infty)$ such that for every smooth $u: \mathbb{R} \to \mathbb{R}$ with μ-median zero, it holds

$$\int_{\mathbb{R}} |u| d\mu \leq \beta(s) \int_{\mathbb{R}} |u'| d\mu + s \operatorname{Osc}(u) \quad \forall s > 0,$$

where $\operatorname{Osc}(u) := \sup(u) - \inf(u)$ is the oscillation of u. Since $\int_{\mathbb{R}} |u| d\mu \leq \frac{1}{2} \operatorname{Osc}(u)$, observe that only the values $s \in (0, 1/2]$ are relevant in the above definition. Moreover, without loss of generality we can assume that $\beta(s) > 0$ for all $s \in (0, 1/2)$. Such inequalities were introduced by Bobkov [7], inspired by a notion of weak Poincaré inequality (an L^2 analogue) due to Röckner and Wang [37], and further analysis have been done in [13,24].

The relationship between $\beta(s)$ and the isoperimetric profile is explained in the following proposition (that holds in more general situations).

Proposition 5.2. *(See Bobkov [7].) Let $\mu \in \mathcal{F}$. Then, the following two statements are equivalent:*

(1) *there exists a non-increasing function $\beta: (0, \infty) \to [0, \infty)$ such that such that for all $s > 0$ and all bounded and smooth u with μ-median equal to 0,*

$$\int |u| d\mu \leq \beta(s) \int |u'| d\mu + s \operatorname{Osc}(u),$$

(2) *there exists a function $\widetilde{I}: (0, 1) \to [0, \infty)$ symmetric around $1/2$ such that for all Borel set E with $0 < \mu(E) < 1$,*

$$P(E) \geq \widetilde{I}(\mu(E)),$$

where β and $\widetilde{I}$ are related by the duality relation

$$\beta(s) = \sup_{s \le t \le \frac{1}{2}} \frac{t-s}{\widetilde{I}(t)}, \qquad \widetilde{I}(t) = \sup_{0<s\le t} \frac{t-s}{\beta(s)} \quad \text{for } t \le \frac{1}{2}. \tag{5.2}$$

We notice that it is an open problem to find the extremal functions, if any, in the weak Cheeger inequality above, when μ is a log-convex probability measure, even in simple examples such as the distribution m_α defined in (2.1).

In the next proposition, we extend the latter to quantitative isoperimetric and quantitative weak Cheeger inequalities. Then we may apply this result to log-convex probability measures.

Proposition 5.3. *Let $\mu \in \mathcal{F}$ and $\Psi, \Psi' : \mathbb{R} \to \mathbb{R}$ be two convex functions, non-decreasing on $(0,\infty)$. Then, the following two statements are equivalent:*

(1) *there exists a non-increasing function $\beta : (0,\infty) \to [0,\infty)$ such that for all $s > 0$ and all bounded and smooth u with μ-median equal to 0 and such that $0 \notin \operatorname{supp} u$,*

$$\beta(s)\Psi\biggl(\int_{\mathbb{R}} \bigl||u| - u^{\#}\bigr| d\mu(x)\biggr) + \int |u| d\mu \le \beta(s) \int \bigl|u'\bigr| d\mu + 2s \operatorname{Osc}(u), \tag{5.3}$$

(2) *there exists a function $\widetilde{I} : (0,1) \to [0,\infty)$ symmetric around $1/2$ such that for all Borel set $E \not\ni 0$ with $0 < \mu(E) < 1$,*

$$P(E) \ge \widetilde{I}\bigl(\mu(E)\bigr) + \Psi'\bigl(\lambda(E)\bigr). \tag{5.4}$$

Moreover, (1) $\Rightarrow$ (2) with $\widetilde{I}$ symmetric around $1/2$ and $\widetilde{I}(t) := \sup_{0 \le s \le \frac{t}{2}} \frac{t-2s}{\beta(s)}$ for $t \in (0,1/2)$ and $\Psi' = \Psi$; and (2) $\Rightarrow$ (1) with $\beta(s) := \sup_{s \le t \le \frac{1}{2}} \frac{t-s}{\widetilde{I}(t)}$ and $\Psi(\cdot) := 2\Psi'(\frac{\cdot}{2})$.

Remark 5.4. Observe that there is not a pure equivalence between the two statements, as in Proposition 5.2. Indeed, there is a loss of a factor 2, which is technical (there would be no loss if $\Psi(x) = |x|$ for all x). The restriction $0 \notin \operatorname{supp} u$ is also technical and is necessary in order to apply Theorem 4.7.

Before proving Proposition 5.3 let us apply the result to our setting. Assume that $\mu \in \mathcal{F}$ and that $J(t) \in C^2(0, \frac{1}{2})$, J' is concave on $(0, \frac{1}{2})$ with $J'(0^+) = 0$ and J satisfies the ∇_2-condition with $\varepsilon \in (0,1)$ so that the assumptions of point (ii) of Theorem 4.7 are satisfied. Therefore, thanks to the aforementioned theorem, Eq. (5.4) holds with $\widetilde{I}(x) = 2J(x/2)$ (see Remark 3.15) and $\Psi'(x) = c'x^2$ (with c' a universal constant). Hence by Proposition 5.3, the quantitative weak Cheeger inequality (5.3) holds with the corresponding β and $\Psi(x) = c'x^2/2$ (note that in some explicit examples, such as the Cauchy distribution, β can be computed explicitly [13]).

In order to prove Proposition 5.3 we need the following technical lemmas whose proofs can be found at the end of the section. The first lemma relates the symmetric difference of the level sets of $|u|$ and to the level sets of the positive and negative part of u.

Lemma 5.5. *Let $\mu \in \mathcal{F}$ and $u : \mathbb{R} \to \mathbb{R}$ be a smooth function with μ-median zero. Define $u^+ := \max(u, 0)$ and $u^- := \max(-u, 0)$. Then*

$$\lambda\bigl(E_h^{u^+}\bigr) + \lambda\bigl(E_h^{u^-}\bigr) \ge \lambda\bigl(E_h^{|u|}\bigr) \quad \text{for all } h > 0.$$

The second lemma [15] bounds from above the L^1 distance between two functions in terms of the measure of the symmetric difference of their level sets. Since the proof is short and elementary, we shall give it for completeness.

Lemma 5.6. *(See [15].) Let $\mu \in \mathbb{F}$ and $\Psi: \mathbb{R} \to \mathbb{R}$ be a convex function. Then, for any non-negative functions $u, v: \mathbb{R} \to \mathbb{R}$, bounded by 1, it holds*

$$\Psi\left(\int_{\mathbb{R}} |u - v| d\mu\right) \le \int_0^1 \Psi\big(\mu\big(E_h^u \triangle E_h^v\big)\big) dh.$$

We are now in position to prove Proposition 5.3

Proof of Proposition 5.3. We start with the proof of (1) $\Rightarrow$ (2). Fix a Borel set $E \not\ni 0$. By standard approximation of the indicator function $\mathbb{1}_E$ (see [9]), we get from (5.3) that

$$\beta(s)\Psi\left(\int_{\mathbb{R}} \big|\mathbb{1}_E - (\mathbb{1}_E)^{\#}\big| d\mu\right) + \mu(E) \le \beta(s)P(E) + 2s.$$

Since $(\mathbb{1}_E)^{\#} = \mathbb{1}_{E^{\#}}$, we have $\int_{\mathbb{R}} |\mathbb{1}_E - (\mathbb{1}_E)^{\#}| d\mu = \mu(E \Delta E^{\#}) = \lambda(E)$. Thus, for all $s > 0$,

$$\Psi\big(\lambda(E)\big) + \frac{\mu(E) - 2s}{\beta(s)} \le P(E)$$

which leads to (5.4) thanks to the definition of $\widetilde{I}$.

Now we prove that (2) implies (1). Let u be a smooth function whose support does not contain 0 and such that its median is 0. By approximation, we may assume that u is bounded, and without loss of generality that $\operatorname{Osc}(u) = 1$ (by homogeneity). Now set $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. By the coarea formula and (5.4) we have

$$\int \big|u^{\pm\prime}\big| d\mu = \int_0^1 P\big(E_h^{u^\pm}\big) dh \ge \int_0^1 \widetilde{I}\big(\mu\big(E_h^{u^\pm}\big)\big) dh + \int_0^1 \Psi'\big(\lambda\big(E_h^{u^\pm}\big)\big) dh.$$

Hence, adding the two inequalities, using the convexity of Ψ' and Lemmas 5.5 and 5.6 together with the definition of *beta*, we have

$$\begin{aligned}
\int |u'| d\mu &= \int \big|u^{+\prime}\big| d\mu + \int \big|u^{-\prime}\big| d\mu \\
&\ge \int_0^1 \widetilde{I}\big(\mu\big(E_h^{u^+}\big)\big) + \widetilde{I}\big(\mu\big(E_h^{u^-}\big)\big) dh + \int_0^1 \Psi'\big(\lambda\big(E_h^{u^+}\big)\big) + \Psi'\big(\lambda\big(E_h^{u^-}\big)\big) dh \\
&\ge \int_0^1 \frac{\mu(E_h^{u^+}) + \mu(E_h^{u^-})}{\beta(s)} dh - \frac{2s}{\beta(s)} + 2\int_0^1 \Psi'\left(\frac{\lambda(E_h^{u^+}) + \lambda(E_h^{u^-})}{2}\right) dh \\
&\ge \frac{\int |u| d\mu}{\beta(s)} + 2\int_0^1 \Psi'\left(\frac{\lambda(E_h^{|u|})}{2}\right) dh - \frac{2s}{\beta(s)} \\
&\ge \frac{\int |u| d\mu - 2s}{\beta(s)} + 2\Psi'\left(\frac{\int ||u| - u^{\#}| d\mu}{2}\right) - \frac{2s}{\beta(s)}.
\end{aligned}$$

Note that, in the first line of the above computation, we used that, since u is smooth, the set $\{x \in \mathbb{R} : u'(x) \neq 0 \text{ and } u(x) = 0\}$ is μ-negligible (see [4]). Multiplying by $\beta(s)$ leads to the expected result since $\operatorname{Osc}(u) = 1$. This achieves the proof of the proposition. □

It remains to prove Lemma 5.5 and Lemma 5.6.

Proof of Lemma 5.5. Fix $u: \mathbb{R} \to \mathbb{R}$ with μ-median zero. By the very definition of the asymmetry (and since 0 is median of u), we have $\frac{\lambda(E_h^{u^\pm})}{2} = \mu(E_h^{u^\pm}) - \mu(E_h^{u^\pm} \cap (E_h^{u^\pm})^\#)$ for all $h > 0$. Then, we observe that $E_h^{u^+}$ and $E_h^{u^-}$ are disjoint, $E_h^{u^+} \cup E_h^{u^-} = E_h^{|u|}$ and $(E_h^{u^\pm})^\# \subseteq (E_h^{|u|})^\#$. Hence,

$$\begin{aligned}\frac{\lambda(E_h^{u^+})}{2} + \frac{\lambda(E_h^{u^-})}{2} &\geq \mu\big(E_h^{u^+}\big) + \mu\big(E_h^{u^-}\big) - \big[\mu\big(E_h^{u^+} \cap \big(E_h^{|u|}\big)^\#\big) + \mu\big(E_h^{u^-} \cap \big(E_h^{|u|}\big)^\#\big)\big] \\ &= \mu\big(E_h^{|u|}\big) - \mu\big(E_h^{|u|} \cap \big(E_h^{|u|}\big)^\#\big) = \frac{\lambda(E_h^{|u|})}{2}\end{aligned}$$

which is the expected result. □

Proof of Lemma 5.6. By Jensen's inequality we have

$$\begin{aligned}\int_0^1 \Psi\big(\mu\big(E_h^u \triangle E_h^v\big)\big)\,dh &= \int_0^1 \Psi\bigg(\int_{\mathbb{R}} \big|\chi_{E_h^u}(x) - \chi_{E_h^v}(x)\big|\,d\mu(x)\bigg)\,dh \\ &\geq \Psi\bigg(\int_0^1 \int_{\mathbb{R}} \big|\chi_{E_h^u}(x) - \chi_{E_h^v}(x)\big|\,d\mu(x)\,dh\bigg) \\ &= \Psi\bigg(\int_{\mathbb{R}} \bigg|\int_0^1 \big[\chi_{E_h^u}(x) - \chi_{E_h^v}(x)\big]\,dh\bigg|\,d\mu(x)\bigg)\end{aligned}$$

which leads to the expected result. □

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Appendix A. Proof of Proposition 4.3

Proof of Proposition 4.3. Let E be a set of measure $p \in (0, \frac{1}{2})$ and asymmetry λ. By Lemma 3.1 there exists a countable set H such that $E = \bigcup_{h \in H}(a_h, b_h)$ up to a set of measure zero that we assume, without loss of generality, to be the empty set.

Step 1. Arguing as in the proof of Theorem 3.13 and using the shifting property (of Proposition 3.4), preserving not only the measure of the set but also its asymmetry (see Fig. 4 for an illustration), we obtain (details are left to the reader) that

$$P(E) \geq P(\widetilde{E}), \tag{A.1}$$

where

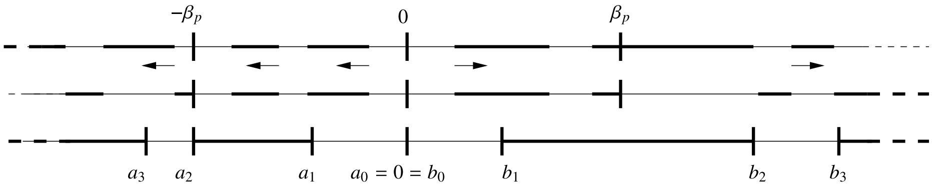


Fig. 4. The reduction from E, when $0 \notin E$, to the set $\widetilde{E}$. The top line represents the set $E = \bigcup_{h \in H}(a_h, b_h)$, the second line is the symmetric difference $E \triangle (-\infty, -\beta_p) \cup (\beta_p, \infty)$. The arrows show how to use the shifting property, preserving the asymmetry. The bottom line is the set $\widetilde{E} = (-\infty, a_3) \cup (a_2, a_1) \cup (a_0, b_0) \cup (b_1, b_2) \cup (b_3, +\infty)$ in the particular case where $a_2 = -\beta_p$ and $a_0 = b_0 = 0$.

$$\widetilde{E} = (-\infty, a_3) \cup (a_2, a_1) \cup (a_0, b_0) \cup (b_1, b_2) \cup (b_3, +\infty)$$

and

$$-\infty \le a_3 \le a_2 \le -\beta_p \le a_1 \le a_0 \le 0 \le b_0 \le b_1 \le \beta_p \le b_2 \le b_3 \le +\infty. \tag{A.2}$$

Depending on the initial set E every inequalities in (A.2) might be either an equality or a strictly inequality. In case of equality, we convey that $(\alpha, \alpha) = \emptyset$ for any $\alpha \in [-\infty, +\infty]$.

Step 2. Examining all[3] the possibilities in (A.2) (equality versus strict inequality) and using again the shifting property, it is possible to further reduce the family of sets with minimal perimeter. Indeed, after reduction (which is an easy (but tedious) exercise left to the reader) one concludes that the minimal perimeter has to be found among only the following 7 sets (see below for an example of such a reduction): E_1, E_2 and $-E_2$, E_3 and $-E_3$, E_4, defined in (4.2)–(4.5), and

$$E_5 = (-\infty, -\beta_{2p-\lambda}) \cup (-\beta_{1-\frac{\lambda}{2}}, \beta_{1-\frac{\lambda}{2}}) \text{ and } -E_5 \quad \text{if } p \le \lambda \le 2p,$$

if either $0 \le \lambda \le p$ and any given $t \in [0, p]$ or $p \le \lambda \le 2p$ and any given $t \in [\lambda - p, 2p - \lambda]$,

$$E_6 = (-\beta_{p-t}, -\beta_{p+\frac{\lambda}{2}}) \cup (\beta_{p+\frac{\lambda}{2}}, \beta_{-p+\lambda+t}) \text{ and } -E_6,$$

and, if either $0 \le \lambda \le p$ and any given $t \in [0, \lambda]$ or $p \le \lambda \le 2p$ and any given $t \in [\lambda - p, p]$,

$$E_7 = (-\beta_{\lambda-t}, -\beta_{p+\lambda}) \cup (\beta_{p-t}, +\infty) \text{ and } -E_7.$$

As an example of the above reduction, let us consider the set $\widetilde{E}$ with $-\infty < a_3 < a_2 < a_1 < a_0 = b_0 < b_1 < b_2 < b_3 < +\infty$ in (A.2). Consider the complementary set $\mathbb{R} \setminus \widetilde{E} = (a_3, a_2) \cup (a_1, b_1) \cup (b_2, b_3)$. Using the shifting property, the perimeter decreases if one moves the interval (a_3, a_2) towards $-\infty$ and the interval (b_2, b_3) towards $+\infty$. Furthermore, the shifting property also guarantees that the perimeter decreases if one symmetrizes the interval (a_1, b_1). All such reductions did not affect neither the measure nor the asymmetry. Finally, considering again the complementary set, we end up with the set E_1 defined in (4.2).

Step 3. At this step, the shifting property becomes useless. To end the proof, one needs to show that E_5, E_6 and E_7 have bigger perimeter than E_1, E_2, E_3, E_4 which will be achieved by using simple analytical computations.

First, we observe that

$$P(E_5) = P\big((-\infty, -\beta_{2p-\lambda})\big) + P\big((-\beta_{1-\frac{\lambda}{2}}, \beta_{1-\frac{\lambda}{2}})\big) > P(E_4)$$

[3] There are 2^{12} of them (but a lot of symmetries!).

since, by the isoperimetric inequality (Theorem 3.13), we are guaranteed that $P((-\infty,-\beta_{2p-\lambda})) > P((-\infty,-\beta_{p-\frac{\lambda}{2}})\cup(\beta_{p-\frac{\lambda}{2}},\infty))$.

We notice that $|\frac{\lambda}{4}-\frac{-p+\lambda+t}{2}| = |\frac{p-t}{2}-\frac{\lambda}{4}|$ so that, by convexity of J (comparing the slopes),

$$\begin{aligned}
&J\left(\frac{p-t}{2}\right)+J\left(\frac{-p+\lambda+t}{2}\right)-2J\left(\frac{\lambda}{4}\right)\\
&=\begin{cases}(J(\frac{p-t}{2})-J(\frac{\lambda}{4}))-(J(\frac{\lambda}{4})-J(\frac{-p+\lambda+t}{2})) & \text{if } t\le p-\frac{\lambda}{2}\\ (J(\frac{-p+\lambda+t}{2})-J(\frac{\lambda}{4}))-(J(\frac{\lambda}{4})-J(\frac{p-t}{2})) & \text{if } t\ge p-\frac{\lambda}{2}\end{cases}\\
&\ge 0
\end{aligned}$$

which, in turn, immediately implies that $P(E_6)\ge P(E_1)$.

Finally, we observe that the map $t\mapsto P(E_7)=J(\frac{\lambda-t}{2})+J(\frac{p+\lambda}{2})+J(\frac{p-t}{2})$ is decreasing, so that (take $t=0$ and $t=\lambda-p$ respectively, which gives the same result)

$$\begin{aligned}
P(E_7)&\ge J\left(\frac{\lambda}{2}\right)+J\left(\frac{p+\lambda}{2}\right)+J\left(\frac{p}{2}\right)\\
&\ge\begin{cases}J(\frac{p+\lambda}{2})+J(\frac{p-\lambda}{2})=P(E_2) & \text{if } 0\le\lambda\le p\\ J(\frac{p+\lambda}{2})+J(\frac{\lambda-p}{2})=P(E_3) & \text{if } p\le\lambda\le 2p.\end{cases}
\end{aligned}$$

This completes the proof of the first part of the proposition (equality cases follows easily by keeping track of equality cases in the various steps in the reduction above).

The second part follows the same lines. One only needs to observe that E_4 need not be considered, since $0\in E_4$, and that $0\notin E_3$ implies $\lambda\le 1-p$ (hence the range $p\le\lambda\le 1-p$). Also, observe that using the shifting lemma never affects the fact that $0\notin E$ during the various step of the reduction above. This achieves the proof. □

Appendix B. Proof of Proposition 4.5

Proof of Proposition 4.5. The proof consists in studying various functions of two variables. Such studies are easy exercises but use various small tricks. For the seek of completeness and in order to help the interesting reader, we give most of the details. We shall use Proposition 4.3 and deal with different cases.

Case $0\le\lambda\le p$. We need to compare the perimeters of E_1, E_2 and E_4. To this aim, consider the function $L_{12}(p,\lambda):=P(E_1)-P(E_2)=2J(\lambda/4)+2J((2p+\lambda)/4)-J((p+\lambda)/2)-J((p-\lambda)/2)$. Considering the partial derivative with respect to λ, and using the sub-linearity of J', one concludes that $\lambda\mapsto L_{12}(p,\lambda)$ is increasing. Since $L_{12}(p,0)=0$, it finally follows that $P(E_2)\le P(E_1)$.

Consider now the function $L_{42}(p,\lambda):=P(E_4)-P(E_2)=2J((2p-\lambda)/4)+2J((2-\lambda)/4)-J((p+\lambda)/2)-J((p-\lambda)/2)$. One has $2\partial_\lambda L_{24}(p,\lambda)=-J'((2p-\lambda)/4)-J'((2-\lambda)/4)-J'((p+\lambda)/2)+J'((p-\lambda)/2)\le 0$ since J' is non-decreasing and $p-\lambda\le p+\lambda$ (here and below we use the shorthand notation $\partial_\lambda,\partial_p$ for the partial derivatives with respect to λ,p). Therefore, $L_{24}(p,\lambda)\ge L_{24}(p,p)=2J(p/4)+2J((2-p)/4)-J(p)$. Taking the derivative with respect to p, one immediately sees that $p\mapsto L_{24}(p)$ is non-increasing so that $L_{24}(p,p)\ge L_{24}(1/4,1/4)=2J(1/8)+2J(3/8)-J(1/2)$. Observing that $t\mapsto J(t)/t^2$ is non-increasing, we have $J(3/8)/(3/8)^2\ge J(1/2)/(1/2)^2$, which leads to

$$2J(3/8)\ge\frac{9}{8}J(1/2)\ge J(1/2). \tag{B.1}$$

Finally we conclude that $L_{24}(p,\lambda)\ge 0$, which guarantees that $P(E_2)\le P(E_4)$. This completes the picture for $0\le\lambda\le p$.

Case $p \le \lambda \le 2p$. We need to compare the perimeters of E_1, E_3 and E_4. We shall first prove that $P(E_1) \ge P(E_3)$ when $p \le \lambda \le 1-p$ and that $P(E_1) \ge P(E_4)$ when $1-p \le \lambda \le 2p$. This will reduce the study to the comparison of the perimeters of E_3 and E_4 only.

Consider first the function $L_{13}(p,\lambda) := P(E_1) - P(E_3) = 2J(\lambda/4) + 2J((2p+\lambda)/4) - J((p+\lambda)/2) - J((\lambda-p)/2)$ with $p \le \lambda \le 1-p$. Since $\lambda \le 2p$, it holds $(\lambda-p)/2 \le \lambda/4$. Hence, $L_{13}(p,\lambda) \ge J(\lambda/4) + 2J((2p+\lambda)/4) - J((p+\lambda)/2)$. Using twice the fact that $t \mapsto J(t)/t^2$ is non-increasing, we have

$$2J\left(\frac{2p+\lambda}{4}\right) \ge \frac{1}{2}\left[\frac{2p+\lambda}{p+\lambda}\right]^2 J\left(\frac{p+\lambda}{2}\right) \quad \text{and} \quad J\left(\frac{\lambda}{4}\right) \ge \left[\frac{\lambda}{2(p+\lambda)}\right]^2 J\left(\frac{p+\lambda}{2}\right),$$

so that, after few rearrangements

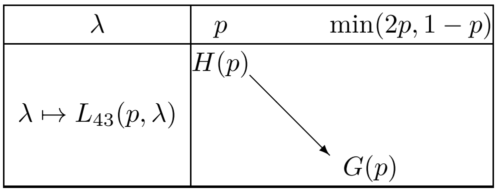
$$J\left(\frac{\lambda}{4}\right) + 2J\left(\frac{2p+\lambda}{4}\right) - J\left(\frac{p+\lambda}{2}\right) \ge \frac{p^2 - \frac{\lambda^2}{4}}{(p+\lambda)^2} J\left(\frac{p+\lambda}{2}\right) \ge 0,$$

since $\lambda \le 2p$. This implies that $P(E_1) \ge P(E_3)$ (when $p \le \lambda \le 1-p$).

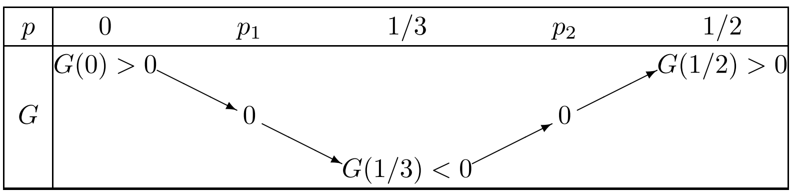
Consider now the function $L_{14}(p,\lambda) =:= P(E_1) - P(E_4) = 2J(\lambda/4) + 2J((2p+\lambda)/4) - 2J((2p-\lambda)/4) - 2J((2-\lambda)/4)$ with $1-p \le \lambda \le 2p$. Since $\lambda \mapsto L_{14}(p,\lambda)$ is non-decreasing, we have $L_{14}(p,\lambda) \ge L_{14}(p,1-p) = 2J((1-p)/4) - 2J((3p-1)/4)$. Last function is non-increasing (in p). Hence, $L_{14}(p,1-p) \ge L_{14}(1/2,1/2) = 0$. This guarantees that, as announced $P(E_1) \ge P(E_4)$ when $1-p \le \lambda \le 2p$.

At this point is remains to compare $P(E_3)$ and $P(E_4)$ when $p \le \lambda \le 2p$, considering the function $L_{43}(p,\lambda) := P(E_4) - P(E_3)$. We will distinguish between two sub-cases.

We start by dealing with $p \le \lambda \le \min(2p, 1-p)$. In that case, $L_{43}(p,\lambda) = 2J((2p-\lambda)/4) + 2J((2-\lambda)/4) - J((p+\lambda)/2) - J((\lambda-p)/2)$ is obviously non-increasing in λ. In order to deduce the sign of L_{43} we need to study the extreme points $H(p) := L_{43}(p,p)$ and $G(p) := L_{43}(p, \min(2p, 1-p))$:



First, we observe that $H(p) := L_{43}(p,p) = 2J(p/4) + 2J((2-p)/4) - J(p)$ is non-increasing (take the derivative) so that $H(p) \ge H(1/2) = 2J(1/8) + 2J(3/8) - J(1/2) > 0$ thanks to (B.1). Then, we notice that $p \mapsto G(p) := L_{43}(p, \min(2p, 1-p))$ is obviously non-increasing on $[0, 1/3]$ and non-decreasing on $[1/3, 1/2]$. Since $G(0) = L_{43}(0,0) = 2J(1/2) > 0$, $G(1/2) = L_{43}(1/2,1/2) = 2J(1/8) + 2J(3/8) - J(1/2) > 0$ thanks to (B.1), and $G(1/3) = L_{43}(1/3,2/3) = 2J(1/3) - J(1/6) - J(1/2) < 0$ (since the slope $[J(1/2) - J(1/3)]/(1/6)$ is larger, by convexity of J, than the slope $[J(1/3) - J(1/6)]/(1/6)$), we end up with the following diagram:



for some $p_1 \in (0, 1/3)$ and some $p_2 \in (1/3, 1/2)$. From this we conclude that $P(E_4) \ge P(E_3)$ when $p \in [0, p_1] \cup [p_2, 1/2]$, and that $P(E_4) - P(E_3)$ changes sign (at a unique point $\lambda_0(p)$) when λ varies and $p \in (p_1, p_2)$ is fixed. This leads to the existence of the function λ_0. This completes the picture for $p \le \lambda \le \min\{2p, 1-p\}$.

Consider finally the range $1-p \le \lambda \le 2p$ (which exists only if $p \in [1/3, 1/2]$). In that case, the function $L_{43}(p, \lambda)$ reads

$$L_{43}(p,\lambda) = 2J\big((2p-\lambda)/4\big) + 2J\big((2-\lambda)/4\big) - J\big(1-(p+\lambda)/2\big) - J\big((\lambda-p)/2\big).$$

Here we used one again the symmetry of J about $1/2$ in order to deal only with variables belonging to $[0, 1/2]$ (observe in particular that $(p+\lambda)/2 \ge 1/2$). The map $p \mapsto L_{43}(p, \lambda)$ is clearly increasing. Hence, we need to study the extremal points $H(\lambda) := L_{43}(\max(\lambda/2, 1-\lambda), \lambda)$ and $G(\lambda) := L_{43}(1/2, \lambda)$:

p	$\max(\lambda/2, 1-\lambda)$	$1/2$
$p \mapsto L_{43}(p,\lambda)$	$H(\lambda)$ $\nearrow$	$G(\lambda)$

Computing G', and using the sub-linearity of J', we conclude that G is decreasing. Since $G(1) = 0$ we are guaranteed that $G(\lambda) > 0$ for any $\lambda \in [1/2, 1)$. On the other hand, note that

$$H(\lambda) = \begin{cases} 2J(\frac{2-3\lambda}{4}) + 2J(\frac{2-\lambda}{4}) - J(\frac{1}{2}) - J(\frac{2\lambda-1}{2}) & \text{if } \frac{1}{2} \le \lambda \le \frac{2}{3} \\ 2J(\frac{2-\lambda}{4}) - J(\frac{4-3\lambda}{4}) - J(\frac{\lambda}{4}) & \text{if } \frac{2}{3} \le \lambda \le 1. \end{cases}$$

Now, in the range $\frac{1}{2} \le \lambda \le \frac{2}{3}$, H this obviously decreasing. While in the range $\frac{2}{3} \le \lambda \le 1$, computing the derivative, and using that $\frac{\lambda}{4} \le \frac{2-\lambda}{4} \le \frac{4-3\lambda}{4}$, we conclude that H is increasing. Then, observe that $H(1/2) = 2J(1/8) + 2J(3/8) - J(1/2) > 0$, by (B.1). Also, $H(2/3) = 2J(1/3) - J(1/2) - J(1/6) = -[J(1/2) - J(1/3)] + [J(1/3) - J(1/6)] < 0$ since the slope $[J(1/2) - J(1/3)]/(1/6)$ is greater, by convexity of J, than the slope $[J(1/3) - J(1/6)]/(1/6)$ and $H(1) = 0$. We end up with the following diagram

λ	$1/2$		λ_1		$2/3$		1
H	$H(1/2) > 0$	$\searrow$	0	$\searrow$	$H(2/3) < 0$	$\nearrow$	0

It follows that $P(E_4) \ge P(E_3)$ when $\lambda \in [1/2, \lambda_1]$, and that $P(E_4) - P(E_3)$ changes sign (at a unique point $p_1(\lambda)$) when p varies and $\lambda \in (\lambda_1, 1)$ is fixed. This leads to the existence of the function p_0 and completes the picture in the range $1-p \le \lambda \le 2p$.

It remains to show that p_0 is $\mathcal{C}^1$, increasing, $p_0(1-p_2) = p_2$ and that $p_0'(1) = 1/2$. That $p_0(1-p_2) = p_2$ follows from the fact that the perimeter is a continuous function of the variables p and λ. The remaining properties follow from the implicit equation $L_{43}(p_0(\lambda), \lambda) = 0$ and the implicit function theorem. This ends the proof. □

Appendix C. Proof of Theorem 4.7

Proof of Theorem 4.7. Fix $p \in [0, 1/2]$ and $\lambda \in [0, 2p]$ and a Borel set E of measure p and asymmetry λ. We start by proving point (i).

By Proposition 4.3, we actually only need to prove that

$$\delta(E_i) \ge c\big[(1-\lambda)^2 + (1-2p)\big]\lambda^2$$

for E_i, $i = 1, 2, 3, 4$, defined in (4.2)–(4.5).

We shall deal with each one of these sets and with the different ranges separately. Also we shall use repeatedly, without any further mention, that $\lambda \in [0,1]$ so that $1 \geq \lambda \geq \lambda^2$ and that $1 \geq \frac{1}{2}[(1-2p)+(1-\lambda)^2]$.

• We start by dealing with the set E_1 and $0 \leq \lambda \leq 2p$. Since J' is non-decreasing, we have

$$\delta(E_1) = P(E_1) - P\big((-\infty,-\beta_p)\cup(\beta_p,+\infty)\big) = 2\left(J\left(\frac{p}{2}+\frac{\lambda}{4}\right)+J\left(\frac{\lambda}{4}\right)-J\left(\frac{p}{2}\right)\right)$$

$$\geq 2\int_{\frac{p}{2}}^{\frac{p}{2}+\frac{\lambda}{4}} J'(t)dt \geq \frac{J'(\frac{p}{2})}{2}\lambda \geq \frac{J'(\frac{p}{2})}{4}\big[(1-2p)+(1-\lambda)^2\big]\lambda^2. \tag{C.1}$$

• Consider now the set E_2 with $0 \leq \lambda \leq p$. One has

$$\delta(E_2) = P(E_2) - P\big((-\infty,-\beta_p)\cup(\beta_p,+\infty)\big) = J\left(\frac{p}{2}+\frac{\lambda}{2}\right)+J\left(\frac{p}{2}-\frac{\lambda}{2}\right)-2J\left(\frac{p}{2}\right)$$

$$= \int_{\frac{p}{2}}^{\frac{p}{2}+\frac{\lambda}{2}}\int_{t-\frac{\lambda}{2}}^{t} J''(u)dudt \geq \int_{\frac{p}{2}+\frac{\lambda}{4}}^{\frac{p}{2}+\frac{\lambda}{2}}\int_{t-\frac{\lambda}{4}}^{t} J''(u)dudt \geq M(p)\left(\frac{\lambda}{4}\right)^2$$

$$\geq \frac{M(p)}{32}\big[(1-2p)+(1-\lambda)^2\big]\lambda^2. \tag{C.2}$$

• We turn to the set E_3 with $p \leq \lambda \leq 2p$. We need to distinguish between two different cases, namely $p \leq \lambda \leq \min(1-p,2p)$ and $\min(1-p,2p) \leq \lambda \leq 2p$ (which holds only if $p \geq 1/3$). For $p \leq \lambda \leq \min(1-p,2p)$, by monotonicity (take $\lambda = p$), we have

$$\delta(E_3) = P(E_3) - P\big((-\infty,-\beta_p)\cup(\beta_p,+\infty)\big) = J\left(\frac{p}{2}+\frac{\lambda}{2}\right)+J\left(\frac{\lambda}{2}-\frac{p}{2}\right)-2J\left(\frac{p}{2}\right)$$

$$\geq J(p) - 2J\left(\frac{p}{2}\right) \tag{C.3}$$

$$\geq \left[J(p)-2J\left(\frac{p}{2}\right)\right]\big[(1-2p)+(1-\lambda)^2\big]\lambda^2. \tag{C.4}$$

When $\min(1-p,2p) \leq \lambda \leq 2p$ (which implies $p \geq 1/3$), it holds $(\lambda+p)/2 \geq 1/2$. Hence, by symmetry of J about $1/2$ (and in order to deal only with increments belonging to $[0,1/2]$), we have $\delta(E_3) = J(1-\frac{p}{2}-\frac{\lambda}{2}) + J(\frac{\lambda}{2}-\frac{p}{2}) - 2J(\frac{p}{2})$. Using that J' is non-decreasing, we have,

$$\delta(E_3) = \int_{\frac{p}{2}}^{1-\frac{\lambda+p}{2}} J'(t)dt - \int_{\frac{\lambda-p}{2}}^{\frac{p}{2}} J'(t)dt$$

$$\geq J'\left(\frac{p}{2}\right)\left(1-p-\frac{\lambda}{2}\right) - J'\left(\frac{p}{2}\right)\left(p-\frac{\lambda}{2}\right) = J'\left(\frac{p}{2}\right)(1-2p) \geq J'\left(\frac{1}{6}\right)(1-2p). \tag{C.5}$$

On the other hand, since $p \mapsto \delta(E_3) = J(1-\frac{p}{2}-\frac{\lambda}{2}) + J(\frac{\lambda}{2}-\frac{p}{2}) - 2J(\frac{p}{2})$ is non-increasing, we get (take $p = 1/2$) by convexity of J on $[0,1/2]$, and using that $J(\frac{\lambda}{2}-\frac{1}{4}) \geq 0$,

$$\delta(E_3) \geq J\left(\frac{3}{4}-\frac{\lambda}{2}\right)+J\left(\frac{\lambda}{2}-\frac{1}{4}\right)-2J\left(\frac{1}{4}\right) \geq J\left(\frac{1}{2}\right)-\frac{2\lambda-1}{4}J'\left(\left(\frac{1}{2}\right)^-\right)-2J\left(\frac{1}{4}\right)$$

Now by Lemma 2.4, $c := J(1/2) - 2J(1/4) > 0$ so that, for $\lambda \leq \frac{1}{2} + \frac{c}{J'(1/2^-)}$[4] we get $\delta(E_3) \geq \frac{c}{2}$, while for $\lambda \geq \frac{1}{2} + \frac{c}{J'(1/2^-)}$ (condition that might be empty), we have

$$\delta(E_3) \geq J\left(\frac{3}{4} - \frac{\lambda}{2}\right) + J\left(\frac{\lambda}{2} - \frac{1}{4}\right) - 2J\left(\frac{1}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}-\frac{\lambda}{2}} \int_{t+\frac{\lambda-1}{2}}^{t} J''(u)dudt$$

$$\geq M\left(\lambda - \frac{1}{2}\right)\left(\frac{1-\lambda}{2}\right)^2 \geq \frac{1}{4}M\left(\frac{c}{J'(1/2^-)}\right)(1-\lambda)^2.$$

Combining these results, we finally get, in the regime $\min(1-p, 2p) \leq \lambda \leq 2p$,

$$\delta(E_3) \geq \frac{1}{8}\min\left(4J'(1/6), 2c, M\left(\frac{c}{J'(1/2)}\right)\right)\left[(1-2p) + (1-\lambda)^2\right]\lambda^2. \tag{C.6}$$

- Finally we deal with E_4. Using that J' is non-decreasing, we have,

$$\begin{aligned}
\delta(E_4) &= P(E_4) - P\big((-\infty, -\beta_p) \cup (\beta_p, +\infty)\big) \\
&= 2J\left(\frac{p}{2} - \frac{\lambda}{4}\right) + 2J\left(\frac{1}{2} - \frac{\lambda}{4}\right) - 2J\left(\frac{p}{2}\right) \geq 2\left(J\left(\frac{1}{2} - \frac{\lambda}{4}\right) - J\left(\frac{p}{2}\right)\right) \\
&= 2\int_{\frac{p}{2}}^{\frac{1}{2}-\frac{\lambda}{4}} J'(t)dt \geq 2J'\left(\frac{p}{2}\right)\left(\frac{1}{2} - \frac{\lambda}{4} - \frac{p}{2}\right) = \frac{J'(\frac{p}{2})}{2}\left[(1-\lambda) + (1-2p)\right] \\
&\geq \frac{J'(\frac{p}{2})}{2}\left[(1-\lambda)^2 + (1-2p)\right]\lambda^2.
\end{aligned} \tag{C.7}$$

The expected result of point (i) follows collecting (C.1), (C.2), (C.6) and (C.7).

Now we turn to the proof of point (ii). By Proposition 4.5, we only need to prove that $\delta(E_i) \geq c'\lambda(E)^2$ for E_i, $i = 1, 2, 3$. As for point (i), we shall deal with each one of these sets in the appropriate ranges given in (4.7) (observe that such ranges may differ from point (i)).

By monotonicity of J', we observe that, for $1 - p \leq \lambda \leq 2p$ (which guarantees that $p \geq 1/3$), (C.1) implies $\delta(E_1) \geq J'(1/6)\lambda^2/2$. On the other hand, back to the computation leading to Eq. (C.2), we have, by monotonicity of J'',

$$\delta(E_2) = \int_{\frac{p}{2}}^{\frac{p}{2}+\frac{\lambda}{2}} \int_{t-\frac{\lambda}{2}}^{t} J''(u)dudt \geq J''\left(\frac{p}{2} + \frac{\lambda}{2}\right) \times \left(\frac{\lambda}{2}\right)^2 \geq \frac{J''(1/2^-)}{4}\lambda^2,$$

where in the last inequality we used that $p + \lambda \leq 1$. Finally, thanks to (C.3), the ∇_2-condition and the fact that $t \mapsto J(t)/t^2$ is non-increasing (a consequence of the assumption J' concave), we have

$$\begin{aligned}
\delta(E_3) &\geq J(p) - 2J(p/2) \geq \varepsilon J(p/2) \geq \varepsilon J(1/2)p^2 \\
&\geq \varepsilon J(1/2)\begin{cases} \frac{\lambda^2}{4} & \text{if } p \leq 1/3 \text{ and } p \leq \lambda \leq 2p \\ \frac{\lambda^2}{9} & \text{if } p \geq 1/3 \text{ and } p \leq \lambda \leq 1-p \end{cases} \geq \frac{\varepsilon J(1/2)}{4}\lambda^2,
\end{aligned}$$

[4] Observe that, by convexity, $c \leq J(1/2) - J(1/4) \leq J'(1/2^-)/4$ so that $\frac{1}{2} + \frac{c}{J'(1/2^-)} \in [1/2, 3/4]$.

where in the last line we used that $\lambda \leq 2/3$ (in the range $p \leq \lambda \leq 1-p$). Collecting the previous computations leads to

$$\min_{i=1,2,3} \delta(E_i) \geq \varepsilon \min\left(\frac{J'(1/6)}{2}, \frac{J''(1/2^-)}{4}, \frac{J(1/2)}{4}\right)\lambda^2.$$

Using that $xJ'(x) \geq J(x)$ (a consequence of the fact that $J(0) = 0$ and that J is convex), and that $t \mapsto J'(t)/t$ is non-increasing (since J' is concave and $J'(0^+) = 0$), we have $\frac{1}{2}J'(1/6) \geq \frac{1}{6}J'(1/2) \geq \frac{1}{3}J(1/2)$. Hence, $\min(\frac{J'(1/6)}{2}, \frac{J''(1/2^-)}{4}, \frac{J(1/2)}{4}) = \min(\frac{J''(1/2^-)}{4}, \frac{J(1/2)}{4})$. Then, since J' is concave and $J'(0^+) = 0$, $xJ''(x) \leq J'(x)$, $x \in (0, 1/2)$. Also, since $t \mapsto J(t)/t^2$ is non-increasing, we have $xJ'(x) \leq 2J(x)$, $x \in (0, 1/2)$. In turn, $J''(x) \leq 2J(x)/x^2$ so that $J(1/2) \geq J''(1/2^-)/8$. As a conclusion, $\min(\frac{J''(1/2^-)}{4}, \frac{J(1/2)}{3}) \geq J''(1/2^-)/32$.

This ends the proof of the theorem. □

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