AGING THROUGH HIERARCHICAL COALESCENCE IN THE EAST MODEL

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ABSTRACT. We rigorously analyze the low temperature non-equilibrium dynamics of the East model, a special example of a one dimensional oriented kinetically constrained particle model, when the initial distribution is different from the reversible one and for times much smaller than the global relaxation time. This setting has been intensively studied in the physics literature to analyze the slow dynamics which follows a sudden quench from the liquid to the glass phase. In the limit of zero temperature (*i.e.* a vanishing density of vacancies) and for initial distributions such that the vacancies form a renewal process we prove that the density of vacancies, the persistence function and the two-time autocorrelation function behave as staircase functions with several plateaux. Furthermore the two-time autocorrelation function displays an aging behavior. We also provide a sharp description of the statistics of the domain length as a function of time, a domain being the interval between two consecutive vacancies. When the initial renewal process has finite mean our results confirm (and generalize) previous findings of the physicists for the restricted case of a product Bernoulli measure. However we show that a different behavior appears when the initial domain distribution is in the attraction domain of a α-stable law. All the above results actually follow from a more general result which says that the low temperature dynamics of the East model is very well described by that of a certain *hierarchical coalescence process*, a probabilistic object which can be viewed as a hierarchical sequence of suitably linked coalescence processes and whose asymptotic behavior has been recently studied in [14].

Mathematics Subject Classification: 60K35, 82C20.

Keywords: kinetically constrained models, non-equilibrium dynamics, coalescence, metastability, aging, interacting particle systems.

1. INTRODUCTION

Facilitated or kinetically constrained spin (particle) models (KCSM) are interacting particle systems which have been introduced in the physics literature [16, 17, 19] to model liquid/glass transition and more generally "glassy dynamics" (see e.g. [26, 20]). A configuration is given by assigning to each vertex x of a (finite or infinite) connected graph $\mathcal{G}$ its occupation variable $\eta(x) \in \{0, 1\}$ which corresponds to an empty or filled site, respectively. The evolution is given by a Markovian stochastic dynamics of Glauber type. Each site with rate one refreshes its occupation variable to a filled or to an empty state with probability $1 - q$ or q respectively provided that the current configuration around it satisfies an a priori specified constraint. For each site x the corresponding

Work supported by the European Research Council through the "Advanced Grant" PTRELSS 228032.

constraint does not involve $\eta(x)$, thus detailed balance w.r.t. the Bernoulli($1-q$) product measure π can be easily verified and the latter is an invariant reversible measure for the process.

One of the most studied KCSM is the East model [19]. It is a one-dimensional model ($\mathcal{G} = \mathbb{Z}$ or $\mathcal{G} = \mathbb{Z}_+ = \{0, 1, \dots\}$) and particle creation/annihilation at a given site x can occur only if its right neighbor $x+1$ is empty. The model is ergodic for any $q \neq 0, 1$ with a positive spectral gap [1, 4] and it relaxes to the equilibrium reversible measure exponentially fast even when started from e.g. any non-trivial product measure [5]. However, as $q \downarrow 0$, the relaxation time $T_{\rm relax}(q)$ diverges very fast, $T_{\rm relax} \sim \big(\exp(\lambda \log(1/q)^2)\big)$ with a sharp constant λ (see [4]). A key issue, both from the mathematical and the physical point of view, is therefore that of describing accurately the evolution at $q \ll 1$ when the initial distribution is different from the reversible one and for time scales which are large but still much smaller than $T_{\rm relax}(q)$ when the exponential relaxation to the reversible measure takes over.

An initial distribution which is often considered in the physics literature is the Bernoulli distribution at a density $1/2$ [28, 27]. We refer the interested reader to [26, 21, 8, 18, 7] for the relevance of this setting in connection with the study of the liquidglass transition as well as for details for KCMS different from East model.

Let us give a rough picture of the non-equilibrium dynamics of the East model as $q \downarrow 0$. Since the equilibrium vacancy density is very small, most of the non-equilibrium evolution will try to remove the excess of vacancies present in the initial distribution and will thus be dominated by the coalescence of domains corresponding to the intervals separating two consecutive vacancies. Of course this process must necessarily occur in a kind of cooperative way because, in order to remove a vacancy, other vacancies must be created nearby (to its right). Since the creation of vacancies requires the overcoming of an *energy barrier*, in a first approximation the non-equilibrium dynamics of the East model for $q \ll 1$ is driven by a non-trivial energy landscape.

In order to better explain the structure of this landscape suppose that we start from a configuration with only two vacancies located at the sites a and $a + \ell$, with $\ell \in [2^{n-1} + 1, \dots, 2^n]$. In this case a nice combinatorial argument (see [6] and also [27]) shows that, in order to remove the vacancy at a within time t, there must exists $s \le t$ such that the number of vacancies inside the interval $(a, a+\ell)$ at time s is at least n. It is rather easy to show that at any given time s the probability of observing n vacancies in $(a, a+\ell)$ is $O(q^n)$ so that, in order to have a non negligible probability of observing the disappearance of the vacancy at a, we need to wait an *activation time* $t_n = O(1/q^n)$. In a more physical language the energy barrier which the system must overcome is $O(\log_2 \ell)$. As it is the case in many metastable phenomena, once the system decides to overcome the barrier and kill the vacancy, it does it in a time scale much smaller than the activation time. In our case this scale is $t_{n-1} = 1/q^{n-1}$.

The above argument indicates the following heuristic picture.

(i) A *hierarchical structure* of the activation times $t_n = 1/q^n$ (and of the energy landscape) well separated one from the other for $q \ll 1$.
(ii) A kind of metastable behavior of the dynamics which removes vacancies in a hierarchical fashion.
(iii) Since the characteristic time scales t_n are well separated one from the other, the evolution should show *active* and *stalling* periods. During the n^{th}-active period,

identified with e.g. the interval $[t_n^{1-\epsilon}, t_n^{1+\epsilon}]$, $\epsilon \ll 1$, only the vacancies with another vacancy to their right at distance less than 2^n can be removed. At the end of an active period no vacancies with distance less than $2^n + 1$ are present anymore as well as no extra (*i.e.* not present at time $t = 0$) vacancies. During the n^{th}-stalling period $[t_n^{1+\epsilon}, t_{n+1}^{1-\epsilon}]$ nothing interesting happens in the sense that none of the vacancies present at the beginning of the period are destroyed and no new vacancies are created at the end of the period.

Clear the above scenario, and particularly the presence of active and stalling periods, implies that physical quantities like the persistence function or the density of vacancies should behave as a staircase function with several *plateaux* and that *aging* should occur for two-time quantities as the two time-autocorrelation.

Such a general picture was somehow suggested in two interesting physics papers [28, 27] and some of the conclusions (properties (iv) above) were indeed observed in numerical simulations [28, 21]. In [28] the true East dynamics was replaced with that of a certain *hierarchical coalescence model* mimicking the features (i)–(iii) described above. In turn, under the assumption that the interval between two consecutive vacancies (domain) in the n-th stalling period rescaled by 2^n has a well defined limiting distribution as $n \to \infty$, the form of this limiting distribution when the initial distribution is a Bernoulli product measure has been computed for the coalescence model.

Partly motivated by the above discussion and partly by other coalescence models in statistical physics with a mean field structure (see e.g. [10, 11, 12, 3]), the present authors introduced in [14] a large class of hierarchical coalescence models and: (1) proved the existence of a scaling limit under very general assumptions, (2) proved the universality of the scaling limit depending only on general features of the initial distribution and not on the details of the model. We refer the reader to Section 3 for more details and to [14] for a much more general setting.

In this paper, besides providing a mathematical derivation of the above mentioned heuristic picture, we rigorously establish aging and plateau behavior (Theorem 2.5). Furthermore (Theorem 2.6) we prove a scaling limit for: i) the inter-vacancy distance and ii) the position of the first vacancy for the model on the positive half line. In particular we prove that this scaling limit is universal if the initial renewal process has finite mean. If instead the initial distribution is the domain of attraction of an α-stable law, $\alpha \in (0, 1)$, the scaling limit is different and falls in another universality class depending on α. In order to establish the above results we actually prove a result which is more fundamental and of independent interest. Namely we show that with probability tending to one as $q \downarrow 0$, the non equilibrium dynamics of the East model starting from a renewal process is well approximated (in variation distance) by a suitable hierarchical coalescence process with rates depending on suitable large deviation probabilities of the East model (Theorem 3.8).

CONTENTS

2. The East process: definition and main results

In what follows we will use the notation $\mathbb{N} := 1, 2, \dots$ and $\mathbb{Z}_+ := 0, 1, 2, \dots$. We will focus on the East process on $\mathbb{Z}_+$ and explain in Section 7 how the result can be extended to the process defined on $\mathbb{Z}$. The East process on $\mathbb{Z}_+$ with parameter $q \in [0, 1]$ is an interacting particle systems with a Glauber type dynamics on the configuration space $\Omega := \{0, 1\}^{\mathbb{Z}_+}$, reversible with respect to the product probability measure $\pi := \prod_{x \in \mathbb{Z}_+} \pi_x$, π_x being the Bernoulli$(1 - q)$ measure. Since we are interested in the small q regime throughout the following we will assume $q \le 1/2$.

Remark 2.1. *Sometimes in the physical literature the parameter q is written as $q = \frac{e^{-\beta}}{1+e^{-\beta}}$ where β is the inverse temperature so that the limit $q \downarrow 0$ corresponds to the zero temperature limit.*

Elements of Ω will usually be denoted by the Greek letters $\sigma, \eta, \dots$ and $\sigma(x)$ will denote the occupancy variable at the site x. The restriction of a configuration σ to a subset Λ of $\mathbb{Z}_+$ will be denoted by σ_Λ. The set of empty sites (or zeros in the sequel) of a configuration σ will be denoted by $\mathcal{Z}(\sigma)$ and they will often be referred to as $x_0 < x_1 < \dots$ without the specification of the configuration if clear from the context.

The East process can be informally described as follows. Each vertex x waits an independent mean one exponential time and then, provided that the current configuration σ satisfies the constraint $\sigma(x + 1) = 0$, the value of $\sigma(x)$ is refreshed and set equal to 1 with probability $1 - q$ and to 0 with probability q. Formally (see [22]) the process is uniquely specified by the action of its infinitesimal Markov generator $\mathcal{L}$ on local (*i.e.* depending on finitely many variables) functions $f : \Omega \mapsto \mathbb{R}$ that is given by

$$\begin{aligned} \mathcal{L}f(\sigma) &= \sum_{x \in \mathbb{Z}_+} c_x(\sigma) \left[\pi_x(f) - f(\sigma)\right] \\ &= \sum_{x \in \mathbb{Z}_+} c_x(\sigma) \left[(1 - \sigma(x))(1 - q) + \sigma(x) q\right] \left(f(\sigma^x) - f(\sigma)\right) \end{aligned} \tag{2.1}$$

where $c_x(\sigma) := 1 - \sigma(x+1)$ encodes the constraint, $\pi_x(f)$ denotes the conditional mean $\pi(f \mid \{\sigma(y)\}_{y \neq x})$ and σ^x is obtained from σ by flipping its value at x, *i.e.*

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ 1 - \sigma(x) & \text{if } y = x \end{cases}.$$

When the initial distribution at time $t = 0$ is Q the law and expectation of the process on the Skohorod space $D([0, \infty), \Omega)$ will be denoted by $\mathbb{P}_Q$ and $\mathbb{E}_Q$ respectively. If $Q = \delta_\sigma$ we write simply $\mathbb{P}_\sigma$. In the sequel we will often write $x_k(t)$ for the k^{th}-zero for the process σ_t at time t if no confusion arises.

Definition 2.1. *Given two probability measures μ on $\mathbb{N} := [1, 2, \dots)$ and ν on $\mathbb{Z}_+$ we will write $Q = Ren(\nu, \mu)$ if, under Q, the first zero x_0 has law ν and it is independent of the random variables $\{x_k - x_{k-1}\}_{k=1}^{\infty}$ which, in turn, form a sequence of i.i.d random variables with common law μ. If $\nu = \delta_0$ then we will write $Q = \mathrm{Ren}(\mu \,|\, 0)$.*

Remark 2.2. *In most of the present paper the initial distribution Q will always be assumed to be of the above form. For further generalizations we refer to Section 7.*

The East process can also be defined on finite intervals $\Lambda := [a, b] \subset \mathbb{Z}_+$ provided that a suitable *zero* boundary condition is specified at the site $b + 1$. More precisely one defines the finite volume generator

$$\mathcal{L}_\Lambda f(\sigma) = \sum_{x \in [a,b-1]} c_x(\sigma)\,[\mu_x(f) - f(\sigma)] + [\mu_b(f) - f(\sigma)] \equiv \sum_{x \in \Lambda} c_x^\Lambda(\sigma)\,[\mu_x(f) - f(\omega)]\ ,$$

$$\text{where} \quad c_x^\Lambda(\sigma) = \begin{cases} 1 - \sigma(x+1) & \text{for } x \in [a, b-1] \\ 1 & \text{if } x = b \end{cases} \tag{2.2}$$

In particular there is no constraint at site b, a fact that pictorially we can interpret by saying that there is a *frozen zero* at site $b + 1$. This frozen zero is the above mentioned boundary condition. In this case the process is nothing but a continuous time Markov chain reversible w.r.t. the product measure $\pi_\Lambda := \prod_{x \in [a,b]} \pi_x$ and, due to the "East" character of the constraint, for any initial condition η its evolution coincides with that of the East process in $\mathbb{Z}_+$ (restricted to Λ) starting from the configuration

$$\tilde\eta(x) := \begin{cases} \eta(x) & \text{if } x \in [a, b]\,, \\ 0 & \text{if } x = b + 1 \\ 1 & \text{otherwise}\,. \end{cases} \tag{2.3}$$

We will use the self-explanatory notation $\mathbb{P}^\Lambda_Q$ (or $\mathbb{P}^\Lambda_\sigma$) for the law of the process starting from the law Q (from σ).

2.0.1. *Additional notation.* In the sequel Λ will always denote a finite interval of $\mathbb{Z}_+$ with endpoints $0 \le a < b < \infty$.

It will also be quite useful to isolate some special configurations in Ω_Λ. We denote by $\sigma_{0\mathbb{1}}$ the configuration in $\Omega_\Lambda := \{0, 1\}^\Lambda$ such that $\mathcal{Z}(\sigma) = \{a\}$ and by $\sigma_{\mathbb{1}}$ the configuration with $\mathcal{Z}(\sigma) = \emptyset$. In words, $\sigma_{0\mathbb{1}}$ is the configuration with a single zero located at the left extreme of the interval, while $\sigma_{\mathbb{1}}$ is the configuration with no zeros. We also let, with a slight abuse of notation, $\mathbb{P}^\Lambda_{0\mathbb{1}} := \mathbb{P}^\Lambda_{\sigma_{0\mathbb{1}}}$ and $\mathbb{P}^\Lambda_{\mathbb{1}} := \mathbb{P}^\Lambda_{\sigma_{\mathbb{1}}}$.

2.1. **Graphical construction.** Here we recall a standard graphical construction which allows to define on the same probability space the finite volume East process for *all* initial conditions. Using a standard percolation argument [13, 23] together with the fact that the constraints c_x are uniformly bounded and of finite range, it is not difficult to see that the graphical construction can be extended without problems also to the infinite volume case. Given a finite interval $\Lambda \subset \mathbb{Z}_+$ we associate to each $x \in \Lambda$ a Poisson process of parameter one and, independently, a family of independent Bernoulli$(1-q)$ random variables $\{s_{x,k} : k \in \mathbb{N}\}$. The occurrences of the Poisson process associated to x will be denoted by $\{t_{x,k} : \ k \in \mathbb{N}\}$. We assume independence as x varies in Λ. Notice that with probability one all the occurrences $\{t_{x,k}\}_{k \in \mathbb{N},\, x \in \mathbb{Z}_+}$ are different. This defines the probability space. The corresponding probability measure will be denoted by $\mathbb{P}$. Given an initial configuration $\eta \in \Omega$ we construct a Markov process $(\sigma_t^{\Lambda,\eta})_{t\geq 0}$ on the above probability space satisfying $\sigma_{t=0}^{\Lambda,\eta} = \eta$ according to the following rules. At each time $t = t_{x,n}$ the site x queries the state of its own constraint c_x^{Λ}. If the constraint is satisfied, *i.e.* if $\sigma_{t-}^{\Lambda,\eta}(x+1) = 0$, then $t_{x,n}$ will be called a *legal ring* and at time t the configuration resets its value at site x to the value of the corresponding Bernoulli variable $s_{x,n}$. We stress here that the rings and coin tosses at x for $s \leq t$ have no influence whatsoever on the evolution of the configuration at the sites which enter in its constraint (here $x+1$) and thus they have no influence of whether a ring at x for $s > t$ is legal or not. It is easy to check that the above construction actually gives a continuous time Markov chain with generator (2.2).

A first immediate consequence is the following *decoupling* property.

Lemma 2.2. *Fix $c < a < b < d$ with $a, b, c, d \in \mathbb{Z}_+ \cup \{\infty\}$ and let $\Lambda = [c, d]$, $\Lambda' = [a, b]$, $V = [b+1, d]$. Take two events $\mathcal{A}$ and $\mathcal{B}$, belonging respectively to the σ–algebra generated by $\{\sigma_s(x)\}_{s \leqslant t,\, x\in\Lambda'}$ and $\{\sigma_s(x)\}_{s \leqslant t,\, x\in V}$. Then, for any $\sigma \in \Omega_\Lambda$,*
(i) $\mathbb{P}_\sigma^\Lambda(\mathcal{B}) = \mathbb{P}_{\sigma_V}^V(\mathcal{B})$;
(ii) $\mathbb{P}_\sigma^\Lambda(\mathcal{A} \cap \mathcal{B} \cap \{\sigma_s^\Lambda(b+1) = 0 \; \forall s \leqslant t\}) = \mathbb{P}_{\sigma_{\Lambda'}}^{\Lambda'}(\mathcal{A}) \mathbb{P}_{\sigma_V}^V(\mathcal{B} \cap \{\sigma_s^V(b+1) = 0 \; \forall s \leqslant t\})$.

The last, simple but quite important consequence of the graphical construction is the following one. Assume that the zeros of the starting configuration σ are labeled in increasing order as $x_0, x_1, \dots, x_n$ and define τ as the first time at which one the x_i's is killed, *i.e.* the occupation variable there flips to one. Then, up to time τ the East dynamics factorizes over the East process in each interval $[x_i, x_{i+1})$.

2.2. **Ergodicity.** The finite volume East process is trivially ergodic because of the frozen zero boundary condition (see 2.2). The infinite volume process in $\mathbb{Z}_+$ is ergodic in the sense that 0 is a simple eigenvalue of the generator $\mathcal{L}$ thought of as a selfadjoint operator on $L^2(\Omega, \pi)$ [4]. As far as more quantitative results are concerned we recall the following (see [4] for part (i) and [5] for part (ii)).

Theorem 2.3.

(i) *The generator* (2.1) *has a positive spectral gap, denoted by* $\mathrm{gap}(\mathcal{L})$*, such that*

$$\lim_{q\downarrow 0} \log(\mathrm{gap}(\mathcal{L})^{-1}) / \left(\log(1/q)\right)^2 = (2\log 2)^{-1}.$$

Moreover, for any interval Λ, the spectral gap of the finite volume generator $\mathcal{L}_\Lambda$ is not smaller than $\mathrm{gap}(\mathcal{L})$.

(ii) *Assume that the initial distribution Q is a product Bernoulli(α) measure, $\alpha \in (0,1)$. Then there exists $m \in (0, \mathrm{gap}(\mathcal{L})]$ and for any local function f there exists a constant C_f such that*

$$|\mathbb{E}_Q(f(\sigma_t)) - \pi(f)| \leq C_f e^{-mt}$$

The above results show that relaxation to equilibrium is indeed taking place at an exponential rate on a time scale $T_{\text{relax}} = \mathrm{gap}(\mathcal{L})^{-1}$ which however, for small values of q, is very large and of the order of $e^{c\log(1/q)^2}$ with $c = (2\log 2)^{-1}$.

2.3. **Main results: plateau behavior, aging and scaling limits.** We are now ready to state our first set of results (Theorem 2.5 and 2.6) which details the non equilibrium behavior of the East process for small values of q (small temperature) and for time scales much smaller that $T_{\rm relax}$. The prove of both theorems is detailed in Section 6 and is obtained thanks to the approximation of the East model with a suitable coalescence process. The definition of this coalescence process and the approximation result (Theorem 3.8), which is indeed the heart of our paper, is instead stated in Section 3 and proven in Section 5.

Definition 2.4. *Given ϵ, $q \in (0,1)$, we set*

$$t_0 := 1;\ t_0^- := 0;\ t_0^+ = \left(\tfrac{1}{q}\right)^{\epsilon}$$
$$t_n := \left(\tfrac{1}{q}\right)^{n};\ t_n^- := t_n^{1-\varepsilon};\ t_n^+ = t_n^{1+\varepsilon} \quad \forall n \geqslant 1\,. \tag{2.4}$$

The time interval $[t_n^-, t_n^+]$ and $[t_n^+, t_{n+1}^-]$ will be called respectively the n^{th}-active period and the n^{th}-stalling period.

Theorem 2.5 (Persistence, vacancy density and two-time autocorrelations during stalling periods: plateau and aging)**.** *Assume that the initial distribution Q is a renewal measure $Q = \mathrm{Ren}(\mu\,|\,0)$ with μ such that, for any $k \in \mathbb{N}$, $\mu([k,\infty)) > 0$ and either one of the following holds:*

a) μ has finite mean;

b) μ belongs to the domain of attraction of a α-stable law or, more generally, $\mu((x,+\infty)) = x^{-\alpha}L(x)$ where $L(x)$ is a slowly varying function at $+\infty$, $\alpha \in [0,1]$[1].

Then, if $o(1)$ denotes an error term depending only on n, m and tending to zero as both tend to infinity,

(i)

$$\lim_{q\downarrow 0} \sup_{t\in[t_n^+, t_{n+1}^-]} \left| \mathbb{P}_Q(\sigma_t(0)=0) - \left(\frac{1}{2^n+1}\right)^{c_0(1+o(1))} \right| = 0\,, \tag{2.5}$$

$$\lim_{q\downarrow 0} \sup_{t\in[t_n^+, t_{n+1}^-]} \left| \mathbb{P}_Q(\sigma_s(0)=0\ \forall s\leqslant t) - \left(\frac{1}{2^n+1}\right)^{c_0(1+o(1))} \right| = 0\,, \tag{2.6}$$

where $c_0 = 1$ in case (a) and $c_0 = \alpha$ in case (b).

(ii) Let $t, s : [0,1/2] \to [0,\infty)$ with $t(q) \geqslant s(q)$ for all $q \in [0,1/2]$. Then

$$\overline{\lim_{q\downarrow 0}}\ \mathbb{P}_Q(\sigma_{t(q)}(0)=0) \leqslant \overline{\lim_{q\downarrow 0}}\ \mathbb{P}_Q(\sigma_{s(q)}(0)=0).$$

The same bound holds with $\underline{\lim}_{q\downarrow 0}$ instead of $\overline{\lim}_{q\downarrow 0}$.

(iii) For $x \in \mathbb{Z}_+$ consider the time auto-correlation function $C_Q(s,t,x) := \mathrm{Cov}_Q(\sigma_t;\sigma_s)$. Then, for any n, m,

$$\lim_{q\downarrow 0} \sup_{\substack{t\in[t_n^+, t_{n+1}^-]\\ s\in[t_m^+, t_{m+1}^-]}} \left| C_Q(s,t,x) - \rho_x \left(\frac{1}{2^n+1}\right)^{c_0(1+o(1))} \left(1 - \rho_x\left(\frac{1}{2^m+1}\right)^{c_0(1+o(1))}\right) \right| = 0$$

[1] A function L is said to be slowly varying at infinity, if, for all $c>0$, $\lim_{x\to\infty} L(cx)/L(x) = 1$.

where $\rho_x = Q(\sigma(x) = 0)$.

The picture that emerges from points (i) and (ii) is depicted in Figure 1

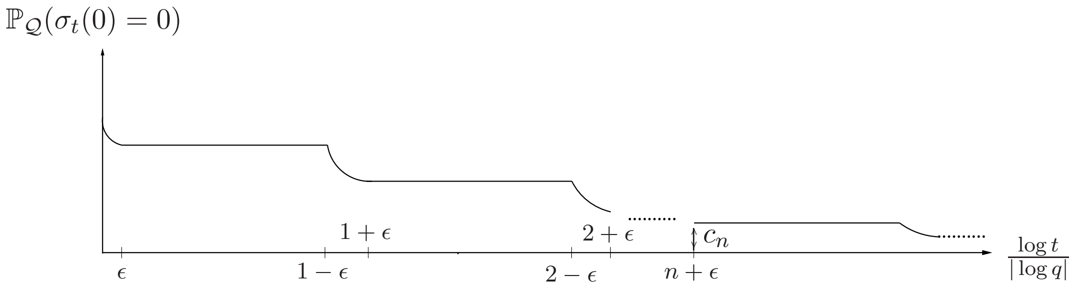

FIGURE 1. Plateau behavior in the limit $q \to 0$, where we set $c_n := (1/(2^n+1))^{c_0(1+o(1))}$ with c_0 defined in Theorem 2.5 and $o(1)$ going to zero as $n \to \infty$.

Remark 2.3.
(1.) Parts (ii)-(iii) hold also if $Q = Ren(\nu, \mu)$ (the proof given in Section 6 remains unchanged). On the other hand part (i) holds for $Q = Ren(\nu, \mu)$ multiplying the asymptotic value by the factor $Q(\sigma_0(0) = 0)$. Alternatively, part (i) holds for $Q = Ren(\nu, \mu)$ if the site $x = 0$ is replaced by the position x_k of the k^{th}-zero at time $t = 0$, $k \geq 0$. In fact, because of the renewal property of Q and of the "East" feature of the process (see e.g. (2.3)), for any $a \in \mathbb{N}$ it holds that $\mathbb{P}_Q(\sigma_t(a) = 0 \,|\, x_k(t=0) = a) = \mathbb{P}_{\hat{Q}}(\sigma_t(0) = 0)$ where $\hat{Q} = Ren(\mu \,|\, 0)$.
(2.) For small values of q the time auto-correlation function $C_Q(s,t,x)$ does depend in a non trivial way on s,t and not just on their difference $t-s$. Hence the word "aging" in the title. Of course, for times much larger than the relaxation time gap^{-1}, the time auto-correlation will be very close to that of the equilibrium process which in turn, by reversibility, depends only on $t-s$.

The next theorem describes the statistics of the interval (domain) between two consecutive zeros in a stalling period.

In order to state it let, for any $c_0 \in (0,1]$, $\tilde{X}^{(\infty)}_{c_0} \geq 1$ be a random variable with Laplace transform given by

$$\mathbb{E}(e^{-s\tilde{X}^{(\infty)}_{c_0}}) = 1 - \exp\Big\{ - c_0 \int_1^\infty \frac{e^{-sx}}{x} dx \Big\} = 1 - \exp\Big\{ - c_0 \,\text{Ei}(s) \Big\}. \tag{2.7}$$

The corresponding probability density is of the form $p_{c_0}(x)\mathbb{I}_{x \geqslant 1}$ where p_{c_0} is the continuous function on $[1,\infty)$ given by

$$p_{c_0}(x) = \sum_{k=1}^\infty \frac{(-1)^{k+1} c_0^k}{k!} \rho_k(x) \mathbb{I}_{x \geqslant k}, \tag{2.8}$$

where $\rho_1(x) = 1/x$ and

$$\rho_{k+1}(x) = \int_1^\infty dx_1 \cdots \int_1^\infty dx_k \frac{1}{x - \sum_{i=1}^k x_i} \prod_{j=1}^k \frac{1}{x_j}, \qquad k \geqslant 1. \tag{2.9}$$

Let also $\tilde{Y}_{c_0}^{(\infty)}$ be a non-negative random variable with Laplace transform given by

$$\mathbb{E}(e^{-s\tilde{Y}_{c_0}^{(\infty)}}) := 1 - \exp\Big\{ - c_0 \int_0^1 \frac{e^{-sx}}{x} dx \Big\} \tag{2.10}$$

Theorem 2.6 (Limiting behavior of the domain length and of the position of the first zero in the stalling periods)**.** *In the same assumptions of Theorem 2.5, let*

$$\bar{X}^{(n)}(t) := (x_1(t) - x_0(t))/(2^{n-1}+1) \quad ; \quad \bar{Y}^{(n)}(t) := x_0(t)/(2^{n-1}+1).$$

Then, for any bounded function f,

$$\lim_{n\uparrow\infty} \lim_{q\downarrow 0} \sup_{t\in[t_n^+, t_{n+1}^-]} \Big| \mathbb{E}_Q\big(f(\bar{X}^{(n+1)}(t))\big) - E\big(f(\tilde{X}_{c_0}^{(\infty)})\big| = 0 \tag{2.11}$$

$$\lim_{n\uparrow\infty} \lim_{q\downarrow 0} \sup_{t\in[t_n^+, t_{n+1}^-]} \Big| \mathbb{E}_Q\big(f(\bar{Y}^{(n+1)}(t))\big) - E\big(f(\tilde{Y}_{c_0}^{(\infty)})\big| = 0 \tag{2.12}$$

where again $c_0 = 1$ if μ has finite mean and $c_0 = \alpha$ if μ belongs to the domain of attraction of a α-stable law.
The result (2.11) holds for f satisfying $|f(x)| \leqslant C(1+|x|)^m$, $m = 1, 2, \dots$, if the $(m+\delta)^{th}$-moment of μ and ν is finite for some $\delta > 0$.

Remark 2.4. *The above result holds also for $Q = \text{Ren}(\nu, \mu)$, which can be obtained from $Q = \text{Ren}(\mu \,|\, 0)$ by a random shifting of law ν. Trivially, the effect of this random shift disappears in the scaling limit. Moreover the moment condition can be relaxed (see the proof of Proposition 4.13).*

3. Hierarchical Coalescence and the East process

In this section we introduce a *hierarchical coalescence process* (in the sequel HCP) which belongs to a much larger class of processes whose definition, asymptotic behavior and scaling limits are stated and analyzed in [14]. We will then state a result (Theorem 3.8) which had been conjectured in [28] which says that in the low temperature limit $q \downarrow 0$ the East process is well described by HCP. This result, together with the knowledge of the asymptotic behavior for HCP detailed in Section 3.4, will be the key to prove our main results for the East model announced in the previous section (Theorem 2.5 and 2.6).

Before giving a formal definition of HCP we start by saying that of the main features of HCP is that time has a hierarchical nature. There is an infinite sequence of *epochs* and inside each epoch the time runs from 0 to ∞. The HCP inside one epoch is just a suitable coalescence process *dependent* on the label of the epoch. The overall evolution is obtained by suitably linking consecutive epochs in the obvious way: the end (*i.e.* the limit $t \to \infty$) of one epoch coincides with the beginning of the next one. The key link between the HCP we propose below and the East process is provided by the very specific choice of the coalescence rates for the n^{th}-epoch process. As it will be apparent below these rates are expressed in terms of suitable large deviation probabilities of the East process.

3.1. **Domains and classes.** In order to define our HCP we need to fix some notation and introduce some basic geometric concepts.

Definition 3.1. *Given a configuration $\sigma \in \Omega$ we say that the interval $[c,d] \subset \mathbb{Z}_+$, $c < d$, is a* domain *of σ if $\sigma(x) = 1$ for any x, $c < x < d$, and $\sigma(c) = \sigma(d) = 0$. If $\sigma \in \Omega_\Lambda$ for some finite or infinite interval $\Lambda \subset \mathbb{Z}_+$ then the domains of σ are defined as the domains of the extended configuration $\tilde{\sigma} \in \Omega$, given in (2.3), which are contained in $[a, b+1]$. In particular, if $-\infty < a \leqslant b < \infty$, the domains of σ are the finite intervals $[c,d]$ where σ appears as $0, 1, \dots, 1, 0$ as well as the interval $[u, b+1]$, where u is the rightmost zero of σ. Given a domain $[c,d]$ its length is defined as $d - c$, while given a zero (empty site) x of σ the length of the domain having x as left extreme is denoted by d_x.*

Next we partition $\mathbb{N}$ by the sets $\mathcal{C}_n$ defined by

$$\mathcal{C}_0 = \{1\} \qquad \mathcal{C}_n = [2^{n-1}+1, 2^n] \ \text{ for } n \geqslant 1\,. \tag{3.1}$$

and we set

$$\mathcal{C}_{\geqslant n} := \left(\cup_{m \geqslant n} \mathcal{C}_m\right), \qquad \mathcal{C}_{>n} := \left(\cup_{m>n} \mathcal{C}_m\right), \qquad \mathcal{C}_{\leqslant n} := \cup_{m=0}^{n} \mathcal{C}_n\,.$$

Definition 3.2. *Given a configuration σ in Ω or Ω_Λ, we say that a domain of σ is of* class *n (respectively, at least n, larger than n, at most n) if its length belongs to $\mathcal{C}_n$ (respectively, $\mathcal{C}_{\geqslant n}$, $\mathcal{C}_{>n}$, $\mathcal{C}_{\leqslant n}$). We also say that a zero (empty site) x of σ is of class n if $d_x \in \mathcal{C}_n$. Similar definitions hold for $\mathcal{C}_{\geqslant n}, \mathcal{C}_{>n}, \mathcal{C}_{\leqslant n}$.*

Finally, we point out a simple property of the sets $\mathcal{C}_n$, which will be crucial in our investigation:

$$d, d' \in \mathcal{C}_n \ \Rightarrow\ d + d' \in \mathcal{C}_{>n}\,, \qquad \forall n \geqslant 0\,. \tag{3.2}$$

In what follows we introduce (hierarchical) coalescence processes as jump stochastic dynamics on Ω, where jumps correspond to filling an empty site. The term coalescence is justified. Indeed, a configuration $\sigma \in \Omega$ is univocally determined by the set $\mathcal{Z}(\sigma)$ of its zeros. Filling the empty site x in a configuration σ corresponds to removing the point x from the set $\mathcal{Z}(\sigma)$. Since a domain is simply the interval between consecutive zeros in $\mathcal{Z}(\sigma)$, removing the point $x \in \mathcal{Z}(\sigma)$ corresponds to the coalescence of the domains on the left and on the right of x.

3.2. **The n^{th}-epoch coalescence process.** We describe here the one–epoch coalescence process associated to the n^{th}-epoch (shortly n^{th}-CP), which depends also on the parameters $q, \varepsilon \in (0,1)$. Fixed these parameters, we define

$$T_0 := q^{(1-\epsilon)/2}\,, \qquad T_1 := 1/q^{3\epsilon}\,, \qquad T_n := (1/q)^{(n-1)(1+3\varepsilon)} \text{ for } n \geqslant 2\,. \tag{3.3}$$

Then for each $n \geqslant 0$ we define the function $\lambda_n : \mathbb{N} \to [0, \infty)$ as

$$\lambda_n(d) := \begin{cases} -T_n^{-1} \log\left(\mathbb{P}_{0\mathbb{1}}^{[0,d-1]}\left(\sigma_s(0) = 0\,\forall s \in [0, T_n]\right)\right) & \text{if } d \in \mathcal{C}_n\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.4}$$

where, we emphasize, $\mathbb{P}_{0\mathbb{1}}^{[0,d-1]}(\cdot)$ refers to the East process in $\Lambda = [0, d-1]$, starting from the configuration $\sigma_{0\mathbb{1}}$ and evolving with parameter q.

Finally we write $\Omega^{(\geq n)}$ for the set of configurations in Ω whose domains are all of class at least n. Then the n^{th}-CP is a Markov process with paths in the Skohorod space $D([0,\infty),\Omega^{(\geq n)})$ whose infinitesimal generator $\mathcal{L}_n$ acts on local functions as

$$\mathcal{L}_n f(\sigma) = \sum_{x\in\mathbb{Z}_+:\sigma(x)=0} \lambda_n(d_x)\big(f(\sigma^x) - f(\sigma)\big)\,. \tag{3.5}$$

Above, σ^x is the configuration obtained from σ by flipping its value in x, i.e. by filling the empty site x (we refer to the case $\sigma(x) = 0$). We will write $\mathbb{P}^{C,n}_\sigma$ for the law of the n^{th}-CP starting from the configuration σ.

As observed in [14], for almost all random paths $\{\sigma_s\}_{s\geqslant 0}$ of the n^{th}-CP, the asymptotic configuration σ_∞ defined as $\sigma_\infty(x) = \lim_{s\uparrow\infty}\sigma_s(x)$ exists and it belongs to $\Omega^{(\geq n+1)} \subset \Omega^{(\geq n)}$. Hence, in what follows, trajectories of the n^{th}-CP will be thought of up to time $t=\infty$ included.

3.3. **The hierarchical coalescence process (HCP).** Fix the parameters $q,\varepsilon\in(0,1)$.

Definition 3.3. *The HCP starting from the configuration $\sigma\in\Omega$ is the stochastic process whose evolution is described by a sequence of random paths*

$$(\sigma^{(n)}_s \;:\; s\in[0,\infty])_{n\in\mathbb{Z}_+} \in D([0,+\infty],\Omega)^{\mathbb{Z}_+}\,,$$

such that (inductively over n) $\{\sigma^{(n)}_s\}_{s\geq 0}$ is a random path of the n^{th}-CP starting from σ if $n=0$ and from $\sigma^{(n-1)}_\infty$ if $n\geqslant 1$.

If the initial configuration (*i.e.* at time $t=0$ in the first epoch) has law Q then the corresponding law and expectation for the HCP will be denoted by $\mathbb{P}^H_Q$ and $\mathbb{E}^H_Q$ respectively. In Section 3.3.1 below we present a refined graphical construction, allowing to define on the same probability space all the HCP as the initial configuration varies in Ω.

Remark 3.1. *We point out that the HCP defined here corresponds to the one in* [14] *with the choice $\lambda^{(n)}_\ell := 0$, $\lambda^{(n)}_r := \lambda_{n-1}$ and $d^{(n)} = 2^{n-2}+1$ for $n\geqslant 2$, $d^{(1)} = 1$. Note that the index of epochs in the formulation of* [14] *runs over $\mathbb{N}$ while here it runs over $\mathbb{Z}_+$ (the HCP process $(\xi^{(n)}_s \;:\; s\in[0,\infty])_{n\in\mathbb{N}} \in D([0,+\infty],\Omega)^{\mathbb{N}}$ defined in* [14] *and with the former choices of the rates is such that, for any $n\in\mathbb{N}$ and $s\in[0,\infty]$, ξ^n has the same law of σ^{n-1}_s).*

One can define the HCP in the finite volume $\Lambda=[a,b]$ as the process whose evolution is described by a sequence of random paths

$$(\sigma^{(n)}_s \;:\; s\in[0,\infty])_{n\in\mathbb{Z}_+} \in D(\bar{\mathbb{R}}_+,\Omega_\Lambda)^{\mathbb{Z}_+}$$

obtained by observing in the interval Λ the infinite volume HCP starting at the configuration $\tilde\sigma$ defined in (2.3). The corresponding law with initial distribution Q will be denoted by $\mathbb{P}^{\Lambda,H}_Q$.

3.3.1. *Graphical construction.* As for the East process we describe a graphical construction of the HCP in finite volume. A similar construction holds also in the infinite volume case again using the results in [13, 23].

Given an interval $\Lambda \subset \mathbb{Z}_+$ we associate to each $x \in \Lambda$ and to each $n \in \mathbb{Z}_+$ a Poisson process of parameter one and, independently, a family $\{S^{(n)}_{x,k} : k \in \mathbb{N}\}$ of independent random variables uniformly distributed on $[0,1]$. We assume independence as x and n vary. The occurrences of the Poisson process associated to the pair (x,n) will be denoted by $(t^{(n)}_{x,k} : k \geqslant 0)$. The above construction defines the probability space whose probability measure is denoted by $\mathbb{P}$. The construction of the path $(\sigma^n_t : t \in [0,\infty])_{n \geqslant 0}$ of the HCP with initial condition σ then proceeds by induction on n. Set $\sigma^{(0)}_0 = \sigma$. At each time $t = t^{(0)}_{x,k}$, $k = 1,2,\dots$, if $\sigma^{(0)}_{t-}(x) = 0$ and if $S^{(0)}_{x,k} \leqslant \lambda_0(d_x(\sigma^{(0)}_{t-}))$ then the configuration $\sigma^{(0)}_t$ is obtained from $\sigma^{(0)}_{t-}$ by filling the site x. In this case, the occurrence $t^{(0)}_{x,k}$ is called *legal ring*. Otherwise $\sigma^{(0)}_t := \sigma^{(0)}_{t-}$. Clearly the limiting configuration $\sigma^{(0)}_\infty$ is well defined a.s.. The path $(\sigma^{(1)}_t)_{t\in[0,\infty]}$ is then defined exactly in the same way by replacing the initial configuration σ with $\sigma^{(0)}_\infty$. The construction is then repeated inductively.

3.3.2. *Characteristic time scales.* Before moving on with the main results for the HCP we pause for a moment and establish some quantitative bounds on the characteristic time scales of the process. Although such results are completely irrelevant for the asymptotic as $n \to \infty$ of the HCP they will play a crucial role when we will compare the HCP with the East process.

Lemma 3.4. *Fix $N \in \mathbb{N}$. Consider the n^{th}-CP with parameters q and $\varepsilon := 1/8N$ in the definition (3.3). Then there exists a finite constant $c = c(N,L)$ such that, for any $n \le N$,*

$$\frac{c}{t_n} \leqslant \min_{d\in\mathcal{C}_n} \lambda_n(d) \leqslant \max_{d\in\mathcal{C}_n} \lambda_n(d) \le \frac{1}{c\,t_n}\,. \tag{3.6}$$

Proof. The proof is based on the results of Section 4 and it can be skipped on a first reading.

Let $\tilde\tau$ be the hitting time of the set $\{\sigma(0) = 1\}$ for the East process with parameter q and let $\Lambda = [0, d-1]$ with $d \in \mathcal{C}_n$. Fix $n \le N$, then using (4.7) we get

$$\mathbb{P}^\Lambda_{01}(\tilde\tau \ge T_n) = 1 - \mathbb{P}^\Lambda_{01}(\tilde\tau \leqslant T_n) \geqslant 1 - c\,T_n/t_n \tag{3.7}$$

and the second half of (3.6) follows. In order to prove the lower bound we write

$$\begin{aligned} &\mathbb{P}^\Lambda_{0\mathbb{1}}(\sigma_s(0) = 0\,\forall s \in [0,T_n]) = \mathbb{P}^\Lambda_{0\mathbb{1}}(\tilde\tau \geqslant T_n) \\ &= \mathbb{P}^\Lambda_{0\mathbb{1}}(\{\tilde\tau \geqslant T_n\} \cap \{\sigma_{T_n} = \sigma_{0\mathbb{1}}\}) + \mathbb{P}^\Lambda_{0\mathbb{1}}(\{\tilde\tau \geqslant T_n\} \cap \{\sigma_{T_n} \neq \sigma_{0\mathbb{1}}\})\,. \end{aligned} \tag{3.8}$$

Notice that, as $q \downarrow 0$, the the second one is $O(q)$ thanks to Lemma 4.2, thus the first term must be of order $O(1)$ since (3.7) guarantees that their sum is of order $O(1)$. Moreover, thanks to the Markov property and to (iv) of Lemma 4.4, for any $t \geqslant t_{n+1}$ such that $t/T_n \in \mathbb{Z}_+$,

$$\mathbb{P}^\Lambda_{0\mathbb{1}}(\{\tilde\tau \geqslant T_n\} \cap \{\sigma_{T_n} = \sigma_{0\mathbb{1}}\})^{t/T_n} \le \mathbb{P}^\Lambda_{0\mathbb{1}}(\tilde\tau \geqslant t) \le \frac{1}{cq} e^{-ct/t_n}\,.$$

Hence

$$\mathbb{P}^{\Lambda}_{0\mathbb{1}}(\tilde{\tau} \geqslant T_n)^{t/T_n} \leq \frac{1}{cq} e^{-ct/t_n} \left(1 + c'q\right)^{t/T_n} \leq \frac{1}{c''q} e^{-c''t/t_n}$$

i.e.

$$\mathbb{P}^{\Lambda}_{0\mathbb{1}}(\tilde{\tau} \geqslant T_n) \leq e^{-c''T_n/t_n} \tag{3.9}$$

thus proving the first half of (3.6).

□

Corollary 3.5. *Fix $N \in \mathbb{N}$. Then there exists a finite constant $c = c(N, L)$ such that the following holds. For any $0 \leq n \leq N$ consider the n^{th}-CP in the interval $\Lambda = [0, L-1]$ with parameters q and $\varepsilon := 1/8N$ in the definition (3.3). Then for any $\sigma \in \Omega_{\Lambda}^{(\geq n)}$*

(i) for any $x, y \in \Lambda$ satisfying $y - x \in \mathcal{C}_n$

$$\mathbb{P}^{\Lambda,n,C}_{\sigma}(\{x, y\} \subset \mathcal{Z}(\sigma_t)) \leqslant \exp(-ct/t_n) \tag{3.10}$$

(ii)

$$\mathbb{P}^{\Lambda,n,C}_{\sigma}\left(|\mathcal{Z}(\sigma) \setminus \mathcal{Z}(\sigma_t)| \geqslant 1\right) \leqslant c^{-1}\, t/t_n\,, \tag{3.11}$$

$$\mathbb{P}^{\Lambda,n,C}_{\sigma}\left(|\mathcal{Z}(\sigma) \setminus \mathcal{Z}(\sigma_t)| \geqslant 2\right) \leqslant c^{-1}\left(t/t_n\right)^2. \tag{3.12}$$

Remark 3.2. *In particular, with probability tending to one as $q \downarrow 0$, for the n^{th}-CP starting from $\sigma \in \Omega_{\Lambda}^{(\geq n)}$ before time t_n^- no zero has disappeared yet while after time t_n^+ all the zeros of class n have disappeared and therefore the infinite time configuration has been reached (namely for any $s \geqslant t_n^+$ it holds $\lim_{q\downarrow 0} \mathbb{P}^{\Lambda,n,C}_{\sigma}(\sigma_s = \sigma_\infty) = 1$).*

Proof. (i) If $\{x, y\} \subset \mathcal{Z}(\sigma_t)$, then the same holds at time $t = 0$ and there is no extra zero between x, y since otherwise we would have a zero of class smaller than n at $t = 0$. Conditionally on $y \in \mathcal{Z}(\sigma_t)$, the event $x \in \mathcal{Z}(\sigma_t)$ implies that the legal ring at x of the graphical construction has occurred after time t. Since such a ring is an exponential variable of parameter $\lambda_n(y - x)$ we conclude that

$$\mathbb{P}^{\Lambda,n,C}_{\sigma}(\sigma_t(x) = 0, \sigma_t(y) = 0) \leqslant \exp(-t\lambda_n(y - x))$$

and the sought bound follows from Lemma 3.4.

(ii) Thanks to the graphical construction, in order for σ_t to be obtained from σ by killing at least two zeros, it is necessary that at least two out of at most L independent Poisson clocks, each one of rate smaller or equal than $\bar{\lambda}_n := \sup_{d \in \mathcal{C}_n} \lambda_n(d)$, have been able to ring before time t. This observation, together with Lemma 3.4, leads to the bound

$$\mathbb{P}^{\Lambda,n,C}_{\sigma}\left(|\mathcal{Z}(\sigma) \setminus \mathcal{Z}(\sigma_t)| \geqslant 2\right) \leqslant c'\left(1 - e^{-t\bar{\lambda}_n}\right)^2 \leq c''\left(t/t_n\right)^2 \tag{3.13}$$

Similarly one proves (3.11)

□

3.4. **Limiting behavior of HCP.** In this section we recall some asymptotic results as $n \to \infty$ obtained in [14] for the law of $\sigma_0^{(n)}$ starting from $Q = \text{Ren}(\nu, \mu)$.

The first result (see Theorem 2.13 in [14]) says that for any $n \in \mathbb{Z}_+$ and $t \in [0, \infty]$ the law $Q_t^{(n)}$ of $\sigma_t^{(n)}$ is of the same type, i.e $Q_t^{(n)} = \text{Ren}(\nu_t^{(n)}, \mu_t^{(n)})$ for suitable probability measures $\nu_t^{(n)}$ on $\mathbb{Z}_+$ and $\mu_t^{(n)}$ on $\mathbb{N}$.

The second result characterizes inductively $\nu^{(n)} := \nu_{t=0}^{(n)}$ and $\mu^{(n)} := \mu_{t=0}^{(n)}$ (notice that $\nu^{(0)} = \nu$, $\mu^{(0)} = \mu$). These laws are the law of the first zero and the law of the domain length at the beginning of the n^{th}-epoch respectively.

Let, for $s \geqslant 0$ and $n \in \mathbb{Z}_+$,

$$G^{(n)}(s) = \sum_{x \in \mathbb{N}} e^{-sx} \mu^{(n)}(x) , \quad H^{(n)}(s) = \sum_{x \in \mathcal{C}_n} e^{-sx} \mu^{(n)}(x) , \quad L^{(n)}(s) = \sum_{x \in \mathbb{Z}_+} e^{-sx} \nu^{(n)}(x) . \tag{3.14}$$

Then the Laplace transforms $G^{(n)}, H^{(n)}$ and $L^{(n)}$ satisfy

$$1 - G^{(n+1)}(s) = (1 - G^{(n)}(s)) e^{H^{(n)}(s)} , \tag{3.15}$$

$$L^{(n+1)}(s) = L^{(n)}(s) \exp\left(H^{(n)}(s) - H^{(n)}(0) \right) . \tag{3.16}$$

Finally the main result of [14] can be formulated as follows. Define the rescaled variables $\tilde{X}^{(0)} := X^{(0)}$, $\tilde{Y}^{(0)} := Y^{(0)}$ and

$$\tilde{X}^{(n)} := X^{(n)}/(2^{n-1} + 1) , \qquad \tilde{Y}^{(n)} := Y^{(n)}/(2^{n-1} + 1), \quad n \geq 1 .$$

where $X^{(n)}$, $Y^{(n)}$ have law $\mu^{(n)}$ and $\nu^{(n)}$ respectively.

Theorem 3.6 ([14]). *Let $Q = \text{Ren}(\nu, \mu)$ and assume that the limit*

$$c_0 := \lim_{s \downarrow 0} \frac{-s\, G^{(0)'}(s)}{1 - G^{(0)}(s)} \tag{3.17}$$

exists (and then necessarily $c_0 \in [0, 1]$). Assumption (3.17) holds if: a) μ has finite mean and then $c_0 = 1$ or b) for some $\alpha \in (0, 1)$ μ belongs to the domain of attraction of an α–stable law or, more generally, $\mu\big((x, \infty)\big) = x^{-\alpha} L(x)$ where $L(x)$ is a slowly varying function at $+\infty$, $\alpha \in [0, 1]$, and in this case $c_0 = \alpha$.

Then:

(i) The rescaled random variable $\tilde{X}^{(n)}$ weakly converges to the random variable $\tilde{X}_{c_0}^{(\infty)}$ (see the discussion right after remark 2.3) whose Laplace transform is given by

$$\mathbb{E}\big(e^{-s\tilde{X}_{c_0}^{(\infty)}}\big) = 1 - \exp\Big\{ - c_0 \int_1^\infty \frac{e^{-sx}}{x} dx \Big\} = 1 - \exp\Big\{ - c_0\, Ei(s) \Big\}, \quad s \geqslant 0 . \tag{3.18}$$

(ii) The rescaled random variable $\tilde{Y}^{(n)}$ weakly converges to the random variable $\tilde{Y}_{c_0}^{(\infty)}$, whose Laplace transform is given by

$$\mathbb{E}\big(e^{-s\tilde{Y}_{c_0}^{(\infty)}}\big) = \exp\left\{ -c_0 \int_0^1 \frac{1 - e^{-sy}}{y} dy \right\} , \quad s \geqslant 0 . \tag{3.19}$$

(iii) If $Y^{(n)}$ denotes the leftmost point in $\sigma_0^{(n)}$, then

$$\mathbb{P}_Q^H\big(Y^{(n)} = Y^{(0)}\big) = 1/(2^{n-1} + 1)^{c_0(1 + o(1))} , \tag{3.20}$$

where $o(1)$ denotes an error going to zero as $n \to \infty$.

(iv) If $Q = Ren(\mu \,|\, 0)$, then

$$\nu^{(n)}(0) = \mathbb{P}_Q^H\Big(\sigma_0^{(n)}(0) = 0\Big) = 1/(2^{n-1}+1)^{c_0(1+o(1))}\,. \tag{3.21}$$

(v) If furthermore μ has finite k-th moment then for any function $f : [0,\infty) \to \mathbb{R}$ such that $|f(x)| \leqslant C + Cx^k$ for some constant C, it holds

$$\lim_{n\to\infty} \mathbb{E}[f(\tilde{X}^{(n)})] = \mathbb{E}[f(\tilde{X}_1^{(\infty)})]$$

3.5. **East process and HCP: approximation results as $q \downarrow 0$.** The main result of this section, Theorem 3.7 below, states that, as $q \downarrow 0$, the behavior of the East process on a finite volume Λ and up to time $T_N = (1/q)^{(N-1)(1+3\varepsilon)}$ (recall (3.3)), is well approximated by the HCP having the same initial distribution. Our second result (Theorem 3.8) states that the same occurs for the position of the first k zeros when working in $\mathbb{Z}_+$.

Recall Definition 2.4 of the active and stalling periods and define, for any $t > 0$, $n(t)$ and $\tau(t)$ by

$$t \in [t^-_{n(t)}, t^-_{n(t)+1})\,, \qquad \tau(t) := t - t^-_{n(t)}\,. \tag{3.22}$$

That allows us to define a canonical map

$$\phi : D([0,\infty),\Omega)^{\mathbb{Z}_+} \mapsto D([0,\infty),\Omega)\,, \tag{3.23}$$

by

$$\phi\Big(\big(\{\sigma_s^{(n)}\}_{s\geq 0}\big)_{n\in\mathbb{Z}_+}\Big)_t := \sigma_{\tau(t)}^{n(t)}.$$

In the sequel and for notation convenience we will write σ_t^H for the more cumbersome $\phi\Big(\big(\{\sigma_s^{(n)}\}_{s\geq 0}\big)_{n\in\mathbb{Z}_+}\Big)_t$ and, if confusion does not arise, we will denote by $x_k^H(t)$ the k-th zero of σ_t^H . In order to have compact formulas we introduce the following convention. For $i = 1, 2$ let F_i be a random variable with values in some set E (the same for $i = 1, 2$) on some probability space $(\Theta_i, \mathcal{F}_i, \mathbb{P}_i)$. Then we define

$$d_{TV}(\{F_1,\mathbb{P}_1\};\{F_2,\mathbb{P}_2\}) := d_{TV}(\mathfrak{p}_1,\mathfrak{p}_2)\,,$$

where $\mathfrak{p}_i$ denotes the law of F_i and $d_{TV}(\cdot,\cdot)$ denotes the total variation distance.

Theorem 3.7. *For any $N \in \mathbb{N}$ let $\epsilon_N := 1/8N$ and choose the parameter ϵ appearing in Definition 2.4 and in (3.3) equal to ϵ_N. Then, for any finite interval Λ and any probability measure Q on Ω_Λ,*

$$\lim_{q\downarrow 0}\ \sup_{t\in[0,t_N^+]}\ d_{TV}\big(\{\sigma_t,\,\mathbb{P}_Q^\Lambda\};\,\{\sigma_t^H,\,\mathbb{P}_Q^{\Lambda,H}\}\big) = 0\,. \tag{3.24}$$

The next theorem gives the approximation result for the laws of the first zero $x_0(t)$ and of the domain interval $x_1(t) - x_0(t)$ of the East process in $\mathbb{Z}_+$ up to times T_N.

Theorem 3.8. *Let $Q = Ren(\nu,\mu)$ with μ such that, for any $n \geq 1$, $\mu([n,\infty)) > 0$. In the same assumption of Theorem 3.7 and for any $k \geq 0$*

$$\lim_{q\downarrow 0}\ \sup_{t\in[0,t_N^+]}\ d_{TV}\big(\{(x_0(t),\ldots,x_k(t)),\,\mathbb{P}_Q\}\,;\,\{(x_0^H(t),\ldots,x_k^H(t)),\,\mathbb{P}_Q^H\}\big) = 0\,. \tag{3.25}$$

Assume that the $(m+\delta)^{th}$*-moment of* μ *and* ν *is finite for some* $\delta > 0$*. Then*

$$\lim_{q\downarrow 0} \sup_{t\in[0,t_N^+]} \left|\mathbb{E}_Q\big([x_{k+1}(t) - x_k(t)]^m\big) - \mathbb{E}_Q^H\big([x_{k+1}^H(t) - x_k^H(t)]^m\big)\right| = 0\,. \tag{3.26}$$

and

$$\lim_{q\downarrow 0} \sup_{t\in[0,t_N^+]} \left|\mathbb{E}_Q\big([x_k(t)]^m\big) - \mathbb{E}_Q^H\big([x_k^H(t)]^m\big)\right| = 0\,. \tag{3.27}$$

Remark 3.3. *The above theorem is really a corollary of Theorem 3.7 once we prove a strong finite volume approximation result both for the East process and for the HCP on* $\mathbb{Z}_+$*. More precisely we will show that, up to time* t_N^+*, the law (and the moments) of the first* k *zeros, in the limit* $q \downarrow 0$*, is very well approximated by the corresponding law (and moments) of the finite volume processes provided that the chosen volume is large enough (see Propositions 4.12, 4.13 in Section 4.2). The assumption* $\mu([k,\infty)) > 0$ *for any integer* k *is there exactly in order to greatly simplify the proof of such an approximation. Without it the result still holds but its proof requires more lengthy arguments which will appear elsewhere* [15].

4. Preliminary results for the low temperature East process

In this section we establish some results for the low temperature East process which will be crucial to prove the approximation with HCP in the following section. Unless otherwise specified, we set $\Lambda := [0, L-1]$ with $L \geqslant 1$ which is fixed once and for all and does not change as $q \downarrow 0$. Let us begin by reviewing some known properties (Lemma 4.1 and Remark 4.1) which will have a fundamental role in what follows.

Recall that a site $x_j \in \mathcal{Z}(\sigma)$ is said to be of class n if $x_{j+1} - x_j \in \mathcal{C}_n = [2^{n-1}+1, 2^n]$. The next combinatorial lemma (see also [28, 27]) says that the minimal number of extra zeros that we have to create in order to kill a zero of class n is n.

Lemma 4.1. [6] *Consider the East process on* $\Lambda := [0, L-1]$ *starting from the completely filled configuration* $\sigma_{\mathbb{1}}$*. For* $n \geqslant 1$ *let* $V(n)$ *be the set of configurations that the process can reach under the condition that, at any given time, no more than* n *zeros are present. Define*

$$\ell(n) := \sup_{\sigma\in V(n)} (L - x_0)$$

where $x_0 = x_0(\sigma)$ *is the smallest element of* $\mathcal{Z}(\sigma)$*. Then* $\ell(n) = 2^n - 1$ *for all* $L \geq 2^n - 1$*.*

Remark 4.1. *If instead of considering a frozen zero boundary condition at* L *we consider a deterministic time-dependent boundary condition* $\sigma_t(L) \in \{0,1\}$*, the above lemma implies that* $\ell(n) \leq 2^n - 1$ *for* $L \geq 2^n + 1$*. A key consequence is the following. Let* $x_j \in \mathcal{Z}(\sigma)$ *be of class* $n \geqslant 1$*. By definition* $d_j = x_{j+1} - x_j$ *belongs to the interval* $[2^{n-1}+1, 2^n]$*. Define* T_j *to be the first time the site* x_j *is filled, then at* T_j *there must be a zero at* $x_j + 1$*. Thus there must exists an intermediate time* $t \in [0, T_j]$ *such that, at time* t *there are at least* n *zeros in the interval* $[x_j + 1, x_{j+1} - 1]$*. Not surprisingly that will force the characteristic time scale of* T_j *to be of the order of* $1/q^n$*.*

4.1. **Energy barriers and characteristic time scales.** We start by establishing two results which say that, in the limit $q \downarrow 0$ and at any given time, the probability of observing k zeros which were not present at $t = 0$ is $O(q^k)$ (see Lemma 4.2). Therefore the probability of the event $\{\sigma_t(x) = 0\}$ coincides, for $q \approx 0$, with the probability of the event that a zero has persisted at x for the whole interval $[0, t]$ (Lemma 4.3).

Then we analyze the East process starting from the special configuration $\sigma_{0\mathbb{1}}$ having a single zero located at the origin. We study two important stopping times. The first, $\tilde{\tau}$, is the first time that the origin is filled (i.e. the zero at the origin is removed) while the second, $\tau_{\mathbb{1}}$, is the hitting time of the completely filled configuration $\sigma_{\mathbb{1}}$. We obtain upper and lower bounds on the characteristic time scales of these random times (see Lemma 4.4), which are optimal in the limit $q \downarrow 0$ (Remark 4.2). As a consequence we establish upper bounds on the probability of observing at time t a zero of class n (Corollary 4.5) and of killing at least one or at least two zeros of class n in a time interval t (Corollary 4.6 and 4.7, respectively). Finally we prove that $\tilde{\tau}$, after a proper rescaling, weakly converges in the limit $q \downarrow 0$ to an exponential variable of parameter 1 (Lemma 4.11 and Remark 4.4).

Lemma 4.2. *Fix $\sigma \in \Omega_\Lambda$, $t \geqslant 0$ and $k \in \mathbb{N}$. Let $V = [0, a] \subset \Lambda$ and let $\{y_1, \dots, y_k\} \subset V \setminus \mathcal{Z}(\sigma)$. Let finally $\mathcal{F}$ be the σ-algebra generated by the Poisson processes and coin tosses in $\Lambda \setminus V$. Then*

$$\mathbb{P}^\Lambda_\sigma\big(\{y_1, \dots, y_k\} \subset \mathcal{Z}(\sigma_t) \,|\, \mathcal{F}\big) \leqslant q^k \,. \tag{4.1}$$

Moreover

$$\mathbb{P}^\Lambda_\sigma\big(\exists\, s \le t: \; \{y_1, \dots, y_k\} \subset \mathcal{Z}(\sigma_t) \,|\, \mathcal{F}\big) \leqslant atq^k \,. \tag{4.2}$$

The same results hold when Λ is replaced by $\mathbb{Z}_+$.

Proof. We appeal to the graphical construction of Section 2.1. Let $y_1, \dots, y_k$ be as in the lemma, labeled in increasing order. Given $m \geqslant 0$, we write $\mathcal{A}_m$ for the event that the last legal ring at y_1 before time t, which is well defined because $y_1 \in \mathcal{Z}(\sigma_t) \setminus \mathcal{Z}(\sigma)$, occurs at time $t_{y_1,m}$. Recall that (i) at the time $t_{y_1,m}$ the current configuration resets its value at y_1 to the value of an independent Bernoulli$(1-q)$ random variable $s_{y_1,m}$ and (ii) that $\mathcal{A}_m$ depends only on the Poisson processes associated to sites $x \ge y_1$ and on the Bernoulli variables associated to sites $x > y_1$. Hence we conclude that

$$\begin{aligned} &\mathbb{P}^\Lambda_\sigma\big(\,\{y_1, \dots, y_k\} \subset \mathcal{Z}(\sigma_t) \,|\, \mathcal{F}\big) \\ &= \mathbb{P}\Big(\cup_{m=1}^\infty \big(\mathcal{A}_m \cap \{s_{y_1,m} = 0\}\big) \cap \big\{\{y_2, \dots, y_k\} \subset \mathcal{Z}(\sigma_t) \,|\, \mathcal{F}\big\}\Big) \\ &= \sum_{m=1}^\infty \mathbb{P}(s_{y_1,m} = 0)\mathbb{P}\left(\mathcal{A}_m \cap \big\{\{y_2, \dots, y_k\} \subset \mathcal{Z}(\sigma_t) \,|\, \mathcal{F}\big\}\right) \\ &\leqslant q\,\mathbb{P}^\Lambda_\sigma\left(\{y_2, \dots, y_k\} \subset \mathcal{Z}(\sigma_t)) \,|\, \mathcal{F}\right) \,. \end{aligned} \tag{4.3}$$

A simple iteration clings the proof of (4.1).

Let us now consider the set $N_t = \{t_1, t_2, \dots\}$ of all the occurrences up to time t of the Poisson processes in V and let $|N_t|$ be its cardinality. Conditioned on N_t and $\mathcal{F}$ the probability of seeing zeros located at $\{y_1, y_2, \dots, y_k\}$ at a given time $s \in N_t$ is bounded from above by q^k because of exactly the same arguments that led to (4.1). Therefore

$$\mathbb{P}^\Lambda_\sigma\big(\exists\, s \le t: \; \{y_1, \dots, y_k\} \subset \mathcal{Z}(\sigma_s) \,|\, \mathcal{F}\big) \leqslant q^k \mathbb{E}(|N_t| \,|\, \mathcal{F}) = Ltq^k \,.$$

because $|N_t|$ is independent from $\mathcal{F}$. Thus (4.2) follows. The last statement in the lemma is trivial. $\square$

Lemma 4.3 (Persistence of zeros). *Fix $t > 0$, $\sigma \in \Omega_\Lambda$ and k sites $y_1, \dots y_k \in \Lambda$. Then for any event $\mathcal{A}$ on $D([0,\infty), \Omega_\Lambda)$*

$$\mathbb{P}^\Lambda_\sigma(\mathcal{A} \cap \{\forall i \; y_i \in \mathcal{Z}(\sigma_t)\}) \leqslant kq + \mathbb{P}^\Lambda_\sigma(\mathcal{A} \cap \{\forall s \in [0,t] \; \forall i \; y_i \in \mathcal{Z}(\sigma_s)\}) \,.$$

The same result holds when Λ is replaced by $\mathbb{Z}_+$.

Proof. We can bound

$$\mathbb{P}^\Lambda_\sigma(\mathcal{A} \cap \{\forall i \; y_i \in \mathcal{Z}(\sigma_t)\})$$
$$\leqslant \mathbb{P}^\Lambda_\sigma(\mathcal{A} \cap \{\forall s \in [0,t] \; \forall i \; y_i \in \mathcal{Z}(\sigma_s)\}) + \sum_{i=1}^k \mathbb{P}^\Lambda_\sigma(\exists s < t : y_i \in \mathcal{Z}(\sigma_t) \setminus \mathcal{Z}(\sigma_s)) \,.$$

For the i^{th}-th term in the above sum we define the stopping time $\tau \geqslant 0$ as the first time such that $\sigma_\tau(y_i) = 1$. Then

$$\mathbb{P}^\Lambda_\sigma(\exists s < t : y_i \in \mathcal{Z}(\sigma_t) \setminus \mathcal{Z}(\sigma_s)) \leq q$$

because of the strong Markov property and Lemma 4.2. Finally, we observe that the proof for the infinite volume East process is completely similar. $\square$

We now move to the study of the following hitting times:

$$\tilde{\tau} := \inf\{t \geqslant 0 : \sigma_t(0) = 1\} \,, \tag{4.4}$$
$$\tau_n := \inf\{t \geqslant 0 : |\mathcal{Z}(\sigma_t) \setminus \{0\}| = n\} \,, \tag{4.5}$$
$$\tau_{\mathbb{1}} := \inf\{t \geqslant 0 : \mathcal{Z}(\sigma_t) = \emptyset\} \,. \tag{4.6}$$

Recall that $\sigma_{0\mathbb{1}} \in \Omega_\Lambda$ is the configuration with only one zero at the origin and that $t_n = 1/q^n$ (see (2.4)).

Lemma 4.4.

(i) *If $L = 1 \in \mathcal{C}_0$, then $\mathbb{P}^\Lambda_{0\mathbb{1}}(\tau_{\mathbb{1}} > t) = e^{-t(1-q)}$ for all $t \geqslant 0$ i.e. the hitting time $\tau_{\mathbb{1}} = \tilde{\tau}$ is an exponential time of parameter $1-q$.*

(ii) *If $L \in \mathcal{C}_{\geqslant n}$, $n \geqslant 0$, then $\mathbb{P}^\Lambda_{0\mathbb{1}}(\tau_n \leqslant \tilde{\tau} \leqslant \tau_{\mathbb{1}}) = 1$.*

(iii) *If $L \in \mathcal{C}_{\geqslant n}$ and $n \geqslant 1$ (i.e. $L \geqslant 2^{n-1}+1$), then there exists a positive constant $c = c(n, L, \bar{q})$, such that it holds*

$$\mathbb{P}^\Lambda_{0\mathbb{1}}(\tau_{\mathbb{1}} < t) \leqslant \mathbb{P}^\Lambda_{0\mathbb{1}}(\tilde{\tau} < t) \leqslant \mathbb{P}^\Lambda_{0\mathbb{1}}(\tau_n < t) \leq ct/t_n \,, \qquad \forall t \geqslant 0 \,. \tag{4.7}$$

(iv) *If $L \in \mathcal{C}_n$ and $n \geqslant 1$ (i.e. $L \in [2^{n-1}+1, 2^n]$), then there exists a constant $c = c(n, L)$ such that, for any $\sigma \in \Omega_\Lambda$,*

$$\mathbb{P}^\Lambda_\sigma(\tilde{\tau} > t) \leqslant \mathbb{P}^\Lambda_\sigma(\tau_{\mathbb{1}} > t) \leqslant \frac{1}{\pi_\Lambda(\sigma)} \exp\{-ct/t_n\} \,, \qquad \forall t \geqslant 0 \,. \tag{4.8}$$

In particular

$$\mathbb{P}^\Lambda_{0\mathbb{1}}(\tilde{\tau} > t) \leqslant \mathbb{P}^\Lambda_{0\mathbb{1}}(\tau_{\mathbb{1}} > t) \leqslant \frac{1}{c\,q} \exp\{-c\,t/t_n\} \,, \qquad \forall t \geqslant 0 \,. \tag{4.9}$$

We postpone the proof of the above lemma to Section 4.1.1.

Remark 4.2. *For $L \in \mathcal{C}_n$ it follows from (4.8) that $\tau_{\mathbb{1}}, \tilde{\tau}$ with high probability are smaller than $t_n^{1+\delta}$, $\delta > 0$. However, due to (4.7), we also have*

$$\lim_{q\downarrow 0} \mathbb{P}^{\Lambda}_{0\mathbb{1}} \left(\tau_{\mathbb{1}} < t_n^{1-\delta}\right) = \lim_{q\downarrow 0} \mathbb{P}^{\Lambda}_{0\mathbb{1}} \left(\tilde{\tau} < t_n^{1-\delta}\right) = 0\,.$$

We state three useful consequences of the previous results. The first one (see Corollary 4.5 below) gives an upper bound on the probability of seeing a domain of class smaller or equal than n at any fixed time independent of n. The second and third one (see Corollary 4.6 and Corollary 4.7) upper bound the probability that one zero or at least two zeros disappear in a fixed time interval when the initial configuration has only zeros of class at most n.

Corollary 4.5 (Domain survival probability)**.** *Fix $n \geqslant 1$. Then there exists a positive constant $c = c(n, L)$ such that, for any $\sigma \in \Omega_\Lambda$ and any $x, y \in \Lambda$ with $x < y$ and $y - x \leqslant 2^n$,*

$$\mathbb{P}^{\Lambda}_{\sigma}(\{x, y\} \subset \mathcal{Z}(\sigma_t)) \leqslant \frac{1}{\min(q, 1-q)^{2^n}} \exp\{-ct/t_n\} + 2q\,, \qquad \forall t \geqslant 0\,.$$

Proof. By using Lemma 4.3 we get

$$\mathbb{P}^{\Lambda}_{\sigma}(\{x, y\} \subset \mathcal{Z}(\sigma_t)) \leqslant 2q + \mathbb{P}^{\Lambda}_{\sigma}(\{x, y\} \subset \mathcal{Z}(\sigma_s)\ \forall s \in [0, t])\,.$$

Let $\bar{\sigma} := \sigma_{[x,y-1]}$. Then Lemma 2.2 and (iv) of Lemma 4.4 imply that

$$\begin{aligned}\mathbb{P}^{\Lambda}_{\sigma}\left(\{x, y\} \subset \mathcal{Z}(\sigma_s)\ \forall s \in [0, t]\right) &\leqslant \mathbb{P}^{[x,y-1]}_{\bar{\sigma}}(\tilde{\tau} > t) \\ &\leqslant \frac{1}{\min(q, 1-q)^{y-x}} \exp\{-ct/t_n\}\,. \end{aligned}$$

□

Corollary 4.6 (Killing at least one zero of class at least n)**.** *Fix $n \geq 1$. Consider the East process on Λ starting from a configuration σ with a zero at $x \in \Lambda$ of class $n_x \geq n$. Then there exists a positive constant $c = c(n_x)$ such that*

$$\mathbb{P}^{\Lambda}_{\sigma}\left(x \notin \mathcal{Z}(\sigma_t)\right) \leqslant ct/t_n\,, \qquad \forall t \geqslant 0\,.$$

The same result holds with Λ replaced by $\mathbb{Z}_+$.

Proof. Let $[x, y]$ be the domain in σ whose left boundary is x. For simplicity of notation we restrict ourselves to the case $y \leq L - 1$. Let $\mathcal{F}$ be the σ-algebra generated by the Poisson processes and coin tosses associated to $[y, L-1]$. Then, thanks to Remark 4.1, in order to remove x within time t there must exists a time $s \leq t$ such that $|\mathcal{Z}(\sigma_s) \cap (x, y)| \geq n$. At this point the thesis follows from Lemma 4.2. The infinite volume case is similar. □

Corollary 4.7. *(Killing at least two zeros of class at least n) Fix $n \geqslant 1$ and consider the East process on Λ starting from a configuration σ with two zeros $x < y$ each of class n. Define $\tau_n(x)$ to be the first time such that in the interval $(x, x + d_x)$ there are n zeros. Similarly for y. Then there exists a constant $c = c(n)$ such that*

$$\mathbb{P}^{\Lambda}_{\sigma}(\tau_n(x) \leq t\,,\ \tau_n(y) \leq t) \leq c\,(t/t_n)^2\,. \tag{4.10}$$

The same for the infinite volume case $\Lambda \to \mathbb{Z}_+$. In particular, if the initial configuration σ has only zeros of class at least n then

$$\mathbb{P}^\Lambda_\sigma\left(\{\mathcal{Z}(\sigma_t) \subset \mathcal{Z}(\sigma)\} \cap \{|\mathcal{Z}(\sigma) \setminus \mathcal{Z}(\sigma_t)| \geqslant 2\}\right) \leqslant c' \left(t/t_n\right)^2 .$$

for some constant $c' = c(\Lambda, n)$.

Proof. Call z the next zero of σ immediately to the right of x. Of course it is possible that $z = y$ and by assumption $z = x + d_x$ with $d_x \geq 2^{n-1} + 1$. Let $\mathcal{F}$ be the σ-algebra of the Poisson processes and coin tosses associated to sites in $[z, L-1]$. Then

$$\mathbb{P}^\Lambda_\sigma(\tau_n(x) \leq t\ ,\ \tau_n(y) \leq t) = \mathbb{E}\left(\mathbb{1}_{\{\tau_n(y)\leq t\}} \mathbb{P}^\Lambda_\sigma(\tau_n(x) \leq t \,|\, \mathcal{F})\right) \leq c(t/t_n)^2$$

because of (4.7). Similarly when Λ is replaced by $\mathbb{Z}_+$. The second conclusion is now immediate once we appeal to Remark 4.1. Indeed, in order to remove within time t two zeros x, y of class at least n, their respective stopping times $\tau_n(x), \tau_n(y)$ must have occurred before time t.

□

Before moving on we summarize the overall picture that emerges from the above results into a single proposition.

Proposition 4.8. *Consider the East process on Λ starting from a configuration σ. Fix a large integer N, fix $\epsilon \leqslant (8N)^{-1}$ and let $t_n^\pm := (1/q)^{n(1\pm\epsilon)}$. Notice that $t_n^- \gg t_{n-1}^+$ for $n \leqslant N$ and small q. Then the following picture up to scale N holds with probability tending to one as $q \downarrow 0$.*

- *The typical time necessary to kill a zero of class n is of order $t_n^{1+o(1)}$ so that at time t_n^- the zeros of σ of class at least n are still present.*
- *At time $t \geqslant t_n^+$ all zeros are of class at least $n+1$ (and thus at time $t \geqslant t_n^-$ of class at least n).*
- *Split the* active period *$[t_n^-, t_n^+]$ into disjoint sub-periods of width T_n defined in (3.3). Then in each sub-period at most one zero of class n is killed.*

The last property follows by a simple application of Corollary 4.7.

4.1.1. *Proof of Lemma 4.4.* We are now left with the proof of Lemma 4.4 and for this purpose we need first two preliminary results. Recall that π_Λ denotes the product Bernoulli measure on Ω_Λ with density $1-q$.

Lemma 4.9. *For any $A \subset \Omega_\Lambda$, the hitting time $\tau_A = \inf\{t \geqslant 0 : \sigma_t \in A\}$ satisfies*

$$\mathbb{P}^\Lambda_{\pi_\Lambda}(\tau_A > t) \leqslant e^{-t\,\mathrm{gap}(\mathcal{L}_\Lambda)\pi_\Lambda(A)} .$$

Proof. It is well known (see e.g. [2]) that $\mathbb{P}^L_{\pi_\Lambda}(\tau_A > t) \leqslant e^{-t\lambda_A}$, where

$$\lambda_A := \inf\left\{\mathcal{D}_\Lambda(f) : \pi_\Lambda(f^2) = 1 \text{ and } f \equiv 0 \text{ on } A\right\} ,$$

$\mathcal{D}_\Lambda(f) = -\pi_\Lambda(f, \mathcal{L}_\Lambda f)$ being the Dirichlet form of f. Since $\mathrm{Var}_\Lambda(f)/\pi_\Lambda(f^2) \geq \pi_\Lambda(A)$ if $f \equiv 0$ on A, from the definition of the spectral gap

$$\mathrm{gap}(\mathcal{L}_\Lambda) := \inf_{f:\ \pi_\Lambda(f)=0} \frac{\mathcal{D}_\Lambda(f)}{\pi_\Lambda(f^2)} , \tag{4.11}$$

it follows immediately that $\lambda_A \geqslant \mathrm{gap}(\mathcal{L}_\Lambda)\pi_\Lambda(A)$. □

In order to use the former result we will need in turn a sharp lower bound on the spectral gap on finite volume.

Lemma 4.10. *Recall that $\Lambda = [0, L-1]$. Then, for any integers n, L with $1 \leqslant L \leqslant 2^n$,*

$$\text{gap}(\mathcal{L}_\Lambda) \geqslant \left(\frac{q}{2}\right)^n .$$

Proof. Fix n, L as above. We first observe that, by monotonicity of the gap for the East model (see [4, Lemma 2.11]) it is enough to consider the case $L = 2^n$. For lightness of notation we denote by γ_n the inverse gap on $[0, 2^n - 1]$ (with a frozen zero at $L = 2^n$). The result is obtained by induction, following the bisection constrained method introduced in [4].

Let $A = [0, 2^{n-1} - 1]$ and $B = [2^{n-1}, 2^n - 1]$ and set $a = 2^{n-1}$. Notice that $\Lambda := [0, 2^n - 1] = A \cup B$, $A \cap B = \emptyset$ and that $\text{gap}(\mathcal{L}_A)^{-1} = \text{gap}(\mathcal{L}_B)^{-1} = \gamma_{n-1}$. We will denote by $\text{Var}_\Lambda(f)$, $\pi_\Lambda(f)$ and $\mathcal{D}_\Lambda(f)$ the variance, mean and Dirichlet form of f on the interval Λ. Analogous notation hold for the same quantities restricted to the intervals A and B.

Consider the following auxiliary "constrained block dynamics". The block B waits a mean one exponential random time and then the current configuration inside it is refreshed with a new one sampled from π_B. The block A does the same but now the configuration is refreshed only if the current configuration σ is such that $\sigma(a) = 0$. The Dirichlet form of this new chain is

$$\mathcal{D}_{\text{block}}(f) = \pi_\Lambda \left(c_A \operatorname{Var}_A(f) + \operatorname{Var}_B(f) \right)$$

with $c_A(\sigma) = 1 - \sigma(a)$. If $\text{gap}_{\text{block}}$ is the corresponding spectral gap then Proposition 4.4 in [4] gives

$$\text{gap}_{\text{block}} \geqslant 1 - \sqrt{1-q} \geqslant \frac{q}{2} .$$

Hence,

$$\operatorname{Var}_\Lambda(f) \leqslant \frac{2}{q} \pi_\Lambda \left(c_A \operatorname{Var}_A(f) + \operatorname{Var}_B(f) \right) .$$

Now $\operatorname{Var}_B(f) \leqslant \gamma_{n-1} \mathcal{D}_B(f)$ and, by construction, (see (2.2)) $c_x^B = c_x^\Lambda$ for any $x \in B$. Therefore

$$\pi_\Lambda \left(\operatorname{Var}_B(f) \right) \leqslant \gamma_{n-1} \sum_{x \in B} c_x^\Lambda \operatorname{Var}_x(f)$$

which is nothing but the contribution carried by B to the full Dirichlet form $\mathcal{D}_\Lambda(f)$. As far as the block A is concerned one observes that $c_x^A \cdot c_A = c_x^\Lambda$ for any $x \in A$. In conclusion

$$\operatorname{Var}_\Lambda(f) \leqslant \frac{2}{q} \gamma_{n-1} \mathcal{D}_\Lambda(f) \quad \textit{i.e.} \quad \gamma_n \leqslant \frac{2}{q} \gamma_{n-1}.$$

Since $\gamma_0 = 1$ the result follows immediately. □

Remark 4.3. *The above result leads to the lower bound $gap(\mathcal{L}_\Lambda) = \left(\frac{q}{2}\right)^{\log_2(1/q)}$ when n is of the order $\log 2 / \log(1/q)$, namely when the length of Λ becomes of the order of the equilibrium domains. We stress that this is not the correct scaling on this length, which is instead $(q)^{\log_2(1/q)/2}$ as proven in Section 6 of [4] for the lower bound and in Appendix 5 of [5] for the upper bound. Indeed, the above described bisection-technique should be refined as described in [4] to capture the correct scaling up to this length. However, for the* *purpose of this paper the above easier bound is enough (and this because we do analyze here the evolution at times much smaller than the global relaxation time).*

Proof of Lemma 4.4.
(i) The proof is straightforward.
(ii) Trivially $\tau_{\mathbb{1}} \leqslant \tilde{\tau}$. On the other hand, if the starting configuration is $\sigma_{0\mathbb{1}}$, Remark 4.1 implies at once that $\tau_n \leqslant \tilde{\tau}$.
(iii) Thanks to (ii) we have

$$\mathbb{P}^{\Lambda}_{0\mathbb{1}}(\tau_{\mathbb{1}} < t) \leqslant \mathbb{P}^{\Lambda}_{0\mathbb{1}}(\tilde{\tau} < t) \leqslant \mathbb{P}^{\Lambda}_{0\mathbb{1}}(\tau_n < t)\,.$$

The last probability is bounded from above by ct/t_n with $c = \binom{L-1}{n}$ by (4.2) in Lemma 4.2.
(iv) Trivially (4.9) follows from (4.8). The proof of (4.8) is based on Lemma 4.9 with $A = \{\sigma_{\mathbb{1}}\}$, and on Lemma 4.10. We get

$$\mathbb{P}^{\Lambda}_{\sigma}(\tau_{\mathbb{1}} > t) \leqslant \frac{1}{\pi_\Lambda(\sigma)} \mathbb{P}^{\Lambda}_{\pi_\Lambda}(\tau_{\mathbb{1}} > t)$$
$$\leqslant \frac{1}{\pi_\Lambda(\sigma)} \exp(-t\,\text{gap}(\mathcal{L}_\Lambda)\pi_\Lambda(\mathbb{1})) \leqslant \frac{1}{\pi_\Lambda(\sigma)} \exp(-c\,t/t_n)$$

with $c := (1/2)^{2^n}/2^n$. □

4.1.2. *Approximate exponentiality of the hitting time $\tilde{\tau}$.* As it is very often the case for systems showing metastable behavior (see e.g. [24]), a loss of memory mechanism produces activation times that become exponential variables after appropriate rescaling. In our case the appropriate activation time is the hitting time $\tilde{\tau} := \inf\{t \geqslant 0 : \sigma_t(0) = 1\}$.

Lemma 4.11. *Let $f(t) := \mathbb{P}^{\Lambda}_{0\mathbb{1}}(\tilde{\tau}/\gamma > t)$, where $\gamma = \gamma(q, \Lambda)$ is such that $f(1) = e^{-1}$. Then, for any $t, s \geqslant 0$, it holds*

$$|f(t+s) - f(s)f(t)| \leq cq \tag{4.12}$$

for some constant $c > 0$ independent of q. In particular, for any $t > 0$, $\lim_{q\downarrow 0} f(t) = e^{-t}$.

Proof. That $\lim_{q\downarrow 0} f(t) = e^{-t}$ follows from (4.12) by standard arguments (see e.g. [25]). We now prove (4.12). For any $s, t \geqslant 0$ we have

$$f(t+s) = \mathbb{P}^{\Lambda}_{0\mathbb{1}}(\tilde{\tau} > \gamma(t+s)\,|\,\tilde{\tau} > \gamma s) f(s).$$

Moreover, thanks to the Markov property,

$$\begin{aligned}\mathbb{P}^{\Lambda}_{0\mathbb{1}}(\tilde{\tau} > \gamma(t+s)\,|\,\tilde{\tau} > \gamma s) &= f(t)\,\mathbb{P}^{\Lambda}_{0\mathbb{1}}(\sigma_{s\gamma} = \sigma_{0\mathbb{1}}\,|\,\tilde{\tau} > \gamma s) \\ &+ \mathbb{P}^{\Lambda}_{0\mathbb{1}}(\tilde{\tau} > \gamma(t+s)\,|\,\tilde{\tau} > \gamma s\,;\,\sigma_{\gamma s} \neq \sigma_{0\mathbb{1}})\mathbb{P}^{\Lambda}_{0\mathbb{1}}(\sigma_{\gamma s} \neq \sigma_{0\mathbb{1}}|\tilde{\tau} > \gamma s)\,,\end{aligned} \tag{4.13}$$

and therefore

$$|f(t+s) - f(t)f(s)| \leq 2\mathbb{P}^{\Lambda}_{0\mathbb{1}}(\{\sigma_{s\gamma} \neq \sigma_{0\mathbb{1}}\} \cap \{\tilde{\tau} > \gamma s\}) \leq 2Lq\,.$$

where we used Lemma 4.2 in the last inequality. □

Remark 4.4. *If $\Lambda = [0, L-1]$ with $L \in \mathcal{C}_n$, then by using Remark 4.2 we obtain $\gamma = t_n^{1+o(1)}$ as $q \downarrow 0$.*

4.2. **Finite volume approximation.** In this section we prove a finite volume approximation result of the infinite volume East process with initial distribution $Q = \text{Ren}(\nu, \mu)$, provided that $\mu([n,\infty)) > 0$ for any n.

Proposition 4.12. *Let $Q = Ren(\nu, \mu)$ and suppose that $\mu([n,\infty)) > 0$ for any n. Then, for any ℓ and any N,*

$$\lim_{L\uparrow\infty} \limsup_{q\downarrow 0} \sup_{t\in[0,t_N^+]} d(t,\ell,L) = 0$$

where $d(t,\ell,L)$ denotes the variation distance between the laws of the vector $(\sigma(0),\dots,\sigma(\ell))$ at time t for the East process in $\Lambda = [0, L-1]$ and the East process in $\mathbb{Z}_+$ with initial distribution Q.

Proof. Let $n_0 \geq N+1$ be such that $\mu(2^{n_0}) > 0$ and let

$$\mathcal{A}_L := \{\sigma \in \Omega:\ \exists\, x_j \in \mathcal{Z}(\sigma) \cap [\ell+1, L-2^{n_0}] \text{ of class } n_0\}\,.$$

From the renewal property of Q it follows that $\lim_{L\to\infty} Q(\mathcal{A}_L) = 1$. For any $\sigma \in \mathcal{A}_L$ let $x_* \in [\ell+1, L-2^{n_0}]$ be the smallest zero of σ of class n_0. Since $n_0 \geq N+1$ Corollary 4.6 (see also Proposition 4.8) implies that $x_* \in \mathcal{Z}(\sigma_s)\ \forall s \leq t_N$ with probability tending to one as $q \downarrow 0$ both for the finite and infinite volume East processes starting from σ. Finally, conditioned to the event that for both processes x_* is not killed up to time t_N, the graphical construction implies that their evolutions in $[0,\ell]$ coincide up to time t_N. □

In order to state the next result it is convenient to define, for any $\Lambda = [0, L-1]$, $x_k^\Lambda := x_k(\sigma)$ if $|\mathcal{Z}(\sigma)\cap\Lambda| \geq k$ and $x_k^\Lambda = L$ otherwise.

Proposition 4.13. *Let $Q = Ren(\nu, \mu)$ and suppose that $\mu([n,\infty)) > 0$ for any n. Then for any $N \in \mathbb{N}$ and $k \geq 0$ it holds:*
(i)

$$\lim_{L\uparrow\infty} \limsup_{q\downarrow 0} \sup_{t\in[0,t_N^+]} d_{TV}\big(\{(x_0(t),\dots,x_k(t)),\,\mathbb{P}_Q\}\,;\,\{(x_0^\Lambda(t),\dots,x_k^\Lambda(t)),\,\mathbb{P}_Q^\Lambda\}\big) = 0\,. \tag{4.14}$$

(ii) If the $(m+\delta)^{th}$-moments of ν,μ are both finite for some $\delta > 0$ then

$$\lim_{L\uparrow\infty} \limsup_{q\downarrow 0} \sup_{t\in[0,t_N^+]} \big|\mathbb{E}_Q\big([x_{k+1}(t)-x_k(t)]^m\big) - \mathbb{E}_Q^\Lambda\big([x_{k+1}^\Lambda(t)-x_k^\Lambda(t)]^m\big)\big| = 0 \tag{4.15}$$

and

$$\lim_{L\uparrow\infty} \limsup_{q\downarrow 0} \sup_{t\in[0,t_N^+]} \big|\mathbb{E}_Q\big([x_k(t)]^m\big) - \mathbb{E}_Q^\Lambda\big([x_k^\Lambda(t)]^m\big)\big| = 0\,. \tag{4.16}$$

Proof. Let us fix N and $n_0 \geq N+1$ such that $\mu_0 := \mu(2^{n_0}) > 0$.
(i) Fix k and $L \gg k2^{n_0}$, and consider the event

$$\mathcal{B}_{k,L} := \{\sigma:\ \exists\, \{x_{j_i}\}_{i=0}^k \subset \mathcal{Z}(\sigma)\cap[0, L-2^{n_0}] \text{ such that each } x_{j_i} \text{ is of class } n_0\}\,.$$

Clearly $\lim_{L\uparrow\infty} Q(\mathcal{B}_{k,L}) = 1$. For $\sigma \in \mathcal{B}_{k,L}$ let $\{x_*^{(i)}\}_{i=0}^k$ be the smallest zeros with the properties described in $\mathcal{B}_{k,L}$. Conditionally on the event that none of the $\{x_*^{(i)}\}_{i=0}^k$ has been killed for both processes within time t_N^+, the first $k+1$ zeros of the East process in $\Lambda = [0, L-1]$ starting from σ necessarily coincide with those of the infinite volume East process. Since the conditioning event has probability tending to one as $q \downarrow 0$ thanks to

Corollary 4.6 (both for the East process on Λ and the infinite volume East process) the thesis follows.

(ii)-(4.15) For simplicity and without loss of generality we only discuss the case $k = 0$. The argument here is conceptually similar to that employed in the proof of (i) but more involved. The reason is that, in order to have a good control on the probability $\mathbb{P}_Q(x_1(t) - x_0(t) \geq \ell)$, ℓ very large even compared to e.g. $1/q$, a single zero at time $t = 0$ of class at least n_0 is no longer enough. What we really need are enough (typically $O(\log(\ell))$ such zeros in order to be sure that with not too small probability at least one of them has survived up to time t. Thus the argument will be split into a part in which the Q-large deviations of the number of such zeros dominate and a second part in which the "resistance" of each zero will play a key role.

We write

$$\begin{aligned}
&\big|\mathbb{E}_Q\big([x_1(t) - x_0(t)]^m\big) - \mathbb{E}_Q^\Lambda\big([x_1^\Lambda(t) - x_0^\Lambda(t)]^m\big)\big| \\
&\leq \big|\mathbb{E}_Q\big([x_1^\Lambda(t) - x_0^\Lambda(t)]^m\big) - \mathbb{E}_Q^\Lambda\big([x_1^\Lambda(t) - x_0^\Lambda(t)]^m\big)\big| \qquad (4.17)\\
&\quad + \big|\mathbb{E}_Q\big([x_1(t) - x_0(t)]^m - [x_1^\Lambda(t) - x_0^\Lambda(t)]^m\big)\big|\,. \qquad (4.18)
\end{aligned}$$

Let us deal first with the term (4.17).

Let

$$\mathcal{D} = \{\sigma:\ x_1(\sigma) \leq L/2\}$$
$$\mathcal{B} = \{\sigma:\ \exists x < y < z \in \mathcal{Z}(\sigma) \cap [0, L] \text{ with } x \geqslant x_1(\sigma) \text{ and } [x, y],\ [y, z] \text{ of class at least } n_0\}.$$

Then we can split the integration w.r.t. Q over the set $\mathcal{D} \cap \mathcal{B}$ and the set $\mathcal{D}^c \cup \{\mathcal{D} \cap \mathcal{B}^c\}$.

For any $\sigma \in \mathcal{D} \cap \mathcal{B}$

$$\lim_{q\downarrow 0}\ \sup_{t\in[0,t_N]} \big|\mathbb{E}_\sigma\big([x_1^\Lambda(t) - x_0^\Lambda(t)]^m\big) - \mathbb{E}_\sigma^\Lambda\big([x_1^\Lambda(t) - x_0^\Lambda(t)]^m\big)\big| = 0$$

exactly by the same argument that was used in (i). Thus

$$\limsup_{q\downarrow 0}\ \sup_{t\in[0,t_N]} \big|\mathbb{E}_Q\big([x_1^\Lambda(t) - x_0^\Lambda(t)]^m\big) - \mathbb{E}_Q^\Lambda\big([x_1^\Lambda(t) - x_0^\Lambda(t)]^m\big)\big| \leq L^m\left(Q(\mathcal{D}^c) + Q(\mathcal{D} \cap \mathcal{B}^c)\right).$$

The boundedness of the m^{th}-moment of μ, ν implies that

$$\lim_{L\uparrow\infty} L^m Q(\mathcal{D}^c) \leq \lim_{L\to\infty} L^m\left(\nu([L/4, \infty) + \mu([L/4, \infty)\right) = 0\,.$$

Consider now the contribution $L^m Q(\mathcal{D} \cap \mathcal{B}^c)$. Note that if we set $d_i(\sigma) = x_{i+1}(\sigma) - x_i(\sigma)$ then $\mathcal{D} \cap \mathcal{B}^c \subset \mathcal{A}_1 \cup \mathcal{A}_2$ where

$$\mathcal{A}_1 = \{\sigma:\ d_i(\sigma) < 2^{n_0},\ \forall\, 1 \leqslant i \leqslant L/2^{n_0+1}\}\,,$$
$$\mathcal{A}_2 := \bigcup_{x=0}^{L/2}\ \bigcup_{\ell=2^{n_0}}^{L-x}\ \bigcup_{j=1}^{L-\ell-x} \{\sigma:\ x_1(\sigma) = x,\ d_j(\sigma) = \ell,\ d_i(\sigma) < 2^{n_0}\ \forall i \neq j \text{ with } 1 \leqslant i \leqslant C(x, \ell)\}$$

where $C(x, \ell) = 2^{-n_0}[L - x - \ell]$. It is immediate to verify that $Q(\mathcal{A}_1)$ is smaller than $(1 - \mu_0)^{L/2^{n_0+1}}$. Hence

$$\lim_{L\uparrow\infty} L^m Q(\mathcal{A}_1) = 0.$$

By a union bound we get

$$
\begin{aligned}
Q(\mathcal{A}_2) &\leqslant \sum_{x=0}^{L/2} Q(x_1 = x) \sum_{\ell=2^{n_0}}^{L-x} (L-\ell-x)\mu(\ell)\mu([1,2^{n_0}])^{2^{-n_0}[L-x-\ell]-1} \\
&\leqslant \sum_{x=0}^{L/2} Q(x_1 = x) \left[\sum_{\ell=2^{n_0}}^{L-x-\lambda\log L} \frac{L^{1-a}}{\mu([1,2^{n_0}])} + \sum_{\ell=L-x-\lambda\log L+1}^{L-x} (\lambda \log L)\mu(\ell) \right] \\
&\leqslant \frac{L^{2-a}}{\mu([1,2^{n_0}])} + \lambda^2 (\log L)^2 \mu[L/4,\infty]
\end{aligned}
$$

where $a := \lambda 2^{-n_0} |\log \mu([1,2^{n_0}])|$ and λ is a positive constant chosen so that $a > m+2$. Therefore by the boundedness of the $(m+\delta)$-th moment of μ we get

$$
\lim_{L\uparrow\infty} L^m Q(\mathcal{A}_2) = 0.
$$

We now examine the term (4.18).

It is immediate to verify that

$$
\begin{aligned}
\left|\mathbb{E}_Q\left([x_1(t)-x_0(t)]^m - [x_1^\Lambda(t)-x_0^\Lambda(t)]^m\right)\right| &\le \mathbb{E}_Q\left(x_1(t)^m \mathbb{1}_{x_1(t)\ge L}\right) \\
&\leqslant L^m \mathbb{P}_Q(x_1(t)\ge L) + c_m \sum_{j=L}^{\infty} j^{m-1} \mathbb{P}_Q(x_1(t)\ge j)
\end{aligned}
\tag{4.19}
$$

for a suitable constant c_m depending on m.

Consider a generic term $\mathbb{P}_Q(x_1(t)\ge j)$, $j \ge L$. Once again we split the Q-integration over $\mathcal{D}_j = \{\sigma :\ x_1(\sigma) \le j/2\}$ and $\mathcal{D}_j^c$ to get

$$
\mathbb{P}_Q(x_1(t)\ge j) \le Q(\mathcal{D}_j^c) + \int_{\mathcal{D}_j} dQ(\sigma)\mathbb{P}_\sigma(x_1(t)\ge j)\,. \tag{4.20}
$$

The contribution to (4.19) of the first term in the r.h.s. of (4.20) is fine, *i.e.*

$$
\lim_{L\uparrow\infty} \left\{ L^m\, Q(\mathcal{D}_L^c) + c_m \sum_{j=L}^{\infty} j^{m-1} Q(\mathcal{D}_j^c) \right\} = 0
$$

again because the m^{th}-moment of μ,ν is finite. We further split the last term in (4.20) as

$$
\begin{aligned}
\int_{\mathcal{D}_j} dQ(\sigma)\mathbb{P}_\sigma(x_1(t)\ge j) &= \int_{\mathcal{D}_j\cap\mathcal{B}_j} dQ(\sigma)\mathbb{P}_\sigma(x_1(t)\ge j) + \int_{\mathcal{D}_j\cap\mathcal{B}_j^c} dQ(\sigma)\mathbb{P}_\sigma(x_1(t)\ge j) \\
&\le \int_{\mathcal{D}_j\cap\mathcal{B}_j} dQ(\sigma)\mathbb{P}_\sigma(x_1(t)\ge j) + Q(\mathcal{D}_j\cap\mathcal{B}_j^c)
\end{aligned}
$$

where

$$
\mathcal{B}_j = \{\sigma :\ \exists \{x_{j_i}\}_{i=0}^{\ell} \in \mathcal{Z}(\sigma)\cap[x_1(\sigma),j] \text{ and each } x_{j_i} \text{ is of class at least } n_0\},
$$

with $\ell = \lfloor \lambda\, m \log(j)/\mu_0 \rfloor$, λ being a numerical constant to be chosen later on.

Let $\mathcal{N}(\sigma) = |\mathcal{Z}(\sigma)\cap[x_1(\sigma),j]|$ and let $\mathcal{N}_0(\sigma) = |\{1 \leqslant i \le 2\ell/\mu_0+1 :\ x_i(\sigma) \text{ is of class at least } n_0\}|$. Then

$$
Q(\mathcal{D}_j\cap\mathcal{B}_j^c) \le Q(\{\mathcal{N} \le 2\ell/\mu_0\}\cap\mathcal{D}_j) + Q(\mathcal{N}_0 \le \ell) \tag{4.21}
$$

where we used $\{\mathcal{N} > 2\ell/\mu_0\} \cap \mathcal{B}_j^c \subset \{\mathcal{N}_0 \le \ell\}$.

In turn, by standard binomial large deviations,

$$Q(\mathcal{N}_0 \le \ell) \le e^{-c\ell\mu_0} \le j^{-\lambda cm}$$

for some numerical constant c.
As far as the term $Q(\mathcal{N} \le 2\ell/\mu_0\,;\,\mathcal{D}_j)$ is concerned we have

$$Q(\{\mathcal{N} \le 2\ell/\mu_0\} \cap \mathcal{D}_j) \le Q\Big(\sum_{i=1}^{\lfloor 2\ell/\mu_0 \rfloor} d_i(\sigma) \ge j/2\Big) \le \frac{2\ell}{\mu_0}\mu\left(\left[j/(4\ell/\mu_0)\right)\right).$$

In conclusion, if λ is taken large enough and by using the assumption on the finiteness of the $(m+\delta)^{th}$-moment of μ, we conclude that

$$\lim_{L\uparrow\infty}\left(L^m Q(\mathcal{D}_L \cap \mathcal{B}_L^c) + c_m \sum_{j\ge L} j^{m-1} Q(\mathcal{D}_j \cap \mathcal{B}_j^c)\right) = 0\,.$$

We are left with the analysis of

$$\int_{\mathcal{D}_j\cap\mathcal{B}_j} dQ(\sigma)\mathbb{P}_\sigma(x_1(t) \ge j) \leqslant \sup_{\sigma\in\mathcal{D}_j\cap\mathcal{B}_j} \mathbb{P}_\sigma(x_1(t) \ge j)\,.$$

Given $\sigma \in \mathcal{D}_j \cap \mathcal{B}_j$ let $\{[a_i, b_i],\ b_i \le a_{i+1}\}_{i=1}^{\ell}$ be the first ℓ domains of class n_0 in σ, contained in $[x_1(\sigma),\ j]$, whose existence is guaranteed by σ being in $\mathcal{B}_j$. Then

$$\{x_1(t) \ge j\} \subset \cup_{k=0}^{\ell} \cap_{\substack{i=1\\ i\neq k}}^{\ell} \{\sigma_t(x) = 1\ \forall x \in [a_i, b_i)\} := \cup_{k=0}^{\ell} \cap_{\substack{i=1\\ i\neq k}}^{\ell} A_i\,.$$

i.e.

$$\mathbb{P}_\sigma(x_1(t) \ge j) \le \mathbb{P}\big(\cap_{\substack{i=1\\ i\neq k}}^{\ell} A_i\big)$$

We claim that

$$\mathbb{P}_\sigma\big(\cap_{\substack{i=1\\ i\neq k}}^{\ell} A_i\big) \le \begin{cases} \beta^{\ell-1} & \text{if } k \ge 1 \\ \beta^{\ell} & \text{if } k = 0 \end{cases}. \tag{4.22}$$

with $\beta(q) = ct_N/t_{n_0}$. Notice that $\lim_{q\downarrow 0}\beta(q) = 0$ since $n_0 > N$. Moreover the r.h.s. of (4.22) is smaller than an arbitrarily large inverse power of j for q small enough because $\ell = O(\log(j))$. Hence, assuming (4.22), the proof of (ii) is finished since

$$\lim_{q\downarrow 0}\sup_{t\in[0,t_N^+]}\left[L^m \sup_{\sigma\in\mathcal{D}_L\cap\mathcal{B}_L} \mathbb{P}_\sigma(x_1(t) \ge L) + c_m \sum_{j=L}^{\infty} j^{m-1} \sup_{\sigma\in\mathcal{D}_j\cap\mathcal{B}_j} \mathbb{P}_\sigma(x_1(t) \ge j)\right] = 0\,.$$

For simplicity we prove the claim 4.22 only for $k = 0$ but the general case is the same. We observe that the event $\cap_{i=2}^{\ell} A_i$ is measurable w.r.t. to the σ-algebra $\mathcal{F}$ generated by the Poisson processes and coin tosses at the sites in $[b_1, \infty)$. Therefore

$$\mathbb{P}_\sigma(\cap_{i=1}^{\ell} A_i) = \mathbb{E}_\sigma\left(\prod_{i=2}^{\ell} \mathbb{1}_{A_i}\, \mathbb{P}_\sigma(A_1 \,|\, \mathcal{F})\right).$$

Thus, by iteration, it is enough to prove that

$$\mathbb{P}_\sigma(A_1 \,|\, \mathcal{F}) \le \beta$$

with β as above. That follows, as usual, from Corollary 4.6 and Remark 4.1.
(ii)-(4.16) The proof is just a trivial adaptation of the proof of (4.15). □

Remark 4.5. *The same results of Propositions 4.12 and 4.13 hold for the hierarchical coalescence process HCP and the proof is practically the same with one big simplification. As soon as a zero of class bigger than N occurs in the initial configuration, then, up to time t_N, the zero cannot be erased. Thus, in this case, the uniform bound on, say, $\mathbb{E}_Q(x_1(t)^m)$ are all obtained through a control of the corresponding m^{th}-moment of Q without any need of "dynamical" estimates.*

5. East process and HCP: proof of Theorems 3.7 and 3.8

As already mentioned at the end of Section 3.5, the proof of Theorem 3.8 follows at once from Theorem 3.7 together with Propositions 4.12 and 4.13 and their analog for the HCP process (see Remark 4.5). Thus the key point here is to prove Theorem 3.7.

Without loss of generality we can assume that (the label of) the largest epoch N which we will observe is larger than one and that $\epsilon = 1/8N$. We will show below that the proof of the theorem can be reduced to the proof of the following claim.

Claim 5.1. *Let $\sigma \in \Omega_\Lambda$ be such that any zero in $\mathcal{Z}(\sigma)$ is of class at least n. Then*

$$\lim_{q\downarrow 0} \sup_{t\in[0,t_n^+-t_n^-]} d_{TV}\left(\{\sigma_t, \mathbb{P}^\Lambda_\sigma\}\,;\,\{\sigma_t, \mathbb{P}^{\Lambda,n,C}_\sigma\}\right) = 0$$

where we recall $\mathbb{P}^{\Lambda,n,C}_\sigma$ denotes the law of the n^{th}-coalescence process on Λ starting from σ and defined in Section 3.2 with the choice $\epsilon = 1/8N$.

Let us explain how to derive Theorem 3.7 assuming the claim. Fixed $t \le t_N^+$, there are two cases to be examined:

(a) t belongs to an *active period i.e.* $t \in [t_n^-, t_n^+]$;
(b) t belongs to a *stalling period i.e.* $t \in [t_{n-1}^+, t_n^-]$.

We first observe that during a stalling period nothing happens with probability tending to one as $q \downarrow 0$. More precisely, for any $\sigma \in \Omega_\Lambda$, by using Proposition 4.8 we get

$$\lim_{q\downarrow 0} \sup_{n\le N} \sup_{t\in[t_{n-1}^+,t_n^-]} d_{TV}\left(\{\sigma_t, \mathbb{P}^\Lambda_\sigma\}\,;\,\{\sigma_{t_{n-1}^+}, \mathbb{P}^\Lambda_\sigma\}\right) = 0 \tag{5.1}$$

and similarly for the HCP process by using Corollary 3.5.

Thus, by a simple triangular inequality for the variation distance, it is enough to consider only case (a). For this purpose we first observe that, thanks to the Markov property, to the fact that with probability tending to one as $q \downarrow 0$ all the zeros of $\sigma_{t_n^-}$ are of class at least n (Corollary 3.5 and Proposition 4.8) and to Claim 5.1,

$$\lim_{q\downarrow 0} \sup_{s\in[0,t_n^+-t_n^-]} d_{TV}\left(\{\sigma_{t_n^-+s}, \mathbb{P}^\Lambda_\sigma\}\,;\,\{\sigma_{t_n^-+s}, \mathbb{P}^{\Lambda,H}_\sigma\}\right) = 0$$

if

$$\lim_{q\downarrow 0} d_{TV}\left(\{\sigma_{t_n^-}, \mathbb{P}^\Lambda_\sigma\}\,;\,\{\sigma_{t_n^-}, \mathbb{P}^{\Lambda,H}_\sigma\}\right) = 0\,.$$

In turn, thanks to (5.1), the above holds if

$$\lim_{q\downarrow 0} d_{TV}\left(\{\sigma_{t_{n-1}^+}, \mathbb{P}^\Lambda_\sigma\}\,;\,\{\sigma_{t_{n-1}^+}, \mathbb{P}^{\Lambda,H}_\sigma\}\right) = 0\,. \tag{5.2}$$

If we recursively iterate the above argument (note that when $n-1=0$ (5.2) holds from Claim 5.1 since $t_0^- = 0$) we get the sought conclusion. Thus the proofs of Theorem 3.7 and 3.8 are completed once we prove Claim 5.1.

5.1. **Proof of Claim 5.1.** Recall that (see (3.3))

$$T_0 = q^{(1-\epsilon)/2}, \quad T_1 = 1/q^{3\epsilon}, \quad T_n = (1/q)^{(n-1)(1+3\varepsilon)} \text{ for } n \geqslant 2.$$

Fix $n \le N$ and divide the time interval $[0, t_n^+ - t_n^-]$ into $M_n = (t_n^+ - t_n^-)/T_n$ *active sub–periods* $[t^{(\ell)}, t^{(\ell+1)})$ ($[t^{(\ell)}, t^{(\ell+1)}]$ if $\ell = M_n - 1$) where $t^{(\ell)} := \ell T_n$. Here we are neglecting the integer part for lightness of notation. Thus $M_0 = q^{-(1+\epsilon)/2}$, $M_1 = \frac{1-q^{2\epsilon}}{q^{1-2\epsilon}}$ and $M_n = \frac{1-q^{2n\epsilon}}{q^{1+3\epsilon-2n\epsilon}}$ if $n \geqslant 2$.

Definition 5.2 (t-trajectories and good t-trajectories).
Fix $t \in [0, t_n^+ - t_n^-]$ and $\sigma \in \Omega_\Lambda$ such that all its zeros are of class at least n. Let $\mathcal{T} := \{t^{(\ell)} : t^{(\ell)} \leqslant t\,,\ 0 \leqslant \ell \leqslant M_n\} \cup \{t\}$. The t–trajectory $\vec\sigma$ of a path $\{\sigma_s\}_{s\,\geqslant 0} \in D([0,\infty), \Omega_\Lambda)$, such that $\sigma_0 = \sigma$, is obtained restricting σ_s to $s \in \mathcal{T}$. We will often write $\vec\sigma_\ell$ for $\sigma_{t^{(\ell)}}$. A t–trajectory $\vec\sigma$ is called good *if given two arbitrary consecutive times $s < s'$ in $\mathcal{T}$ then either $\sigma_{s'} = \vec\sigma_s$ or $\sigma_{s'}$ is obtained from $\vec\sigma_s$ by removing a single zero of class n. The set of all good t–trajectories is denoted by $\mathcal{G}_t(\sigma)$.*

It follows from Corollary 3.5 and Proposition 4.8 that the set of good t-trajectories has probability tending to one as $q \downarrow 0$ both for the East process and for the n^{th}-CP. The key to prove the claim will be the following result.

Proposition 5.3. *For any $\sigma \in \Omega_\Lambda$ such that all its zeros are of class at least n*

$$\lim_{q\downarrow 0} \sup_{t \le t_n^+ - t_n^-} \sum_{\vec\sigma \in \mathcal{G}_t(\sigma)} |\mathbb{P}_\sigma^\Lambda(\vec\sigma) - \mathbb{P}_\sigma^{\Lambda,n,C}(\vec\sigma)| = 0.$$

Assuming the proposition we conclude the proof of Claim 5.1 as follows. Let $\mathcal{E} \subset \Omega_\Lambda$ and write

$$\mathbb{P}_\sigma^\Lambda(\sigma_t \in \mathcal{E}) = \sum_{\substack{\vec\sigma \in \mathcal{G}_t(\sigma) \\ \sigma_t \in \mathcal{E}}} \mathbb{P}_\sigma^\Lambda(\vec\sigma) + \sum_{\substack{\vec\sigma \in \mathcal{G}_t^c(\sigma) \\ \sigma_t \in \mathcal{E}}} \mathbb{P}_\sigma^\Lambda(\vec\sigma)$$

and similarly for the n^{th}-CP. Thus

$$\begin{aligned} &|\mathbb{P}_\sigma^\Lambda(\sigma_t \in \mathcal{E}) - \mathbb{P}_\sigma^{\Lambda,n,C}(\sigma_t \in \mathcal{E})| \\ &\le \sum_{\vec\sigma \in \mathcal{G}_t(\sigma)} |\mathbb{P}_\sigma^\Lambda(\vec\sigma) - \mathbb{P}_\sigma^{\Lambda,n,C}(\vec\sigma)| + \mathbb{P}_\sigma^\Lambda(\mathcal{G}_t(\sigma)^c) + \mathbb{P}_\sigma^{\Lambda,n,C}(\mathcal{G}_t(\sigma)^c)\,. \end{aligned} \tag{5.3}$$

As observed before Proposition 5.3, the last two terms in the r.h.s. tend to zero as $q \downarrow 0$ in a strong sense, namely

$$\lim_{q\downarrow 0} \sup_{t \le t_n^+ - t_n^-} \Big[\mathbb{P}_\sigma^\Lambda(\mathcal{G}_t(\sigma)^c) + \mathbb{P}_\sigma^{\Lambda,n,C}(\mathcal{G}_t(\sigma)^c)\Big] = 0\,.$$

The first term in the r.h.s of (5.3) tends to zero because of Proposition 5.3. Claim 5.1 is proved.

Proof of Proposition 5.3. For simplicity we restrict to times t of the form $t = t^{(\ell)}$ for some $\ell \leqslant M$. The general case can be treated similarly. Moreover for lightness of notation we will drop the superscript Λ and n from our notation $\mathbb{P}^{\Lambda}_{\sigma}$, $\mathbb{P}^{\Lambda,n,C}_{\sigma}$. Recall that $\Omega^{(\geq n)}_{\Lambda}$ denotes the set of configurations σ such that each $x \in \mathcal{Z}(\sigma)$ is of class at least n and define

$$\delta = \sup_{\sigma\in\Omega^{(\geq n)}_{\Lambda}} \max\left(\Big| \frac{\mathbb{P}_{\sigma}(\sigma_{T_n} = \sigma)}{\mathbb{P}^{C}_{\sigma}(\sigma_{T_n} = \sigma)} - 1 \Big|, \Big| \frac{\mathbb{P}^{C}_{\sigma}(\sigma_{T_n} = \sigma)}{\mathbb{P}_{\sigma}(\sigma_{T_n} = \sigma)} - 1 \Big| \right) \tag{5.4}$$

$$\gamma = \sup_{\sigma\in\Omega^{(\geq n)}_{\Lambda}} \sup_{\substack{x\in\mathcal{Z}(\sigma)\\ x \text{ of class } n}} \max\left(\Big| \frac{\mathbb{P}_{\sigma}(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\})}{\mathbb{P}^{C}_{\sigma}(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\})} - 1 \Big|, \Big| \frac{\mathbb{P}^{C}_{\sigma}(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\})}{\mathbb{P}_{\sigma}(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\})} - 1 \Big| \right). \tag{5.5}$$

Then, by the Markov property, given $\vec{\sigma} \in \mathcal{G}_t(\sigma)$ it holds

$$\begin{aligned}\Big|\frac{\mathbb{P}_{\sigma}(\vec{\sigma})}{\mathbb{P}^{C}_{\sigma}(\vec{\sigma})} - 1\Big| &= \Big| \prod_{\ell=0}^{M_n-1} \frac{\mathbb{P}_{\vec{\sigma}_\ell}(\sigma_{T_n} = \vec{\sigma}_{\ell+1})}{\mathbb{P}^{C}_{\vec{\sigma}_\ell}(\sigma_{T_n} = \vec{\sigma}_{\ell+1})} - 1\Big| \\ &\leq (1+\delta)^{M_n}(1+\gamma)^{c} - 1\end{aligned}$$

for some constant c depending on (Λ, n), because the number of transitions $\vec{\sigma}_\ell \to \vec{\sigma}_{\ell+1}$ in which a zero is removed is uniformly bounded (e.g. by the cardinality of Λ). Above we have used (5.4) and (5.5) because $\vec{\sigma}_\ell \in \Omega^{(\geq n)}$ for all ℓ since $\sigma \in \Omega^{(\geq n)}$ and $\vec{\sigma} \in \mathcal{G}_t(\sigma)$. Hence it is sufficient to show that $\lim_{q\downarrow 0} M_n\delta = 0$ and $\lim_{q\downarrow 0}\gamma = 0$.

5.1.1. *Bounding* δ, γ. It follows from Corollary 3.5 and Proposition 4.8 that $\mathbb{P}^{C}_{\sigma}(\sigma_{T_n} = \sigma) \geq 1/2$ for q small enough uniformly in $\sigma \in \Omega^{(\geq n)}_{\Lambda}$ and similarly for $\mathbb{P}_{\sigma}(\sigma_{T_n} = \sigma)$. Thus

$$\begin{aligned}\delta &\leq 2 \sup_{\sigma\in\Omega^{(\geq n)}_{\Lambda}} \Big|\mathbb{P}_{\sigma}(\sigma_{T_n} = \sigma) - \mathbb{P}^{C}_{\sigma}(\sigma_{T_n} = \sigma)\Big| \\ &= 2 \sup_{\sigma\in\Omega^{(\geq n)}_{\Lambda}} \Big|\mathbb{P}_{\sigma}(\sigma_{T_n} \neq \sigma) - \mathbb{P}^{C}_{\sigma}(\sigma_{T_n} \neq \sigma)\Big|\end{aligned}$$

$$\leqslant 2 \sup_{\sigma\in\Omega^{(\geq n)}_{\Lambda}} \left[\mathbb{P}_{\sigma}\big(\mathcal{Z}(\sigma_{T_n}) \not\subseteq \mathcal{Z}(\sigma)\big) + \mathbb{P}_{\sigma}\big(|\mathcal{Z}(\sigma)\setminus\mathcal{Z}(\sigma_{T_n})| \geq 2\big) + \mathbb{P}^{C}_{\sigma}\big(|\mathcal{Z}(\sigma)\setminus\mathcal{Z}(\sigma_{T_n})| \geq 2\big)\right] \tag{5.6}$$

$$+2 \sup_{\sigma\in\Omega^{(\geq n)}_{\Lambda}} \sum_{x\in\mathcal{Z}(\sigma)} \Big|\mathbb{P}_{\sigma}\big(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\}\big) - \mathbb{P}^{C}_{\sigma}\big(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\}\big)\Big|. \tag{5.7}$$

The contribution in (5.6) can be bounded, using Corollary 3.5, Lemma 4.2 and Corollary 4.7 for $n \geqslant 1$ and by an easy calculation in the case $n = 0$, by $c\left(q + (T_n/t_n)^2\right)$ for some constant $c = c(L, N)$ and therefore, when multiplied by $M_n \leq t^{+}_n/T_n$, vanishes as $q \downarrow 0$.

The contribution in (5.7) is instead bounded from above by

$$c \sup_{\sigma\in\Omega^{(\geq n)}_{\Lambda}} \left(\gamma \sup_{x\in\mathcal{Z}(\sigma)} \mathbb{P}^{C}_{\sigma}\big(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\}\big) + \sup_{\substack{x\in\mathcal{Z}(\sigma)\\ x \text{ of class } \geq n+1}} \mathbb{P}_{\sigma}\big(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\}\big)\right)$$

by the definition (5.5) of γ (recall that any zero x of class at least $n+1$ cannot be erased during the n–th coalescence process). Because of Corollary 4.6, uniformly in $\sigma \in \Omega_\Lambda^{(\geq n)}$,

$$\sup_{\substack{x\in\mathcal{Z}(\sigma)\\ x \text{ of class } \geq n+1}} \mathbb{P}_\sigma\big(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\}\big) \leq cT_n/t_{n+1}$$

and therefore, when multiplied by M_n tends to zero as $q \downarrow 0$. Similarly, using Lemma 3.4,

$$\sup_{x\in\mathcal{Z}(\sigma)} \mathbb{P}^C_\sigma\big(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\}\big) \leq cT_n/t_n\,.$$

which, once it is multiplied by M_n, is bounded from above by ct_n^+/t_n.

In conclusion, in order to show that $\lim_{q\downarrow 0} M_n\delta = 0$ and $\lim_{q\downarrow 0}\gamma = 0$, it is enough to show that $\lim_{q\downarrow 0}\gamma\, t_n^+/t_n = 0$ uniformly in $\sigma \in \Omega_\Lambda^{(\geq n)}$. For this purpose, given $x \in \mathcal{Z}(\sigma)$ with domain $d_x \in \mathcal{C}_n$, we assume that the closest zero of σ to the left of x is also of class n. Call z its position. The case in which this assumption is not verified can be treated analogously.

Then we write

$$\mathbb{P}^C_\sigma\big(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\}\big) = \text{Prob}\,(\xi_x \leq T_n,\ \xi_x \leq \xi_z) \prod_{\substack{y\in\mathcal{Z}(\sigma)\\ y\neq x,z \text{ is of class } n}} \text{Prob}\,(\xi_y \geq T_n)$$

where $\{\xi_y\}$, $y \in \mathcal{Z}(\sigma)$, are independent exponential variables with parameter $\lambda_n(d_y)$ (recall the graphical construction in Section 3.3.1).

Using the definition (3.4) of the rates λ_n and (4.7) of Lemma 4.4, for any $y \in \mathcal{Z}(\sigma)$ of class n

$$\text{Prob}\,(\xi_y \geq T_n) = \mathbb{P}^{[0,d_y-1]}_{0\mathbb{1}}(\tilde\tau \geqslant T_n) = 1 + O(T_n/t_n)$$

where $\tilde\tau$ is the hitting time of the set $\{\sigma : \sigma(0) = 0\}$. On the other hand, for the same reasons,

$$\begin{aligned}\text{Prob}(\xi_x \leq T_n) \geq \text{Prob}\,(\xi_x \leq T_n,\ \xi_x \leq \xi_z) \geq \text{Prob}\,(\xi_x \leq T_n) - \text{Prob}\,(\xi_x \leq T_n,\ \xi_z \leq T_n)\\ = \mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}(\tilde\tau \leqslant T_n)(1 + O(T_n/t_n))\,.\end{aligned}$$

We conclude that, uniformly in σ,

$$\mathbb{P}^C_\sigma\big(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\}\big) = \mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}(\tilde\tau \leqslant T_n)(1 + O(T_n/t_n))\,. \tag{5.8}$$

Similarly, with $\tau_n(y)$ the first time such that there are n zeros strictly inside the domain of $y \in \mathcal{Z}(\sigma)$, we can write

$$\begin{aligned}&\mathbb{P}_\sigma\big(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\}\big)\\ &= \mathbb{P}_\sigma\left(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\},\ \tau_n(y) > T_n\ \forall y \neq x\right)\\ &+\mathbb{P}_\sigma\left(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma)\setminus\{x\},\ \tau_n(y) \leq T_n \text{ for some } y \neq x\right)\end{aligned} \tag{5.9}$$

The last term, thanks to Corollary 4.7, is bounded from above by $c(T_n/t_n)^2$. Thanks to Lemma 2.2 the first term in the r.h.s. factorizes as

$$\mathbb{P}^V_\eta\left(\sigma_{T_n} = \sigma_{0\mathbb{1}}\right) \prod_{\substack{y\in\mathcal{Z}(\sigma)\\ y\neq x,z}} \mathbb{P}^{[0,d_y-1]}_{0\mathbb{1}}(\tau_n > T_n,\ \sigma_{T_n} = \sigma_{0\mathbb{1}})$$

where $V = [0, d_z + d_x - 1]$ and $\eta \in \Omega_V$ is such that $\mathcal{Z}(\eta) = \{0, d_z\}$. Indeed, if $\tau_n(y) > T_n \ \forall y \neq x$, all the zeros in $\mathcal{Z}(\sigma)$ different from x are frozen thanks to Remark 4.1. By Lemma 4.2 and (4.7) in Lemma 4.4

$$\prod_{\substack{y\in\mathcal{Z}(\sigma)\\ y\neq x,z}} \mathbb{P}^{[0,d_y-1]}_{0\mathbb{1}}(\tau_n > T_n,\ \sigma_{T_n} = \sigma_{0\mathbb{1}}) = 1 + O(q) + O(T_n/t_n) = 1 + O(T_n/t_n)$$

On the other hand, conditioned on the σ-algebra $\mathcal{F}$ of the Poisson processes and coin tosses associated to $[d_z, d_z + d_x - 1]$,

$$\mathbb{P}^V_\eta\left(\mathcal{Z}(\sigma_{T_n}) \cap [0, d_z - 1] = \{0\} \,|\, \mathcal{F}\right) = 1 + O(q) + O(T_n/t_n) = 1 + O(T_n/t_n)$$

because of (4.1) and (4.2) in Lemma 4.2 and Remark 4.1. Hence

$$\mathbb{P}^V_\eta\left(\sigma_{T_n} = \sigma_{0\mathbb{1}}\right) = \mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}\left(\sigma_{T_n} = \sigma_{\mathbb{1}}\right)(1 + O(T_n/t_n)).$$

Finally

$$\mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}\left(\sigma_{T_n} = \sigma_{\mathbb{1}}\right) = 1 - \mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}\left(\sigma_{T_n} \neq \sigma_{\mathbb{1}}\right) \geq 1 - \mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}\left(\tilde\tau \geq T_n\right) + O(q) \geq cT_n/t_n$$

because of (3.9).

Going back to (5.9) and collecting the above estimates we conclude that

$$\mathbb{P}_\sigma\big(\mathcal{Z}(\sigma_{T_n}) = \mathcal{Z}(\sigma) \setminus \{x\}\big) = \mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}\left(\sigma_{T_n} = \sigma_{\mathbb{1}}\right)(1 + O(T_n/t_n)) \tag{5.10}$$

If we put together (5.8) and (5.10) we get

$$\gamma \leq \left| \frac{\mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}(\sigma_{T_n} = \sigma_{\mathbb{1}})}{\mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}(\tilde\tau \leqslant T_n)} - 1 \right| + O(T_n/t_n).$$

The contribution of the error term $O(T_n/t_n)$ to $\gamma\, t_n^+/t_n$ tends to zero as $q \downarrow 0$. As far as the first term is concerned we can write

$$1 - \frac{\mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}(\sigma_{T_n} = \sigma_{\mathbb{1}})}{\mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}(\tilde\tau \leqslant T_n)}$$
$$= \frac{\mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}(\{\tilde\tau \leqslant T_n\} \cap \{\sigma_{T_n} \neq \sigma_{\mathbb{1}}\})}{\mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}(\tilde\tau \leqslant T_n)} \leq c\, q / \mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}(\tilde\tau \leqslant T_n) \leq c\, q \frac{t_n}{T_n}. \tag{5.11}$$

In the first inequality we used the bound (see Lemma 4.2)

$$\mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}(\{\tilde\tau \leqslant T_n\} \cap \{\sigma_{T_n}(y) = 0\}) \leq \begin{cases} \mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}(\sigma_{T_n}(y) = 0) \leq cq & \text{if } y \neq 0 \\ \mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}(\{\tilde\tau \leqslant T_n\} \cap \{\sigma_{T_n}(0) = 0\}) \leq cq \end{cases}$$

where, for the case $y = 0$, the estimate follows from the strong Markov property and Lemma 4.2 applied to the starting configuration $\sigma_{\tilde\tau}$. In the second inequality we used (3.9) to get $\mathbb{P}^{[0,d_x-1]}_{0\mathbb{1}}(\tilde\tau \leqslant T_n) \geq cT_n/t_n$.

Since $\lim_{q\downarrow 0} \frac{t_n^+}{t_n}\, q\, \frac{t_n}{T_n} = 0$ we can conclude that $\lim_{q\downarrow 0} \gamma \frac{t_n^+}{t_n} = 0$ and the proof is complete. □

6. Proof of Theorems 2.5 and 2.6

Proof of Theorem 2.5.
(i) Thanks to Lemma 4.3

$$\overline{\lim_{q\downarrow 0}} \sup_{t\in[t_n^+,t_{n+1}^-]} |\mathbb{P}_Q(\sigma_t(0)=0)-\mathbb{P}_Q(\sigma_s(0)=0\ \forall s\le t)|=0\,.$$

Hence it is enough to prove that

$$\overline{\lim_{q\downarrow 0}} \sup_{t\in[t_n^+,t_{n+1}^-]} \left|\mathbb{P}_Q(\sigma_t(0)=0)-\left(\frac{1}{2^n+1}\right)^{c_0(1+o(1))}\right|=0$$

where $o(1)$ is an error term going to zero as $n\to\infty$. Equation (3.25) of Theorem 3.8 tells us that

$$\overline{\lim_{q\downarrow 0}} \sup_{t\in[t_n^+,t_{n+1}^-]} \left|\mathbb{P}_Q(\sigma_t(0)=0)-\mathbb{P}_Q^H(x_0(t)=0)\right|=0.$$

In turn, thanks to Remark 3.2 it holds

$$\overline{\lim_{q\downarrow 0}} \sup_{t\in[t_n^+,t_{n+1}^-]} \left|\mathbb{P}_Q^H(x_0(t)=0)-\mathbb{P}_Q^H(\sigma_0^{(n+1)}(0)=0)\right|=0.$$

and (iv) of Theorem 3.6 says that

$$\mathbb{P}_Q^H(\sigma_0^{(n+1)}(0)=0)=\frac{1}{(2^n+1)^{c_0(1+o(1))}}$$

and the sought result follows.
(ii) The result follows immediately by using Lemma 4.3.
(iii) Fix $x\in\mathbb{Z}_+$, $m<n$ and $s\in[t_m^+,t_{m+1}^-]$, $t\in[t_n^+,t_{n+1}^-]$. Because of Lemma 4.2

$$\overline{\lim_{q\downarrow 0}} \sup_{\substack{t\in[t_n^+,t_{n+1}^-]\\ s\in[t_m^+,t_{m+1}^-]}} \mathbb{P}_Q(\sigma_t(x)=0\,|\,\sigma_s(x)=1)=0\,.$$

Hence

$$\begin{aligned} C_Q(s,t,x)&=\mathbb{P}_Q(\sigma_t(x)=0\cap\sigma_s(x)=0)-\mathbb{P}_Q(\sigma_t(x)=0)\mathbb{P}_Q(\sigma_s(x)=0)\\ &=\mathbb{P}_Q(\sigma_t(x)=0)\left(1-\mathbb{P}_Q(\sigma_s(x)=0)\right)+\delta(s,t,q)\end{aligned}$$

with

$$\overline{\lim_{q\downarrow 0}} \sup_{\substack{t\in[t_n^+,t_{n+1}^-]\\ s\in[t_m^+,t_{m+1}^-]}} \delta(s,t,q)=0\,.$$

Similarly

$$\overline{\lim_{q\downarrow 0}} \sup_{t\in[t_n^+,t_{n+1}^-]} |\mathbb{P}_Q(\sigma_t(x)=0)-\mathbb{P}_Q(\sigma_t(x)=0\,,\ \sigma_0(x)=0)|=0$$

and the same at time s. Since $\mathbb{P}_Q(\sigma_t(x)=0\,|\,\sigma_0(x)=0)=\mathbb{P}_Q(\sigma_t(0)=0)$ because of the renewal property of Q the proof follows at once from part (i). □

Proof of Theorem 2.6. The proof follows at once from Theorems 3.6, 3.8 and Remark 3.2. □

7. Extensions

In this section we present some extensions of our results. As already mentioned, the technical assumption that the interval law μ satisfies $\mu([k,\infty)) > 0$ for all $k \in \mathbb{N}$ can be removed as discussed in [15]. As a consequence, in what follows we disregard this assumption.

7.1. **The East process on $\mathbb{Z}$ with renewal stationary initial distribution.** We say that a random subset σ of $\mathbb{Z}$ is stationary if its law Q is left invariant by any translation along a vector $x \in \mathbb{Z}$. In addition, we say that it is renewal if the law $Q(\cdot\,|\,0 \in \sigma)$ equals $\mathrm{Ren}(\mu\,|\,0)$ for some probability measure on $\mathbb{N}$ (note that $Q(0 \in \sigma)$ must be positive due to the stationarity). For simplicity, we write $Q = \mathrm{Ren}(\mu)$ and call μ the interval law. Note that under the Bernoulli probability on $\{0,1\}^{\mathbb{Z}}$ with parameter p, the set of zeros has law $\mathrm{Ren}(\mu\,|\,0)$ with $\mu(n) = p^{n-1}(1-p)$.

It can be proved (see [9]) that μ must have finite mean. Moreover, Q–a.s. the random subset σ is given by infinite points $\{x_k\}_{k\in\mathbb{Z}}$ with $\lim_{k\to\pm\infty} x_k = \pm\infty$. In what follows, we enumerate the points in σ with the convention that $x_k < x_{k+1}$ and $x_0 \leqslant 0 < x_1$.

Due to formula (C3) in [14, Appendix C] the set σ with law $Q = \mathrm{Ren}(\mu)$ is characterized by the following properties:

(i) the points $x_0 \leqslant 0 < x_1$ have law

$$
\begin{aligned}
Q(x_0 \leqslant -m\,,\; x_1 \geqslant n) &= (1/\bar{\mu}) \sum_{\ell=n+m}^{\infty} \mu(\ell)(\ell - n - m + 1) \\
&= (1/\bar{\mu}) \sum_{\ell=n+m}^{\infty} \mu([\ell,\infty))
\end{aligned}
\tag{7.1}
$$

for all $n \geqslant 1, m \geqslant 0$, where $\bar{\mu}$ denotes the average of μ: $\bar{\mu} = \sum_{\ell=1}^{\infty} \mu(\ell)\ell$;

(ii) the domain lengths $(x_{k+1} - x_k)$, $k \in \mathbb{Z}\setminus\{0\}$, are i.i.d. random variables with law μ and are also independent from x_0, x_1.

Note that (7.1) implies that $Q(0 \in \xi) = 1/\bar{\mu}$ and that $|x_0| + 1$ has the same law of x_1. Due to the above characterization, under $Q = \mathrm{Ren}(\mu)$ the law of $\sigma \cap \mathbb{Z}_+$ is given by $Q_+ = \mathrm{Ren}(\nu, \mu)$, where ν is the probability measure on $\mathbb{Z}_+$ such that

$$
\nu(n) = (1/\bar{\mu})\mu([n+1,\infty))\,, \qquad n \in \mathbb{Z}_+
$$

(indeed by stationarity ν coincides with the law of $x_1 - 1$ since x_1 is the leftmost point of $\sigma \cap \mathbb{N}$). Note that ν has finite m^{th}–moment if and only if μ has finite $(m+1)^{th}$-moment.

The above observation implies that Theorem 2.5 can be adapted to the stationary case following the guidelines of Remark 2.3.

The East process starting from Q must be compared with the hierarchical coalescence process on $\mathbb{Z}$ starting from Q (the definition is a straightforward extension of the one given for the HCP on $\mathbb{Z}_+$). In [14] it is proved that, considering the HCP on $\mathbb{Z}$ starting with distribution $Q = \mathrm{Ren}(\mu)$, the law at the beginning of the n–th epoch is simply $\mathrm{Ren}(\mu_n)$ with μ_n defined from μ as in Subsection 3.4.

By slightly modifications in the proof, Theorem 3.8 becomes:

Theorem 7.1. *For any $N \in \mathbb{N}$ let $\epsilon_N := 1/8N$ and choose the parameter ϵ appearing in Definition 2.4 and in (3.3) equal to ϵ_N. Let $Q = Ren(\mu)$ with μ probability measure on $\mathbb{N}$ having finite mean. Then for any $k \in \mathbb{Z}$*

$$\lim_{q\downarrow 0} \sup_{t\in[0,t_N^+]} d_{TV}\big(\{(x_{-k}(t),\dots,x_k(t)),\, \mathbb{P}_Q\}\,;\,\{(x^H_{-k}(t),\dots,x^H_k(t)),\, \mathbb{P}^H_Q\}\big) = 0\,.$$

Assume that the $(m+\delta)^{th}$-moment of μ is finite for some $\delta > 0$. Then, for each $k \in \mathbb{Z}\setminus\{0\}$ it holds

$$\lim_{q\downarrow 0} \sup_{t\in[0,t_N^+]} \big|\mathbb{E}_Q\big([x_{k+1}(t)-x_k(t)]^m\big) - \mathbb{E}^H_Q\big([x^H_{k+1}(t)-x^H_k(t)]^m\big)\big| = 0\,.$$

Assume that the $(m+1+\delta)^{th}$-moment of μ is finite for some $\delta > 0$. Then the above equation (3.26) is valid also for $k = 0$ and moreover, for all $k \in \mathbb{Z}$, it holds

$$\lim_{q\downarrow 0} \sup_{t\in[0,t_N^+]} \big|\mathbb{E}_Q\big([x_k(t)]^m\big) - \mathbb{E}^H_Q\big([x^H_k(t)]^m\big)\big| = 0\,.$$

Using the above approximation result and the scaling limits discussed in [14], Theorem 2.6 remains valid in the stationary case by setting

$$\bar X^{(n)}(t) := (x_{k+1}(t)-x_k(t))/(2^{n-1}+1) \quad ; \quad \bar Y^{(n)}(t) := x_1(t)/(2^{n-1}+1),$$

where k is any integer in $\mathbb{Z}\setminus\{0\}$. $\bar Y^{(n)}(t)$ can also be defined as $|x_0(t)|/(2^{n-1}+1)$.

Remark 7.1. *The above extensions to the stationary case, and their derivation, will be discussed in more detail in [15]. There we will present other results, including the aging through hierarchical coalescence in the East process on the half–line $\{-1,-2,\dots\}$ with frozen zero at site 0.*

7.2. **The East process on $\mathbb{Z}_+$ with exchangeable initial distribution.** Our main results, with suitable modifications, can be formulated also when the initial distribution in an *exchangeable* one. We say that the law Q of a random set of points $\{x_i\}_{i=0}^{\infty}$ in $\mathbb{Z}_+$ containing the origin is exchangeable if this set has infinite cardinality a.s. and the law of the random sequence $x_1 - x_0 = x_1$, $x_2 - x_1$, $x_3 - x_2$,... is invariant w.r.t. finite permutations. By De Finetti Theorem, Q can be expressed as $Q = \int_\Upsilon \mathfrak{p}(d\zeta) Q_\zeta$, where $Q_\zeta = \mathrm{Ren}(\mu_\zeta\,|\,0)$ and the parameter ζ varies on a probability space $(\Upsilon, \mathfrak{p})$ [14, Appendix D].

Considering the East process on $\mathbb{Z}_+$ with initial distribution Q, suppose that for $\mathfrak{p}$–a.a. $\zeta \in \Upsilon$ the law μ_ζ satisfies condition (a) or (b) in Theorem 2.5, set $c_0(\zeta) = 1$ and $c_0(\zeta) = \alpha$ respectively. Then Theorem 2.5 remains valid introducing in the asymptotic values the average $\int_\Upsilon \mathfrak{p}(d\zeta)$ and replacing c_0 with $c_0(\zeta)$ and ρ_x with $Q_\zeta(\sigma(x) = 0)$. By similar modifications, also Theorem 2.6 remains valid for an exchangeable Q. Clearly, the average over $\mathfrak{p}(d\zeta)$ may lead to new asymptotic behaviors. Finally, Theorem 3.8 still holds provided that the $(m+\delta)^{th}$-moment w.r.t Q of $(x_{k+1}-x_k)$ (which is k–independent by exchangeability) is finite.

Acknowledgements. We thank the Laboratoire de Probabilités et Modèles Aléatoires, the University Paris VII and the Department of Mathematics of the University of Roma Tre for the support and the kind hospitality. C. Toninelli acknowledges the partial support of the French Ministry of Education through the ANR BLAN07-2184264 grant.

References

1. D. Aldous and P. Diaconis, *The asymmetric one-dimensional constrained ising model: rigorous results*, J. Stat. Phys. **107** (2002), no. 5-6, 945–975.
2. A. Asselah and P. Dai Pra, *Quasi-stationary measures for conservative dynamics in the infinite lattice*, Ann. Probab. **29** (2001), no. 4, 1733–1754.
3. A.J. Bray, B. Derrida, and C. Godrèche, *Non–trivial algebraic decay in a soluble model of coarsening*, Europhys. Lett. **27** (1994), no. 3, 175–180.
4. N. Cancrini, F. Martinelli, C. Roberto, and C. Toninelli, *Kinetically constrained spin models*, Probability Theory and Related Fields **140** (2008), no. 3-4, 459–504.
5. N. Cancrini, F. Martinelli, R. Schonmann, and C. Toninelli, *Facilitated oriented spin models: some non equilibrium results*, J. Stat. Phys. **138** (2010), no. 6, 1109–1123.
6. F. Chung, P. Diaconis, and R. Graham, *Combinatorics for the east model*, Adv. in Appl. Math. **27** (2001), no. 1, 192–206.
7. F. Corberi and L. F. Cugliandolo, *Out-of-equilibrium dynamics of the spiral model*, J. Stat. Mech. (2009), P09015.
8. A. Crisanti, F. Ritort, A. Rocco, and M. Sellitto, *Inherent structures and non-equilibrium dynamics of 1d constrained kinetic models: a comparison study*, J. Chem. Phys. **113** (2000), 10615–10647.
9. D.J. Daley and D. Vere-Jones, *An introduction to the theory of point processes.*, second ed., Probability and its Applications (New York), Springer, New York, 2008, General theory and structure.
10. B. Derrida, *Coarsening phenomena in one dimension*, Complex systems and binary networks (Guanajuato, 1995), Lecture Notes in Physics, vol. 461, Springer, Berlin, 1995, pp. 164–182.
11. B. Derrida, C. Godrèche, and I. Yekutieli, *Stable distributions of growing and coalescing droplets*, Europhys. Lett. **12** (1990), no. 5, 385–390.
12. ______, *Scale-invariant regimes in one-dimensional models of growing and coalescing droplets*, Physical Review A **44** (1991), no. 10, 6241–6251.
13. R. Durrett, *Lecture notes on particle systems and percolation*, Lecture Notes in Mathematics (1995), no. 1608.
14. A. Faggionato, F. Martinelli, C. Roberto, and C. Toninelli, *Universality in one dimensional hierarchical coalescence processes*, preprint (2010).
15. A. Faggionato, F. Martinelli, and C. Roberto, C. Toninelli, in preparation.
16. G.H. Fredrickson and H.C. Andersen, *Kinetic ising model of the glass transition*, Phys. Rev. Lett. **53** (1984), 1244–1247.
17. ______, *Facilitated kinetic ising models and the glass transition*, J. Chem. Phys. **83** (1985), 5822–5831.
18. J. P. Garrahan and M. E. J. Newman, *Inherent structures and non-equilibrium dynamics of 1d constrained kinetic models: a comparison study*, Phys. Rev. E **62** (2000), 7670–7680.
19. J. Jäckle and S. Eisinger, *A hierarchically constrained kinetic ising model*, Z. Phys. B: Condens. Matter **84** (1991), no. 1, 115–124.
20. C.Toninelli J.P. Garrahan, P.Sollich, *Kinetically constrained models*, to appear in "Dynamical heterogeneities in glasses, colloids, and granular media", Oxford Univ.Press, Eds.: L. Berthier, G. Biroli, J-P Bouchaud, L. Cipelletti and W. van Saarloos. Preprint arXiv:1009.6113.
21. S. Leonard, P. Mayer, P. Sollich, L. Berthier, and J.P. Garrahan, *Non-equilibrium dynamics of spin facilitated glass models*, J. Stat. Mech. (2007), P07017.
22. T.M. Liggett, *Interacting particle systems*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 276, Springer-Verlag, New York, 1985.
23. ______, *Stochastic interacting systems: contact, voter and exclusion processes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 324, Springer-Verlag, Berlin, 1999.
24. Fabio Martinelli, Enzo Olivieri, and Elisabetta Scoppola, *Small random perturbations of finite- and infinite-dimensional dynamical systems: unpredictability of exit times*, J. Statist. Phys. **55** (1989), no. 3-4, 477–504.
25. E. Olivieri and M.E. Vares, *Large deviations and metastability*, Encyclopedia of Mathematics and its Applications, vol. 100, Cambridge University Press, Cambridge, 2005.

26. F. Ritort and P. Sollich, *Glassy dynamics of kinetically constrained models*, Advances in Physics **52** (2003), no. 4, 219–342.
27. P. Sollich and M.R. Evans, *Glassy time-scale divergence and anomalous coarsening in a kinetically constrained spin chain*, Phys. Rev. Lett **83** (1999), 3238–3241.
28. ______, *Glassy dynamics in the asymmetrically constrained kinetic ising chain*, Phys. Rev. E (2003), 031504.

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