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POINCARÉ INEQUALITY AND THE L^P CONVERGENCE OF SEMI-GROUPS.

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Abstract

We prove that for symmetric Markov processes of diffusion type admitting a "carré du champ", the Poincaré inequality is equivalent to the exponential convergence of the associated semi-group in one (resp. all) $L^p(\mu)$ spaces for $1 < p < +\infty$. We also give the optimal rate of convergence. Part of these results extends to the stationary, not necessarily symmetric situation.

1 Introduction and main results.

Let X_t be a general Markov process taking values in some Polish space E. We denote by L its infinitesimal generator. We assume that the extended domain of L contains a nice core $\mathscr{D}$ of uniformly continuous functions which is an algebra. Functions in $\mathscr{D}$ will be called "smooth", and constant functions are assumed to be in $\mathscr{D}$. Next, we define the "carré du champ" operator

$$\Gamma(f,g) = \frac{1}{2}\left(L(fg) - fLg - gLf\right) .$$

For a given probability measure μ on E, the associated Dirichlet form can thus be defined for smooth f's as

$$\mathscr{E}(f,f) := -\int f\,Lf\,d\mu = \int \Gamma(f,f)\,d\mu\,.$$

In the sequel we assume that μ is an invariant measure. Thus L generates a μ stationary semi-group P_t, which is a contraction semi-group on all $\mathbb{L}^p(\mu)$ for $1 \leq p \leq +\infty$. We shall also look at cases where the following $\mathbb{L}^2$ ergodic theorem holds (this is automatic in the symmetric case): for all $f \in \mathbb{L}^2(\mu)$,

$$\lim_{t\to+\infty} \| P_t f - \int f\,d\mu \|_{\mathbb{L}^2(\mu)} = 0\,.$$

For the above, see *e.g.* [3].
Here and in the sequel, for any $p \in [1,\infty)$, $\| f \|_{\mathbb{L}^p(\mu)}$ or shortly $\| f \|_p$, stands for the $\mathbb{L}^p(\mu)$-norm of f with respect to μ: $\| f \|_p^p := \int |f|^p\,d\mu$.
In the sequel we shall assume that Γ comes from a derivation, i.e.

$$\Gamma(fg,h) = f\,\Gamma(g,h) + g\,\Gamma(f,h)\,,$$

i.e. (in the terminology of [1]) that $X_.$ is a diffusion. We also recall the chain rule: if φ is a C^2 function,

$$L(\varphi(f)) = \varphi'(f)\,Lf + \varphi''(f)\,\Gamma(f,f)\,.$$

It is well known that the following two statements are equivalent

(H-Poinc). μ satisfies a Poincaré inequality, i.e. there exists a constant C_P such that for all smooth f,

$$\mathrm{Var}_\mu(f) := \int f^2 d\mu - \left(\int f\,d\mu\right)^2 \leq C_P \int \Gamma(f,f)\,d\mu\,.$$

(H-2-1). There exists a constant λ_2 such that

$$\mathrm{Var}_\mu(P_t f) \leq e^{-2\lambda_2 t}\,\mathrm{Var}_\mu(f)\,.$$

If one of these assumptions is satisfied we have $\lambda_2 = 1/C_P$.

The symmetric situation is peculiar due to the following well known reinforcement of the previous equivalence

Lemma 1.1. *Assume that μ is symmetric. If there exists $\beta > 0$ such that for all $f \in \mathscr{C}$, $\mathscr{C}$ being an everywhere dense subset of $\mathbb{L}^2(\mu)$ the following property holds $Var_\mu(P_t f) \leq c_f\,e^{-2\beta t}$, then $Var_\mu(P_t f) \leq e^{-2\beta t}\,Var_\mu(f)$ for all $f \in \mathbb{L}^2(\mu)$, i.e. the Poincaré inequality holds with $C_P \leq 1/\beta$.*

There are several proofs of this lemma (using the spectral resolution or some convexity tools), see e.g. [8] lemma 2.6. In particular any exponential decay of the Variance

$$\textbf{(H-2)} \quad \mathrm{Var}_\mu(P_t f) \leq K_2\,e^{-2\beta t}\,\mathrm{Var}_\mu(f)\,,$$

implies **(H-2-1)** with $\beta = \lambda_2$, i.e. we may take $K_2 = 1$.

A similar statement is false in the non-symmetric case. Examples are known where (**H-2**) holds with $K_2 > 1$ but where the Poincaré inequality does not hold (see the kinetic Ornstein-Uhlenbeck process studied in [6] (also see [2] section 6)).

Our aim in this note will be be to investigate the $\mathbb{L}^p$ situation. For $f \in \mathbb{L}^p(\mu)$ define

$$N_p(f) := \| f - \int f \, d\mu \|_p \ .$$

Consider the operators defined on $\mathbb{L}^p$ by $T_t(f) = P_t(f - \mu(f))$, where $\mu(f) = \int f \, d\mu$. T_t is a bounded operator in $\mathbb{L}^p$ with operator norm denoted by $\| T_t \|_p$. For all $t \geq 0$, $\| T_t \|_p \leq 2$, and for $p = 2$ we have $\| T_t \|_2 \leq 1$. According to the Riesz-Thorin interpolation theorem, using the pairs $(1, 2)$ or $(2, +\infty)$ depending whether $1 \leq p \leq 2$ or $p \geq 2$, we thus have

$$\| T_t \|_p \leq 2^{1-r_p} \quad \text{with} \quad r_p = 2 \min\left(\frac{1}{p}, 1 - \frac{1}{p}\right) . \tag{1.2}$$

The same interpolation argument (with the pairs $(1, p)$, $(p, +\infty)$ and possibly $p = 2$) yields

Theorem 1.3. *The following statements are equivalent.*

(1) ***(H-2)*** *is satisfied.*

(2) There exist some $1 < p < +\infty$ and positive constants λ_p and K_p such that for all $f \in \mathbb{L}^p(\mu)$ and all $t > 0$,

$$N_p(P_t f) \leq K_p \, e^{-\lambda_p t} \, N_p(f) .$$

(3) for all $1 < p < +\infty$, there exist positive constants λ_p and K_p such that for all $f \in \mathbb{L}^p(\mu)$ and all $t > 0$,

$$N_p(P_t f) \leq K_p \, e^{-\lambda_p t} \, N_p(f) .$$

We shall denote by ***(H-p)*** *the property (2) for a given p.*
Furthermore, if μ is symmetric, these statements are also equivalent to ***(H-Poinc)****.*

In addition if (**H-2**) is satisfied, we have

$$K_p \leq 2^{1-r_p} K_2 \quad \text{and} \quad \lambda_p \geq r_p \, \lambda_2 . \tag{1.4}$$

Note that one cannot directly use the Riesz-Thorin theorem on the closed subspace of functions with zero mean, so the prefactor 2^{1-r_p} cannot be (at least immediately) avoided.
Our goal will be to obtain the best possible constants λ_p and K_p, assuming that the Poincaré inequality holds (i.e. $K_2 = 1$).
There are two interests in such a result. In the first case it is obviously the rate of convergence at infinity. In this case, it is important to get the largest possible λ_p, despite the (reasonable) value of K_p. In the second case it is the opposite: to obtain the result with $K_p = 1$ so that the inequality becomes an equality at time $t = 0$. In this case, one may use the result in order to obtain (for example) isoperimetric controls. The ideal situation is, of course, the one where we can reach these goals simultaneously (as for $p = 2$). Unfortunately we did not succeed in obtaining this ideal result. Nevertheless, we will improve on the result obtained via interpolation in both directions. More precisely, we will prove the two results described below.

Theorem 1.5. *If (H-Poinc) is satisfied, then for all $p \geq 2$, (**H**-p) holds with $K_p = 1$ and*

$$\lambda_p \geq \frac{1}{8^{2^k}\, C_P}, \qquad \text{if } p \leq 2^k \text{ for some } k \geq 2\,.$$

Note that for $p = 2$ ($k = 1$) we recover a worse constant than the known $\lambda_2 = 1/C_P$.

Theorem 1.6. *If (**H-Poinc**) is satisfied, then for all $p \geq 1$, (**H-p**) holds with*

$$\lambda_p \geq \frac{4(p-1)}{p^2\, C_P} = 4\,\lambda_2\, \frac{1}{p}\left(1 - \frac{1}{p}\right) \qquad \text{and an appropriate } K_p\,.$$

For $p \geq 4$ one may choose $K_p = 2^{2-\frac{4}{p}}$ and for $2 < p \leq 4$ one may choose $K_p = p^2/(p-2)^2$. For $1 \leq p < 2$ we have $K_p \leq 2^{\frac{2}{p}-1} K_{p/(p-1)}$.

Note that the constant λ_p obtained in Theorem 1.6 satisfies

$$\lambda_p = 2\,\max\left(\frac{1}{p}\,,\, 1 - \frac{1}{p}\right)\, r_p\, \lambda_2\,,$$

and so improves upon the one obtained by interpolation (see (1.4)) by a factor almost equal to 2 for large $p's$. Actually we believe that $\lambda_p = 4\,\lambda_2\, \frac{1}{p}\left(1 - \frac{1}{p}\right)$ is the best constant for all p's, but we did not succeed in giving a rigorous proof of this fact.
Once it was seen that the Poincaré inequality implies the same exponential decay for both P_t and its adjoint P_t^*, the case $1 < p < 2$ is obtained via a simple duality argument, so that we shall only consider the case $p \geq 2$ in the sequel.
Note also that for $p = 2^k$ for $k \geq 1$, our proof gives $K_p = 2^{1-2/p}$ and even a better one for large p.
There is something surprising in the previous theorem, namely the explosion of K_p when p goes to 2. Actually the proof will furnish a "better" result for small t's, namely

Theorem 1.7. *If (H-Poinc) is satisfied, then for all $2 \leq p \leq 4$, (**H-p**) holds with*

$$\lambda_p = \frac{1}{C_P}\left(\frac{1}{p} + \frac{1}{2}\right) \qquad \text{and} \qquad K_p = 2^{2-\frac{4}{p}}\,.$$

In this situation it also holds

$$N_p(P_t f) \leq \left(1 + \frac{4(p-1)t}{p C_P}\right)^{1/p} e^{-4\lambda_2 \frac{1}{p}\left(1-\frac{1}{p}\right)t}\, N_p(f)\,.$$

The latter result suggests that one can expect a "reasonable" K_p for p close to 2 and $\lambda_p = 4\,\lambda_2\, \frac{1}{p}\left(1 - \frac{1}{p}\right)$. The same duality argument allows us to consider the cases $p \leq 2$. One can also remark that the final bound yields, fortunately, $N_p(P_t f)/N_p(f) \leq 1$ for all $t \geq 0$ contrary to the previous one obtained in Theorem 1.6.

The case $p = 1$ is extensively studied in [4] and the Poincaré inequality is no longer sufficient in general to obtain an exponential decay in $\mathbb{L}^1(\mu)$. Replacing $\mathbb{L}^p$ norms by Orlicz norms (weaker than any N_p for $p > 1$) is possible provided one reinforces the Poincaré inequality into a F-Sobolev inequality (see [4] Theorem 3.1 and also [5]) as it is well known in the case $F = \log$ for the Orlicz space $\mathbb{L}\log\mathbb{L}$.

We could not find the statement of our results in the literature, perhaps because the interpolation argument appeared to be immediate. However recall that in [7], F.Y. Wang used the equivalent Beckner type formulation of Poincaré inequality to give a partial answer to the problem i.e., a Poincaré inequality with constant C_P is equivalent to the following: for any $1 < p \leq 2$ and for any non-negative f,

$$\int (P_t f)^p \, d\mu - \left(\int f \, d\mu\right)^p \leq e^{-\frac{4(p-1)t}{p\,C_P}} \left(\int (f)^p \, d\mu - \left(\int f \, d\mu\right)^p\right).$$

One has to take care with the constants since a factor 2 may or may not appear in the definition of Γ, depending on authors and of papers by the same authors. This result cannot be used to study the decay to the mean in $\mathbb{L}^p$ norm, but it is of particular interest when studying densities of probability.
Note that the decay rate we obtain in Theorem 1.6 is the same as the one in Wang's result.

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2 Poincaré inequalities and $\mathbb{L}^p$ spaces.

This section is dedicated to the proof of our results.

2.1 Proof of Theorem 1.5.

A natural idea is to study the time derivative of $N_p(P_t f)$, namely

$$\frac{d}{dt} N_p^p(P_t f) = p \int sign(P_t f - \mu(f))\, |P_t f - \mu(f)|^{p-1}\, LP_t f \, d\mu.$$

It follows that the following staments are equivalent:

There exists a constant $C(p)$ such that for all f,

$$N_p^p(P_t f) \leq e^{-\frac{pt}{C(p)}}\, N_p^p(f). \tag{2.1}$$

There exists a constant $C(p)$ such that for all $f \in \mathscr{D}$ with $\mu(f) = 0$,

$$N_p^p(f) \,\leq\, -C(p) \int sign(f)\, |f|^{p-1}\, Lf \, d\mu. \tag{2.2}$$

In order to compare all the inequalities (2.2) to the Poincaré inequality (i.e. $p=2$) one is tempted to make the change of function $f\mapsto sign(f)|f|^{2/p}$ (or $f\mapsto sign(f)|f|^{p/2}$) and to use the chain rule. Unfortunately, first $\varphi(u)=u^{2/p}$ is not C^2, and secondly $\mu(sign(f)|f|^{2/p})\neq 0$ (the same for $p/2$ for the second argument).
However, for $p\geq 2$, one can integrate by parts in (2.2) which thus becomes

$$N_p^p(f)\leq C(p)(p-1)\int |f|^{p-2}\,\Gamma(f,f)\,d\mu = C(p)\,\frac{4(p-1)}{p^2}\int \Gamma(|f|^{p/2},|f|^{p/2})\,d\mu. \tag{2.3}$$

Thus, it remains to show that the Poincaré inequality implies (2.3) for all $p\geq 2$. This will be done in three steps. First we will show that (2.3) holds for $p=4$. Next we shall show that it holds for all $2\leq p\leq 4$. Finally we shall show that if (2.3) holds for p then it holds for $2p$. This will complete the proof by an induction argument. In this procedure, starting with $p=2$, only the last two steps are necessary. Anyway, we believe that the details for $2p=4$ will help the reader to follow the scheme of proof for the general cases.

The case $p=4$. We proceed with the proof for $p=4$, *i.e.* we prove that the Poincaré inequality implies (2.3) for $p=4$. Assume that $\mu(f)=0$. First, applying the Poincaré inequality to f^2 we get

$$\int f^4 d\mu\leq\left(\int f^2 d\mu\right)^2+4\,C_P\int f^2\,\Gamma(f,f)\,d\mu\,,$$

so that it remains to prove that

$$\left(\int f^2 d\mu\right)^2\leq C\int f^2\,\Gamma(f,f)\,d\mu\,,$$

for some constant C. For any $u>0$ let $\varphi=\varphi_u:\mathbb{R}\mapsto\mathbb{R}$ be the 2-Lipschitz function defined by $\varphi(s)=0$ if $|s|\leq u$, $\varphi(s)=s$ if $|s|\geq 2u$ and linear in between. Applying Poincaré inequality to $\varphi(f)$ yields

$$\int(\varphi(f))^2\,d\mu\leq\left(\int\varphi(f)\,d\mu\right)^2+4\,C_P\int_{\{|f|\geq u\}}\Gamma(f,f)\,d\mu\,.$$

But

$$\int(\varphi(f))^2\,d\mu\geq\int_{\{|f|\geq 2u\}}f^2\,d\mu\geq\int f^2 d\mu\,-\,4u^2\,,$$

and since $\mu(f)=0$,

$$\begin{aligned}\left|\int\varphi(f)\,d\mu\right|\leq\int|\varphi(f)-f|d\mu&=\int_{\{|f|\leq u\}}|f|d\mu+\int_{\{u\leq|f|\leq 2u\}}|\varphi(f)-f|d\mu\\&\leq u\left(\mu(\{|f|\leq u\})+\mu(\{u\leq|f|\leq 2u\})\right)\leq u\end{aligned}\tag{2.4}$$

where we have used that $|\varphi(s)-s|\leq u$ for any $s\in[-2u,-u]\cup[u,2u]$. Summarizing, it follows that

$$\int f^2\,d\mu\leq 5u^2+4\,C_P\int_{\{|f|\geq u\}}\Gamma(f,f)\,d\mu\leq 5u^2+\frac{4}{u^2}\,C_P\int f^2\,\Gamma(f,f)\,d\mu\,.$$

Optimizing in u^2 finally yields

$$\left(\int f^2\,d\mu\right)^2 \le 90\,C_P \int f^2\,\Gamma(f,f)\,d\mu\,,$$

and in turn

$$N_4^4(f) \le 94\,C_P \int f^2\,\Gamma(f,f)\,d\mu\,.$$

The constant 94 is clearly not optimal. Note that replacing the constant 2 by $(1+\rho)$, $\rho > 0$, in the definition of φ would yield a worse constant.

The case $2 \le p \le 4$**.** In the previous part we used the ordinary form of the Poincaré inequality. Actually if $1 \le a \le 2$ we have

$$\int |f|^2\,d\mu - \left(\int |f|^a\,d\mu\right)^{2/a} \le \mathrm{Var}_\mu(f) \le C_P \int \Gamma(f,f)\,d\mu\,. \tag{2.5}$$

Hence if $2 \le p \le 4$, applying (2.5) with $|f|^{p/2}$, and choosing $a = 4/p$ we have

$$\int |f|^p\,d\mu \le \left(\int f^2\,d\mu\right)^{p/2} + C_P\,(p^2/4) \int |f|^{p-2}\,\Gamma(f,f)\,d\mu\,. \tag{2.6}$$

Now recall that, in the previous step, we have shown

$$\int f^2\,d\mu \le 5u^2 + 4\,C_P \int_{\{|f|\ge u\}} \Gamma(f,f)\,d\mu\,,$$

hence

$$\int f^2\,d\mu \le 5u^2 + \frac{4}{u^{p-2}}\,C_P \int |f|^{p-2}\,\Gamma(f,f)\,d\mu\,.$$

Again, we may optimize in u, and obtain

$$\int |f|^p\,d\mu \le \left(\frac{p^2}{4} + 2^{2+\frac{p}{2}}\,5^{\frac{p}{2}-1}\right) C_P \int |f|^{p-2}\,\Gamma(f,f)\,d\mu\,, \tag{2.7}$$

so that (2.3) holds for $2 \le p \le 4$, with

$$C(p) \le \frac{\frac{p^2}{4} + 2^{2+\frac{p}{2}}\,5^{\frac{p}{2}-1}}{p-1}\,C_P\,.$$

From p **to** $2p$**.** Now assume that (2.3) holds for some $p \ge 2$ and that the Poincaré inequality also holds with constant C_P. First we apply Poincaré inequality to the function $|f|^p$ to get

$$\int |f|^{2p}\,d\mu \le \left(\int |f|^p\,d\mu\right)^2 + C_P\,p^2 \int |f|^{2p-2}\,\Gamma(f,f)\,d\mu\,.$$

Our aim is to bound the quantity $\int |f|^p\,d\mu$ using (2.3). On the one hand we have

$$\int |\varphi(f)|^p\,d\mu \ge \int |f|^p - 2^p u^p.$$

On the other hand, using that $|a+b|^p \le 2^{p-1}(|a|^p+|b|^p)$ and (2.4), it follows from (2.3) applied to $\varphi(f)-\mu(\varphi(f))$ that (recall that φ is 2-Lipschitz)

$$\begin{aligned}\int |\varphi(f)|^p d\mu &\le 2^{p-1}|\mu(\varphi(f))|^p + 2^{p-1}\int |\varphi(f)-\mu(\varphi(f))|^p d\mu \\ &\le 2^{p-1}u^p + 2^{p-1}(p-1)C(p)\int_{\{|f|\ge u\}} |\varphi(f)-\mu(\varphi(f))|^{p-2}\Gamma(\varphi(f),\varphi(f))d\mu \\ &\le 2^{p-1}u^p + 2^{p+1}(p-1)C(p)\int_{\{|f|\ge u\}} |\varphi(f)-\mu(\varphi(f))|^{p-2}\Gamma(f,f)d\mu.\end{aligned}$$

Now, again by (2.4) we have

$$\begin{aligned}\int_{\{|f|\ge u\}} |\varphi(f)-\mu(\varphi(f))|^{p-2}\Gamma(f,f)d\mu &\le \int_{\{|f|\ge u\}} ||f|+u|^{p-2}\Gamma(f,f)d\mu \\ &= \frac{1}{u^p}\int_{\{|f|\ge u\}} u^p|f|^{p-2}\left|1+\frac{u}{|f|}\right|^{p-2}\Gamma(f,f)d\mu \\ &\le \frac{2^{p-2}}{u^p}\int |f|^{2p-2}\Gamma(f,f)d\mu.\end{aligned}$$

Summarizing we end up with

$$\int |f|^p d\mu \le \left(2^p+2^{p-1}\right)u^p + \frac{2^{2p-1}}{u^p}(p-1)C(p)\int |f|^{2p-2}\Gamma(f,f)d\mu.$$

Since $2^p+2^{p-1}=3\cdot 2^{p-1}$, optimizing over u leads to

$$\left(\int |f|^p d\mu\right)^2 \le 3\cdot 2^{3p}(p-1)C(p)\int |f|^{2p-2}\Gamma(f,f)d\mu.$$

In turn,

$$\int |f|^{2p} d\mu \le \left(3\cdot 2^{3p}(p-1)C(p)+p^2C_P\right)\int |f|^{2p-2}\Gamma(f,f)d\mu.$$

This is equivalent to say that

$$C(2p)(2p-1) \le 3\cdot 2^{3p}(p-1)C(p)+p^2C_P. \tag{2.8}$$

Note that the previous computation applied to $p=2$ (recall that $C(2)=C_P$) leads to a worse constant than 94. From (2.8) we easily deduce that

$$C(2^k) \le 2^k \cdot 2^{3\cdot 2^k}$$

from which the expected result of Theorem 1.5 follows.

2.2 Proof of Theorem 1.6

The proof of Theorem 1.6 will use the following two lemmata

Lemma 2.9. *Assume that (**H-Poinc**) holds, and that (**H-p**) holds for some $p \geq 1$ with $\lambda_p = \theta_p / p\, C_P$ and K_p.*
*Then, provided $p\,\theta_p > 2p-1$, (**H-2p**) holds with*

$$\lambda_{2p} = 4(2p-1)/(2p)^2 C_P \quad \text{and} \quad K_{2p}^{2p} = 1 + \frac{2p-1}{p\,\theta_p - (2p-1)}\, K_p^{2p}\,.$$

Lemma 2.10. *Assume that (**H-Poinc**) holds. Then for all $2 \leq p \leq 4$, (**H-p**) holds with*

$$\lambda_p = 4(p-1)/p^2 C_P \quad \text{and} \quad K_p^p = \frac{p^2}{(p-2)^2}\,.$$

In this situation we also have

$$N_p(P_t f) \leq \left(1 + \frac{4(p-1)t}{pC_P}\right)^{1/p} e^{-\frac{4(p-1)t}{p^2 C_P}}\, N_p(f)\,.$$

Proof of Lemma 2.9.
Without loss of generality we assume that $\int f\, d\mu = 0$ and denote by $U_p(t) := N_p^p(P_t f)$. Recall that, since $2p \geq 2$,

$$\begin{aligned} U'_{2p}(t) &= 2p \int sign(P_t f)\, |P_t f|^{2p-1}\, LP_t f\, d\mu \\ &= -2p\,(2p-1) \int (P_t f)^{2p-2}\, \Gamma(P_t f, P_t f)\, d\mu \\ &= -\frac{4(2p-1)}{2p} \int \Gamma(|P_t f|^p, |P_t f|^p)\, d\mu\,. \end{aligned}$$

In addition the Poincaré inequality, applied to $|P_t f|^p$, yields

$$U_{2p}(t) \leq U_p^2(t) + C_P \int \Gamma(|P_t f|^p, |P_t f|^p)\, d\mu\,.$$

Putting these inequalities together we thus have

$$U'_{2p}(t) \leq -\frac{4(2p-1)}{2p\, C_P}\, U_{2p}(t) + \frac{4(2p-1)}{2p\, C_P}\, U_p^2(t)\,. \tag{2.11}$$

We may thus apply Gronwall's lemma and obtain

$$U_{2p}(t) \leq e^{-\frac{4(2p-1)t}{2p\, C_P}} \left(U_k(0) + \frac{4(2p-1)}{2p\, C_P} \int_0^t e^{\frac{4(2p-1)s}{2p\, C_P}}\, U_p^2(s)\, ds\right)\,.$$

Applying (**H-p**) we obtain

$$\begin{aligned} U_{2p}(t) &\leq e^{-\frac{4(2p-1)t}{2p\,C_P}}\left(U_{2p}(0)+\frac{4(2p-1)}{2p\,C_P}\int_0^t e^{\frac{4(2p-1)s}{2p\,C_P}}\left(K_p^p\,e^{-\frac{p\,\theta_p\,s}{p\,C_P}}\,U_p(0)\right)^2 ds\right)\\ &\leq e^{-\frac{4(2p-1)t}{2p\,C_P}}\left(U_{2p}(0)+\frac{4(2p-1)}{2p\,C_P}\int_0^t e^{-\frac{2(p\,\theta_p-(2p-1))s}{p\,C_P}}\,K_p^{2p}\,U_p^2(0)\,ds\right)\\ &\leq e^{-\frac{4(2p-1)t}{2p\,C_P}}\left(U_{2p}(0)+\frac{2p-1}{p\,\theta_p-(2p-1)}\,K_p^{2p}\,U_p^2(0)\right)\\ &\leq e^{-\frac{4(2p-1)t}{2p\,C_P}}\,U_{2p}(0)\left(1+\frac{2p-1}{p\,\theta_p-(2p-1)}\,K_p^{2p}\right) \end{aligned}$$

since $U_p^2(0)\leq U_{2p}(0)$ thanks to the Cauchy-Schwarz inequality.

Proof of Lemma 2.10.
If $2\leq p\leq 4$, write $p=2q$ with $1\leq q\leq 2$. We have

$$U_p'(t)=-\frac{4(p-1)}{p}\int \Gamma(|P_tf|^q,|P_tf|^q)\,d\mu\,.$$

As in the second step of the proof of theorem 1.5, we may use the following consequence of the Poincaré inequality,

$$N_{2q}^{2q}(|P_tf|)-N_2^{2q}(P_tf)\leq C_P\int \Gamma(|P_tf|^q,|P_tf|^q)\,d\mu\,. \tag{2.12}$$

As in the previous proof we thus obtain

$$U_p'(t)\leq -\frac{4(p-1)}{p\,C_P}\,U_p(t)+\frac{4(p-1)}{p\,C_P}\,U_2^q(t)\,.$$

Using Gronwall's lemma and applying the Poincaré inequality for U_2, we thus have

$$\begin{aligned} U_p(t) &\leq e^{-\frac{4(p-1)t}{p\,C_P}}\left(U_p(0)+\frac{4(p-1)}{p\,C_P}\int_0^t e^{\frac{4(p-1)s}{p\,C_P}}\,e^{-\frac{sp}{C_P}}\,U_2^q(0)\,ds\right)\\ &\leq e^{-\frac{4(p-1)t}{p\,C_P}}\left(U_p(0)+\frac{4(p-1)}{p\,C_P}\int_0^t e^{-\frac{s(p-2)^2}{p\,C_P}}\,U_2^q(0)\,ds\right)\\ &\leq e^{-\frac{4(p-1)t}{p\,C_P}}\,U_p(0)\left(1+\frac{4(p-1)}{(p-2)^2}\right), \end{aligned}$$

yielding the first part of the lemma. The second one immediately follows from the second inequality above.

Proof of Theorem 1.6 and of Theorem 1.7.
The two previous lemmata yield $\lambda_4=3/4C_P$ and $K_4=\sqrt{2}$.
It follows that T_t is bounded in $\mathbb{L}^4$ with an operator norm smaller than or equal to $2\,e^{-3t/4C_P}$.
Interpolating between $\mathbb{L}^2$ and $\mathbb{L}^4$ yields for $2\leq p\leq 4$,

$$\lambda_p=\frac{1}{C_P}\left(\frac{1}{p}+\frac{1}{2}\right) \quad\text{and}\quad K_p=2^{(2-\frac{4}{p})}\,.$$

Of course this value of λ_p is smaller, for $2 < p < 4$, than the expected $4(p-1)/p^2 C_P$, but it is already better than the one obtained via direct interpolation. K_p is exactly $(2^{1-\frac{2}{p}})^2$.
It is immediate that for $2 \leq p \leq 4$, $\theta_p = 1 + \frac{p}{2}$ satisfies $p\theta_p > 2p - 1$, so that we may apply Lemma 2.9 for $2 \leq p \leq 4$. Now if $\theta_p = 4(p-1)/p$ and $p \geq 2$ it is immediate that $p\theta_p \geq 2p - 1$. Hence we have obtained that for all $p \geq 4$ we may choose $\lambda_p = 4(p-1)/p^2 C_P$.
It is not difficult, although tedious, to check that if $K_p \leq 2^{2-\frac{4}{p}}$ then $K_{2p} \leq 2^{2-\frac{4}{2p}}$ (one has to deal first with the case $2 \leq p \leq 4$ with $p\theta_p = p + \frac{p^2}{2}$, and then with the case $p \geq 4$).

References

[1] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto and G. Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses*. Société Mathématique de France, Paris, 2000.

[2] D. Bakry, P. Cattiaux and A. Guillin. Rate of convergence for ergodic continuous Markov processes : Lyapunov versus Poincaré. *J. Func. Anal.*, 254:727–759, 2008. MR2381160

[3] P. Cattiaux. A pathwise approach of some classical inequalities. *Pot. Anal.*, 20:361–394, 2004. MR2032116

[4] P. Cattiaux and A. Guillin. Trends to equilibrium in total variation distance. *Ann. Inst. Henri Poincaré. Prob. Stat.*, 45(1):117–145, 2009.
also see the more complete version available on ArXiv.Math.PR/0703451, 2007. MR2500231

[5] C. Roberto and B. Zegarlinsky. Orlicz-Sobolev inequalities for sub-Gaussian measures and ergodicity of Markov semi-groups. *J. Funct. Anal.*, 243(1):28–66, 2007. MR2289793

[6] C. Villani. Hypocoercivity. *Mem. Amer. Math. Soc.* 202(no. 950), 2009. MR2562709

[7] F. Y. Wang. Probability distance inequalities on Riemannian manifolds and path spaces. *J. Func. Anal.*, 206:167–190, 2004. MR2024350

[8] L. Wu. Poincaré and transportation inequalities for Gibbs measures under the Dobrushin uniqueness condition. *Ann. Prob.*, 34(5):1960–1989, 2006. MR2271488