The East Model: Recent Results and New Progresses[*]

A. Faggionato[1] **, F. Martinelli**[2] **, C. Roberto**[3] **and**
C. Toninelli[4]

[1] Dip. Matematica "G. Castelnuovo", Univ. "La Sapienza". P. le Aldo Moro 2, 00185 Roma, Italy. E-mail: faggiona@mat.uniroma1.it
[2] Dip. Matematica, Univ. Roma Tre, Largo S.L.Murialdo 00146, Roma, Italy. E-mail: martin@mat.uniroma3.it
[3] Modal'X, Univ. Paris Ouest Nanterre, 200 av. République 92000 Nanterre, France. E-mail: cyril.roberto@math.cnrs.fr
[4] L.P.M.A. and CNRS-UMR 7599, Univ. Paris VI-VII 4, Pl. Jussieu 75252 Paris, France. E-mail: cristina.toninelli@upmc.fr

Abstract. The East model is a particular one dimensional interacting particle system in which certain transitions are forbidden according to some constraints depending on the configuration of the system. As such it has received particular attention in the physics literature as a special case of a more general class of systems referred to as *kinetically constrained models*, which play a key role in explaining some features of the dynamics of glasses. In this paper we give an extensive overview of recent rigorous results concerning the equilibrium and non-equilibrium dynamics of the East model together with some new improvements.

Contents

1. Introduction

Facilitated or kinetically constrained spin (particle) models (KCSM) are interacting particle systems which have been introduced in the physics literature [23, 24, 37] to model liquid/glass transition and more generally "glassy dynamics" (see e.g. [34, 45]). A configuration is given by assigning to each vertex x of a (finite or infinite) connected graph $\mathcal{G}$ its occupation variable $\eta(x) \in \{0, 1\}$ which corresponds to an empty or filled site, respectively. The evolution is given by a Markovian stochastic dynamics of Glauber type. Each site with rate one refreshes its occupation variable to a filled or to an empty state with probability $1 - q$ or q respectively provided that the current configuration around it satisfies an a priori specified constraint. For each site x the corresponding constraint does not involve $\eta(x)$, thus detailed balance w.r.t. the Bernoulli$(1 - q)$ product

measure π can be easily verified and the latter is an invariant reversible measure for the process.

One of the most studied KCSM is the East model [37][1]. It is a one-dimensional model ($\mathcal{G} = \mathbb{Z}$ or $\mathcal{G} = \mathbb{Z}_+ = \{0, 1, \dots\}$) and particle creation/anni-hilation at a given site x can occur only if the *East neighbor* of x, namely the vertex $x + 1$, is empty. The model is ergodic for any $q \in (0, 1)$ with a positive spectral gap [2, 13] and it relaxes to the equilibrium reversible measure expo-nentially fast even when started from e.g. any non-trivial product measure [14]. However, due to the fact that the rates can be zero, the East model has specific features quite different from those of more common systems. For example the relaxation time $T_{\mathrm{relax}}(q)$ diverges very fast as $q \downarrow 0$, $T_{\mathrm{relax}} \sim (1/q)^{(1/2)\log_2(1/q)}$ (see [13]), and several coercive inequalities stronger than the Poincaré inequality (e.g. the logarithmic Sobolev inequality) fail (see Section 3.3 for more details).

A key issue, both from the mathematical and the physical point of view, is therefore that of describing accurately the evolution at $q \ll 1$ when the initial distribution is different from the reversible one and for time scales which are large but still much smaller than $T_{\mathrm{relax}}(q)$ when the exponential relaxation to the reversible measure takes over. A typical case, often referred to in the physics literature as a quench from high to low density of vacancies, is to take as starting distribution i.i.d. occupancy variables with density $1/2$. We refer the interested reader to [10, 16, 30, 40, 45] for the relevance of this setting in connection with the study of the liquid/glass transition as well as for details for KCMS different from East model.

As first suggested in the non-rigorous work [46, 47] and recently mathemat-ically established in [26], the non-equilibrium dynamics of the East model as $q \downarrow 0$ is dominated by a metastable type of evolution in a energy landscape with a hierarchical structure. Such metastable dynamics can in turn be very well described by a hierarchical coarsening process [26] for the excess vacan-cies whose long time behavior can be analyzed rigorously. Remarkably such a hierarchical coalescence process (i) has exactly the same general structure of other coalescence processes introduced in the physics literature for very different situations (see e.g. [17–19]) and (ii) the form of its universality classes can be mathematically established and computed [25]. As a consequence one is able to draw almost exact conclusions on the out-of-equilibrium dynamics of the East model [26].

In this paper we mostly try to provide an extensive self-contained review of the existing mathematical theory of the East model in the various regimes. We also provide the analysis of the logarithmic Sobolev inequality and some of its recently introduced modifications, as well as some extension of the main theorems proved in [26] for the low density non-equilibrium dynamics.

---

[1]Quite interestingly the East model plays a key role in certain random walks over the upper triangular matrices with entries in the field $\mathbb{Z}_n$ for n prime [44].

Finally, we stress that some of the results and/or the techniques that we present are valid for more general KCSM (e.g. spectral gap, persistence, log-Sobolev); while others are related to the oriented and/or one-dimensional character of the East Model.

2. The East process: definition and construction

2.1. Notation

Throughout all the paper we will use the notation $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$. The configuration space for the East model is either $\Omega := \{0, 1\}^{\mathbb{Z}}$ or $\Omega_\Lambda = \{0, 1\}^\Lambda$ for some (finite or infinite) subset $\Lambda \subset \mathbb{Z}$. Given a parameter $q \in [0, 1]$, for any $x \in \mathbb{Z}$, π_x denotes a Bernoulli $(1 - q)$ measure, $\pi := \prod_{x \in \mathbb{Z}} \pi_x$ and $\pi_\Lambda := \prod_{x \in \Lambda} \pi_x$ for $\Lambda \subset \mathbb{Z}$. Also, we set $p := 1 - q$.

Elements of Ω will usually be denoted by the Greek letters $\sigma, \eta, \dots$ and $\sigma(x)$ will denote the occupancy variable at the site x: when $\sigma(x) = 1$ we say that site x is occupied or filled (by a particle), while when $\sigma(x) = 0$ we say that there is a vacancy (or no particle) at site x, or also that x is empty. The restriction of a configuration σ to a subset Λ of $\mathbb{Z}$ will be denoted by σ_Λ. Given two sets Λ, V and two configurations σ, σ', $\sigma_\Lambda \sigma'_V$ denotes the configuration equal to σ on Λ and to σ' on V. The set of empty sites (or zeros) of a configuration σ will be denoted by $\mathcal{Z}(\sigma)$.

The mean with respect to π of a function f on Ω is denoted by $\pi(f)$, while its variance is denoted by $\mathrm{Var}(f)$. Similar definitions hold for $\pi_\Lambda(f), \mathrm{Var}_\Lambda(f)$ and f a function on Ω_Λ. If f is a function on Ω we denote by $\pi_\Lambda(f)$ and $\mathrm{Var}_\Lambda(f)$ the mean and the variance of f with respect to the conditional probability $\pi\big(\cdot \mid \{\sigma(y)\}_{y \in \Lambda}\big) = \pi\big(\cdot \mid \sigma_\Lambda\big)$. Namely, $\pi_\Lambda(f)$ is the mean of f with respect to π_Λ computed keeping fixed the variables $\sigma(y)$, $y \notin \Lambda$. Similarly for $\mathrm{Var}_\Lambda(f)$. For simplicity, we set $\pi_x(f) := \pi_{\{x\}}(f)$ and $\mathrm{Var}_x(f) := \mathrm{Var}_{\{x\}}(f)$.

Finally we introduce the entropy functional $\mathrm{Ent}(f) := \pi(f \log(f/\pi(f))$ for any non-negative function f, say in $\mathbb{L}^2(\Omega, \pi)$, and similarly $\mathrm{Ent}_\Lambda(f) = \pi_\Lambda(f \times \log(f/\pi_\Lambda(f))$.

Remark 2.1. In the physical literature, the parameter q, which represents the density of vacancies as will become clear later, is written as

$$q = \frac{e^{-\beta}}{1 + e^{-\beta}}$$

where β is the inverse temperature. In particular, the limit $q \downarrow 0$ corresponds to the zero temperature limit.

2.2. Infinitesimal generator of the East process

The East process can be informally described as follows. Each vertex x waits an independent mean one exponential time and then, provided that the current configuration σ satisfies the constraint $\sigma(x+1) = 0$, the value of $\sigma(x)$ is refreshed according to π_x, i.e. it is set equal to 1 with probability $p = 1 - q$ and to 0 with probability q. The process can be rigorously constructed in a standard way, see [38]. Formally, it is univocally specified by the action of its infinitesimal Markov generator $\mathcal{L}$ on local (i.e. depending on finitely many variables) functions $f \colon \Omega \mapsto \mathbb{R}$, given by

$$\mathcal{L}f(\sigma) = \sum_{x \in \mathbb{Z}} c_x(\sigma)\big[\pi_x(f) - f(\sigma)\big] \tag{2.1}$$

$$= \sum_{x \in \mathbb{Z}} c_x(\sigma)\big[(1 - \sigma(x))p + \sigma(x)q\big]\big(f(\sigma^x) - f(\sigma)\big)$$

where $c_x(\sigma) := 1 - \sigma(x+1)$ encodes the constraint, and σ^x is obtained from σ by flipping its value at x, i.e.

$$\sigma^x(y) = \begin{cases} \sigma(y), & \text{if } y \neq x, \\ 1 - \sigma(x), & \text{if } y = x. \end{cases}$$

The domain of $\mathcal{L}$ is denoted by $\mathrm{Dom}(\mathcal{L})$. When the initial distribution at time $t = 0$ is Q, the law and expectation of the process on the Skorohod space $D([0, \infty), \Omega)$ will be denoted by $\mathbb{P}_Q$ and $\mathbb{E}_Q$ respectively. If $Q = \delta_\sigma$ we write simply $\mathbb{P}_\sigma$. The process at time t will be denoted by σ_t.

The East process can also be defined on intervals $\Lambda = [a, b] \subset \mathbb{Z}$ or half-lines $\Lambda = (-\infty, b] \subset \mathbb{Z}$ provided that a suitable *zero* boundary condition is specified at the site $b + 1$. More precisely one defines the generator $\mathcal{L}_\Lambda$ (called finite volume generator if Λ is finite), acting on any local function $f \colon \Omega_\Lambda \to \mathbb{R}$ as

$$\mathcal{L}_\Lambda f(\sigma) = \sum_{x \in \Lambda} c_x(\sigma)[\pi_x(f) - f(\sigma)] + [\pi_b(f) - f(\sigma)]$$

$$\equiv \sum_{x \in \Lambda} c_x^\Lambda(\sigma)[\pi_x(f) - f(\sigma)],$$

$$\text{where} \quad c_x^\Lambda(\sigma) = \begin{cases} 1 - \sigma(x+1) & \text{for } x \in \Lambda \setminus \{b\}, \\ 1, & \text{if } x = b. \end{cases} \tag{2.2}$$

In particular there is no constraint at site b as a *frozen zero* lies at site $b + 1$. One can define the East process also on half-lines $\Lambda = [b, \infty)$, in this case the generator is given by

$$\mathcal{L}_\Lambda f(\sigma) = \sum_{x \in [b, \infty)} c_x(\sigma)[\pi_x(f) - f(\sigma)].$$

Due to the "East" character of the constraint, taking Λ as above (i.e. $\Lambda = [a, b]$, $(-\infty, b]$, $[b, \infty)$), for any initial condition $\eta \in \Omega_\Lambda$ the evolution on the interval Λ coincides with that of the East process on $\mathbb{Z}$ (restricted to Λ) starting from the configuration

$$\tilde{\eta}(x) := \begin{cases} \eta(x), & \text{if } x \in \Lambda, \\ 0, & \text{if } x = b + 1, \quad b = \max \Lambda, \\ 1 & \text{otherwise.} \end{cases} \tag{2.3}$$

We will use the self-explanatory notation $\mathbb{P}_Q^\Lambda$ (or $\mathbb{P}_\sigma^\Lambda$) for the law of the process starting from the law Q (from σ) and σ_t^Λ for the process at time t.

Note that, by construction, the East process on $\mathbb{Z}$ (respectively on Λ) is reversible with respect to π (respectively π_Λ). Analytically this is equivalent to saying that $\mathcal{L}$ (respectively $\mathcal{L}_\Lambda$) is a self-adjoint operator in $\mathbb{L}^2(\pi)$ (respectively $\mathbb{L}^2(\pi_\Lambda)$). Moreover, for any $f, g \in \text{Dom}(\mathcal{L})$ (in particular, for any local functions f, g), the Dirichlet form associated to the generator $\mathcal{L}$ is given by

$$\mathcal{D}(f, g) := \frac{1}{2} \sum_{x \in \mathbb{Z}} \sum_{\sigma \in \Omega} \pi(\sigma) c_x(\sigma) \big[(1 - \sigma(x))p + \sigma(x)q\big]$$
$$\times (f(\sigma^x) - f(\sigma))(g(\sigma^x) - g(\sigma)).$$

Similarly the Dirichlet form associated to the generator $\mathcal{L}_\Lambda$ is given by

$$\mathcal{D}_\Lambda(f, g) := \frac{1}{2} \sum_{x \in \Lambda} \sum_{\sigma \in \Omega_\Lambda} \pi_\Lambda(\sigma) c_x^\Lambda(\sigma) \big[(1 - \sigma(x))p + \sigma(x)q\big]$$
$$\times (f(\sigma^x) - f(\sigma))(g(\sigma^x) - g(\sigma)).$$

It is simple to check that

$$\mathcal{D}(f) := \mathcal{D}(f, f) = \sum_{x \in \mathbb{Z}} \pi\big(c_x \, \text{Var}_x(f)\big), \quad f \in \text{Dom}(\mathcal{L}), \tag{2.4}$$

$$\mathcal{D}_\Lambda(f) := \mathcal{D}_\Lambda(f, f) = \sum_{x \in \Lambda} \pi_\Lambda\big(c_x^\Lambda \, \text{Var}_x(f)\big), \quad f \in \text{Dom}(\mathcal{L}_\Lambda). \tag{2.5}$$

In what follows, when considering a local function f on Ω we denote by $\mathcal{D}_\Lambda(f)$ the Dirichlet form of f with respect to $\mathcal{L}_\Lambda$ and π_Λ computed holding fixed the variables $\{\sigma(y)\}_{y \notin \Lambda}$. In particular, (2.5) still holds due to our definition of $\pi_\Lambda(\cdot)$ and $\text{Var}_x(\cdot)$ for functions defined on Ω.

Finally we introduce the associated Markov semigroup $P_t = \exp\{t\mathcal{L}\}$ which satisfies $P_t f(\sigma) = \mathbb{E}_\sigma(f(\sigma_t))$ and similarly $P_t^\Lambda = \exp\{t\mathcal{L}_\Lambda\}$.

2.3. Graphical construction

In this section we briefly recall a standard graphical construction which allows to define on the same probability space the finite volume East process

for *all* initial conditions. Using a standard percolation argument, see [22, 39], together with the fact that the constraints c_x are uniformly bounded and of finite range, it is not difficult to see that the graphical construction can be extended to any infinite volume.

Given a finite interval $\Lambda \subset \mathbb{Z}$ we associate to each $x \in \Lambda$ a Poisson process of parameter one and, independently, a family of independent Bernoulli$(1-q)$ random variables $\{s_{x,k} : k \in \mathbb{N}\}$ (coin tosses). The occurrences of the Poisson process associated to x will be denoted by $\{t_{x,k} : k \in \mathbb{N}\}$. We assume independence as x varies in Λ. Notice that with probability one all the occurrences $\{t_{x,k}\}_{k \in \mathbb{N}, x \in \Lambda}$ are different. This defines the probability space. The corresponding probability measure will be denoted by $\mathbb{P}$. Given an initial configuration $\eta \in \Omega$ we construct a Markov process $(\sigma_t^{\Lambda, \eta})_{t \geq 0}$ on the above probability space satisfying $\sigma_{t=0}^{\Lambda, \eta} = \eta$ according to the following rules. At each time $t = t_{x,n}$ the site x queries the state of its own constraint c_x^Λ. If the constraint is satisfied, i.e. if $\sigma_{t_-}^{\Lambda, \eta}(x+1) = 0$, then $t_{x,n}$ will be called a *legal ring* and at time t the configuration resets its value at site x to the value of the corresponding Bernoulli variable $s_{x,n}$. We stress here that the rings and coin tosses at x for $s \leq t$ have no influence whatsoever on the evolution of the configuration at the sites which enter in its constraint (here $x+1$) and thus they have no influence of whether a ring at x for $s > t$ is legal or not. It is classical to see that the above construction actually gives a continuous time Markov chain with generator (2.2).

A simple but important consequence of the graphical construction is the following one. Assume that the zeros of the starting configuration σ inside Λ are labeled in increasing order as $x_0, x_1, \ldots, x_n$ and define τ as the first time at which one the x_i's is killed, i.e. the occupation variable there flips to one. Then, up to time τ the East dynamics factorizes over the East process in each interval $[x_i, x_{i+1})$.

3. Main results

In this section we collect the most relevant rigorous results on the East model. The main references are [2, 9, 13, 14, 25, 26], while two results are new: the analysis of the α-log-Sobolev inequalities (see Section 3.3) and an extension of the results of [26] (see Section 3.5).

3.1. Spectral gap

The finite volume East process is trivially ergodic because of the frozen zero boundary condition (see (2.2)). The infinite volume process in $\mathbb{Z}$ is also ergodic in the sense that 0 is a simple eigenvalue of the generator $\mathcal{L}$, as proved in [13]. This last property implies that the process converges to equilibrium, in $\mathbb{L}^2(\pi)$. More precisely (see e.g. [38, Theorem 4.13, Chapter IV]), the following classical equivalence holds.

Theorem 3.1. *The following properties are equivalent:*

(a) $\lim_{t\to\infty} \mathbb{E}_\sigma(f(\sigma_t)) = \pi(f)$ *in* $\mathbb{L}^2(\pi)$ *for all* $f \in \mathbb{L}^2(\pi)$;

(b) 0 *is a simple eigenvalue for* $\mathcal{L}$.

The next natural question that arises is how fast the process converges to equilibrium. The classical tool to answer such a question is a spectral gap estimate. We recall that the spectral gap (or inverse of the relaxation time) of the generator $\mathcal{L}$ is defined as

$$\mathrm{gap}(\mathcal{L}) := \inf_{\substack{f\in\mathrm{Dom}(\mathcal{L})\\f\neq\mathrm{const}}} \frac{\mathcal{D}(f)}{\mathrm{Var}(f)} = \inf_{\substack{f\text{ local}\\f\neq\mathrm{const}}} \frac{\mathcal{D}(f)}{\mathrm{Var}(f)}. \tag{3.1}$$

Similarly one defines the spectral gap, $\mathrm{gap}(\mathcal{L}_\Lambda)$, of the generator $\mathcal{L}_\Lambda$ for $\Lambda \subset \mathbb{Z}$. It is well-known (see e.g. [1, Chapter 2]) that $\mathrm{gap}(\mathcal{L}) \geq \gamma$ for some $\gamma > 0$ is equivalent to the following exponential decay of the semigroup:

$$\mathrm{Var}(P_t f) = \int \left[\mathbb{E}_\sigma(f(\sigma_t)) - \pi(f)\right]^2 d\pi(\sigma) \leq \mathrm{Var}(f)e^{-2\gamma t} \quad \forall t > 0, \ \forall f \in \mathrm{Dom}(\mathcal{L}).$$

The next result asserts that $\mathrm{gap}(\mathcal{L}) > 0$ so that the process indeed converges to equilibrium exponentially fast, in $\mathbb{L}^2(\pi)$. Moreover one can compute the precise asymptotic of $\mathrm{gap}(\mathcal{L})$ in the limit $q \downarrow 0$.

Theorem 3.2 ([2, 13, 14]). *The following holds:*

(i) *The generator* $\mathcal{L}$ *has a positive spectral gap, i.e.* $\mathrm{gap}(\mathcal{L}) > 0$.

(ii) *The asymptotic of* $\mathrm{gap}(\mathcal{L})$ *for* $q \downarrow 0$ *is given by*

$$\lim_{q\downarrow 0} \log(\mathrm{gap}(\mathcal{L})^{-1})/ \left(\log(1/q)\right)^2 = (2\log 2)^{-1}.$$

(iii) *For any interval* $\Lambda \subset \mathbb{Z}$, *the spectral gap of the finite volume generator* $\mathcal{L}_\Lambda$ *is not smaller than* $\mathrm{gap}(\mathcal{L})$, *i.e.* $\mathrm{gap}(\mathcal{L}_\Lambda) \geq \mathrm{gap}(\mathcal{L})$.

Remark 3.1. Points (i) and (iii) have been proven for the first time by Aldous and Diaconis in [2]. These authors also showed the correct upper bound in (ii) together with a lower bound that is off by a factor $1/2$. This wrong factor $1/2$ also appeared in the conjectured behavior of the relaxation time suggested in the physics literature [46, 47] and based on simple energy barriers considerations. The discrepancy with the correct asymptotic as given in (ii) is mainly due to neglecting an important contribution coming from the entropy (i.e. number of ways to overcome the energy barrier). The matching lower bound was proven in [13] by a completely novel approach while an alternative (and somehow very

natural) proof of the upper bound can also be found in [14]. The necessary techniques will be developed in Section 5 where the reader will find the complete proof of Theorem 3.2.

The techniques developed in [13] to prove the positivity of the spectral gap (Item (i)) are actually valid for a wide class of KCSM (not necessarily one-dimensional).

3.2. Persistence function

We now consider the persistence function $F(t)$ which represents the probability for the equilibrium process that the occupation variable at the origin does not change before time t. More precisely (see e.g. [35, 46]) the persistence function is defined by

$$F(t) := \int d\pi(\eta) \, \mathbb{P}_\eta\big(\sigma_0(s) = \eta_0, \ \forall s \leq t\big). \tag{3.2}$$

In [13], using a Feynman–Kac formula approach, it is proved for general KCMS that, if the spectral gap is positive, then $F(t)$ decays exponentially fast as predicted in the physics literature.

Theorem 3.3 ([13]). *It holds*

$$F(t) \leq 2 \exp\Big\{ -\frac{\mathrm{gap}(\mathcal{L}) \min(p, q)}{4} t \Big\} \quad \forall t \geq 0.$$

The proof of the latter is given in Section 6.

3.3. Log-Sobolev constant

The next step in understanding the long-time behavior of the East model is the study of the log-Sobolev constant. This coercive constant is usually used to prove exponential decay in the sup-norm (and therefore in a stronger sense compared to $\mathbb{L}^2(\pi)$, see [36, 41]), by means of the celebrated hypercontractivity property. Unfortunately, for the East Model the log-Sobolev constant in infinite volume does not exist.

In fact, a whole family of Sobolev type inequalities does not hold. This family is called α-log-Sobolev inequalities, $\alpha \in [0, 2]$ being a parameter. In a finite interval Λ they are defined as follows.

Given $\alpha \in (0, 2] \setminus \{1\}$, one says that π_Λ satisfies the α-log-Sobolev inequality if there exists some constant $C_\alpha(\Lambda) \in (0, \infty)$ such that, for any $f : \Omega_\Lambda \to \mathbb{R}$, it holds

$$\mathrm{Ent}_\Lambda(f) \leq \frac{\alpha\alpha' C_\alpha(\Lambda)}{4} \mathcal{D}_\Lambda(f^{1/\alpha}, f^{1/\alpha'}) \tag{3.3}$$

where α' is the dual exponent of α, i.e. such that $1/\alpha + 1/\alpha' = 1$. Observe that, since α may belong to $(0, 1)$, α' may be negative. However, due to the

multiplicative factor α', the right-hand side is always non-negative. The α-log-Sobolev inequalities with $\alpha = 1$ or $\alpha = 0$ are defined by a limiting procedure.

Such a family has been introduced in [43] as an interpolating family from the log-Sobolev inequality to the Poincaré inequality. Indeed, for $\alpha = 2$ inequality (3.3) reduces to the standard log-Sobolev inequality of Gross [33]. Also, in [43, Section 4], it is proved that the 0-log-Sobolev inequality, with constant C_0, is precisely equivalent to the standard Poincaré inequality with constant $C_0/2$. Moreover, the limiting case $\alpha = 1$ is equivalent to the following inequality independently studied in the literature:

$$\mathrm{Ent}_\Lambda(f) \leq \frac{C_1(\Lambda)}{4} \mathcal{D}_\Lambda(f, \log f). \tag{3.4}$$

The latter[2] has been introduced in [6] to study the concentration phenomenon of birth and death processes on the integer line. It is known [21] that (3.4) is actually equivalent to the following exponential decay to equilibrium of the semi-group, in the entropy sense:

$$\mathrm{Ent}_\Lambda(P_t f) \leq \exp\left\{-\frac{4t}{C_1(\Lambda)}\right\} \mathrm{Ent}_\Lambda(f) \quad \forall t \geq 0.$$

Hence a control on the constants $C_\alpha(\Lambda)$ may reveal to be crucial in the study of the long-time behavior of the dynamics, specially for $\alpha = 1, 2$.

In the next theorem, we prove that, for any $\alpha \in (0, 2]$ the constant $C_\alpha(\Lambda) = \Omega(|\Lambda|)$ compares to $|\Lambda|$. Thus, in order to get exponential decay to equilibrium either in the sup-norm or in the entropy sense, one has to use alternative strategies. One of them will be developed in the next section. Also, Theorem 3.4 below answers partially to a question asked to us by Krzysztof Oleszkiewicz (see [43, Section 12]), namely about the existence of an example for which the Poincaré Inequality holds while none of the α-log-Sobolev inequalities, $\alpha \in (0, 2]$, hold.

Theorem 3.4. *Fix $\alpha \in (0, 2]$ and a finite interval Λ of $\mathbb{Z}$. Let $C_\alpha(\Lambda)$ be the best possible (i.e. the smallest) constant in inequality (3.3). Then, there exists a constant c (that may depend on q and α) such that*

$$\frac{1}{c}|\Lambda| \leq C_\alpha(\Lambda) \leq c|\Lambda|.$$

The proof of Theorem 3.4 can be found in Section 7. As a conclusion it is natural to ask whether one can find the precise asymptotic behavior, as $q \downarrow 0$, of the log-Sobolev constant, and more generally of any α-log-Sobolev inequality.

---

[2]Inequality (3.4) is sometimes called "modified logarithmic Sobolev inequality" [6, 8, 31, 32] or "entropy inequality" [12, 15, 20]. Recently a yet new name, 1-log-*Sobolev inequality*, has been introduced always for the same object.

3.4. Out of equilibrium I: long time behavior

In this section we address the following questions: does the law of the process at time t converge to the reversible measure π as $t \to \infty$ if it starts from some non-equilibrium measure $Q \neq \pi$? And if it converges, how fast?

As already explained in the previous section, one usually answers such questions studying the log-Sobolev constant and using the so-called hypercontractivity property of the semigroup $\exp\{t\mathcal{L}\}$. Unfortunately, the log-Sobolev constant of a segment of size L compares to L as stated in Theorem 3.4 (and in particular is not uniformly bounded in the size of the system) so that the (now) usual Holley–Stroock strategy does not apply.

Taking advantage of the oriented character of the East process, one can anyway prove the following result.

Theorem 3.5 ([14]). *Fix $\alpha \in (0,1) \setminus \{p\}$ and assume that the initial distribution Q is a Bernoulli (α) measure. Then for any local function f,*

$$\int dQ(\sigma) \left| \mathbb{E}_\sigma(f(\sigma_t)) - \pi(f) \right| \leq C_f e^{-mt}$$

where $C_f = \|f\|_\infty (p \wedge q)^{-|\mathrm{supp}(f)|}/|q - \alpha|$ (with $|\mathrm{supp}(f)|$ the cardinality of the support of f), and

$$m = \frac{1}{2} \mathrm{gap}(\mathcal{L}) \min\left(1, \frac{\log(1/\alpha)}{\log(\alpha/p \wedge q)}\right).$$

The above result, proved in Section 8, shows that relaxation to equilibrium is indeed taking place at an exponential rate on a time scale $T_{\mathrm{relax}} = \mathrm{gap}(\mathcal{L})^{-1}$ which, for small values of q, is very large and of the order of $\exp\{c \log(1/q)^2\}$ with $c = (2 \log 2)^{-1}$.

Remark 3.2. Although the above result is quite natural it should be noted that it cannot hold for *any* initial law Q (as one could naively expect). Consider for example starting the East dynamics from i.i.d. on the negative part of $\mathbb{Z}$ and from identically equal to one on the positive part. Then clearly the positive part of $\mathbb{Z}$ will never relax to equilibrium just because there are no vacancies around! We refer to [14] for a complete classification of the allowed initial distributions.

In higher dimensional models like the North-East model the non-equilibrium dynamics should exhibit an even richer structure because of the possible presence of a critical density above which the spectral gap becomes zero, infinite blocked configurations appear etc. We refer the interested reader to the introduction of [14] for a quick review.

Finally we observe that the oriented character of the East Model is essential in the proof of Theorem 3.5. However the asymptotic convergence to the reversible measure should hold in the ergodic regime for more general KCSM. A particular step in this direction can be found in [5].

3.5. Out of equilibrium II: plateau behavior, aging and scaling limits

In this section, we give a set of results (Theorem 3.6 and 3.7) which details the non-equilibrium behavior of the East process for small values of q (small temperature) and, in contrast with the previous section, for time scales much smaller that $T_{\text{relax}} = \text{gap}(\mathcal{L})^{-1}$. The proofs of both theorems are quite involved. Hence they will not be given in full detail. We mention that one of the main ingredient is an approximation of the East model by means of a suitable hierarchical coalescence process introduced in the physics literature [47] and rigorously studied in [25] (see also [27] for extensions). The definition of this coalescence process and the approximation result will not be given here but can be found in [26].

Definition 3.1. Given $\varepsilon, q \in (0, 1)$, we set

$$t_0 := 1; \qquad t_0^- := 0; \qquad t_0^+ = \left(\frac{1}{q}\right)^{\varepsilon};$$

$$t_n := \left(\frac{1}{q}\right)^n; \qquad t_n^- := t_n^{1-\varepsilon}; \qquad t_n^+ = t_n^{1+\varepsilon} \quad \forall n \geq 1. \qquad (3.5)$$

The time interval $[t_n^-, t_n^+]$ and $[t_n^+, t_{n+1}^-]$ will be called respectively the *nth-active period* and the *nth-stalling period*.

In the next theorem we deal with the persistence, the vacancy density and the two-time autocorrelations during stalling periods and prove plateau and aging behavior.

Given a configuration σ we denote by $\{x_k\} = \{x_k(\sigma)\}$ the position of the empty sites of σ, with the rules that $x_0 \leq 0 < x_1$ and $x_k < x_{k+1}$ for all integers k. Then, given a probability measure μ on $\mathbb{N}$, we write $Q = \text{Ren}(\mu \mid 0)$ if, under Q, the first zero x_0 is located at the origin 0, the random variables $\{x_k - x_{k-1}\}_{k=1}^{\infty}$ form a sequence of i.i.d. random variables with common law μ, and there is no other empty site on the left of the origin.

Theorem 3.6. (Persistence, vacancy density and two-time autocorrelations.) *Fix a probability measure μ on $\mathbb{N}$, $d := \inf\{a : \mu(a) > 0\}$ and let n_d be the smallest integer n such that $d \in [2^{n-1} + 1, 2^n]$. Assume that the initial distribution Q is a renewal measure $Q = \text{Ren}(\mu \mid 0)$ and either one of the following holds:*

a) *the measure μ has finite mean;*

b) *the measure μ belongs to the domain of attraction of a α-stable law or, more generally, $\mu((x, +\infty)) = x^{-\alpha}L(x)$ where $L(x)$ is a slowly varying function at $+\infty$, $\alpha \in [0, 1]$[3].*

---

[3]A function L is said to be slowly varying at infinity, if, for all $c > 0$, $\lim\limits_{x \to \infty} L(cx)/L(x) = 1$.

Then, if $o(1)$ denotes an error term depending only on n and tending to zero as both tend to infinity, for any $n \geq n_d$

(i)

$$\lim_{q \downarrow 0} \sup_{t \in [t_n^+, t_{n+1}^-]} \left| \mathbb{P}_Q(\sigma_t(0) = 0) - \left(\frac{1}{2^n + 1} \right)^{c_0(1+o(1))} \right| = 0, \quad (3.6)$$

$$\lim_{q \downarrow 0} \sup_{t \in [t_n^+, t_{n+1}^-]} \left| \mathbb{P}_Q(\sigma_s(0) = 0 \ \forall s \leq t) - \left(\frac{1}{2^n + 1} \right)^{c_0(1+o(1))} \right| = 0, \quad (3.7)$$

where $c_0 = 1$ in case (a) and $c_0 = \alpha$ in case (b).

(ii) *Let $t, s : [0, 1/2] \to [0, \infty)$ with $t(q) \geq s(q)$ for all $q \in [0, 1/2]$. Then*

$$\overline{\lim_{q \downarrow 0}} \, \mathbb{P}_Q\big(\sigma_{t(q)}(0) = 0\big) \leq \overline{\lim_{q \downarrow 0}} \, \mathbb{P}_Q\big(\sigma_{s(q)}(0) = 0\big).$$

The same bound holds with $\underline{\lim}_{q \downarrow 0}$ instead of $\overline{\lim}_{q \downarrow 0}$.

(iii) *For $x \in \mathbb{Z}_+$ let $C_Q(s, t, x) = \mathrm{Cov}_Q(\sigma_t(x); \sigma_s(x))$ be the two-time autocorrelation function. Then, for any $n, m \geq n_d$*

$$\lim_{q \downarrow 0} \sup_{t \in [t_n^+, t_{n+1}^-], s \in [t_m^+, t_{m+1}^-]} \left| C_Q(s, t, x) - \rho_x \left(\frac{1}{2^n + 1} \right)^{c_0(1+o(1))} \right.$$
$$\left. \times \left(1 - \rho_x \left(\frac{1}{2^m + 1} \right)^{c_0(1+o(1))} \right) \right| = 0$$

where $\rho_x = Q(\sigma(x) = 0)$.

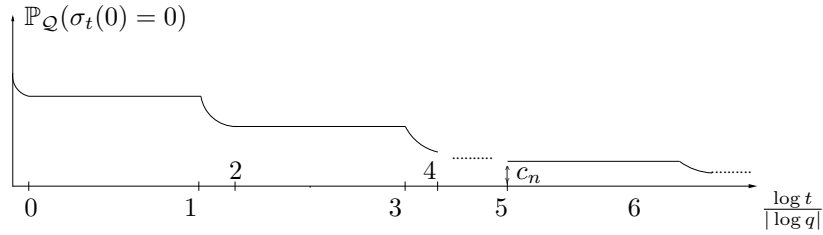The picture that emerges from points (i) and (ii) is depicted in Figure 1.

Figure 1. Plateau behavior in the limit $q \to 0$ (with, for simplicity, $d = 1$), where we set $c_n := (1/(2^n + 1))^{c_0(1+o(1))}$ with c_0 defined in Theorem 3.6 and $o(1)$ going to zero as $n \to \infty$.

Remark 3.3. Theorem 3.6 is stated in [26] in a less general form. Indeed, in [26], μ must satisfy the following technical condition: for any $k \in \mathbb{N}$, $\mu\left([k, \infty)\right) > 0$. In order to remove this assumption we will need a new technical fact described by Lemma 9.1 in Section 9. Having such a lemma, it is simple to adapt the proof of [26]. On the other hand, Theorem 3.6 holds now only starting from scale n_d simply because, on any smaller scale, nothing interesting happens since the filled sites are essentially frozen.

Finally, we observe that, for small values of q, the two-time autocorrelation function $C_Q(s, t, x)$ depends in a non trivial way on s, t and not just on their difference $t - s$ (see Point (iii)). This explains the word "aging". Clearly, for times much larger than the relaxation time $\mathrm{gap}^{-1}(\mathcal{L})$, the time auto-correlation will be very close to that of the equilibrium process which in turn, by reversibility, depends only on $t - s$.

The next theorem describes the statistics of the interval (domain) between two consecutive zeros in a stalling period. In order to state it let, for any $c_0 \in (0, 1]$, $\tilde{X}_{c_0}^{(\infty)} \geq 1$ be a random variable with Laplace transform given by

$$\mathbb{E}\left(\exp\{-s\tilde{X}_{c_0}^{(\infty)}\}\right) = 1 - \exp\left\{-c_0 \int_1^\infty \frac{e^{-sx}}{x}\, dx\right\} = 1 - \exp\left\{-c_0\, \mathrm{Ei}(s)\right\}. \quad (3.8)$$

The corresponding probability density is of the form $p_{c_0}(x)\mathbb{1}_{x \geq 1}$ (see [25]) where p_{c_0} is the continuous function on $[1, \infty)$ given by

$$p_{c_0}(x) = \sum_{k=1}^\infty \frac{(-1)^{k+1} c_0^k}{k!} \rho_k(x)\, \mathbb{1}_{x \geq k}, \quad (3.9)$$

where $\rho_1(x) = 1/x$ and

$$\rho_{k+1}(x) = \int_1^\infty dx_1 \cdots \int_1^\infty dx_k\, \frac{1}{x - \sum_{i=1}^k x_i} \prod_{j=1}^k \frac{1}{x_j}, \quad k \geq 1. \quad (3.10)$$

Let also $\tilde{Y}_{c_0}^{(\infty)}$ be a non-negative random variable with Laplace transform given by

$$\mathbb{E}\left(\exp\{-s\tilde{Y}_{c_0}^{(\infty)}\}\right) := 1 - \exp\left\{-c_0 \int_0^1 \frac{1 - e^{-sx}}{x}\, dx\right\}. \quad (3.11)$$

Starting from an initial law $Q = \mathrm{Ren}(\mu \mid 0)$ denote by $x_0(t) = x_0(\sigma_t)$ the position of the first zero at time t, and by $x_1(t) = x_1(\sigma_t)$ the position of the second zero.

Theorem 3.7. (Limiting behavior of the domain length and of the first zero.) *Under the same assumptions of Theorem 3.6, let*

$$\bar{X}^{(n)}(t) := \frac{x_1(t) - x_0(t)}{2^{n-1} + 1}; \quad \bar{Y}^{(n)}(t) := \frac{x_0(t)}{2^{n-1} + 1}.$$

Then, for any bounded function f,

$$\lim_{n\uparrow\infty} \lim_{q\downarrow 0} \sup_{t\in[t_n^+, t_{n+1}^-]} \left| \mathbb{E}_Q\big(f(\bar{X}^{(n+1)}(t))\big) - E\big(f(\tilde{X}_{c_0}^{(\infty)})\big) \right| = 0, \qquad (3.12)$$

$$\lim_{n\uparrow\infty} \lim_{q\downarrow 0} \sup_{t\in[t_n^+, t_{n+1}^-]} \left| \mathbb{E}_Q\big(f(\bar{Y}^{(n+1)}(t))\big) - E\big(f(\tilde{Y}_{c_0}^{(\infty)})\big) \right| = 0, \qquad (3.13)$$

where again $c_0 = 1$ if μ has finite mean and $c_0 = \alpha$ if μ belongs to the domain of attraction of a α-stable law.

The result (3.12) holds for f satisfying $|f(x)| \le C(1 + |x|)^m$, $m = 1, 2, \ldots$, if the $(m + \delta)$th moment of μ is finite for some $\delta > 0$.

Remark 3.4. The above result holds for a wider class of initial measure Q. Moreover, the moment condition can be relaxed, see [26]. The proof of the above theorem will not be given here and can be found in [26]. In general, the asymptotics of the first k zeros can be deduced from the results of [26].

We refer the reader to Remark 4.1 in Section 4 for an heuristic interpretation of the scaling $t_n = (1/q)^n$ and the renormalization length 2^n.

3.6. Large deviations of the activity

Let us introduce some notation. For simplicity, in this section, we set $\Lambda_N = [1, N]$, $\sigma_t^N = \sigma_t^{\Lambda_N}$ and $\pi_N := \pi_{\Lambda_N}$ (recall the notation introduced two lines after (2.3)) and we denote by $\langle\ \rangle$ the mean over the evolution of the process and over the initial configuration which is distributed with π_N. We also define the total activity as

$$\mathcal{A}(t) := \sum_{x\in\Lambda_N} \mathcal{A}_x(t)$$

where $\mathcal{A}_x(t) := \#\big\{s \le t : \lim_{\varepsilon\downarrow 0} \sigma_{t-\varepsilon}^N(x) \ne \lim \sigma_t^N(x)\big\}$ is the random variable that counts the number of configuration changes at site x during the time interval $[0, t]$.

Let us explain why the total activity is a relevant quantity. The East model, as it is common for kinetically constrained models and more generally for glassy systems, is characterized by a spatially heterogeneous dynamics, namely by the mixture of frozen and mobile areas (see for example Section 1.5 in [34]). The occurrence of these heterogeneities has led to the idea that the dynamics takes place on a first-order coexistence line between active and inactive dynamical

phases [28,29,34,42]. In order to exploit this idea the activity has been proposed as a relevant order parameter to discern active and inactive dynamics and a dynamical approach has been devised to define a suitable notion of free energy. In this dynamical approach the role of the free energy is played by the large deviation function of the activity which, as will be detailed below, undergoes a first order transition in the thermodynamic limit.

Since $\mathcal{A}_x(t) - \int_0^t c_x^{\Lambda_N}(\sigma_s^N)\, ds$ is a martingale, it can be proved that $\mathcal{A}(t)$ satisfies the following law of large numbers

$$\lim_{N\to\infty}\ \lim_{t\to\infty}\ \frac{\mathcal{A}(t)}{Nt} = 2p(1-p)^2.$$

In the sequel we set $\mathbb{A} := 2p(1-p)^2$. Thus one could expect that the probability $\mathcal{P}(a)$ of observing a deviation from the mean value $\mathcal{A}(t)/Nt \simeq a$ scales as

$$\lim_{N\to\infty}\ \lim_{t\to\infty}\ \frac{1}{Nt}\log \mathcal{P}(a) = -f(a), \qquad\qquad (3.14)$$

with $0 < f(a) < \infty$ for $a \neq \mathbb{A}$ as it occurs in absence of the kinetic constraint. However, as it has been observed in [28,29], due to the presence of the constraint it is possible to realize at a low cost a trajectory with zero activity by starting from a completely filled configuration and imposing that a single site does not change its state. Analogously one can obtain an activity smaller than the mean one by blocking for a fraction of time a single site. As a consequence of this sub-extensive cost for lowering the activity it holds $f(a) = 0$ for $a < \mathbb{A}$. For the same reason, the moment generating function controlling the fluctuation of the total activity

$$\psi(\lambda) = \lim_{N\to\infty}\ \lim_{t\to\infty}\ \frac{1}{Nt}\log \big\langle \exp\{\lambda\mathcal{A}(t)\}\big\rangle \qquad\qquad (3.15)$$

is non analytic at $\lambda = 0$ with a discontinuous first order derivative [28, 29].

In [9] the authors study the finite size scaling of this first order transition by analyzing this generating function with a refined thermodynamic scaling, namely

$$\varphi(\alpha) := \limsup_{N\to\infty}\ \lim_{t\to\infty}\ \frac{1}{t}\log \Big\langle \Big(\exp\Big\{ \frac{\alpha}{N}\mathcal{A}(t)\Big\}\Big)\Big\rangle, \quad \alpha \in \mathbb{R},$$

which corresponds to a blow up of the region $\lambda = \alpha/N \sim 0$ and prove the following result

Theorem 3.8 ([9]). *For any $p \in (0,1)$, there exist $\alpha_1 < \alpha_0 < 0$ and a constant $\Sigma > 0$ such that*

(i) *for $\alpha > \alpha_0$, it holds $\varphi(\alpha) = \mathbb{A}\alpha$;*

(ii) *for $\alpha < \alpha_1$ it holds $\varphi(\alpha) = -\Sigma$.*

The results of this theorem are illustrated in Figure 2 below. In particular it shows that, in a range of λ of order $1/N$, the transition is shifted from 0. As a corollary, the authors give the following estimates on the large deviations for a reduced activity:

Corollary 3.1 ([9]). *For any $u \in [0,1)$, it holds*

$$-\Sigma(1 - u) \leq \lim_{\varepsilon \to 0} \liminf_{N \to \infty} \lim_{t \to \infty} \frac{1}{t} \log \left\langle \left(\frac{\mathcal{A}(t)}{Nt} \in [u\mathbb{A} - \varepsilon, u\mathbb{A} + \varepsilon] \right) \right\rangle$$

$$\leq \lim_{\varepsilon \to 0} \limsup_{N \to \infty} \lim_{t \to \infty} \frac{1}{t} \log \left\langle \left(\frac{\mathcal{A}(t)}{Nt} \in [u\mathbb{A} - \varepsilon, u\mathbb{A} + \varepsilon] \right) \right\rangle$$

$$\leq \alpha_0 \mathbb{A}(1 - u).$$

This scaling is anomalous compared to the extensive scaling in N of the unconstrained model and it is a direct consequence of the sub-extensive cost for lowering the activity (while the large deviations for increasing the activity above $\mathbb{A}$ remain extensive in N). As detailed in Section 2.2 of [9], by analogy with equilibrium phase transitions, one can interpret Σ as a surface tension between the inactive and the active region (per unit of time) and this quantity is obtained from a variational problem (see Section 6 of [9]).
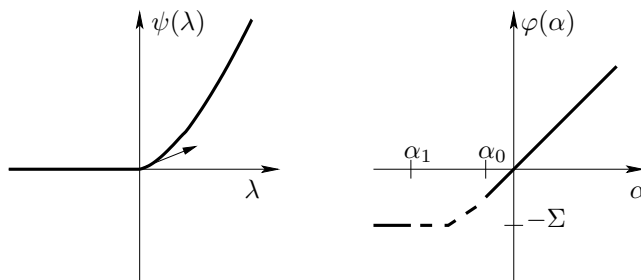

Figure 2. The functions ψ and φ. The results of Theorem 3.8 are depicted on the right, in thick line. The dashed lines correspond to a conjectured behavior.

The results in [9] do not provide the entire phase diagram for the generating function $\varphi(\alpha)$ (only for $\alpha \notin [\alpha_1, \alpha_0]$). A first open problem is to prove the conjecture of [9] that there is a unique critical value α_c and that the two regimes remain valid up to α_c (as depicted in the dashed line of Figure 2), namely $\varphi = -\Sigma$ for $\alpha \leq \alpha_c$ and $\varphi = \alpha\mathbb{A}$ for $\alpha \geq \alpha_c$, which would imply by continuity $\alpha_c = -\Sigma/\mathbb{A}$. This conjecture is supported by numerical simulations [7]. If this conjecture is verified, then Corollary 3.1 can be improved and the large

deviations for reducing the activity would be given by

$$\forall u \in [0,1], \quad \lim_{N \to \infty} \lim_{t \to \infty} \frac{1}{t} \log \left\langle \frac{\mathcal{A}(t)}{Nt} \simeq u\mathbb{A} \right\rangle = -\Sigma(1-u).$$

Another interesting open problem is to describe the interface between the active and inactive regions in the inactive regime $\alpha < \alpha_c$, in particular to prove that this interface is localized close to the boundary. Indeed in [7] the interface has been conjectured to fluctuate within a region of the order of $N^{1/3}$, a result which is supported by numerical results and by the study of an effective model where the boundary between active and inactive regions is described by a Brownian interface. Finally, a very interesting issue from the physicists point of view is to understand if and how the dynamical phase transition can lead to quantitative predictions on the model at $\lambda = 0$.

4. Combinatorics

In this section we collect, and partially prove, some useful combinatorial results, which are intimately related to the oriented character of the East process.

Let us fix some notation. Consider the East process on $\Lambda = (-\infty, -1]$ starting from the completely filled configuration (i.e. the configuration $\sigma_{\mathbb{1}}$ satisfying $\sigma_{\mathbb{1}}(x) = 1$ for any $x \in (-\infty, -1]$) with a frozen vacancy at the origin 0. Denote by $V(n)$ the set of all configurations that the process can reach using *at most n* zeros. Define the absolute value of the position of the leftmost zero in all the configurations of $V(n)$, namely

$$\ell(n) := \sup \left\{ -y : y \in (-\infty, -1], \ \exists \eta \in V(n) \text{ s.t. } \eta(y) = 0 \right\}$$

with the convention that $\sup \emptyset = 0$.

Proposition 4.1 ([11, 47]). *The following holds for $n \geq 0$:*

(i) $\ell(n) = 2^n - 1$;

(ii) $|V(n)| \leq 2^{\binom{n}{2}} n! c^n$ *with $c \leq 0,7$.*

Remark 4.1. The result of Point (i) holds true if one replaces Λ by any finite interval $[-L, -1]$ with $L \geq 2^n - 1$.

More precise statements on the cardinality $|V(n)|$ of $V(n)$ (and in particular a lower bound of the same order with a different constant c) can be found in [11].

Consider the East process on $[-\ell, -1]$ with a frozen zero at the origin 0. Let $n \in \mathbb{N}$ be such that $\ell \in [2^{n-1} + 1, 2^n]$ and $n \geq 1$, or $\ell = 1$ and $n = 0$. Suppose the East process starts from the configuration having a single vacancy located at $-\ell$ (i.e. starts from $01111\ldots1$). Then, due to the above proposition, the system must pass through a configuration with n extra zeros before killing

the vacancy at $-\ell$. This occurs with probability of order q^n and thus with an activation time $t_n = (1/q)^n$. This explains the scale t_n introduce in Section 3.5, and the renormalization length 2^n.

Point (i), stated in [11, Fact 1 (i)], had already been noticed in [46], [47, Section B]. In order to be self-contained and to clarify a mechanism which will be at the heart of the behavior of the East process when $q \downarrow 0$, we provide its proof. With respect to [11], we present a slightly different approach based on the arguments of [47] .

The proof of Proposition 4.1 is based on the following lemma:

Lemma 4.1. *Let $V(n,k)$, $0 \le k \le n$, be the subset of $V(n)$ of those configurations which contain exactly k zeros. Let*

$$\tilde{\ell}(n,k) := \sup \left\{ |y| : y \in (-\infty, -1], \exists \eta \in V(n,k) \ s.t. \ \eta(y) = 0 \right\}$$

be the position of the leftmost zero considering all the configurations of $V(n,k)$.
Then the following holds:

(i) *$\tilde{\ell}(n,k)$ is increasing in k for $n \ge k \ge 0$,*

(ii) *$\ell(n) = \tilde{\ell}(n,n)$ for $n \ge 0$,*

(iii) *$\tilde{\ell}(n,1) = \tilde{\ell}(n-1,n-1) + 1$ for $n \ge 1$,*

(iv) *$\tilde{\ell}(n,k) \ge \sum_{j=1}^{k} \tilde{\ell}(n-j+1,1)$ for $n \ge k \ge 1$.*

Proof. To prove Item (i) we only need to exhibit, for $k < n$, a configuration in $V(n, k+1)$ with a zero at $-[\tilde{\ell}(n,k)+1]$, i.e. a configuration with $k+1$ zeros, one of which at $-[\tilde{\ell}(n,k)+1]$, that the system can reach from $\sigma_{\mathbb{1}}$ without exceeding quota n zeros. Such a configuration can be obtained as follows: starting from $\sigma_{\mathbb{1}}$ the system reaches the configuration which realizes $\tilde{\ell}(n,k)$ without exceeding quota n zeros. This configuration has $k < n$ zeros, therefore the system is allowed to create an additional zero at $-[\tilde{\ell}(n,k) + 1]$.

From the fact that $\tilde{\ell}(n,k)$ is increasing in k, we get the identity $\tilde{\ell}(n,n) = \ell(n)$ stated in Item (ii).

To prove Item (iii) we observe that, just before creating the zero at $-\tilde{\ell}(n,1)$, the system should have a zero at $-\tilde{\ell}(n,1)+1$. Moreover, after creating the zero at $-\tilde{\ell}(n,1)$, the system should remove all zeros on the right of $-\tilde{\ell}(n,1)$ without exceeding quota n zeros and therefore without exceeding $n-1$ zeros on $[-\tilde{\ell}(n,1)+1, -1]$. Reversing this last part of the evolution and disregarding what happens outside $[-\tilde{\ell}(n,1) + 1, -1]$, we get a trajectory starting from the fully filled configuration $\sigma_{\mathbb{1}}$ and realizing the zero at $-\tilde{\ell}(n,1) + 1$ without exceeding quota $n-1$ zeros. The conclusion then follows by applying Item (i).

It remains to prove Item (iv). In order to realize the path which leads to a configuration with k zeros, one of which at $-\tilde{\ell}(n,k)$ and without exceeding

quota n zeros, the system can first create a single zero at $-\tilde{\ell}(n,1)$ without exceeding quota n zeros, then it can use this zero as an anchor to create a further zero at $-\tilde{\ell}(n,1) - \tilde{\ell}(n-1,1)$ by means of a path with at most n simultaneous zeros including the one at $-\tilde{\ell}(n,1)$. By continuing this procedure we get the inequality of Item (iv).   □

We can now come back to Proposition 4.1:

Proof of Proposition 4.1. Combining Items (iii) and (iv) of the above lemma, we get

$$\tilde{\ell}(n,n) \geq \sum_{j=1}^{n} \tilde{\ell}(n-j, n-j) + n = \sum_{j=0}^{n-1} \tilde{\ell}(j,j) + n.$$

Then, using the fact that $\tilde{\ell}(0,0) = 0$, we get by induction that

$$\tilde{\ell}(n,n) \geq 2^n - 1.$$

Using Item (ii) of the above lemma, we conclude that $\tilde{\ell}(n) \geq 2^n - 1$.

The proof of the reverse inequality $\tilde{\ell}(n) = \tilde{\ell}(n,n) \leq 2^n - 1$ goes by induction. It is trivially fulfilled for $n = 0$. Suppose the inequality holds up to $n-1$, where $n \geq 1$. Then, Item (iii) in the above Lemma implies that $\tilde{\ell}(k,1) \leq 2^{k-1}$ for all $k : 1 \leq k \leq n$. Then consider a configuration η which realizes $\tilde{\ell}(n,n)$. By definition η should contain n zeros and by using reversibility we know there should exist a path which kills all these zeros with at each step at most n zeros. Thus at least one of the n zeros of η should be such that it can be killed without creating further zeros, namely it should have the next zero to the right at distance $\tilde{\ell}(1,1) = 1$. Let for simplicity η contain only one such zero (otherwise the strategy can be easily adapted) and consider the configuration η' obtained by killing this zero. Then η' should contain at least a zero which has the next zero at distance at most $\tilde{\ell}(2,1)$ (it should be killed by adding at most one extra zero) and the configuration obtained by killing this zero should contain at least a zero which has the next zero at distance at most $\tilde{\ell}(3,1)$ and so on. By using this observation and the iterative assumption which guarantees that $\tilde{\ell}(k,1) \leq 2^{k-1}$ for all $k \leq n$ we get

$$\tilde{\ell}(n,n) \leq 1 + 2^1 + \cdots 2^{n-1} = 2^n - 1$$

which, together with the above lower bound leads to the desired result $\ell(n) = \tilde{\ell}(n,n) = 2^n - 1$.

The proof of Point (ii) can be found in [11, Theorem 2, 4 and 5].   □

5. Spectral gap: proof of Theorem 3.2

In this section we prove Theorem 3.2. We start with the proof of Point (iii) which is the easiest. Actually we will prove the following stronger useful result (from which Point (iii) immediately follows, details are left to the reader).

Proposition 5.1 (Monotonicity of the spectral gap [13]). *Let $V \subset \Lambda$ be two intervals of $\mathbb{Z}$ (possibly infinite). Then*

$$\mathrm{gap}(\mathcal{L}_\Lambda) \leq \mathrm{gap}(\mathcal{L}_V).$$

Proof. For any local function $f : \Omega_V \to \mathbb{R}$ we have $\mathrm{Var}_V(f) = \mathrm{Var}_\Lambda(f)$ because of the product structure of the measure π_Λ and $\mathcal{D}_\Lambda(f) \leq \mathcal{D}_V(f)$ because, for any $x \in V$ and any $\sigma \in \Omega_\Lambda$, $c_x^\Lambda(\sigma) \leq c_x^V(\sigma)$. The result follows at once from the variational characterization of the spectral gap. $\qquad \square$

Point (i) and the upper bound in Point (ii) of Theorem 3.2 follow from the next result. We use the following standard notation: $\log_2 a = \log a / \log 2$, $a > 0$.

Theorem 5.1 ([13]). *For any $\delta \in (0,1)$ there exists $C_\delta > 0$ such that*

$$\mathrm{gap}(\mathcal{L}) \geq q^{C_\delta} q^{\log_2(1/q)/(2-\delta)}. \tag{5.1}$$

In particular

$$\limsup_{q \to 0} \log(1/\mathrm{gap}(\mathcal{L}))/(\log(1/q))^2 \leq (2\log 2)^{-1}. \tag{5.2}$$

The lower bound in Point (ii) of Theorem 3.2 is proven in [2] by a subtle but somehow intricate argument based on path techniques. In [14, Theorem 5.1] the authors give an alternative (softer) proof. Below we reproduce such proof in order to clarify the role played by energy barriers.

Theorem 5.2 ([14]). *For any $\delta \in (0,1)$, there exists $C_\delta > 0$ such that*

$$\mathrm{gap}(\mathcal{L}) \leq C_\delta q^{\log_2(1/q)(1-\delta)/2}.$$

In particular

$$\liminf_{q \to 0} \log(1/\mathrm{gap}(\mathcal{L}))/(\log(1/q))^2 \geq (2\log 2)^{-1}.$$

Proof of Theorem 5.1. The limiting result (5.2) follows at once from (5.1).

In order to get the lower bound (5.1) we will apply the bisection-constrained method introduced in [13], which extends the classical bisection method [41]. Observe first that, due to Proposition 5.1, $\mathrm{gap}(\mathcal{L}) \geq \inf_\Lambda \mathrm{gap}(\mathcal{L}_\Lambda)$ where the infimum runs over all possible finite intervals $\Lambda = [a, b] \subset \mathbb{Z}$. Hence, our aim is to prove a lower bound on $\mathrm{gap}(\mathcal{L}_\Lambda)$, uniformly in Λ.

Fix $\delta \in (0, 1/6)$ and define $l_k = 2^k$, $\delta_k = \lfloor l_k^{1-3\delta} \rfloor$ and $s_k := \lfloor l_k^\delta \rfloor$, $\lfloor \cdot \rfloor$ denoting the integer part. Let also $\mathbb{F}_k$ be the set of intervals which, modulo translations, have the form $[0, \ell]$ with $\ell \in [0, l_k + l_k^{1-\delta}]$. As in [4], given $\Lambda = [a, b] \in \mathbb{F}_k \setminus \mathbb{F}_{k-1}$, for $i = 1, \ldots, s_k$ we define

$$\Lambda_1^{(i)} := \left[a, \frac{b+a}{2} + \frac{2i}{8}\delta_k \right] \quad \text{and} \quad \Lambda_2^{(i)} := \left[\frac{b+a}{2} + \frac{2i-1}{8}\delta_k, b \right],$$

so that the sequence $\{\Lambda_1^{(i)}, \Lambda_2^{(i)}\}_{i=1}^{s_k}$ satisfies the following properties:

(i) $\Lambda = \Lambda_1^{(i)} \cup \Lambda_2^{(i)}$,

(ii) $d\big(\Lambda \backslash \Lambda_1^{(i)}, \Lambda \backslash \Lambda_2^{(i)}\big) \geq \delta_k/8$,

(iii) $\big(\Lambda_1^{(i)} \cap \Lambda_2^{(i)}\big) \cap \big(\Lambda_1^{(j)} \cap \Lambda_2^{(j)}\big) = \emptyset$, if $i \neq j$,

(iv) $\Lambda_1^{(i)}, \Lambda_2^{(i)} \in \mathbb{F}_{k-1}$.

Above, $d(\cdot, \cdot)$ denotes the Euclidean distance.

Remark 5.1. In other words, any set of $\mathbb{F}_k \setminus \mathbb{F}_{k-1}$ can be obtained as a "slightly overlapping union" of two intervals in $\mathbb{F}_{k-1}$.

Define
$$\gamma_k = \sup_{\Lambda \in \mathbb{F}_k} \mathrm{gap}(\mathcal{L}_\Lambda)^{-1}.$$

Due to Proposition 5.1 the above supremum is attained on the intervals $\Lambda_k = [a, a + l_k + l_k^{1-\delta}]$. Applying the bisection–constrained method introduced in [14], we want to establish a recursive inequality between γ_k and γ_{k-1}. To this aim we fix $\Lambda \in \mathbb{F}_k \setminus \mathbb{F}_{k-1}$ and write it as $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_1, \Lambda_2 \in \mathbb{F}_{k-1}$ as above (we drop the superscript (i), recall that Λ_1 is on the left of Λ_2). Moreover, we set $I \equiv \Lambda_1 \cap \Lambda_2$. We now run the following constrained "block dynamics" on Ω_Λ with blocks $B_1 := \Lambda \setminus \Lambda_2$ and $B_2 := \Lambda_2$ (see Figure 3 below). The block B_2 waits a mean one exponential random time and then the current configuration inside it is refreshed with a new one sampled from π_{Λ_2}. The block B_1 does the same but now the configuration is refreshed only if the current configuration σ contains *at least* one zero inside the strip I.

The Dirichlet form of this auxiliary chain is simply
$$\mathcal{D}_{\mathrm{block}}(f) = \pi_\Lambda \big(c_1 \, \mathrm{Var}_{B_1}(f) + \mathrm{Var}_{B_2}(f) \big)$$

where $c_1(\sigma)$ is just the indicator of the event that σ contains at least one zero inside the strip I (as an illustration, $c_1(\sigma) = 1$ for the configuration given in Figure 3). Recall that $\mathrm{Var}_{B_1}(f)$, $\mathrm{Var}_{B_2}(f)$ depends only on $\sigma_{B_1^c}$, $\sigma_{B_2^c}$ respectively.

Denote by $\gamma_{\mathrm{block}}(\Lambda)$ the inverse spectral gap of the auxiliary Markov chain on Ω_Λ with block dynamics. The following bound, whose proof can be found in [13] and at the end of this section for completeness, is not difficult to prove.

Proposition 5.2 ([13]). *Let* $\varepsilon_k \equiv \max_I \{\pi(\forall x \in I, \sigma(x) = 1)\} = p^{\min_I |I|}$ *where the* $\max_I$ *and the* $\min_I$ *are taken over the* s_k *possible choices of the pairs* (Λ_1, Λ_2) *and* $I = \Lambda_1 \cap \Lambda_2$. *Then*
$$\gamma_{\mathrm{block}}(\Lambda) \leq \frac{1}{1 - \sqrt{\varepsilon_k}}.$$

As a consequence of the above result, by writing down the standard Poincaré inequality for the block auxiliary chain, we get for any $f : \Omega_\Lambda \to \mathbb{R}$ that

$$\text{Var}_\Lambda(f) \leq \frac{1}{1 - \sqrt{\varepsilon_k}} \pi_\Lambda \big(c_1 \text{Var}_{B_1}(f) + \text{Var}_{B_2}(f) \big). \tag{5.3}$$

The second term, using the definition of γ_{k-1} and the fact that $B_2 \in \mathbb{F}_{k-1}$, is bounded from above by

$$\pi_\Lambda \big(\text{Var}_{B_2}(f) \big) \leq \gamma_{k-1} \sum_{x \in B_2} \pi_\Lambda \big(c_x^{B_2} \text{Var}_x(f) \big). \tag{5.4}$$

Notice that, by construction, $c_x^{B_2}(\sigma) = c_x^\Lambda(\sigma)$ for all $x \in B_2$ and all σ. Therefore the term $\sum_{x \in B_2} \pi_\Lambda \big(c_x^{B_2} \text{Var}_x(f) \big)$ is nothing but the contribution carried by the set B_2 to the full Dirichlet form $\mathcal{D}_\Lambda(f)$.

Next we examine the more complicated term $\pi_\Lambda \big(c_1 \text{Var}_{B_1}(f) \big)$ with the goal in mind to bound it with the missing term of the full Dirichlet form $\mathcal{D}_\Lambda(f)$. Assume that $I = [a, b-1]$. For any configuration σ, define the random variable ξ as the distance between the rightmost empty site in the strip I and the right boundary of I, namely

$$\xi(\sigma) := \min_{x \in I : \sigma(x) = 0} \{ b - x \}$$

with the convention that $\min(\emptyset) = +\infty$ (see Figure 3). The indicator function c_1 guarantees that, for any configuration σ with $c_1(\sigma) = 1$, $\xi(\sigma) \in [1, b - a]$.
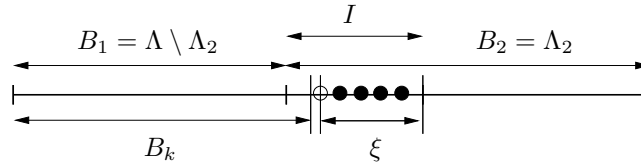

Figure 3. The set Λ divided into the blocks B_1 and B_2 and the random variable ξ. The configuration is such that $c_1(\sigma) = 1$. Empty circles correspond to empty sites.

Set for simplicity $\ell := b - a$ and decompose $c_1(\sigma)$ according to ξ so that (observe that $\text{Var}_{B_1}(f)$ depends only on σ_{B_2}, such a dependence is understood below)

$$\pi_\Lambda \big(c_1 \text{Var}_{B_1}(f) \big) = \sum_{k=1}^\ell \pi_\Lambda \big(\mathbb{1}_{\{\xi = k\}} \text{Var}_{B_1}(f) \big) \tag{5.5}$$

$$= \sum_{k=1}^\ell \sum_{\sigma_{B_2 \setminus I}} \pi_\Lambda(\sigma_{B_2 \setminus I}) \sum_{\sigma_I} \pi_\Lambda(\sigma_I) \mathbb{1}_{\{\xi = k\}}(\sigma_I) \text{Var}_{B_1}(f)$$

$$= \sum_{k=1}^{\ell} \sum_{\sigma_{B_2 \setminus I}} \pi_\Lambda(\sigma_{B_2 \setminus I}) \sum_{\sigma_{I \setminus I_k}} \pi_\Lambda(\sigma_{I \setminus I_k}) \mathbb{1}_{\{\xi = k\}}(\sigma_{I \setminus I_k})$$

$$\times \sum_{\sigma_{I_k}} \pi_\Lambda(\sigma_{I_k}) \operatorname{Var}_{B_1}(f)$$

where $I_k := [a, b - k - 1]$. In the last identity we used the independence of $\mathbb{1}_{\{\xi = k\}}$ from σ_{I_k} (this comes from the fact that ξ is the rightmost empty site inside I, hence, in order to decide that $\xi(\sigma) = k$, one has to know $\sigma(x)$ only for $x \in I \setminus I_k$).

Set $B_k = B_1 \cup I_k$ (see Figure 3). Then, the convexity of the variance implies, for any k, that

$$\sum_{\sigma_{I_k}} \pi_\Lambda(\sigma_{I_k}) \operatorname{Var}_{B_1}(f) \le \operatorname{Var}_{B_k}(f). \tag{5.6}$$

Then, the Poincaré inequality together with Proposition 5.1 finally gives

$$\operatorname{Var}_{B_k}(f) \le \operatorname{gap}(\mathcal{L}_{B_k})^{-1} \sum_{x \in B_k} \pi_{B_k} \big(c_x^{B_k} \operatorname{Var}_x(f) \big)$$

$$\le \operatorname{gap}(\mathcal{L}_{\Lambda_1})^{-1} \sum_{x \in B_k} \pi_{B_k} \big(c_x^{B_k} \operatorname{Var}_x(f) \big). \tag{5.7}$$

Recall that $B_1 = \Lambda \setminus \Lambda_2$, $B_2 = \Lambda_2$, $I = \Lambda_1 \cap \Lambda_2$. The role of the event $\{\xi = k\}$ should at this point be clear. Indeed, thanks to the empty site given by ξ, we have

$$c_x^{B_k}(\sigma) \mathbb{1}_{\{\xi = k\}}(\sigma) \le c_x^\Lambda(\sigma) \mathbb{1}_{\{\xi = k\}}(\sigma) \quad \forall x \in B_k, \ \sigma \in \Omega_\Lambda. \tag{5.8}$$

Let us come back to (5.5). Using (5.6), (5.7), (5.8) we conclude that the last member of (5.5) is bounded from above by

$$\sum_{k=1}^{\ell} \sum_{\sigma_{B_2 \setminus I}} \pi_\Lambda(\sigma_{B_2 \setminus I}) \sum_{\sigma_{I \setminus I_k}} \pi_\Lambda(\sigma_{I \setminus I_k}) \mathbb{1}_{\{\xi = k\}}(\sigma_{I \setminus I_k}) \operatorname{gap}(\mathcal{L}_{\Lambda_1})^{-1}$$

$$\times \sum_{x \in B_k} \pi_{B_k} \big(c_x^\Lambda \operatorname{Var}_x(f) \big)$$

$$= \operatorname{gap}(\mathcal{L}_{\Lambda_1})^{-1} \sum_{k=1}^{\ell} \pi_\Lambda \Big(\mathbb{1}_{\{\xi = k\}} \sum_{x \in B_k} c_x^\Lambda \operatorname{Var}_x(f) \Big).$$

Since $B_k = B_1 \cup I_k \subset B_1 \cup I = \Lambda_1 \in \mathbb{F}_{k-1}$, from the above bound and (5.5) we conclude that

$$\pi_\Lambda \big(c_1 \operatorname{Var}_{B_1}(f) \big) \le \gamma_{k-1} \pi_\Lambda \Big(\sum_{x \in \Lambda_1} c_x^\Lambda \operatorname{Var}_x(f) \Big). \tag{5.9}$$

In conclusion (cf. (5.3), (5.4) and (5.9)) we have shown that

$$\text{Var}_\Lambda(f) \le \frac{1}{1 - \sqrt{\varepsilon_k}} \gamma_{k-1} \Big(\mathcal{D}_\Lambda(f) + \sum_{x \in \Lambda_1 \cap \Lambda_2} \mu_\Lambda \big(c_x^\Lambda \, \text{Var}_x(f) \big) \Big).$$

Averaging over the s_k possible choices of the sets Λ_1, Λ_2 gives (recall property (iii) at the beginning of the proof)

$$\text{Var}_\Lambda(f) \le \frac{1}{1 - \sqrt{\varepsilon_k}} \gamma_{k-1} \Big(1 + \frac{1}{s_k} \Big) \mathcal{D}_\Lambda(f)$$

which implies that

$$\gamma_k \le \frac{1}{1 - \sqrt{\varepsilon_k}} (1 + \frac{1}{s_k}) \gamma_{k-1} \tag{5.10}$$

$$\le \gamma_{k_0 - 1} \prod_{j=k_0}^{k} \frac{1}{1 - \sqrt{\varepsilon_j}} \Big(1 + \frac{1}{s_j} \Big) \le \gamma_{k_0 - 1} \prod_{j=k_0}^{\infty} \frac{1}{1 - \sqrt{\varepsilon_j}} \Big(1 + \frac{1}{s_j} \Big)$$

where k_0 is the smallest integer such that $\delta_{k_0} > 1$.

By definition of the quantity ε_k given in Proposition 5.2 and by construction of the $\Lambda_{1,2}^{(i)}$'s, $|I| \ge \delta_k/8$, so that $\varepsilon_k \le p^{\delta_k/8}$. The convergence of the product in (5.10) is thus guaranteed and the positivity of the spectral gap follows.

Let us now discuss the asymptotic behavior of the gap as $q \downarrow 0$. We first observe that $\gamma_{k_0-1} \le \gamma_{k_0} < (1/q)^{\alpha_\delta}$ for some finite α_δ. That follows e.g. from a coupling argument: in a time lag one and with probability larger than q^{α_δ} for suitable α_δ, any configuration in $\Lambda_{k_0} \in \mathbb{F}_{k_0}$ can reach the empty configuration by just flipping one after another the spins starting from the right boundary. In other words, under the maximal coupling, two arbitrary configurations will couple in a time lag one with probability larger than q^{α_δ}, i.e. $\gamma_{k_0} < (1/q)^{\alpha_\delta}$. We now analyze the infinite product (5.10) which we rewrite as

$$\prod_{j=k_0}^{\infty} \Big(\frac{1}{1 - \sqrt{\varepsilon_j}} \Big) \prod_{j=k_0}^{\infty} \Big(1 + \frac{1}{s_j} \Big).$$

The second factor, due to the exponential growth of the scales, is bounded by a constant independent of q.

To bound the first factor define $j_* = \min\{j : \varepsilon_j \le e^{-1}\}$ and observe that, for q small enough

$$-2 + \frac{\log_2(1/q)}{1 - 3\delta} \le j_* \le 2 + \frac{\log_2(1/q)}{1 - 3\delta}.$$

Then write

$$\prod_{j=k_0}^{\infty} \Big(\frac{1}{1 - \sqrt{\varepsilon_j}} \Big) \le \prod_{j=1}^{j_*} \Big(\frac{1 + \sqrt{\varepsilon_j}}{1 - \varepsilon_j} \Big) \prod_{j > j_*}^{\infty} \Big(\frac{1}{1 - \sqrt{\varepsilon_j}} \Big) \le e^C \, 2^{j_*} \prod_{j=1}^{j_*} \Big(\frac{1}{1 - \varepsilon_j} \Big) \tag{5.11}$$

where we used the bound $1/(1 - \sqrt{\varepsilon_i}) \leq 1 + (e/(e+1))\sqrt{\varepsilon_j}$ valid for any $j \geq j_*$ together with

$$\sum_{j > j_*}^{\infty} \log\left(1 + \frac{e}{e+1}\sqrt{\varepsilon_j}\right)$$

$$\leq \frac{e}{e+1}\sum_{j > j_*}^{\infty}\sqrt{\varepsilon_j} \leq \frac{e}{e+1}\int_{j_*-1}^{\infty} dx \, \exp\left(-\frac{q(2^{x(1-3\delta)})}{16}\right)$$

$$= A_\delta \int_{2^{(j_*-1)(1-3\delta)}}^{\infty} dz \, \frac{\exp(-qz/16)}{z}$$

$$\leq 16 A_\delta 2^{-(j_*-1)(1-3\delta)} q^{-1} \exp\left(-\frac{q2^{(j_*-1)(1-3\delta)}}{16}\right) \leq C$$

for some constant C independent of q.

Observe now that $1 - \varepsilon_j \geq 1 - \exp(-q\delta_j/8) \geq Aq\delta_j$ for any $j \leq j_*$ and some constant A independent of q. Thus the right-hand side of (5.11) is bounded from above by

$$e^C\left(\frac{2}{Aq}\right)^{j_*}\prod_{j=1}^{j_*}\delta_j^{-1} \leq \frac{1}{q^a}\left(\frac{1}{q}\right)^{j_*} 2^{-(1-3\delta)j_*^2/2} \leq \frac{1}{q^{a'}}\left(\frac{1}{q}\right)^{\log_2(1/q)/(2-6\delta)}$$

for some constants a, a' (independent of q). This ends the proof of Theorem 5.1.
$\square$

Proof of Theorem 5.2. Since it is always true that $\mathrm{gap}(\mathcal{L}) \leq 1$ (take $f = \sigma_0$ in (3.1)), we can assume without loss of generality that q is small. Then, thanks to Proposition 5.1, $\mathrm{gap}(\mathcal{L}) \leq \mathrm{gap}(\mathcal{L}_\Lambda)$ with $\Lambda = [0, \ell)$ and $\ell = 1/q$ that we assume, for simplicity, to be an integer. In order to bound from above $\mathrm{gap}(\mathcal{L}_\Lambda)$, we will make use of the following general result.

Lemma 5.1. *For any $A \subset \Omega_\Lambda$, the hitting time $\tau_A = \inf\{t \geq 0 : \sigma_t^\Lambda \in A\}$ satisfies*

$$\mathbb{P}_{\pi_\Lambda}^\Lambda(\tau_A > t) \leq \exp\left(-t\,\mathrm{gap}(\mathcal{L}_\Lambda)\pi_\Lambda(A)\right).$$

Proof. It is well known (see e.g. [3]) that $\mathbb{P}_{\pi_\Lambda}^\Lambda(\tau_A > t) \leq \exp(-t\lambda_A)$, where

$$\lambda_A := \inf\left\{\mathcal{D}_\Lambda(f) : \pi_\Lambda(f^2) = 1 \text{ and } f \equiv 0 \text{ on } A\right\}.$$

If $f \equiv 0$ on A we can bound

$$\mathrm{Var}_\Lambda(f) = \frac{1}{2}\sum_{\sigma \in \Omega_\Lambda}\sum_{\sigma' \in \Omega_\Lambda}\pi_\Lambda(\sigma)\pi_\Lambda(\sigma')\big(f(\sigma) - f(\sigma')\big)^2 \qquad (5.12)$$

$$\geq \sum_{\sigma \in A}\sum_{\sigma' \in \Omega_\Lambda}\pi_\Lambda(\sigma)\pi_\Lambda(\sigma')\big(f(\sigma) - f(\sigma')\big)^2 = \pi_\Lambda(A)\pi_\Lambda(f^2).$$

Hence, for such a function f it holds

$$\mathcal{D}_\Lambda(f) \geq \operatorname{gap}(\mathcal{L}_\Lambda) \operatorname{Var}_\Lambda(f) \geq \operatorname{gap}(\mathcal{L}_\Lambda)\pi_\Lambda(A)\pi_\Lambda(f^2).$$

It then follows that $\lambda_A \geq \operatorname{gap}(\mathcal{L}_\Lambda)\pi_\Lambda(A)$. $\qquad\square$

Denote by τ the first time there are $n := \lfloor \log_2 \ell \rfloor$ empty sites in $[0, \ell)$, and by τ_0 the first time there is an empty site at the origin. Thanks to Point (i) of Proposition 4.1 and since $2^n \leq \ell \leq 2^{n+1}-1$, starting from the filled configuration in order to end up at time τ_0 with an empty site located at the origin, the system must have created before $n + 1 \geq n$ empty sites in $[0, \ell)$. Hence $\tau \leq \tau_0$ when starting from $\mathbb{1}$. In turn, Lemma 5.1 applied to the set $A = \{\eta : \eta_0 = 0\}$ implies that

$$\begin{aligned}
\exp\left(-t\operatorname{gap}(\mathcal{L}_\Lambda)q\right) = \exp\left(-t\operatorname{gap}(\mathcal{L}_\Lambda)\pi_\Lambda(A)\right) \qquad (5.13)\\
\geq \mathbb{P}^\Lambda_{\pi_\Lambda}(\tau_0 > t) \geq \pi_\Lambda(\mathbb{1})\mathbb{P}^\Lambda_{\mathbb{1}}(\tau > t).
\end{aligned}$$

Recall the definition of $V(n)$ introduced in Section 4 and denote by $\Omega_n \subset V(n)$ those configurations with exactly n zeros. Thanks to Point (ii) of Proposition 4.1, for a suitable constant $c \leq 0.7$ it holds

$$\pi_\Lambda(\mathbb{1})\mathbb{P}^\Lambda_{\mathbb{1}}(\tau \leq t) \leq \mathbb{P}^\Lambda_{\pi_\Lambda}(\tau \leq t) \leq \mathbb{P}^\Lambda_{\pi_\Lambda}\left(\exists s \leq t : \eta_s \in \Omega_n\right) \leq \frac{t}{q}\pi_\Lambda(\Omega_n) \quad (5.14)$$

$$\leq \frac{t}{q}\,q^n|\Omega_n| \leq \frac{t}{q}\,q^n 2^{\binom{n}{2}}n!c^n \leq tq^{(n/2)(1+o(1))}.$$

Here $o(1)$ tends to zero as n goes to infinity and therefore when $q \downarrow 0$. To prove the third inequality in (5.14), observe that we can write η_t as time change of a discrete time Markov chain. More precisely, it holds $\eta_t = \eta^{d.t.}_{N_t}$ where $\eta^{d.t.}$ is the discrete time Markov chain on Ω_Λ whose transition matrix $\mathbb{P}$ satisfies $\mathbb{I} - \mathbb{P} = |\Lambda|^{-1}\mathcal{L}_\Lambda$, and where $(N_t)_{t\geq 0}$ represents a Poisson process with mean $\mathbb{E}(N(t)) = |\Lambda|t$. Trivially, π_Λ is reversible for $\eta^{d.t.}$. We write P for the law of $\eta^{d.t.}$ having π_Λ as initial distribution. Then

$$\begin{aligned}
\mathbb{P}^\Lambda_{\pi_\Lambda}&\left(\exists s \leq t : \eta_s \in \Omega_n\right)\\
&= \mathbb{P}^\Lambda_{\pi_\Lambda}\left(\exists s < t : \eta_s \in \Omega_n\right)\\
&= \sum_{k=0}^\infty \mathbb{P}(N(t) = k)P_{\pi_\Lambda}\left(\exists j : 0 \leq j < k \text{ and } \eta^{d.t.}_j \in \Omega_n\right)\\
&\leq \sum_{k=0}^\infty \mathbb{P}(N(t) = k)k\pi_\Lambda(\Omega_n) = \mathbb{E}(N(t))\pi_\Lambda(\Omega_n) = \frac{t}{q}\pi_\Lambda(\Omega_n).
\end{aligned}$$

The above inequality (5.14) together with (5.13) implies that

$$\exp\left(-t\operatorname{gap}(\mathcal{L}_\Lambda)q\right) \geq \pi_\Lambda(\mathbb{1}) - tq^{(n/2)(1+o(1))}. \qquad (5.15)$$

Since $\pi_L(\mathbb{1}) = (1-q)^{1/q} \sim 1$, we choose $t = q^{-(n/2)(1-\delta/2)}$ so that the above right-hand side is at least $1/2$ for q small enough. Finally, we conclude from (5.15) that for q small, $t\,\mathrm{gap}(\mathcal{L}_\Lambda)q \le \log 2$. The expected result follows.   □

We end this section with the proof of Proposition 5.2.

Proof of Proposition 5.2. For any mean zero function $f \in L^2(\Omega_\Lambda, \pi_\Lambda)$ let

$$\pi_1 f := \pi_{B_2}(f), \quad \pi_2 f := \pi_{B_1}(f)$$

be the natural projections onto $L^2(\Omega_{B_i}, \pi_{B_i})$, $i = 1, 2$. Obviously $\pi_1 \pi_2 f = \pi_2 \pi_1 f = 0$. The generator of the block dynamics can then be written as:

$$\mathcal{L}_{\mathrm{block}} f = c_1(\pi_2 f - f) + \pi_1 f - f$$

and the associated eigenvalue equation as

$$c_1(\pi_2 f - f) + \pi_1 f - f = \lambda f. \tag{5.16}$$

By taking $f(\sigma_\Lambda) = g(\sigma_{B_2})$ with g non zero and with $\pi_1 g = 0$, we see that $\lambda = -1$ is an eigenvalue. Moreover, since $c_1 \le 1$, $\lambda \ge -1$. Assume now $0 > \lambda > -1$ and apply π_2 to both sides of (5.16) to obtain (recall that $c_1 = c_1(\sigma_{B_2})$)

$$-\pi_2 f = \lambda \pi_2 f \implies \pi_2 f = 0. \tag{5.17}$$

For any f with $\pi_2 f = 0$ the eigenvalue equation becomes

$$f = \frac{\pi_1 f}{1 + \lambda + c_1} \tag{5.18}$$

and that is possible only if

$$1 = \pi_1\Big(\frac{1}{1 + \lambda + c_1}\Big) = \frac{1}{1 + \lambda}\pi_{B_2}(c_1 = 0) + \frac{1}{2 + \lambda}\pi_{B_2}(c_1 = 1).$$

We can solve the equation to get

$$\lambda = -1 + \sqrt{1 - \pi_{B_2}(c_1)} \le -1 + \sqrt{\varepsilon_k}.$$

                                                                                            □

6. Persistence function: proof of Theorem 3.3

We follow [13]. Observe first that $F(t) = F_1(t) + F_0(t)$ where

$$F_1(t) = \int d\pi(\eta)\, \mathbb{P}_\eta\big(\sigma_0(s) = 1 \text{ for all } s \le t\big)$$

and similarly for $F_0(t)$. We will prove the exponential decay of $F_1(t)$, the case of $F_0(t)$ being similar.

For any $\lambda > 0$ the exponential Chebyshev inequality gives

$$F_1(t) = \int d\pi(\eta) \, \mathbb{P}_\eta\left(\int_0^t ds \, \sigma_0(s) = t\right) \leq \exp(-\lambda t) \, \mathbb{E}_\pi\left(\exp\left(\lambda \int_0^t ds \, \sigma_0(s)\right)\right)$$

where we recall that $\mathbb{E}_\pi$ denotes the expectation over the process started from the equilibrium distribution π. Consider the self-adjoint operator $H_\lambda := \mathcal{L} + \lambda V$, on $L^2(\pi)$, where V is the multiplication operator by σ_0. By the very definition of the scalar product $\langle f, g \rangle$ in $L^2(\pi)$ and the Feynman–Kac formula, we can rewrite $\mathbb{E}_\pi\left(\exp\left(\lambda \int_0^t ds \, \sigma_0(s)\right)\right)$ as $\langle \mathbf{1}, \exp(tH_\lambda)\mathbf{1} \rangle$. Thus, if β_λ denotes the supremum of the spectrum of H_λ, $\mathbb{E}_\pi\left(\exp\left(\lambda \int_0^t ds \, \sigma_0(s)\right)\right) \leq \exp(t\beta_\lambda)$. In turn,

$$F_1(t) \leq \exp\left(-\lambda t\left(1 - \frac{\beta_\lambda}{\lambda}\right)\right).$$

Hence, in order to complete the proof we need to show that for suitable positive λ the constant β_λ/λ is strictly smaller than one.

For any function f, with $\|f\|_{\mathbb{L}^2(\pi)} = 1$, in the domain of H_λ (which coincides with $\text{Dom}(\mathcal{L})$), write $f = \alpha\mathbf{1} + g$ with $\langle \mathbf{1}, g \rangle = 0$. Thus, by Cauchy–Schwarz inequality and the fact that $\mathcal{L}\mathbf{1} = 0$ and that $|\sigma_0| \leq 1$, we have

$$\langle f, H_\lambda f \rangle = \langle g, \mathcal{L}g \rangle + \alpha^2 \lambda \langle \mathbf{1}, V\mathbf{1} \rangle + \lambda \langle g, Vg \rangle + 2\lambda\alpha \langle \mathbf{1}, Vg \rangle$$

$$\leq (\lambda - \text{gap}(\mathcal{L}))\langle g, g \rangle + \alpha^2 \lambda p + 2\lambda|\alpha|\left(\langle g, g \rangle pq\right)^{1/2}.$$

Since $\|f\|_{\mathbb{L}^2(\pi)} = 1$, $\alpha^2 + \langle g, g \rangle = 1$ and

$$\frac{\beta_\lambda}{\lambda} \leq \sup_{0 \leq \alpha \leq 1}\left\{\left(1 - \frac{\text{gap}(\mathcal{L})}{\lambda}\right)(1 - \alpha^2) + p\alpha^2 + 2\alpha\left((1 - \alpha^2)pq\right)^{1/2}\right\}. \qquad (6.1)$$

If we choose $\lambda = \text{gap}(\mathcal{L})/2$, the right-hand side of (6.1) becomes

$$\sup_{0 \leq \alpha \leq 1}\left\{(1 + p)\alpha^2 - 1 + 2\alpha\left((1 - \alpha^2)pq\right)^{1/2}\right\}$$

$$\leq \sup_{0 \leq \alpha \leq 1}\left\{(1 + p)\alpha^2 - 1 + 2\left((1 - \alpha^2)pq\right)^{1/2}\right\} = \frac{pq}{1 + p} + p < 1$$

since $p \neq 1$. Thus $F_1(t)$ satisfies

$$F_1(t) \leq \exp\left(-t \frac{\text{gap}(\mathcal{L})}{2} \frac{q}{1 + p}\right).$$

A very similar computation (details are left to the reader) shows that

$$F_0(t) \leq \exp\left(-t \frac{\text{gap}(\mathcal{L})}{2} \frac{p}{1 + p}\right).$$

This ends the proof of Theorem 3.3.

7. Log-Sobolev constant: proof of Theorem 3.4

The aim of this section is to prove Theorem 3.4.

Fix a finite interval Λ and assume for simplicity that $\Lambda = [1, L-1]$.

We start with the easy part, namely the upper bound on $C_\alpha(\Lambda)$. We observe first that, thanks to [43, Theorem 1.8], the α-log-Sobolev inequality with constant $C_\alpha(\Lambda)$ implies any β-log-Sobolev inequality with the same constant $C_\alpha(\Lambda)$ as soon as $\beta \leq \alpha$. Hence, we have $C_\alpha(\Lambda) \leq C_2(\Lambda)$ for any $\alpha \in [0,2]$ so that we only have to prove that $C_2(\Lambda) \leq cL$. This in turn is implied by the following well-known property of Diaconis and Saloff-Coste [21, Corollary A.4]:

$$C_2(\Lambda) \leq \frac{1}{\text{gap}(\mathcal{L}_\Lambda)} \frac{\log(1/\pi_\Lambda^*) - 1}{1 - 2\pi_\Lambda^*}$$

where $\pi_\Lambda^* := \min_{\sigma \in \Omega_\Lambda} \pi_\Lambda(\sigma) = \min(p,q)^{L-1}$. The expected upper bound follows at once from Theorem 3.2.

Now we turn to the lower bound on $C_\alpha(\Lambda)$. The result will be achieved by a test function. Define the random variable $\xi \in \{1, \dots, L\}$ as the distance from the leftmost empty site in Λ to the left boundary of Λ, namely (recall that $\Lambda = [1, L-1]$),

$$\xi(\sigma) := \min_{x \in \Lambda : \sigma(x) = 0} \{d(x, 0)\}$$

with the convention that $\min \emptyset = L$. In other words, ξ is the position, in $\{1, \dots, L\}$, of the leftmost empty site (including the boundary condition at site L). Then, for $g : \{1, \dots, L\} \to \mathbb{R}$, let $f : \Omega_\Lambda \to \mathbb{R}$ defined as $f(\sigma) = g(\xi(\sigma))$. Define also the distribution m on the set $\{1, \dots, L\}$ by $m(k) = \pi_\Lambda(\{\xi = k\})$ so that

$$m(k) = \begin{cases} qp^{k-1}, & \text{if } k = 1, 2 \dots, L-1, \\ p^{L-1}, & \text{if } k = L. \end{cases}$$

Hence, the α-log-Sobolev inequality (3.3), applied to f, reads

$$m\left(g \log\left(\frac{g}{m(g)}\right)\right) = \text{Ent}_\Lambda(f) \leq \frac{\alpha \alpha' C_\alpha(\Lambda)}{4} \mathcal{D}_\Lambda\left(g(\xi)^{1/\alpha}, g(\xi)^{1/\alpha'}\right). \qquad (7.1)$$

Let us analyze the right-hand side of the latter. By definition of the Dirichlet form, we have

$$\begin{aligned}
\mathcal{D}_\Lambda &\left(g(\xi)^{1/\alpha}, g(\xi)^{1/\alpha'}\right) \\
&= \sum_{k=1}^{L} \sum_{\sigma : \xi(\sigma) = k} \pi_\Lambda(\sigma) \sum_{x=1}^{L} c_x^\Lambda(\sigma)[(1 - \sigma(x))p + \sigma(x)q] \\
&\quad \times \left(g(\xi(\sigma^x))^{1/\alpha} - g(\xi(\sigma))^{1/\alpha}\right)\left(g(\xi(\sigma^x))^{1/\alpha'} - g(\xi(\sigma))^{1/\alpha'}\right)
\end{aligned}$$

$$= m(L-1)p\big(g(L)^{1/\alpha} - g(L-1)^{1/\alpha}\big)\big(g(L)^{1/\alpha'} - g(L-1)^{1/\alpha'}\big)$$

$$+ \sum_{k=1}^{L-2} m(k)pq\big(g(k+1)^{1/\alpha} - g(k)^{1/\alpha}\big)\big(g(k+1)^{1/\alpha'} - g(k)^{1/\alpha'}\big)$$

$$+ \sum_{k=2}^{L} m(k)q\big(g(k-1)^{1/\alpha} - g(k)^{1/\alpha}\big)\big(g(k-1)^{1/\alpha'} - g(k)^{1/\alpha'}\big),$$

where we used that, given that $\xi = k$, the only possible flips are at site $x = k-1$ (for $k \geq 2$), and at site $x = k$, in which case the flip is admissible only if site $x+1$ is empty (hence the extra factor q), except for $\xi = L-1$ where by definition the constraint is always satisfied, due to the boundary condition at site $x = L$. By a change of variable, and using that $m(k)pq = m(k+1)q$, we arrive at

$$\mathcal{D}_\Lambda\big(g(\xi)^{1/\alpha}, g(\xi)^{1/\alpha'}\big)$$

$$= 2\sum_{k=1}^{L-2} m(k)pq\big(g(k+1)^{1/\alpha} - g(k)^{1/\alpha}\big)\big(g(k+1)^{1/\alpha'} - g(k)^{1/\alpha'}\big)$$

$$+ m(L-1)p(1+q)\big(g(L)^{1/\alpha} - g(L-1)^{1/\alpha}\big)\big(g(L)^{1/\alpha'} - g(L-1)^{1/\alpha'}\big).$$

Observe that the latter corresponds to the Dirichlet form associated to the birth and death process on $\{1, \ldots, L\}$ with reversible measure m and transition rates $p(k, k+1) = 2pq$ for $k = 1, \ldots, L-2$, and $p(L-1, L) = p(1+q)$ (and $p(k+1, k)$ computed so that the detailed balanced condition $m(k)p(k, k+1) = m(k+1)p(k+1, k)$ holds). In turn, inequality (7.1) is nothing but the α-Sobolev inequality corresponding to this birth and death process[4].

Consider now the special choice $g(k) = \lambda^{k-1}$, defined on $\{1, \ldots, L\}$, with $\lambda \in (0, 1/p)$ a parameter that will be chosen later. The above Dirichlet form reduces, after simple algebra, to

$$\mathcal{D}_\Lambda\big(g(\xi)^{1/\alpha}, g(\xi)^{1/\alpha'}\big)$$

$$= (\lambda^{1/\alpha} - 1)(\lambda^{1/\alpha'} - 1)\left(q(1+q)(p\lambda)^{L-1} + 2q^2 \sum_{k=1}^{L-2}(p\lambda)^k\right)$$

$$= (\lambda^{1/\alpha} - 1)(\lambda^{1/\alpha'} - 1)\frac{pq}{1-p\lambda}\big[p(1 - (1+q)\lambda)(p\lambda)^{L-2} + 2q\big].$$

Denote by $\varepsilon = 1 - p\lambda \in (0, 1)$. By a Taylor expansion, as ε goes to 0, we thus have

$$\mathcal{D}_\Lambda\big(g(\xi)^{1/\alpha}, g(\xi)^{1/\alpha'}\big) = \left[\frac{1}{p^{1/\alpha}} - 1\right]\left[\frac{1}{p^{1/\alpha'}} - 1\right]pq(2qL + 1 + q + o(1)). \quad (7.2)$$

---

[4]In fact it is possible to prove that the α-Sobolev constant associated to this birth and death process compares to L for any $\alpha \in (0, 2]$ and that the Poincaré inequality (i.e. the case $\alpha = 0$) holds with a constant C_0 independent of L.

On the other hand,

$$m(g) = \sum_{k=1}^{L} m(k)g(k) = (p\lambda)^{L-1} + q\sum_{k=1}^{L-1} (p\lambda)^{k-1}$$

$$= \frac{(p\lambda)^{L-1}p(1-\lambda) + q}{1 - p\lambda} = 1 + q(L-1) + o(1).$$

Hence,

$$\frac{d}{d\log\lambda}m(g) = \frac{pq\lambda + (p\lambda)^{L-1}\big[p(1-\lambda)(1-p\lambda)(L-1) - pq\lambda\big]}{(1-p\lambda)^2}.$$

Since $d\,m(g)/d\log\lambda = \sum_{k=1}^{L} m(k)kg(k)$, we deduce that

$$m(g\log g) = \log\lambda \sum_{k=1}^{L} m(k)kg(k)$$

$$= \log\lambda \frac{pq\lambda + (p\lambda)^{L-1}[p(1-\lambda)(1-p\lambda)(L-1) - pq\lambda]}{(1-p\lambda)^2}$$

$$= \frac{1}{2}(L-1)(qL+2p)\log\left(\frac{1}{p}\right) + o(1).$$

In turn, there exists a constant c, that may depend on q but that is independent of L and α, such that

$$m\Big(g\log\Big(\frac{g}{m(g)}\Big)\Big) = m(g\log g) - m(g)\log m(g) \geq cL^2 + o(1).$$

The expected result finally follows, using (7.2), from (7.1), in the limit $\varepsilon \to 0$. This achieves the proof of the lower bound on the constants $C_\alpha(\Lambda)$ and therefore of Theorem 3.4.

8. Out of equilibrium I, long time behavior: proof of Theorem 3.5

The aim of this section is to prove Theorem 3.5. To that purpose, we will deeply use the oriented character of the East process. We need some preparation.

In [2], Aldous and Diaconis introduced the following notion of *distinguished zero*:

Definition 8.1 (Distinguished zero [2]). Fix an initial configuration $\sigma \in \Omega$, suppose that $\sigma(x) = 0$ and call the site x distinguished. Then, setting $\xi_0(\sigma) = x$, the position $\xi_s = \xi_s(\sigma) \in \mathbb{Z}$ of the distinguished zero at time $s > 0$ obeys the following iterative rule. $\xi_s = x$ for all times s strictly smaller than the first legal ring (recall the graphical construction of Section 2.3) of the mean one Poisson clock associated to site x when it jumps to site $x + 1$. Then it waits the next legal ring at $x + 1$ and when this occurs it jumps to $x + 2$, and so on.

Thus, with probability one, the path $\{\xi_s\}_{s\in[0,t]}$ is right-continuous, piecewise constant, non decreasing, with at most a finite number of discontinuities at which it increases by (exactly) one. See Figure 4 where $t_1, \ldots, t_4$ are legal rings. Also note that, by definition of the legal rings, necessarily $\sigma_s(\xi_s) = 0$ for all s (hence the name distinguished zero).

Remark 8.1. One important feature of the distinguished zero is the following property that we will often use in the sequel. Fix a starting configuration σ with $\sigma(b) = 0$. Make b distinguished. Then, given the path $\{\xi_s\}_{s\in[0,t]}$, the law of $\sigma_{t[a,\xi_t)}$ (i.e. the restriction of σ_t to $[a,\xi_t)$) depends only on $\sigma_{[a,b)}$ and not on $\sigma_{[a,b)^c}$.

In the following we use the standard notation $f(u-) = \lim_{\varepsilon\downarrow 0} f(u - \varepsilon)$.

By exploiting the oriented character of the East process, and more precisely the fact that the motion of the distinguished zero for $s > t$ cannot be influenced by the clock rings and coin tosses in $(-\infty, \xi_t)$, Aldous and Diaconis proved the following important fact: if one starts with the equilibrium measure π, say on $(-\infty, x)$, with x distinguished, then the process is still at equilibrium, at any time, on $(-\infty, \xi_t)$. More precisely:

Lemma 8.1 ([2]). *Consider the East process on $\mathbb{Z}$. Fix an interval $\Lambda = [a, b]$ with possibly $a = -\infty$ (in which case $\Lambda = (-\infty, b)$). Assume that, at time zero, $\sigma(b) = 0$ while σ_Λ is distributed according to the equilibrium measure π_Λ. Make b distinguished and call ξ_s its position at time s. Then, the conditional distribution of $\sigma_{t[a,\xi_t)}$ (i.e. σ_t restricted to $[a,\xi_t)$) given the path $\{\xi_s\}_{s\leq t}$ is the equilibrium measure $\pi_{[a,\xi_t)}$.*

We prove Lemma 8.1 for completeness.

Proof of Lemma 8.1. Let $0 < t_1 < t_2 < \ldots$ be the (random) times when the distinguished zero jumps, i.e. increases by one. Setting $t_0 = 0$, at time $s \in [t_i, t_{i+1})$ the position of the distinguished zero is precisely $\xi_s = b + i$ (see Figure 4). The proof goes by induction on the interval $[t_i, t_{i+1})$ containing t. The case $i = 0$ follows from the stationarity of π_Λ for the East process on Λ with a frozen zero on the right. Assume that for all $t \in [t_{i-1}, t_i)$ the law of $\sigma_{t[a,\xi_{t_i}-)}$, given the path $\{\xi_s\}_{s\leq t}$, is the equilibrium measure $\pi_{[a,\xi_{t_i}-)}$. Given the legal ring at site $b + i - 1$ at time t_i, we conclude that the law of $\sigma_{t_i[a,\xi_{t_i})}$ is the equilibrium measure $\pi_{[a,\xi_{t_i})}$: indeed, the new configuration $\sigma_{t_i[a,\xi_{t_i})}$ consists of the existing configuration $\sigma_{t_i-[a,\xi_{t_i}-)}$ (distributed according to $\pi_{[a,\xi_{t_i}-)} = \pi_{[a,b+i-1)}$) together with the new configuration at site $b + i - 1$ created by the legal ring, which is also Bernoulli$(1 - q)$, thus making the whole configuration distributed according to $\pi_{[a,\xi_{t_i})} = \pi_{[a,b+i)}$. Then, at any subsequent time $t \in [t_i, t_{i+1})$, knowing that there is no legal ring at site $b + i$ and by stationarity, the law of $\sigma_{t[a,\xi_t)} = \sigma_{t[a,\xi_{t_i})}$ remains i.i.d. Bernoulli$(1 - q)$. This carries the induction forward and ends the proof of the lemma. $\square$
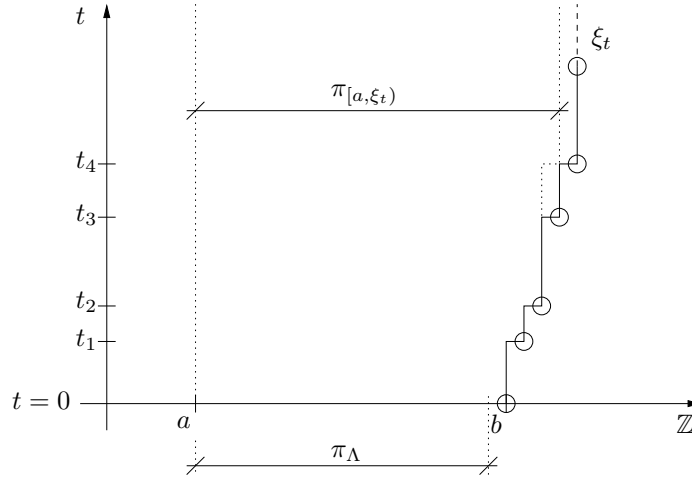
Figure 4.   The path $\{\xi_s\}_{s\geq 0}$ of the distinguished zero b.   Illustration of Lemma 8.1 on the interval $\Lambda = [a,b)$.

We are now in position to prove Theorem 3.5.

Proof of Theorem 3.5. Let f be a local function, and assume that its support is included in $[a,a']$. Assume for simplicity that $\pi(f) = 0$. Given a configuration σ, let $b = b(\sigma) = \inf\{x \geq a'+1 \text{ s.t. } \sigma(x) = 0\}$ be the position of the first empty site in σ on the right of a' (it exists Q a.s.). Make b distinguished and denote by ξ_s its position at time s. Given the path $\{\xi_s\}_{s\leq t}$, let $0 < t_1 < t_2 < \cdots < t_{n-1} < t$ be the times when the distinguished zero jumps, and set $t_0 = 0$, $t_n = t$. By construction, $\xi_s = b + i$ for any $s \in [t_i, t_{i+1})$, see Figure 4.

Since the support of f is included in $[a,a'] \subset [a, \xi_t)$, thanks to Lemma 8.1 it holds that

$$\pi_{[a,b)}\big(\mathbb{E}.\big(f(\sigma_t) \mid \{\xi_s\}_{s\leq t}\big)\big) = \int d\pi_{[a,b)}(\sigma) \, \mathbb{E}_\sigma\big(f(\sigma_{t\,[a,\xi_t)}) \mid \{\xi_s\}_{s\leq t}\big)$$
$$= \pi_{[a,\xi_t)}(f) = 0. \tag{8.1}$$

Note that the notation used in the above first member is justified by Remark 8.1. The same remark will be frequently used below in our notational choice.

For any σ, thanks to (8.1) and the Cauchy–Schwarz inequality,

$$\big|\mathbb{E}_\sigma\big(f(\sigma_t)\big)\big| \leq \mathbb{E}_\sigma\big(\big|\mathbb{E}_\sigma(f(\sigma_t) mid\{\xi_s\}_{s\leq t})\big|\big) \tag{8.2}$$
$$\leq \frac{1}{(p \wedge q)^{b-a}}\mathbb{E}_\sigma\Big(\int d\pi_{[a,b)}(\eta) \, \big|\mathbb{E}_\eta\big(f(\eta_t) \mid \{\xi_s\}_{s\leq t}\big)\big|\Big)$$
$$\leq \frac{1}{(p \wedge q)^{b-a}}\mathbb{E}_\sigma\big(\mathrm{Var}_{\pi_{[a,b)}}\big(\mathbb{E}.(f(\eta_t) \mid \{\xi_s\}_{s\leq t})\big)^{1/2}\big).$$

Now our aim is to control the right-hand side of the latter, using the Poincaré Inequality. Define $V_i := [a, b+i)$ for $i = 0, 1 \ldots, n-1$ (that corresponds to $[a, \xi_s)$ when $s \in [t_i, t_{i+1})$) and denote by $\{P_s^{(i)}\}_{s \in [t_i, t_{i+1})}$ the Markov semigroup associated to the East process in the interval V_i with a fixed zero boundary condition at site $b + i$. Given the path $\{\xi_s\}_{s \leq t}$, thanks to Remark 8.1, $\sigma_{t[0, \xi_t)}$ coincides with the process obtained from the initial configuration σ_{V_0} evolving according to $\{P_s^{(0)}\}_{s \in [t_0, t_1)}$, up to time t_1, then evolving according to $\{P_s^{(1)}\}_{s \in [t_1, t_2)}$, up to time t_2, and so on. Hence, if one writes for simplicity $\sigma \otimes \sigma' = \sigma_{V_0} \sigma'_{\{b\}}$ for the configuration in $\{0, 1\}^{V_1}$ equal to σ on V_0 and to σ' on $\{b\}$, then, for any η,

$$
\begin{aligned}
&\mathbb{E}_\eta \big(f(\eta_t) \mid \{\xi_s\}_{s \leq t} \big) \\
&\quad = \mathbb{E}_{\eta_{V_0}} \big(f(\eta_t) \mid \{\xi_s\}_{s \leq t} \big) \\
&\quad = \sum_{\substack{\sigma' \in \{0,1\} \\ \sigma \in \{0,1\}^{V_0}}} P_{t_1}^{(0)}(\eta_{V_0}, \sigma) \pi_b(\sigma') \mathbb{E}_{\sigma \otimes \sigma'} \big(f\big((\sigma \otimes \sigma')_{t-t_1} \big) \mid \{\xi_s\}_{s \in [t_1, t]} \big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathrm{Var}_{\pi_{V_0}} \big(\mathbb{E}. \big(f(\eta_t) \mid \{\xi_s\}_{s \leq t} \big) \big) \\
&\qquad \leq \exp \big(-2 \operatorname{gap}(\mathcal{L}_{V_0}) t_1 \big) \\
&\qquad\quad \times \mathrm{Var}_{\pi_{V_0}} \Big(\sum_{\sigma' \in \{0,1\}} \pi_b(\sigma') \mathbb{E}_{\sigma \otimes \sigma'} \big(f((\sigma \otimes \sigma')_{t-t_1}) \mid \{\xi_s\}_{s \in [t_1, t]} \big) \Big) \\
&\qquad \leq \exp \big(-2 \operatorname{gap}(\mathcal{L}) t_1 \big) \mathrm{Var}_{\pi_{V_1}} \big(\mathbb{E}. \big(f(\eta_{t-t_1}) \mid \{\xi_s\}_{s \in [t_1, t]} \big) \big)
\end{aligned}
$$

where we used Proposition 5.1 to bound from below $\operatorname{gap}(\mathcal{L}_{V_0})$ by $\operatorname{gap}(\mathcal{L})$, and the convexity of the variance. The same procedure leads to

$$
\begin{aligned}
&\mathrm{Var}_{\pi_{V_1}} \big(\mathbb{E}. \big(f(\eta_{t-t_1}) \mid \{\xi_s\}_{s \in [t_1, t]} \big) \big) \\
&\quad \leq \exp \big(-2 \operatorname{gap}(\mathcal{L})(t_2 - t_1) \big) \mathrm{Var}_{\pi_{V_2}} \big(\mathbb{E}. \big(f((\eta)_{t-t_1-t_2}) \mid \{\xi_s\}_{s \in [t_2, t]} \big) \big)
\end{aligned}
$$

so that, by a simple induction, we get

$$
\mathrm{Var}_{\pi_{V_0}} \big(\mathbb{E}. \big(f(\eta_t) \mid \{\xi_s\}_{s \leq t} \big) \big) \leq e^{-2 \operatorname{gap}(\mathcal{L}) t} \mathrm{Var}_{\pi_{[a, \xi_t]}}(f).
$$

Plugging this bound into (8.2) leads to

$$
\begin{aligned}
\big| \mathbb{E}_\sigma(f(\sigma_t)) \big| &\leq \frac{e^{-\operatorname{gap}(\mathcal{L}) t}}{(p \wedge q)^{b-a}} \mathbb{E}_\sigma \big(\mathrm{Var}_{\pi_{[a, \xi_t]}}(f)^{1/2} \big) \\
&\leq \frac{e^{-\operatorname{gap}(\mathcal{L}) t}}{(p \wedge q)^{b-a}} \|f\|_\infty.
\end{aligned}
$$

Fix $\delta > 0$. From the latter, we finally get

$$\int dQ(\sigma) \left| \mathbb{E}_\sigma(f(\sigma_t)) - \pi(f) \right|$$

$$= \int dQ(\sigma) \left| \mathbb{E}_\sigma(f(\sigma_t)) \right| \mathbb{1}_{b \leq a' + \delta t} + \int dQ(\sigma) \left| \mathbb{E}_\sigma(f(\sigma_t)) \right| \mathbb{1}_{b > a' + \delta t}$$

$$\leq \left(e^{-\operatorname{gap}(\mathcal{L})t} \int \frac{dQ(\sigma)}{(p \wedge q)^{b(\sigma) - a + \delta t}} + Q(b > a' + \delta t) \right) \|f\|_\infty$$

$$= \left(e^{-\operatorname{gap}(\mathcal{L})t} \frac{1 - \alpha}{(p \wedge q)^{a' - a}} \sum_{k=0}^{\delta t} \left(\frac{\alpha}{p \wedge q} \right)^k + \alpha^{\delta t + 1} \right) \|f\|_\infty.$$

Now we distinguish between two cases. (i) If $\alpha < p \wedge q$, then one lets δ tend to $+\infty$ so that the expected result immediately follows. (ii) If $\alpha > p \wedge q$, then one chooses $\delta = \operatorname{gap}(\mathcal{L})/2 \log(\alpha/q)$. The expected result follows after some simple algebra and few simplifications left to the reader. This ends the proof.
$\square$

9. Out of equilibrium II, aging and plateau behavior: proof of Theorem 3.6

The aim of this section is to prove Theorem 3.6. We will not give the complete proof (that is quite long and involved). The interested reader may however found it in [26] under the additional condition on μ that $\mu([k, \infty)) > 0$ for any $k \in \mathbb{N}$. Here, we shall only explain briefly the extra ingredient that we use in order to remove this condition. To this aim, we will use different technical lemmas from [26] that we recall, for completeness, at the end of this section.

Proof. The technical condition on μ is used in Section 4.2 of [26], namely on the finite volume approximation. More precisely, it is used to guarantee the existence of infinitely many empty sites that remain empty up to a final fixed time t_N with N fixed (recall Definition 3.1). Such empty sites then allow to compare the East process on $[0, \infty)$ with the East process on a finite box $[0, L]$.

The strategy, in order to remove the technical condition on μ, that we adopt here is the following. We may prove that the process itself creates infinitely many empty sites that remain empty up to the final fixed time t_N. Given this, the proof remains unchanged with respect to [26]. In turn, this is a consequence of Lemma 9.1 below. Hence one only needs to prove Lemma 9.1.

Recall that $d \in \mathbb{N} \setminus \{0\}$ is the smallest length such that $\mu(\{d\}) > 0$, and that n_d is the smallest integer n such that $d \in [2^{n-1} + 1, 2^n]$ (we can assume $d > 1$ otherwise the setting is the same as in [26]). In the sequel, we fix $n = n_d$, for simplicity of notations.

Observe first that, since Q is renewal, for any $j \geq 1$, almost surely there exist infinitely many sites $\{x^{(m)}\}_{m \in \mathbb{N}}$ such that, $\sigma(x^{(m)} + id) = 0$ for $i = 0, \ldots, j$, and

$\sigma(y) = 1$ for all $y \in [x^{(m)}, x^{(m)} + jd] \setminus \{x^{(m)} + id, i = 0, \ldots j\}$, i.e. infinitely many collections of $j + 1$ consecutive empty sites at distance d one from the next one. Denote for simplicity by x_1 the first positive site satisfying the above property and set $x_i = x_1 + (i - 1)d$, $i = 2, \ldots, j + 1$. Define $\Lambda_i = [x_i + 1, x_{i+1} - 1]$, $i = 1, \ldots, j$ (see Figure 5).
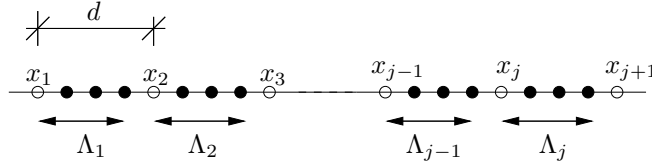

Figure 5. Points $x_1, \ldots, x_{j+1}$ and intervals $\Lambda_1, \ldots, \Lambda_j$.

Finally, introduce the following hitting times:

$$\tilde{\tau}_i = \inf \{t \geq 0 : \sigma_t(x_i) = 1\},$$
$$\tau_i = \inf \{t \geq 0 : \sigma_t(y) = 1 \text{ for any } y \in \Lambda_i \cup \{x_i\}\}.$$

Namely, $\tilde{\tau}_i$ is the first time that the vacancy at x_i is removed, while τ_i is the first time that the box $[x_i, x_{i+1} - 1]$ appears totally filled. Note that, by construction, it must be $\tilde{\tau}_i < \tau_i$ (trivially $\tilde{\tau}_i \leq \tau_i$ and observe that when the zero is removed from x_i for the first time there must be a zero on $x_i + 1$).

Lemma 9.1. *For any positive integer $j \geq 3$, there exists a positive constant $c(j)$ such that, almost surely*

$$\liminf_{q \downarrow 0} \mathbb{P}_\sigma \left(\tilde{\tau}_2 < \tau_2 < \tilde{\tau}_3 < \tau_3 < \cdots < \tilde{\tau}_{j-1} < \tau_{j-1} < \min(\tilde{\tau}_1, \tau_j) \right) \geq c(j).$$

Proof of Lemma 9.1. Recall that, for any interval Λ, $\mathbb{P}_\sigma^\Lambda$ denotes the law of the process on the interval Λ, with empty boundary condition, starting from σ. Fix a parameter $\delta \in (0, 1)$ (that will be chosen later) and, for $i, k = 1, \ldots, j - 1$, let E_i^k be the event that, during the time interval

$$I_k := \left[\frac{\delta(k - 1)}{j - 2} t_n, \frac{\delta k}{j - 2} t_n \right],$$

there have never been simultaneously n empty sites in the interval Λ_i:

$$E_i^k = \left\{ \sum_{y \in \Lambda_i} \{1 - \sigma_s(y)\} \leq n - 1 \ \forall s \in I_k \right\}.$$

Also, set A, B for the events that, in the time interval $[0, \delta t_n]$, there have never been simultaneously n empty sites in the boxes Λ_1, Λ_j respectively:

$$A = \Big\{ \sum_{y \in \Lambda_1} \{1 - \sigma_s(y)\} \leq n - 1 \; \forall s \in [0, \delta t_n] \Big\},$$

$$B = \Big\{ \sum_{y \in \Lambda_j} \{1 - \sigma_s(y)\} \leq n - 1 \; \forall s \in [0, \delta t_n] \Big\}.$$

Finally, let G be the event of the lemma, $F_i^k = \{\tau_i \in I_k\}$ and

$$G_r^k = \Big(\bigcap_{k+2 \leq \ell \leq r} E_\ell^k \Big) \bigcap F_{k+1}^k, \quad k = 1, \ldots, j-2, \; r \geq k+2.$$

By words, the event G_r^k can be described as follows: during the time interval I_k the box $[x_{k+1}, x_{k+2}) = \{x_{k+1}\} \cup \Lambda_{k+1}$ appears totally filled for the first time, while in the same time interval I_k there are always less than n simultaneous empty sites in all other boxes $\Lambda_\ell = (x_\ell, x_{\ell+1})$ with $k + 2 \leq \ell \leq r$.

Figure 6 illustrates the event G of the lemma (a special realization of it) and the evolution of the configuration (with positive probability) from time 0 till time δt_n. This may help the reader to follow the proof.

We now consider the event $\cap_{k=1}^{j-2} G_{j-1}^k$. Due to Remark 4.1, for $1 \leq k \leq j-2$ the empty site x_{k+1} remains empty during the time intervals $I_1 \cup I_2 \cup \cdots \cup I_{k-1}$ and is filled together with the whole box Λ_{k+1} during the time interval I_k. In particular, the event $\cap_{k=1}^{j-2} G_{j-1}^k$ implies that

$$\tilde{\tau}_2 < \tau_2 < \tilde{\tau}_3 < \tau_3 < \cdots < \tilde{\tau}_{j-1} < \tau_{j-1}.$$

Always by Remark 4.1 the events A, B imply that the empty sites x_1, x_j remain empty up to time δt_n. As a consequence we conclude that

$$\Big(\bigcap_{k=1}^{j-2} G_{j-1}^k \Big) \cap A \cap B \subset G,$$

so that

$$\mathbb{P}_\sigma(G) \geq \mathbb{P}_\sigma \Big(A \, \Big| \, \Big(\bigcap_{k=1}^{j-2} G_{j-1}^k \Big) \cap B \Big) \mathbb{P}_\sigma \Big(\bigcap_{k=1}^{j-2} G_{j-1}^k \, \Big| \, B \Big) \mathbb{P}_\sigma(B). \qquad (9.1)$$

By Lemma 9.2 below, it holds that

$$\mathbb{P}_\sigma \Big(A \, \Big| \, \Big(\bigcap_{k=1}^{j-2} G_{j-1}^k \Big) \cap B \Big) \geq 1 - d\delta, \quad \mathbb{P}_\sigma(B) \geq 1 - d\delta. \qquad (9.2)$$
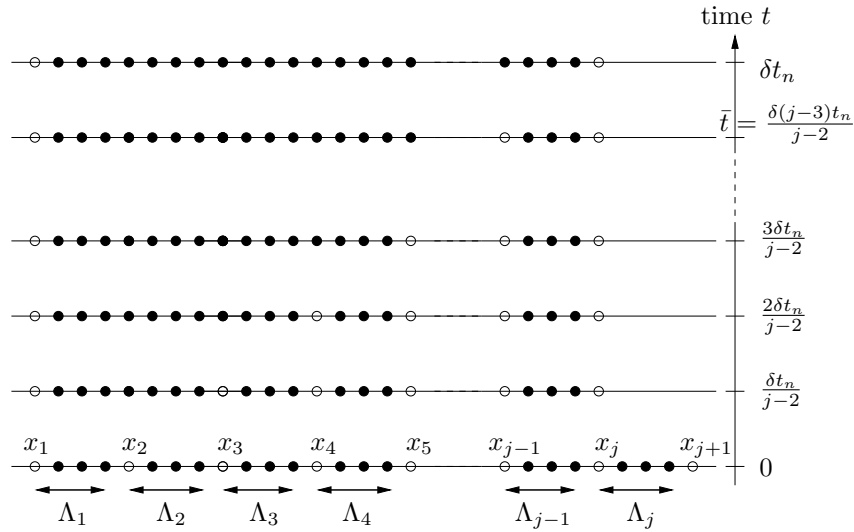
Figure 6. The event G is implied by the evolution illustrated on the picture. Site x_j remains empty (up to time δt_n) and acts as a boundary condition of $[x_1, x_j - 1]$ during this time interval. $\bar{\sigma}$ is the configuration at time $\bar{t} = \delta t_n (j-3)/(j-2)$.

On the other hand, as already explained above, the event B guarantees that $\sigma_s(x_j) = 0$ for all $s \in [0, \delta t_n]$, so that, up to time δt_n, the process in infinite volume restricted to $[x_1, x_j - 1]$ coincides with the process in the finite volume $[x_1, x_j - 1]$ (with empty boundary condition at site x_j). Hence,

$$\mathbb{P}_\sigma\left(\bigcap_{k=1}^{j-2} G_{j-1}^k \,\Big|\, B\right) = \mathbb{P}_\sigma^{[x_1, x_j - 1]}\left(\bigcap_{k=1}^{j-2} G_{j-1}^k\right). \tag{9.3}$$

Our aim is to bound the latter inductively. To that purpose set

$$X = \mathbb{P}_{\sigma_{0\mathbb{1}}}^{[1,d]}\left(\tau \le \frac{\delta t_n}{j-2}\right),$$

i.e. the probability that, starting from $\sigma_{0\mathbb{1}}$, within time $\delta t_n/(j-2)$ the box $[1, d]$ appears completely filled.

Let also $\bar{\sigma}$ be the configuration obtained from σ by removing the empty sites $x_2, x_3, \ldots, x_{j-2}$ (i.e. $\bar{\sigma}(x) = \sigma(x)$ for all $x \ne x_2, x_3, \ldots, x_{j-2}$ and $\bar{\sigma}(x_i) = 1$ for $i = 2, 3, \ldots, j-2$, see Figure 6). Set for simplicity $\bar{t} = \delta(j-3)t_n/(j-2)$. Conditioning on the σ-algebra $\mathcal{F}_{\bar{t}}$ generated by the Poisson processes and coin

tosses up to time $\bar{t}$, and using the Markov Property, we conclude that

$$\mathbb{P}_\sigma^{[x_1,x_j-1]}\left(\bigcap_{k=1}^{j-2} G_{j-1}^k\right) \tag{9.4}$$

$$\geq X\mathbb{P}_\sigma^{[x_1,x_j-1]}\left(\left(\bigcap_{k=1}^{j-3} G_{j-1}^k\right)\bigcap\{\sigma_{\bar{t}}=\bar{\sigma}\}\right)$$

$$\geq X\left[\mathbb{P}_\sigma^{[x_1,x_j-1]}\left(\bigcap_{k=1}^{j-3} G_{j-1}^k\right) - \mathbb{P}_\sigma^{[x_1,x_j-1]}\left(\left(\bigcap_{k=1}^{j-3} G_{j-1}^k\right)\bigcap\{\sigma_{\bar{t}}\neq\bar{\sigma}\}\right)\right].$$

We deal with the two terms of the latter separately. Set $\bar{B}$ for the event that, up to time $\bar{t}$, there have never been simultaneously n empty sites in the interval Λ_{j-1}:

$$\bar{B} = \left\{\forall s\in[0,\bar{t}], \sum_{y\in\Lambda_{j-1}}\{1-\sigma_s(y)\}\leq n-1\right\}.$$

Then, we observe that $\cap_{k=1}^{j-3}G_{j-1}^k = \cap_{k=1}^{j-3}G_{j-1}^k\cap\bar{B}$. Hence, again using Lemma 9.2 below and the fact that the event $\bar{B}$ guarantees that $\sigma_s(x_{j-1})=0$ for all $s\in[0,\bar{t}]$, we have

$$\mathbb{P}_\sigma^{[x_1,x_j-1]}\left(\bigcap_{k=1}^{j-3} G_{j-1}^k\right) = \mathbb{P}_\sigma^{[x_1,x_j-1]}\left(\bigcap_{k=1}^{j-3} G_{j-1}^k \,\Big|\, \bar{B}\right)\mathbb{P}_\sigma^{[x_1,x_j-1]}(\bar{B})$$

$$\geq \mathbb{P}_\sigma^{[x_1,x_{j-1}-1]}\left(\bigcap_{k=1}^{j-3} G_{j-2}^k\right)(1-d\delta). \tag{9.5}$$

Due to (9.4) and (9.5), from Claim 9.1 and Claim 9.2 below, we conclude that, for δ and q small enough,

$$\mathbb{P}_\sigma^{[x_1,x_j-1]}\left(\bigcap_{k=1}^{j-2} G_{j-1}^k\right) \geq c_1\mathbb{P}_\sigma^{[x_1,x_{j-1}-1]}\left(\bigcap_{k=1}^{j-3} G_{j-2}^k\right) - c_2 q$$

for some constants $c_1,c_2>0$ independent on q. A simple iteration (adapting Claim 9.2) allows us to end up with

$$\mathbb{P}_\sigma^{[x_1,x_j-1]}\left(\bigcap_{k=1}^{j-2} G_{j-1}^k\right) \geq c_1'\mathbb{P}_\sigma^{[x_1,x_3-1]}(F_2^1) - c_2 q = c_1'X - c_2 q, \tag{9.6}$$

for some constant c' depending on j but independent of q. The result of Lemma 9.1 follows then from (9.1), (9.2), (9.3) and (9.6).

Claim 9.1. *There exists $\delta_o > 0$ and a constant $c = c(d, j) > 0$ (independent of q) such that, for ant $\delta \in (0, \delta_0)$ and any q small enough*

$$X := \mathbb{P}^{[1,d]}_{\sigma_{01}}\left(\tau \leq \frac{\delta t_n}{j-2}\right) \geq c.$$

Claim 9.2. *There exists a constant $c = c(j)$ (independent of q) such that*

$$\mathbb{P}^{[x_1, x_j - 1]}_{\sigma}\left(\left(\bigcap_{k=1}^{j-3} G^k_{j-1}\right) \cap \{\sigma_{\bar{t}} \neq \bar{\sigma}\}\right) \leq cq.$$

Proof of Claim 9.1. Set $M = \delta t_n/[(j-2)T_n]$ where $T_n := (1/q)^{(n-1)(1+3\varepsilon)}$ is defined in [26, Section 3.2] with some fixed (small) parameter $\varepsilon > 0$. Hence, one has to study $Y_M := \mathbb{P}^{[1,d]}_{\sigma_{01}}(\tau > \delta t_n/(j-2)) = \mathbb{P}^{[1,d]}_{\sigma_{01}}(\tau > MT_n)$. To bound Y_M we use an induction procedure. Namely, by the Strong Markov Property, we have

$$\begin{aligned}
Y_M &= \mathbb{P}^{[1,d]}_{\sigma_{01}}(\tau > (M-1)T_n + T_n) \\
&= \mathbb{E}^{[1,d]}_{\sigma_{01}}\left(\mathbb{1}_{\tau > (M-1)T_n} \mathbb{P}^{[1,d]}_{\sigma_{(M-1)T_n}}(\tau > T_n)\right) \\
&= \mathbb{E}^{[1,d]}_{\sigma_{01}}\left(\mathbb{1}_{\tau > (M-1)T_n} \mathbb{P}^{[1,d]}_{\sigma_{01}}(\tau > T_n)\right) \\
&\quad + \mathbb{E}^{[1,d]}_{\sigma_{01}}\left(\mathbb{1}_{\tau > (M-1)T_n} \mathbb{P}^{[1,d]}_{\sigma_{(M-1)T_n}}(\tau > T_n)\mathbb{1}_{\sigma_{(M-1)T_n} \neq \sigma_{01}}\right) \\
&\leq Y_{M-1}Y_1 + \mathbb{P}^{[1,d]}_{\sigma_{01}}\left(\{\tau > (M-1)T_n\} \cap \{\sigma_{(M-1)T_n} \neq \sigma_{01}\}\right).
\end{aligned}$$

We observe that the event $\{\tau > (M-1)T_n\}$ together with $\{\sigma_{(M-1)T_n} \neq \sigma_{01}\}$ guarantees that the configuration $\sigma_{(M-1)T_n}$ has an empty site in $[2, d]$, which was not present in σ_{01}. Indeed, if that was false then it should be $\sigma_{(M-1)T_n} = \sigma_{01}$ or $\sigma_{(M-1)T_n} = \sigma_{1}$, thus leading to a contradiction. Hence, by Lemma 9.2 we get

$$\mathbb{P}^{[1,d]}_{\sigma_{01}}\left(\{\tau > (M-1)T_n\} \cap \{\sigma_{(M-1)T_n} \neq \sigma_{01}\}\right) \leq q.$$

In turn, $Y_M \leq Y_{M-1}Y_1 + q$. Set $X_k = Y_k - q/(1 - Y_1)$ so that $X_M \leq X_{M-1}Y_1$ which, after iteration, and using that $-q/(1 - Y_1) \leq 0$, leads to $Y_M \leq Y_1^M + q/(1 - Y_1)$.

To end the proof we need to examine the term $Y_1 = \mathbb{P}^{[1,d]}_{\sigma_{01}}(\tau > T_n)$. To that aim, using $\tilde{\tau}$ defined above, we have the following decomposition

$$Y_1 = \mathbb{P}^{[1,d]}_{\sigma_{01}}(\tilde{\tau} > T_n) + \mathbb{P}^{[1,d]}_{\sigma_{01}}(\tau > T_n > \tilde{\tau}).$$

As above, the event $\{\tau > T_n > \tilde{\tau}\}$ implies that, at time T_n there is an empty site in $[2, d]$, which was not present in σ_{01}. Hence, by Lemma 9.2, $\mathbb{P}^{[1,d]}_{\sigma_{01}}(\tau > T_n > \tilde{\tau}) \leq q$. Applying Lemma 9.3 to the first term in the right-hand side of the latter, it follows that $Y_1 \leq \exp\{-cT_n/t_n\} + q$ for some constant c that does not

depend on q. Therefore, as soon as δ is small enough, expanding in the limit $q \to 0$, we get

$$Y_M \leq \left(\exp\left\{ -\frac{cT_n}{t_n} \right\} + q \right)^M + \frac{q}{1 - \exp\{-cT_n/t_n\} - q}$$
$$= \exp\left\{ -\frac{c\delta}{j-2} \right\} + o(1)$$

where $o(1)$ goes to zero as q goes to zero.

All the previous computations lead to

$$X = 1 - Y_M \geq 1 - \exp\left\{ -\frac{c\delta}{j-2} \right\} + o(1).$$

This ends the proof of the claim. □

Proof of Claim 9.2. First we observe that $\sigma_{\bar{t}} \neq \bar{\sigma}$ implies either that (a) there exists an empty site, at time $\bar{t}$ that was not present at time 0 (i.e. there exists an empty site in $[x_1 + 1, x_j - 1] \setminus \{x_2, x_3, \ldots, x_{j-1}\}$), or (b) at least one of the sites x_k, $k \in \{2, 3, \ldots, j-2\}$, is empty. Thanks to Lemma 9.2 below, case (a) has probability less than or equal to q and we can focus on case (b). Assume for simplicity that x_2 is an empty site at time $\bar{t}$ (the other cases can be treated analogously). We follow the lines of [26, Lemma 4.2], appealing to the graphical construction of Section 2.3. Given $m \geq 0$, we write $\mathcal{A}_m$ for the event that the last legal ring at x_2 before time $\bar{t}$ (which is well defined because x_2 has been filled during the time interval $[0, \delta t_n/(j-2)]$) occurs at time $t_{y_1,m}$. Recall that (i) at the time $t_{y_1,m}$ the current configuration resets its value at x_2 to the value of an independent Bernoulli$(1-q)$ random variable $s_{x_2,m}$ and (ii) that $\mathcal{A}_m$ depends only on the Poisson processes associated to sites $x \geq x_2$ and on the Bernoulli variables associated to sites $x > x_2$. Hence we conclude that (we drop the superscript $[x_1, x_j - 1]$)

$$\mathbb{P}_\sigma \left(\left(\bigcap_{k=1}^{j-3} G_{j-1}^k \right) \cap \{\sigma_{\bar{t}}(x_2) = 0\} \right)$$
$$= \mathbb{P}_\sigma \left(\bigcup_{m=1}^{\infty} \left(\mathcal{A}_m \cap \{s_{x_2,m} = 0\} \right) \bigcap \left(\bigcap_{k=1}^{j-3} G_{j-1}^k \right) \right)$$
$$\leq \sum_{m=1}^{\infty} \mathbb{P}_\sigma(s_{x_2,m} = 0) \, \mathbb{P}_\sigma(\mathcal{A}_m) \leq q.$$

The claim follows. □

The proof of Lemma 9.1 is complete. □

The proof of Theorem 3.6 is complete.   □

Below we recall some useful technical facts borrowed from [26].

Given a configuration σ, let $\mathcal{Z}(\sigma) = \{x \in \mathbb{Z} : \sigma(x) = 0\}$ be the set of all empty sites of σ.

Lemma 9.2 (Lemma 4.2 of [26]). *Fix $\sigma \in \Omega$, $t \geq 0$ and $k \in \mathbb{N}$. Let $V = [0, a] \subset \mathbb{Z}$ and let $\{y_1, \ldots, y_k\} \subset V \setminus \mathcal{Z}(\sigma)$. Let finally $\mathcal{F}$ be the σ-algebra generated by the Poisson processes and coin tosses in $\mathbb{Z} \setminus V$. Then*

$$\mathbb{P}_\sigma^\Lambda \big(\{y_1, \ldots, y_k\} \subset \mathcal{Z}(\sigma_s) \mid \mathcal{F}\big) \leq q^k, \quad \forall s > 0. \tag{9.7}$$

Moreover

$$\mathbb{P}_\sigma \big(\exists\, s \leq t : \{y_1, \ldots, y_k\} \subset \mathcal{Z}(\sigma_s) \mid \mathcal{F}\big) \leq atq^k. \tag{9.8}$$

Lemma 9.3. *Let $T_n = (1/q)^{(n-1)(1+3\varepsilon)}$ for some fixed parameter $\varepsilon \in (0, 1)$. Let $\sigma_{0\mathbb{1}}$ be the configuration, on $[1, d]$, with only one empty site at 1, and $\tilde{\tau} := \inf\{s : \sigma_s(1) = 1\}$. Then,*

$$\mathbb{P}_{\sigma_{0\mathbb{1}}}^{[1,d]}(\tilde{\tau} > T_n) \leq \exp\left\{ -\frac{cT_n}{t_n} \right\}$$

for some constant $c = c(d, \varepsilon)$ that does not depend on q.

Proof of Lemma 9.3. The result of Lemma 9.3 follows from [26, Lemma 3.4] together with the definition of $\lambda_n(d)$ given in [26, equation (3.4)].   □

Remark 9.1. Note that, in [26, Lemma 4.2], the result holds for configurations living in $\Omega_{\mathbb{Z}_+}$. However, the proof can easily be adapted to $\Omega = \Omega_{\mathbb{Z}}$ as stated in Lemma 9.2.

Acknowledgements

References

[1] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto and G. Scheffer (2000) *Sur les Inégalités de Sobolev Logarithmiques.* Panoramas et Syntheses **10**, Soc. Math. de France, Paris.

[2] D. Aldous and P. Diaconis (2002) The asymmetric one-dimensional constrained Ising model: rigorous results. *J. Stat. Phys.* **107** (5–6), 945–975.

[3] A. ASSELAH AND P. DAI PRA (2001) Quasi-stationary measures for conservative dynamics in the infinite lattice. *Ann. Prob.* **29** (4), 1733–1754.

[4] L. BERTINI, N. CANCRINI AND F. CESI (2002) The spectral gap for a Glauber-type dynamics in a continuous gas. *Ann. Inst. H. Poincaré, Probabilités et Statistiques* **38** (1), 91–108.

[5] O. BLONDEL, N. CANCRINI, F. MARTINELLI, C. ROBERTO AND C. TONINELLI (2013) Fredrickson – Andersen one spin facilitated model out of equilibrium. *Markov Processes and Relat. Fields* **19** (3), 383–406.

[6] S.G. BOBKOV AND M. LEDOUX (1998) On modified logarithmic Sobolev inequalities for Bernoulli and Poisson measures. *J. Func. Anal.* **156** (2), 347–365.

[7] T. BODINEAU, V. LECOMTE AND C. TONINELLI (2012) Finite size scaling of the dynamical free-energy in a kinetically constrained model. *J. Stat. Phys.* **147** (1), 1–17.

[8] S.G. BOBKOV AND P. TETALI (2006) Modified logarithmic Sobolev inequalities in discrete settings. *J. Theor. Probab.* **19** (2), 289–336.

[9] T. BODINEAU AND C. TONINELLI (2012) Activity phase transition for constrained dynamics. *Commun. Math. Phys.* **311** (2), 357–396.

[10] F. CORBERI AND L.F. CUGLIANDOLO (2009) Out-of-equilibrium dynamics of the spiral model. *J. Stat. Mech.* P09015.

[11] F. CHUNG, P. DIACONIS AND R. GRAHAM (2001) Combinatorics for the East model. *Adv. Appl. Math.* **27** (1), 192–206.

[12] P. CAPUTO, P. DAI PRA AND G. POSTA (2009) Convex entropy decay via the Bochner – Bakry – Emery approach. *Ann. Inst. H. Poincaré, Probabilités et Statistiques* **45** (3), 734–753.

[13] N. CANCRINI, F. MARTINELLI, C. ROBERTO AND C. TONINELLI (2008) Kinetically constrained spin models. *Probab. Theory and Relat. Fields* **140** (3–4), 459–504.

[14] N. CANCRINI, F. MARTINELLI, R. SCHONMANN AND C. TONINELLI (2010) Facilitated oriented spin models: some non equilibrium results. *J. Stat. Phys.* **138** (6), 1109–1123.

[15] P. CAPUTO AND G. POSTA (2007) Entropy dissipation estimates in a zero-range dynamics. *Probab. Theory and Relat. Fields* **139** (1–2), 65–87.

[16] A. CRISANTI, F. RITORT, A. ROCCO AND M. SELLITTO (2000) Inherent structures and non-equilibrium dynamics of $1d$ constrained kinetic models: a comparison study. *J. Chem. Phys.* **113**, 10615–10647.

[17] B. DERRIDA (1995) Coarsening phenomena in one dimension. In: *Complex Systems and Binary Networks* (Guanajuato, 1995), Lect. Notes Phys. **461**, Springer, Berlin, 164–182.

[18] B. DERRIDA, C. GODRÈCHE AND I. YEKUTIELI (1990) Stable distributions of growing and coalescing droplets. *Europhys. Lett.* **12** (5), 385–390.

[19] B. DERRIDA, C. GODRÈCHE AND I. YEKUTIELI (1991) Scale-invariant regimes in one-dimensional models of growing and coalescing droplets. *Phys. Rev. A* **44** (10), 6241–6251.

[20] P. DAI PRA, A. M. PAGANONI AND G. POSTA (2002) Entropy inequalities for unbounded spin systems. *Ann. Prob.* **30** (4), 1959–1976.

[21] P. DIACONIS AND L. SALOFF-COSTE (1996) Logarithmic Sobolev inequalities for finite Markov chains. *Ann. Appl. Probab.* **6** (3), 695–750.

[22] R. DURRETT (1995) *Lecture Notes on Particle Systems and Percolation.* Lect. Notes Math. **1608**.

[23] G.H. FREDRICKSON AND H.C. ANDERSEN (1984) Kinetic Ising model of the glass transition. *Phys. Rev. Lett.* **53**, 1244–1247.

[24] G.H. FREDRICKSON AND H.C. ANDERSEN (1985) Facilitated kinetic Ising models and the glass transition. *J. Chem. Phys.* **83**, 5822–5831.

[25] A. FAGGIONATO, F. MARTINELLI, C. ROBERTO AND C. TONINELLI (2012) Universality in one dimensional hierarchical coalescence processes. *Ann. Prob.* **40**, 1377–1435.

[26] A. FAGGIONATO, F. MARTINELLI, C. ROBERTO AND C. TONINELLI (2012) Aging through hierarchical coalescence in the East model. *Commun. Math. Phys.* **309**, 459–495.

[27] A. FAGGIONATO, C. ROBERTO AND C. TONINELLI (2013) Universality for one-dimensional hierarchical coalescence processes with double and triple merges. To appear in *J. Appl. Probab.*

[28] J.P. GARRAHAN, R.L. JACK, V. LECOMTE, E. PITARD, K. VAN DUIJVENDIJK AND K. VAN WIJLAND (2007) Dynamic first-order transition in kinetically constrained models of glasses. *Phys. Rev. Lett.* **98**, P195702.

[29] J.P. GARRAHAN, R.L. JACK, V. LECOMTE, E. PITARD, K. VAN DUIJVENDIJK AND K. VAN WIJLAND (2009) First-order dynamical phase transition in models of glasses: an approach based on ensembles of histories. *J. Phys. A* **42**, P075007.

[30] J.P. GARRAHAN AND M.E.J. NEWMAN (2000) Inherent structures and non-equilibrium dynamics of 1d constrained kinetic models: a comparison study. *Phys. Rev. E* **62**, 7670–7680.

[31] S. GOEL (2004) Modified logarithmic Sobolev inequalities for some models of random walk. *Stoch. Process. Appl.* **114** (1), 51–79.

[32] F. GAO AND J. QUASTEL (2003) Exponential decay of entropy in the random transposition and Bernoulli–Laplace models. *Ann. Appl. Probab.* **13** (4), 1591–1600.

[33] L. GROSS (1975) Logarithmic Sobolev inequalities. *Amer. J. Math.* **97** (4), 1061–1083.

[34] J.P. GARRAHAN, P. SOLLICH AND C. TONINELLI (2011) Kinetically constrained models. In: *Dynamical Heterogeneities in Glasses, Colloids, and Granular Media and Jamming Transitions*, L. Berthier, G. Biroli, J-P. Bouchaud, L. Cipelletti and W. van Saarloos (eds.), International Series of Monographs in Physics, Oxford Univ. Press., 341–369.

[35] P. Harrowell (1993) Visualizing the collective motion responsible for α and β relaxations in a model glass. *Phys. Rev. E* **48**, 4359–4363.

[36] R.A. Holley and D.W. Stroock (1989) Uniform and L^2 convergence in one-dimensional stochastic Ising models. *Commun. Math. Phys.* **123** (1), 85–93.

[37] J. Jäckle and S. Eisinger (1991) A hierarchically constrained kinetic Ising model. *Z. Phys. B: Condens. Matter* **84** (1), 115–124.

[38] T.M. Liggett (1985) *Interacting Particle Systems.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **276**, Springer-Verlag, New York.

[39] T.M. Liggett (1999) *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **324**, Springer-Verlag, Berlin.

[40] S. Leonard, P. Mayer, P. Sollich, L. Berthier and J.P. Garrahan (2007) Non-equilibrium dynamics of spin facilitated glass models. *J. Stat. Mech.* **7**, P07017.

[41] F. Martinelli (1999) Lectures on Glauber dynamics for discrete spin models. In: *Lectures on Probability Theory and Statistics* (Saint-Flour, 1997), Springer, Berlin, 93–191.

[42] M. Merolle, J.P. Garrahan and D. Chandler (2005) Space-time thermo-dynamics of the glass transition. *Proc. Nat. Acad. Sci. USA* **102**, 10837–10840.

[43] E. Mossel, K. Oleszkiewicz and A. Sen (2013) On reverse hypercontractivity. *Geom. Funct. Anal.* **23** (3), 1062–1097.

[44] Y. Peres and A. Sly (2013) Mixing of the upper triangular matrix walk. *Probab. Theory Relat. Fields* **156** (3–4), 581–591.

[45] F. Ritort and P. Sollich (2003) Glassy dynamics of kinetically constrained models. *Adv. Phys.* **52** (4), 219–342.

[46] P. Sollich and M.R. Evans (1999) Glassy time-scale divergence and anomalous coarsening in a kinetically constrained spin chain. *Phys. Rev. Lett.* **83**, 3238–3241.

[47] P. Sollich and M.R. Evans (2003) Glassy dynamics in the asymmetrically constrained kinetic Ising chain. *Phys. Rev. E* **68** (3), 031504.