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Logarithmic Sobolev constant for the dilute Ising lattice gas dynamics below the percolation threshold

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Abstract

We consider a conservative stochastic lattice gas dynamics reversible with respect to the canonical Gibbs measure of the bond dilute Ising model on $\mathbb{Z}^d$ at inverse temperature β. When the bond dilution density p is below the percolation threshold, we prove that, for any $\varepsilon > 0$, any particle density and any β, with probability one, the logarithmic Sobolev constant of the generator of the dynamics in a box of side L centered at the origin cannot grow faster that $L^{2+\varepsilon}$.

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1. Introduction

In this paper we study the logarithmic Sobolev constant for a bond dilute Ising lattice gas with Kawasaki dynamics. The bond dilute Ising lattice gas can be described as follows. Consider the d-dimensional lattice $\mathbb{Z}^d$. At each site x of the lattice we associate an occupation number of particle $\sigma(x) \in \{0, 1\}$. The equilibrium states are

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described by the Gibbs measure on $\mathbb{Z}^d$ characterized (informally) by the Hamiltonian

$$H^{J,\lambda} = -\sum_{[x,y]} J_{xy}(2\sigma(x)-1)(2\sigma(y)-1) + \lambda\sum_{x}\sigma(x)j,$$

where $[x,y]$ denotes a generic bond of the lattice $\mathbb{Z}^d$ and the couplings $\{J_{xy}\}$ are i.i.d. random variables taking only two values, $J_{xy}=0$ (bond *closed*) and $J_{xy}=\beta$ (bond *open*) with probability $1-p$ and p, respectively, where $\beta>0$ can be interpreted as the inverse temperature. When a configuration of the bond r.v. $\{J_{xy}\}$ is given we say that a disorder configuration is given. We denote by C_x the cluster of the site x, namely the set of all sites in $\mathbb{Z}^d$ which are connected to x by a path of open bonds. Let $\theta(p)$ be the probability that the cluster of the origin is infinite. There exists $p_c(d)$ such that $\theta(p)=0$ if $p<p_c$ and $\theta(p)>0$ if $p>p_c$ and p_c is called *percolation threshold* (Grimmett, 1999).

In Kawasaki dynamics each particle performs a random walk such that: (i) jumps to occupied sites are suppressed so that there is at most one particle at any given site; (ii) no creation or annihilation of particles is allowed so that the total number of particles is conserved; (iii) the rate c_{xy} with which a particle at site x jumps to one of its nearest neighbors y, depends on the particle distribution around $x\cup y$ and on some external random field (the disorder) in such a way that the whole process is reversible w.r.t. the canonical Gibbs measure of the bond dilute Ising lattice gas described above.

The main interest is to analyze the relaxation to equilibrium of the dynamics in a finite box Q_L of side L centered at the origin as a function of L, when the thermodynamic parameters and the disorder distribution are such that one has simultaneously subsets of $\mathbb{Z}^d$ in which the jump rates are those of a non-interacting gas, and subsets where instead the jump rates are those of a gas in the phase coexistence region. In the physics literature this situation is sometimes referred to as the Griffiths phase (see, e.g. Grimmett, 1999 and Fröhlich, 1986, see also the introduction of Cancrini and Martinelli, 2001, and references therein for a more detailed discussion).

Two key quantities give an estimate of the rate of convergence to equilibrium of the system: the inverse of the spectral gap of the generator of the dynamics (ISG) and the logarithmic Sobolev constant (LSC), see the definitions in the next section. The first one gives the relaxation time to equilibrium in the L^2-norm, while the second one is connected to the hypercontractivity of the Markov semi-group of the process and to the decay of entropy, see Diaconis and Saloff-Coste (1996) and Ané et al. (2000).

In Cancrini and Martinelli (2001) it has been proved that, when the bond dilution density p is below the percolation threshold, for any particle density and any β, with probability one, the inverse of the spectral gap scales like L^2. Such an estimate is then used to prove the L^2-decay to equilibrium for local observable. The main result of this paper is the following. Given $p<p_c$ and $0<\varepsilon\ll 1$, there exists a subset of disorder configurations of probability one such that, for any particle density and any β, the LSC cannot grow faster than $L^{2+\varepsilon}$.

Results for the logarithmic Sobolev constant up to now were available only when the thermodynamic parameters β and λ are such that a high-temperature mixing condition holds and $p=1$ (pure translation invariant case). In this situation it has been proved that the logarithmic Sobolev constant scales like L^2. We refer the reader to the basic reference Yau (1996) and more recently to Cancrini et al., 2002a. In the limit of small β and arbitrary p this result can be easily extended (see Remark 22). An interesting problem is whether the diffusive scaling of the logarithmic Sobolev constant is affected for β large, i.e. when no high-temperature mixing condition holds inside the clusters. In this case for Kawasaki dynamics the LSC of an isolated (that is with fixed number of particles) cubic cluster of side l may scale like $e^{kl^{d-1}}$ depending on the number of particles (see Cancrini et al., 1999 and Theorem 3.14 below). Nevertheless even if a given cluster has a large LSC when its number of particles is kept fixed, due to the conservation of the number of particles and to the fact that it can exchange particles with its complement, its contribution to the global LSC could be not so large. Moreover when $p < p_c$, with large probability the largest cluster in a box Q_L of side L has volume smaller than $c \log L$, so that its LSC, with its number of particles fixed is smaller than $e^{c'(\log L)^{(d-1)/d}}$ (see Theorem 3.14 below). Let us imagine that only one cluster, denoted by C, is present and the number of particles is fixed in Q_L. Since it is clear that in order to reach equilibrium the particles must diffuse through the whole box, at least in this extreme case we cannot expect a global LSC smaller than L^2 (the LSC of the simple exclusion in the complement of C). Thus, in this non-realistic case, the worst (w.r.t. the choice of the number of particles) LSC for the cluster is much smaller than the expected global relaxation time. It is thus interesting to investigate if the diffusion of the particles is the dominant effect when more clusters are present. It is not difficult to prove that if the number of clusters inside the box Q_L is independent of L this is the case (see Remark 22 and the appendix). In the more realistic case in which clusters on all scales below $c \log L$ appear, our result which (see above or Theorem 2) on the LSC do not exclude the possibility of a cooperative effect of the clusters that eventually leads to a relaxation slower than the diffusive one L^2. We stress here that our estimate on the L dependence of the LSC could be not optimal and that in Cancrini and Martinelli (2001) it has been proved that the ISG scales like L^2. Furthermore in Caputo (2001) it is proven that for unbounded spin systems with convex interaction on $\mathbb{Z}^d$, the diffusive scaling of the ISG and of the LSG of conservative dynamics comes from geometrical considerations.

In the sketch of the proof below we try to explain from where the L^ε factor comes from.

1.1. Sketch of the proof

We here give the main ideas of our proof without rigorous computation and hidden several technical aspects. Let Q_L be a box of side L centered at the origin and let $c_s(l,L)$ be the largest among the logarithmic Sobolev constant in subboxes of Q_L of side l such that $l \in [L^\varepsilon, L]$ for a fixed $\varepsilon > 0$.

The proof can be divided into three steps:

(i) given $p < p_c$ and $0 < \varepsilon \ll 1$, it can be proven that there exists a subset of disorder configurations, say Θ_0, of probability one, such that in any subbox of Q_L of side $l \in [L^\varepsilon, L]$ the system has certain homogeneity properties that make it quite indistinguishable from a usual, translation invariant high-temperature lattice gas;

(ii) fixed a disorder configuration in Θ_0 the recursive method, developed in Cancrini et al. (2002a) (where the high-temperature mixing condition and translation invariance are largely used), can be applied to obtain

$$c_s(2l,L) \leqslant \tfrac{3}{2}c_s(l,L) + kl^2L^{\varepsilon} \quad \text{for all } l \in [L^{\varepsilon},L]. \tag{1}$$

This leads by iteration to

$$c_s(Q_L) \leqslant L^{2+\varepsilon}[k + L^{-\alpha}c_s(L^{\varepsilon},L)]$$

for a suitable constant α independent of ε.

(iii) one can prove that there exists a numerical positive constant k' such that $c_s(L^{\varepsilon},L) \leqslant L^{k'\varepsilon}$. And the result follows.

Point (i) has been established in Cancrini and Martinelli (2001) (see Section 4.2). The inequality of point (iii) can be easily obtained by (Cancrini and Martinelli, 2001, Corollary 5.2) and the fact that $c_s(V) \leqslant |V|\text{gap}(V)$ (see Martinelli, 1999, and Section 4.8 below).

Much more difficult is the proof of inequality (1) and for this reason we explain again the recursive argument presented in Cancrini et al. (2002a). Roughly speaking, in the recursive argument, the idea is to control the LSC of a box of side $2l$ by that one of a box of side l (see inequality (1)). Looking at the definition of the LSC (see (12)) it comes out that to obtain an upper bound of the LSC the starting point is the entropy. Let Λ be a subbox of Q_L of side $2l$. Divide it into two halves Λ_1 and Λ_2.

Because of the conservation of the number of particles inside Λ, there is a constraint on the whole system. Thus, even in the absence of interaction, the Gibbs canonical measure v does not factorize over Λ_1 and Λ_2. A natural way to eliminate such a global constraint is to fix the number of particles inside each subboxes Λ_1 and Λ_2, and, as the number of particles N_Λ is fixed inside Λ, it is enough to fix the number of particles inside one subbox, e.g. Λ_1. Technically it can be achieved by conditioning. Let n_1 be the random variables counting the number of particles inside Λ_1, then we can write

$$\text{Ent}_v(f^2) = v(\text{Ent}(f^2|n_1)) + \text{Ent}_v[v(f^2|n_1)], \tag{2}$$

where $\text{Ent}(f^2|n_1)$ is a shorthand notation for the entropy of f^2 w.r.t. $v(\cdot|n_1)$. Equality (2) can be interpreted as follows in terms of dynamics. The LSC of Kawasaki dynamics in a box Λ is related to the LSC of the modified Kawasaki dynamics in which the number of particles in the two sets Λ_1,Λ_2 is conserved (the first term of (2)) *and* to the LSC of the process of exchange particles between Λ_1,Λ_2 (the last term in the r.h.s. of (2)). In some sense, the heart of our approach is to separate the two effects which are, a priori, strongly interlaced and to analyze them separately. Then, note that $v_{n_1,N_\Lambda-n_1} := v(\cdot|n_1)$ is a multi-canonical measure where the number of particles is fixed inside Λ_1 *and* Λ_2.

Assume now for a moment that there is no interaction ($\beta=0$). Then, the σ-algebras $\mathscr{F}_1 := \mathscr{F}_{\Lambda_1^c}$ and $\mathscr{F}_2 := \mathscr{F}_{\Lambda_2^c}$, namely the σ-algebras generated by the dilute Ising model variables outside Λ_1 and Λ_2, respectively, are independent and $\nu_{n_1,N_\Lambda-n_1}=\nu_{n_1,N_\Lambda-n_1}(\cdot|\mathscr{F}_1)\otimes\nu_{n_1,N_\Lambda-n_1}(\cdot|\mathscr{F}_2)=:\nu_1\otimes\nu_2$. thus, the tensorization property of the entropy gives

$$\text{Ent}(f^2|n_1)\leqslant\nu_{n_1,N_\Lambda-n_1}(\text{Ent}_{\nu_1}(f^2)+\text{Ent}_{\nu_2}(f^2)).$$

Note that ν_1 and ν_2, are canonical measures on the smaller sets Λ_1 and Λ_2, respectively. In particular, the linear dimension of one direction has been halved.

When the interaction is present ($\beta\neq0$) and a high-temperature strong mixing condition holds, the exponential decay of correlations can be used to obtain an almost factorization of the measure if the sets Λ_1 and Λ_2 are far away one from the other. For this purpose divide Λ into two (almost) halves $\tilde{\Lambda}_1,\tilde{\Lambda}_2$ in such a way that the overlap between $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ is a thin layer of width δL, $\delta\ll1$.

Then, by the previous reasoning, one has to fix the number of particles inside the three sets Λ_1,Λ_2 and $\tilde{\Lambda}_1\cap\tilde{\Lambda}_2$ using twice the conditioning formula of the entropy

$$\text{Ent}_\nu(f^2)=\nu(\text{Ent}(f^2|n_1,n_2))+\nu(\text{Ent}(\nu(f^2|n_1,n_2)|n_1))+\text{Ent}_\nu[\nu(f^2|n_1)],\qquad(3)$$

where n_1 and n_2 stand for the random variables counting the number of particles inside Λ_1 and Λ_2.

Consider the first term in (3). Then it can be shown that $\mathscr{F}_1:=\mathscr{F}_{\tilde{\Lambda}_1^c}=\mathscr{F}_{\Lambda_2}$ and $\mathscr{F}_2:=\mathscr{F}_{\tilde{\Lambda}_2^c}=\mathscr{F}_{\Lambda_1}$ are weakly dependent in the sense that there exists $0<\varepsilon(L)\ll1$ such that

$$\|\nu_{n_1,n_2}(g|\mathscr{F}_2)-\nu_{n_1,n_2}(g)\|_\infty\leqslant\varepsilon(L)\nu_{n_1,n_2}(g),$$

for all non-negative function g measurable with respect to $\mathscr{F}_1$ (weak dependence of the boundary condition due to the distance between Λ_1 and Λ_2). Here ν_{n_1,n_2} is a multi-canonical measure in which the number of particles has been fixed in Λ_1 and Λ_2 and thus by the global conservation law, in $\tilde{\Lambda}_1\cap\tilde{\Lambda}_2$. Last inequality allows to use Proposition 2.1 in Cesi (2001) and obtain the almost factorization of the entropy:

$$\nu(\text{Ent}(f^2|n_1,n_2))\leqslant(1+\varepsilon(L))\nu_{n_1,n_2}(\text{Ent}_{\nu_1}(f^2)+\text{Ent}_{\nu_2}(f^2)),$$

where $\nu_1:=\nu_{n_1,n_2}(\cdot|\mathscr{F}_1)$ and $\nu_2:=\nu_{n_1,n_2}(\cdot|\mathscr{F}_2)$. As in the case of no interaction, ν_1 and ν_2 are canonical measures on smaller sets where the linear dimension in one direction has been almost halved. The first term of the r.h.s of (1) and the factor $\frac{3}{2}$ comes from $1+\varepsilon(L)$. The two other terms in (3) are similar so we discuss only the third one which is notationally simpler. Let γ be the distribution of the number of particles inside Λ_1:$\gamma(n_1):=\nu(N_{\Lambda_1}=n_1)$, and $g^2(n_1):=\nu(f^2|n_1)$. Then, $\text{Ent}_\nu\nu(f^2|n_1)=\text{Ent}_\gamma(g^2)$. And we are left with a one-dimensional problem ($n_1\in\{0,\dots,|\Lambda_1|\}$). We have a sufficiently good control on the distribution γ (see Cancrini and Martinelli, 2000b) in such a way that we can prove, using Hardy inequalities (see Miclo, 1999), the following logarithmic Sobolev inequality:

$$\text{Ent}_\gamma(g^2)\leqslant k(N)\sum_{n_1}\gamma(n_1-1)\vee\gamma(n_1)[g(n_1)-g(n_1-1)]^2,$$

where $k(N)=\text{O}(N)$. Note that the discrete gradient on g measures the effect on the function g of the exchange of one particle between the two subboxes. The estimate of the discrete gradient on g is the technical part of the present paper and gives the second term in the r.h.s. of (1). We adapt to our model the techniques developed in Cancrini et al. (2002a) where the high-temperature mixing property and translation invariance are largely used. The L^ε factor comes from the fact that the homogeneity properties which make our system practically equivalent to a system with a translation invariant interaction and a high-temperature mixing condition hold only from scale L to scale L^ε (in Remark 29 we explain why it is less relevant in the case of the spectral gap).

Road map.

- In Section 2 we define the setting and state the main result.
- In Section 3 we recall some very simple large deviation results for independent bond percolation proved in Cancrini and Martinelli (2001).
- In Section 4 we collect several preliminary results that are essential in the proof.
- In Section 5 we prove recursively the main theorem on the LSC. All the technical results are obtained in Sections 6 and 7.

2. Notation and results

In this section we first define the setting in which we will work (spin model, Gibbs measure, dynamics), and then state the main theorem of this work.

2.1. The lattice and the configuration space

The lattice. We consider the d-dimensional lattice $\mathbb{Z}^d$ with *sites* $x=(x_1,\dots,x_d)$ and norms

$$|x|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p} \qquad p \geqslant 1 \quad \text{and } |x| = |x|_\infty = \max_{i\in\{1,\dots,d\}} |x_i|.$$

The associated distance functions are denoted by $d_p(\cdot,\cdot)$ and $d(\cdot,\cdot)$. By Q_L we denote the cube of all $x=(x_1,\dots,x_d)\in\mathbb{Z}^d$ such that $x_i\in\{0,\dots,L-1\}$. If $x\in\mathbb{Z}^d$, $Q_L(x)$ stands for Q_L+x. We also let B_L be the ball (w.r.t $d(\cdot,\cdot)$) of radius L centered at the origin, i.e. $B_L=Q_{2L+1}((-L,\dots,-L))$. If Λ is a finite subset of $\mathbb{Z}^d$ we write $\Lambda\subset\subset\mathbb{Z}^d$. The cardinality of Λ is denoted by $|\Lambda|$. $\mathbb{F}$ is the set of all nonempty finite subsets of $\mathbb{Z}^d$. $[x,y]$ is the *closed segment* with end points x and y. The *bonds* of $\mathbb{Z}^d$ are those $e=[x,y]$ with x,y nearest neighbors in $\mathbb{Z}^d$. By an abuse of notation we will still denote by $\mathbb{Z}^d$ the associated graph. $\hat{\mathbb{F}}$ is the set of all nonempty finite subgraphs of the graph $\mathbb{Z}^d$. Given $A\in\hat{\mathbb{F}}$ we write A_v and A_b for the set of vertices and the set of bonds of A, respectively. On the other hand, given Λ in $\mathbb{F}$, we will always identify Λ with the unique element $\hat{\Lambda}$ of $\hat{\mathbb{F}}$ with vertices the sites of Λ and bonds the set of all bonds of $\mathbb{Z}^d$ such that both end points are in Λ.

Given $\Lambda\subset\mathbb{Z}^d$ we define its interior and exterior boundaries as, respectively, $\partial^-\Lambda=\{x\in\Lambda: d(x,\Lambda^c)\leqslant 1\}$ and $\partial^+\Lambda=\{x\in\Lambda^c\colon\ d(x,\Lambda)\leqslant 1\}$, and more generally we define the boundaries of width n as $\partial_n^-\Lambda=\{x\in\Lambda\ :\ d(x,\Lambda^c)\leqslant n\}$, $\partial_n^+\Lambda=\{x\in\Lambda^c: d(x,\Lambda)\leqslant n\}$. For a fixed small positive number $\varepsilon\in(0,1)$ we define $\mathscr{R}_L^\varepsilon$ be the class of parallelepipeds inside Q_L with sides parallel to the coordinate axes, longest side greater than L^ε and ratio between the shortest and the longest side greater than ε, $\mathscr{R}_L^\varepsilon(l)$ be the class of all those parallelepipeds in $\mathscr{R}_L^\varepsilon$ such that the longest side is smaller than l, and $\bar{\mathscr{R}}_L^\varepsilon(l)$ the class of all parallelepipeds in $\mathscr{R}_L^\varepsilon(l)$ such that the shortest side is greater than L^ε.

The configuration space. Our *configuration space* is $\Omega = S^{\mathbb{Z}^d}$, where $S = \{0,1\}$, or $\Omega_V = S^V$ for some $V \subset \mathbb{Z}^d$. The single spin space S is endowed with the discrete topology and Ω with the corresponding product topology. Given $\sigma \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ we denote by σ_Λ the natural projection over Ω_Λ. If U, V are disjoint, $\sigma_U \tau_V$ is the configuration on $U \cup V$ which is equal to σ on U and τ on V. Given $V \in \mathbb{F}$ we define the *number of particles* N_V: $\Omega \mapsto \mathbb{N}$ as $N_V(\sigma) = \sum_{x\in V} \sigma(x)$ while the *density* is given by $\rho_V = N_V/|V|$.

If f is a function on Ω, Δ_f denotes the smallest subset of $\mathbb{Z}^d$ such that $f(\sigma)$ depends only on σ_{Δ_f}. f is called *local* if Δ_f is finite. $\mathscr{F}_\Lambda$ stands for the σ-algebra generated by the set of projections $\{\pi_x\}$, $x \in \Lambda$, from Ω to $\{0,1\}$, where π_x: $\sigma \mapsto \sigma(x)$. When $\Lambda = \mathbb{Z}^d$ we set $\mathscr{F} = \mathscr{F}_{\mathbb{Z}^d}$ and $\mathscr{F}$ coincides with the Borel σ-algebra on Ω with respect to the topology introduced above. By $\|f\|_\infty$ we mean the supremum norm of f. The *gradient* of a function f is defined as

$$(\nabla_x f)(\sigma) = f(\sigma^x) - f(\sigma),$$

where $\sigma^x \in \Omega$ is the configuration obtained from σ, by flipping the spin at the site x. Finally $\mathrm{Osc}(f) = \sup_{\sigma,\eta} |f(\sigma) - f(\eta)|$.

2.2. The dilute Ising lattice gas

We consider an abstract probability space $(\Theta, \mathscr{B}, \mathbb{P})$ and a set of i.i.d. real-valued random variables indexed by the bonds of $\mathbb{Z}^d$, $J = \{J_{xy}\}_{[x,y]\in\mathbb{Z}^d}$. $\mathbb{E}(\cdot)$ stands for the expectation with respect to $\mathbb{P}$. We assume that the couplings J_{xy} take only two values, $\beta > 0$ and 0, with probability p and $1-p$ respectively.

Given a disorder configuration we declare a bond $[x, y]$ *open* if $J_{xy} = \beta$ and *closed* otherwise. We denote by C_x the cluster of the site x, namely the set of all sites in $\mathbb{Z}^d$ which are connected to x by a path of open bonds, and by $\hat{C}_x$ the connected subgraph of $\mathbb{Z}^d$ whose vertices are the sites in C_x and whose bonds are the open bonds with end points in C_x. Notice that $\hat{C}_x = \{x\}$ if all the bonds with x as one end point are closed.

Given a disorder configuration J, for each $\sigma \in \Omega$ and $\Lambda \in \hat{\mathbb{F}}$ the Hamiltonian or energy function of the particle configuration σ in the graph Λ is given by

$$H_\Lambda^J(\sigma) = -\sum_{[x,y]\in\Lambda_b} J_{xy}(2\sigma(x)-1)(2\sigma(y)-1).$$

Given a collection of real numbers $\underline{\lambda} = \{\lambda_x\}_{x\in\mathbb{Z}^d}$ that in the sequel will be referred to as generalized chemical potential, we define $H_\Lambda^{J,\underline{\lambda}}(\sigma)$ as

$$H_\Lambda^{J,\underline{\lambda}}(\sigma) = H_\Lambda^J(\sigma) - \sum_{x\in\Lambda_v} \lambda_x \sigma(x).$$

Finally, given $\tau \in \Omega$, we let

$$H_\Lambda^{\tau,J,\underline{\lambda}}(\sigma) = H_\Lambda^{J,\underline{\lambda}}(\sigma) - \sum_{\substack{[x,y]\in\mathbb{Z}^d \\ x\in\Lambda_v, y\notin\Lambda_v}} J_{xy}(2\sigma(x)-1)(2\tau(y)-1)$$

and τ is called the *boundary condition*.

For each $\Lambda \in \hat{\mathbb{F}}$ and $\tau \in \Omega$ the (finite volume) grand canonical conditional Gibbs measure on $(\Omega, \mathscr{F})$, is given by

$$d\mu_\Lambda^{\tau,J,\underline{\lambda}}(\sigma) = \begin{cases} (Z_\Lambda^{\tau,J,\underline{\lambda}})^{-1}\exp[-H_\Lambda^{\tau,J,\underline{\lambda}}(\sigma)] & \text{if } \sigma(x)=\tau(x) \text{ for all } x\in\Lambda_v^c, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where $Z_\Lambda^{\tau,J,\underline{\lambda}}$ is the proper normalization factor called partition function.

Remark 1 (Warning). In most notation we will drop the superscript J if that does not generate confusion and the superscript $\underline{\lambda}$ if $\underline{\lambda}=0$. Moreover, for any $\Lambda \subset \mathbb{Z}^d$ we will always write $\mu_\Lambda^{\tau,\underline{\lambda}}$ instead of the more precise notation $\mu_{\hat{\Lambda}}^{\tau,\underline{\lambda}}$. Finally, if the couplings J_{xy} are constant and equal to β for all $[x,y]\in\Lambda_b$ and zero if either x or y are not in Λ_v, then we will write $\mu_\Lambda^{\beta,\underline{\lambda}}$ for the corresponding Gibbs measure. In other words $\mu_\Lambda^{\beta,\underline{\lambda}}$ is the Gibbs measure for the standard Ising model in Λ with inverse temperature β, chemical potentials $\underline{\lambda}$ and free boundary conditions.

We will sometimes refer to this model as the grand canonical dilute Ising model with parameters β, $\underline{\lambda}$ and p.

We finally introduce the *canonical Gibbs measures* on $(\Omega, \mathscr{F})$ defined as

$$\nu_{\Lambda,N}^\tau = \mu_\Lambda^\tau(\cdot|N_\Lambda = N) \quad N\in\{0,1,\ldots,|\Lambda|\}, \tag{5}$$

where N_Λ is the number of particles in Λ.

2.3. The dynamics

We consider the so-called Kawasaki dynamics in which particles ($\sigma(x)=1$) can jump to nearest neighbor empty ($\sigma(x)=0$) locations. For $\sigma\in\Omega$, let σ^{xy} be the configuration obtained from σ by exchanging the spins $\sigma(x)$ and $\sigma(y)$. Let $t_{xy}\sigma = \sigma^{xy}$ and define $(T_{xy}f)(\sigma) = f(t_{xy}\sigma)$. The stochastic dynamics we want to study is determined by the Markov generators L_Λ, Λ a connected finite subgraph of $\mathbb{Z}^d$, defined by

$$(L_\Lambda f)(\sigma) = \sum_{[x,y]\in\Lambda_b} c_{xy}(\sigma)(\nabla_{xy}f)(\sigma) \quad \sigma\in\Omega, \quad f\colon \Omega\mapsto\mathbb{R}, \tag{6}$$

where $\nabla_{xy} = T_{xy} - \mathbb{1}$. The non-negative real quantities $c_{xy}(\sigma)$ are the *transition rates* for the process.

The general assumptions on the transition rates are:

(1) *Finite range*. $c_{xy}(\sigma)$ depends only on the spins $\sigma(z)$ with $d(\{x,y\},z) \leqslant r$.
(2) *Detailed balance*. For all $\sigma \in \Omega$ and $[x,y] \in \mathscr{E}_{\mathbb{Z}^d}$

$$\exp[-H_{\{x,y\}}(\sigma)]c_{xy}(\sigma) = \exp[-H_{\{x,y\}}(\sigma^{xy})]c_{xy}(\sigma^{xy}). \tag{7}$$

(3) *Positivity and boundedness.* There exist positive real numbers $c_m(\beta)$, $c_M(\beta)$ such that

$$c_m \leqslant c_{xy}(\sigma) \leqslant c_M \quad \forall x, y \in \mathbb{Z}^d, \quad \sigma \in \Omega. \tag{8}$$

We denote by $L^{\tau}_{\Lambda,N}$ the operator L_{Λ} acting on $L^2(\Omega, v^{\tau}_{\Lambda,N})$ (this amounts to fix equal to $\tau_{\Lambda^c_v}$ the configuration outside Λ_v and N as the number of particles). Assumptions (1)–(3) guarantee that there exists a unique Markov process whose generator is $L^{\tau}_{\Lambda,N}$, and whose semi-group we denote by $(T_t^{\Lambda,N,\tau})_{t\geqslant 0}$. $L^{\tau}_{\Lambda,N}$ is a bounded operator on $L^2(\Omega, v^{\tau}_{\Lambda,N})$ and $v^{\tau}_{\Lambda,N}$ is its unique invariant measure. Moreover $v^{\tau}_{\Lambda,N}$ is *reversible* with respect to the process, i.e. $L^{\tau}_{\Lambda,N}$ is self–adjoint on $L^2(\Omega, v^{\tau}_{\Lambda,N})$.

A fundamental quantity associated with the dynamics of a reversible system is the gap of the generator, i.e.

$$\operatorname{gap}(L^{\tau}_{\Lambda,N}) = \operatorname{infspec}(-L^{\tau}_{\Lambda,N} \upharpoonright \mathbb{1}^{\perp}), \tag{9}$$

where $\mathbb{1}^{\perp}$ is the subspace of $L^2(\Omega, v^{\tau}_{\Lambda,N})$ orthogonal to the constant functions. We let $\mathscr{E}$ be the Dirichlet form associated with the generator $L^{\tau}_{\Lambda,N}$,

$$\mathscr{E}^{\tau}_{\Lambda,N}(f,f) = \langle f, -L^{\tau}_{\Lambda,N} f\rangle_{L^2(\Omega, v^{\tau}_{\Lambda,N})} = \frac{1}{2}\sum_{[x,y]\in\Lambda_b} v^{\tau}_{\Lambda,N}[c_{xy}(\nabla_{xy} f)^2] \tag{10}$$

and $\operatorname{Var}^{\tau}_{\Lambda,N}$ is the variance relative to the probability measure $v^{\tau}_{\Lambda,N}$. The gap can also be characterized as

$$\operatorname{gap}(L^{\tau}_{\Lambda,N}) = \inf_{\substack{f\in L^2(\Omega, v^{\tau}_{\Lambda,N}):\\ \operatorname{Var}^{\tau}_{\Lambda,N}(f)\neq 0}} \frac{\mathscr{E}^{\tau}_{\Lambda,N}(f,f)}{\operatorname{Var}^{\tau}_{\Lambda,N}(f)}. \tag{11}$$

A second relevant quantity is the logarithmic Sobolev constant $c^{\tau}_{V,N}$ defined as the smallest constant c such that

$$\operatorname{Ent}^{\tau}_{V,N}(f^2) \leqslant \frac{c}{2}\,\mathscr{E}^{\tau}_{V,N}(f,f) \tag{12}$$

for all non-negative functions f with $v^{\tau}_{V,N}(f^2) = 1$, where $\operatorname{Ent}^{\tau}_{V,N}(f^2) = v^{\tau}_{V,N}(f^2 \ln f^2)$ is the *entropy* of f^2 with respect to $v^{\tau}_{V,N}$. For the connection between spectral gap, logarithmic Sobolev constant and speed of relaxation to equilibrium we refer the reader to Diaconis and Saloff-Coste (1996).

2.4. Main results

We are finally in a position to formulate the main results of this paper on the logarithmic Sobolev constant of the generator of Kawasaki dynamics in a finite volume.

Let p_c denote the critical percolation for independent bond percolation in $\mathbb{Z}^d$ (see, e.g., Grimmett, 1999).

Theorem 2. *Assume that $p < p_c$ and fix $\varepsilon > 0$. Then there exists a set $\Theta_0 \subset \Theta$ with $P(\Theta_0) = 1$ and two positive constants c_1, c_2 such that for any $J \in \Theta_0$ and any L large enough*

$$c_1 L^2 \leqslant \min_{N,\tau} c^{\tau,J}_{Q_L,N} \leqslant \max_{N,\tau} c^{\tau,J}_{Q_L,N} \leqslant c_2 L^{2+\varepsilon}. \tag{13}$$

3. Simple large deviations for independent bond percolation

In this section we recall some very simple large deviation results for independent bond percolation below the percolation threshold that allow to prove some sort of homogenization property of the dilute Ising model at large scale. The interested reader can find the proofs in Cancrini and Martinelli (2001). Given an integer n, let f be a real function on the set of all finite connected subgraphs of $\mathbb{Z}^d$ which is translation invariant, that is $f(A) = f(A + x)$ for all $x \in \mathbb{Z}^d$ and for all finite connected subgraph A, and such that $|f(A)| \leqslant |A_v|^n$ where A_v is the set of vertices of the graph A. Let, for a fixed finite set Λ and a given disorder configuration J,

$$\langle f \rangle_{\Lambda,J} = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} f(\hat{C}_x) \tag{14}$$

and let $\bar{f} := \mathbb{E}(f(C_0))$ provided that $\mathbb{E}(|C_0|^n) < \infty$.

Proposition 3. *Assume $p < p_c$ and let $\varepsilon_0 = 1/(2d(n+1)+1)$ where n is the integer governing the growth of f. Let Λ be a parallelepiped with ratio between the shortest and longest side greater than ε. Then for any $\varepsilon \in (0, \varepsilon_0)$ there exist constants $0 < \delta = \delta(n, p, \varepsilon) < 1$, $m_1 = m_1(p) > 0$ and $m_2 = m_2(p, n) > 0$ such that*

(a) $\mathbb{P}\left(\sup_{x \in \Lambda} |C_x| \geqslant v\right) \leqslant |\Lambda| e^{-m_1 v} \quad \forall v \geqslant 0,$

(b) $\mathbb{P}(|\langle f \rangle_{\Lambda,J} - \bar{f}| \geqslant |\Lambda|^{-\varepsilon}) \leqslant e^{-m_2 |\Lambda|^\delta}.$

Here is a simple consequence of the above large deviation results. Given $\varepsilon \in (0, \varepsilon_0)$, an integer L and N real, translation invariant functions $\{f_i\}_{i=1}^N$ on the set of all finite connected subgraphs of $\mathbb{Z}^d$ such that $\max_{i \leqslant N} |f_i(A)| \leqslant |A_v|^4$ for any A, let us consider the event $\Theta(\varepsilon, M, N, L) = \bigcup_{R \in \mathscr{R}_L^\varepsilon} \Theta_R(\varepsilon, M, N, L)$ where

$$\Theta_R(\varepsilon, M, N, L) := \left\{\sup_{x \in R} |C_x| \geqslant M \log L\right\} \cup \left\{\sup_{n \leqslant 4} \sum_{x \in R} |C_x|^n \geqslant M|R|\right\}$$

$$\cup \left\{\sup_{i \leqslant N} |\langle f_i \rangle_{R,J} - \bar{f}_i| \geqslant |R|^{-\varepsilon}\right\}.$$

Then,

Corollary 4. *Assume $p < p_c$. There exists M such that for any $a < \infty$*

$$\sum_{L=1}^{\infty} \sup_{N \leqslant L^a} \mathbb{P}(\Theta(\varepsilon, M, N, L)) < +\infty.$$

4. Preliminary results

In this section we collect several preliminary results that are essential in order to prove that, with large probability, the logarithmic Sobolev constant of the Kawasaki dynamics for the dilute Ising model below p_c, on all scales between L^ε to L can be bounded from above by exactly the same methods employed when a high-temperature mixing condition and a translation invariance property hold. More precisely, we will formulate three conditions on the disorder configuration in the cube Q_L which will ensure that, if satisfied, the corresponding dilute Ising model shares all the relevant (for our purposes) features of the high-temperature standard Ising model. Moreover, our conditions will be meaningful in the sense that the probability of not being all verified simultaneously will be summable in L.

4.1. The general setting

Throughout all this section our setting and notation will be as follows. Let Λ be a parallelepiped in the class $\bar{\mathscr{R}}_L^\varepsilon$ whose longest side is say along the d direction and is L_d. Take $L_1, \ldots, L_{j_{\max}}$ such that $\sum_{j=1}^{j_{\max}} L_j = L_d$ and $L_j \geqslant \varepsilon L_d$ for any $j = 1, \ldots, j_{\max}$. We then take $\Lambda_j = \{x \in \Lambda\colon L_{j-1} \leqslant x_d \leqslant L_j\}$ with $L_0 = 0$, which are elements of $\mathscr{R}_L^\varepsilon$. Let also $\mathbf{N} := \{N_j\}_{j=1}^{j_{\max}}$ be a set of possible values of $\mathrm{N}_\Lambda := \{N_{\Lambda_j}\}_{j=1}^{j_{\max}}$ and let $\rho_j := N_j/|\Lambda_j|$. Given a boundary condition τ and a disorder configuration J, there exists a unique choice of the chemical potential $\underline{\lambda}$, constant on each Λ_j, $j = 1, \ldots, j_{\max}$, such that $\mu_\Lambda^{\tau,\underline{\lambda}}(N_{\Lambda_j}) = N_j$, $j = 1, \ldots, j_{\max}$ (see the appendix in Cancrini and Martinelli (2000a)).

We denote by $\mu := \mu_\Lambda^{\tau,\underline{\lambda}}$ the grand canonical Gibbs measure and by $\nu := \nu_{\Lambda,\mathbb{F}N}^\tau$ the multi-canonical Gibbs measure $\mu_\Lambda^{\tau,\underline{\lambda}}(\cdot|\mathbf{N}_\Lambda = \mathbf{N})$ and by Ω_τ the set of configurations τ' that coincide with τ in the half-space $\{x \in \mathbb{Z}^d\colon x_d < L_d\}$, where L_d is largest among the d-coordinates of the sites in Λ.

4.2. Assumptions on the disorder configurations

Let $\varepsilon_0 = 1/(10d+1)$ and let us fix a small positive number $\varepsilon \in (0, \varepsilon_0)$ and a large positive number M. For any $\Lambda \subset \mathbb{Z}^d$, any integer $N \in \{1, 2, \ldots, |\Lambda|\}$, any boundary condition $\tau \in \Omega$ and any disorder configuration J let also $\lambda = \lambda(\Lambda, N, \tau, J)$ be the (unique) constant chemical potential such that $\mu_\Lambda^{J,\tau,\lambda}(N_\Lambda) = N$ and let $\lambda_0 = \lambda_0(\Lambda, N)$ be such that $\mathbb{E}(|\hat{C}_0|^{-1} \mu_{C_0}^{\beta,\lambda_0}(N_{C_0})) = N/|\Lambda|$, that is the particle density of the cluster of the origin averaged on the disorder is equal to $N/|\Lambda|$. The existence and uniqueness of the chemical potential λ is proved in the appendix of Cancrini and Martinelli (2000a), for λ_0 a similar reasoning can be applied. Then our conditions read as follows.

Assumption 1. For any $R \in \mathscr{R}_L^\varepsilon$

$$\max_{x\in R} |C_x| \leqslant M \log L \quad \text{and} \quad \max_{n\leqslant 4} \sum_{x\in R} |C_x|^n \leqslant M|R|.$$

Assumption 2. For any $R = R_1 \cup R_2$ with $R_1, R_2 \in \mathscr{R}_L^\varepsilon \cup \emptyset$

$$\sup_{\tau} \sup_{N\in[1,\dots,|R|]} |\mu_R^{\lambda_0(R,N)}(N_{R_i}) - |R_i|\, \mathbb{E}(|C_0|^{-1} \mu_{\hat{C}_0}^{\beta,\lambda_0(R,N)}(N_{C_0}))| \leqslant |R_i|^{1-\varepsilon}$$

for $i = 1, 2$.

Assumption 3. Let $h_x := \mathrm{e}^{-\nabla_x H}\sigma(x)$. Then for any $R \in \mathscr{R}_L^\varepsilon$,

$$\sup_{\tau} \sup_{N\in[0,\dots,|R|]} \left| \mu_R^{\tau,\lambda_0}\left(N_R, \sum_{x\in R} h_x\right) - |R|\mathbb{E}\left(|C_0|^{-1}\mu_{\hat{C}_0}^{\beta,\lambda_0}\left(N_{C_0}, \sum_{x\in C_0} h_x\right)\right)\right| \leqslant |R|^{1-\varepsilon}.$$

Similarly for $\hat{h}_x := \mathrm{e}^{-\nabla_x H}(1-\sigma(x))$ and $\tilde{h}_x = \sigma(x)$.

Definition 5. The set of disorder configurations J that satisfy Assumptions 1–3 will be denoted by $\Theta_{\mathrm{good}}(L, M, \varepsilon)$.

Thanks to Corollary 4 we have the following result.

Proposition 6. *Assume $p < p_c$. Then*
(i) *there exists M such that for any $\varepsilon \in (0, \varepsilon_0)$ $\sum_{L=1}^{\infty} \mathbb{P}(\Theta_{\mathrm{good}}(L, M, \varepsilon)^c) < \infty$. In particular, for any large enough M and any $\varepsilon \in (0, \varepsilon_0)$:*
(ii) *there exists a set $\Theta_0 \subset \Theta$ such that $\mathbb{P}(\Theta_0) = 1$ and for any $J \in \Theta_0$ there exists $L(J)$ such that $J \in \Theta_{\mathrm{good}}(L, M, \varepsilon)$ for any $L \geqslant L(J)$;*
(iii) *there exists $\gamma = \gamma(M) > 0$, $\lim_{M\to+\infty} \gamma(M) = +\infty$, such that $\mathbb{P}(L(J) > l) \leqslant l^{-\gamma}$.*

Proof. Once point (i) of the proposition is established point (ii) is nothing but the standard Borel–Cantelli lemma.

To analyze the convergence of the series $\sum_L \mathbb{P}(\Theta_{\mathrm{good}}(L, M, \varepsilon)^c)$ we first observe that, thanks to Proposition 3, the probability that Assumption 1 is violated can be bounded from above by $c\,|\mathscr{R}_L^\varepsilon|\,(L^{-mM} + \mathrm{e}^{-m_2(\varepsilon L^\varepsilon)^{d\delta}})$ (where m is a positive constant depending on p and n and we used $|R| \geqslant (\varepsilon L^\varepsilon)^d$). In order to compute the probability that Assumption 2 is violated in $R = R_1 \cup R_2$ with $R_i \in \mathscr{R}_L^\varepsilon \cup \emptyset$, $i = 1, 2$, we define for any $A \in \hat{\mathbb{F}}$ the function $f_{N,R}(A) := |A|^{-1}\mu_A^{\lambda_0(R,N)}(N_A)$ for $N = 1, \dots, |R|$. With this notation and using the fact that $\mu_R^{\tau,\lambda_0(R,N)}$ is the product measure over the clusters in R, we can write

$$|\mu_R^{\tau,\lambda_0(R,N)}(N_{R_i}) - |R_i|\mathbb{E}(|C_0|^{-1}\mu_{\hat{C}_0}^{\beta,\lambda_0(R,N)}(N_{C_0}))|$$

$$\leqslant \left|\sum_{x\in R_i}[f_{N,R}(C_x) - \bar{f}]\right| + \left|\sum_{x\in R_i: C_x\cap R_i^c\neq\emptyset}\left[\frac{1}{|C_x\cap R_i|}\mu_R^{\tau,\lambda_0(R,N)}(N_{C_x\cap R_i})\right.\right.$$

$$\left.\left. - \frac{1}{|C_x|}\mu_{C_x}^{\lambda_0(R,N)}(N_{C_x})\right]\right| \leqslant |R_i|\,|\langle f_{N,R}\rangle_{R_i,J} - \bar{f}| + C\,R_i^{(d-1)/d}\sup_{x\in R_i}|C_x|^2$$

for a suitable constant $C = C(d,\beta)$. We have used here that any $R\in\mathscr{R}_L^\varepsilon$ has surface smaller than $C''|R|^{(d-1)/d}$. We can at this point use Proposition 3, the fact that the number of sets R that can be constructed is bounded by $|\mathscr{R}_L^\varepsilon|^2$, and the probability that Assumption 1 is violated estimated above to conclude that the probability that Assumption 2 is violated can be bounded from above by $c|\mathscr{R}_L^\varepsilon|^2|R|\mathrm{e}^{-(m_2\varepsilon L^\varepsilon)^{d\delta}}$. The probability that Assumption 3 is violated can be estimated similarly. Using the fact that the cardinality of $\mathscr{R}_L^\varepsilon$ is bounded from above by L^{2d}, point (i) follows, provided that M is taken large enough.

We are left with the proof of point (iii). By the definition of $L(J)$

$$\mathbb{P}(L(J) > l) \leqslant \mathbb{P}(\Theta_{\text{good}}(l,M,\varepsilon)^c)$$

proceeding as for point (ii) the result follows. □

4.3. Bounds on various covariances

Here we report, for completeness, some results which follow immediately from the factorization property of the grand canonical measure over the clusters, since they enter at various levels in the analysis of the Kawasaki dynamics for the dilute Ising model.

Given the setting above, for each set $V\subseteq\Lambda$ and $n\in\mathbb{N}$ we define

$$V_j := V\cap\Lambda_j \quad \bar{V} := \bigcup_{x\in V} C_x \quad \text{and} \quad V^{(n)} := \sum_{x\in V}|C_x\cap V|^n. \tag{15}$$

The following proposition holds for any disorder configuration J.

Proposition 7. *There exists a constant c depending only on β such that for any bounded local function f with support $\Delta_f\subset\Lambda$*

(*a*) $|\mu_\Lambda^{\tau,\underline{\lambda}}(N_{C_x\cap V_i}, N_{C_x\cap V_j})| \leqslant c\min\{\rho_i,\rho_j\}|C_x\cap V_i||C_x\cap V_j|$,

(*b*) $\mu_\Lambda^{\tau,\underline{\lambda}}(\bar{N}^2_{C_x\cap V_i}) \geqslant c^{-1}\rho_i|C_x\cap V_i|$,

(*c*) $\mu_\Lambda^{\tau,\underline{\lambda}}(\bar{N}^2_{V_i}) \leqslant c\rho_i V_i^{(1)}$,

(*d*) $\mu_\Lambda^{\tau,\underline{\lambda}}(\bar{N}^2_{V_i}) \geqslant c^{-1}\rho_i|V_i|$,

$(e)\quad |\mu_\Lambda^{\tau,\lambda}(f,N_{V_i})| \leqslant c\|f\|_\infty \min\{\rho_i|\bar{\Delta}_f \cap V_i|, (\rho_i(\bar{\Delta}_f \cap V_i)^{(1)})^{1/2}\},$

$(f)\quad \mu_\Lambda^{\tau,\lambda}(\bar{N}_{V_i}^4) \leqslant c\max\{(\rho_i V_i^{(1)})^2, (\rho_i V_i^{(3)})\},$

$(g)\quad |\mu_\Lambda^{\tau,\lambda}(f,N_{V_i},N_{V_i})| \leqslant c\|f\|_\infty \rho_i|\bar{\Delta}_f \cap V_i|^2,$

$$(h)\quad |\mu_\Lambda^{\tau,\lambda}(\bar{N}_{V_i}^3)| \leqslant c\rho_i V_i^{(2)}, \tag{16}$$

where $\bar{N}_V = N_V - \mu_\Lambda^{\tau,\lambda}(N_V)$ *for each set* $V \subseteq \Lambda$.

4.4. On the tilting fields

We recall here the following quite general result on the relationship between particle numbers and the chemical potential (see the appendix in Cancrini and Martinelli (2000a)). We assume here $j_{\max}=1$ so that we can set, for notation convenience, $N_1 = n$, $\rho = n/L^d$. In order to be more clear, in the following lemma we will write explicitly the dependence on the boundary conditions and the chemical potential of the grand canonical Gibbs state.

Lemma 8. *Let* $\alpha \in (0,1)$. *Then there exists a constant k independent on L such that for any L large enough and any f with* $\|f\|_\infty = 1$, *if* $\Delta_f \subset \Lambda$

(i) $\|\frac{\mathrm{d}}{\mathrm{d}n}\mu^{\tau,\lambda}(f)\|_\infty \leqslant k\,\frac{|\bar{\Delta}_f|}{|\Lambda|}$,

(ii) $\|\frac{\mathrm{d}^2}{\mathrm{d}n^2}\mu^{\tau,\lambda}(f)\|_\infty \leqslant k\frac{1}{n}\,\frac{|\bar{\Delta}_f|}{|\Lambda|}$.

Proof. We omit the proof since it is practically the same of an almost identical result (Proposition 3.1) of Cancrini and Martinelli (2000b) with the difference that the properties coming from the mixing condition are substituted by Proposition 7. □

4.5. Equivalence of ensembles

Here we recall some fine results on the finite volume comparison of ensembles that will be crucial in most of our future arguments. We refer the reader to Sections 6 and 7.2 of Cancrini and Martinelli (2000a).

With the definition of $\bar{V}$ given above (Section 4.3), we say that a subset $V \subset \Lambda$ is good if $\bar{V} \subset \Lambda_j$ for some $j = 1,\ldots,j_{\max}$, otherwise, it is bad.

Proposition 9. *In the above setting assume* $J \in \Theta_{\text{good}}(L,M,\varepsilon)$. *Then there exists constants* $C = C(M,\varepsilon)$, $C' = C'(M,\varepsilon)$ *and* $L_0 = L_0(M,\varepsilon)$ *such that, if* $L \geqslant L_0$

(a) *for all bounded local functions f with support* $\Delta_f \subset \Lambda$ *satisfying* $|\Delta_f|\,(M\log L)^4 \ll |\Lambda|$

$$|v(f) - \mu(f)| \leqslant C\|f\|_\infty \begin{cases} \dfrac{\bar{\Delta}_f^{(3)}}{|\Lambda|} & \textit{if } \Delta_f \textit{ is good}, \\[2ex] \dfrac{|\Delta_f|\,(M\log L)^4}{|\Lambda|} & \textit{otherwise} \end{cases}$$

(b) *for all local functions f with support $\Delta \subset \Lambda_n$ and $n \leqslant j_{\max}$, satisfying $\bar{\Delta}^{(3)} \leqslant |\Lambda|$ if Δ is good or $|\Delta| \log L \ll |\Lambda|$ if it is bad, we have*

$$\sup_{\tau' \in \Omega_\tau} |\nu^\tau(f) - \nu^{\tau'}(f)| \leqslant C' \|f\|_\infty \varepsilon(\Delta, L),$$

where $\bar{\Delta}^{(3)}$ has been defined in (15) *and*

$$\varepsilon(\Delta, L) \leqslant \frac{\bar{\Delta}^{(3)}}{|\Lambda|} + \left(\frac{\log L}{L}\right)^{k-n+1} [\min\{(\bar{\Delta}^{(1)})^{1/2}, |\bar{\Delta}|\}],$$

if Δ is good,

$$\varepsilon(\Delta, L) \leqslant \frac{|\Delta|(\log L)^4}{|\Lambda|} + \max_{j=n,n\pm 1} \left[\min\{((\bar{\Delta} \cap \Lambda_j)^{(1)})^{1/2}, |\bar{\Delta} \cap \Lambda_j|\} \left(\frac{\log L}{L}\right)^{j_{\max}-j+1}\right],$$

if Δ is bad.

Proof. See Theorem 6.4 and Proposition 7.4 in Cancrini and Martinelli (2000a). □

Remark 10. Actually the first part of Proposition 9 holds in a much more general context (see Section 6 in Cancrini and Martinelli, 2000a).

Proposition 11. *In the same setting assume $J \in \Theta_{\text{good}}(L, M, \varepsilon)$. Let f be such that $|\Lambda_j \setminus \Delta_f| \geqslant \varepsilon |\Lambda_j|$ for any $j = 1, \ldots, j_{\max}$. Then there exists a constant $A = A(M, \varepsilon)$ such that*

$$\nu(|f|) \leqslant A\mu(|f|).$$

In particular

$$\nu(f, f) \leqslant A\mu(f, f).$$

Proof. The proof is identical to that of Proposition 3.3 in Cancrini and Martinelli (2000b) if we observe that, see Cancrini and Martinelli (2000a), for any $J \in \Theta_{\text{good}}(L, M, \varepsilon)$

$$\|\mu_\Lambda^{\tau,\lambda}(\mathrm{e}^{\mathrm{i} \sum_j (t_j/v_j) N_{\Lambda_j}} \mid \mathscr{F}_{\Delta_f})\|_\infty \leqslant \mathrm{e}^{-\alpha \sum_j t_j^2}$$

for a suitable constant $\alpha := \alpha(\varepsilon, M)$, where $v_j^2 := \mu_\Lambda^{\tau,\lambda}(N_{\Lambda_j}, N_{\Lambda_j})$. □

We conclude this paragraph with a final result that plays a crucial role in our approach (see Section 4.6). For simplicity we discuss the next estimates in two dimensions and at the end we explain how to generalize it to higher dimensions.

Assume that the number of layers $j_{\max}$ is greater than 4, fix j_0 such that $3 \leqslant j_0 \leqslant j_{\max}$ and let $A = \bigcup_{j=1}^{j_0} \Lambda_j$, $B = \bigcup_{j_0-1}^{j_{\max}} \Lambda_j$ and $S = \Lambda_{j_0-1} \cup \Lambda_{j_0}$. Define also $\nu_X(\cdot) := \nu(\cdot \mid \mathscr{F}_{X^c})$ for $X = A, B, S$. Notice that ν-almost surely N_A, N_B and N_S are constant.

Lemma 12. *Assume that $J \in \Theta_{\text{good}}(L, M, \varepsilon)$ and let g be a positive function measurable w.r.t. $\mathscr{F}_{A^c}$. Then there exist $k = k(M, \varepsilon, \beta)$ and $L_0 = L_0(\varepsilon, M)$ such that, if $L \geqslant L_0$,*

$$\|\nu_B(g) - \nu(g)\|_\infty \leqslant \varepsilon \nu(g).$$

Proof. Fix a positive function g measurable w.r.t. $\mathscr{F}_{A^c}$. Fix also a configuration η and let $h_x(\eta) := \mathrm{e}^{-\nabla_x H(\eta)}/v_B^\eta(\mathrm{e}^{-\nabla_x H})$, $x \in \partial^+ S \cap A$. Using the definition of h_x and the DLR equations (valid because the numbers of particles in A, B, S are constant) we have

$$\begin{aligned}|v_B^{\eta^x}(g) - v_B^\eta(g)| &= |v_B^\eta(g, h_x)| \\ &= |v_B^\eta(g, v_S(h_x))| \\ &\leqslant v_B^\eta(g) \sup_{\tau,\tau' \in \Omega_\tau} |v_S^\tau(h_x) - v_S^{\tau'}(h_x)|,\end{aligned}$$

where Ω_τ is the set of configurations τ' which differs from τ only on $\partial^+ S \cap B$. By point (b) of Proposition 9 there exists a positive constant $k = k(M, \varepsilon, \beta)$ such that for L large enough

$$\sup_{\tau' \in \Omega_\tau} |v_S^\tau(h_x) - v_S^{\tau'}(h_x)| \leqslant k \frac{(\log L)^3}{L^2} \quad \forall \tau. \tag{17}$$

Thus

$$v_B^{\eta^x}(g) \leqslant \left[1 + k \frac{(\log L)^3}{L^2}\right] v_B^\eta(g).$$

Now, notice that any boundary configurations η and η' differ at most in L sites, by iteration we get

$$v_B^\eta(g) \leqslant \left[1 + k \frac{(\log L)^3}{L^2}\right]^L v_B^{\eta'}(g) \leqslant (1+\varepsilon) v_B^{\eta'}(g). \tag{18}$$

It suffices now to integrate (18) w.r.t. $dv(\eta')$ and use the arbitrariness of η. □

Remark 13. The restriction of $d=2$ comes from point (b) of Proposition 9. In fact, in, e.g. three dimensions, bound (17) becomes useless. The way out is to have the "safety belt" S divided into more layers (just three in $d=3$). It is interesting at this point to observe that a similar problem occurs also in the recursive study of the spectral gap (see Cancrini and Martinelli, 2000b). In that case, however, the safety belt S in $d=2$ consisted of just one atom and not of two as in our case. The reason is that, in the spectral gap analysis, a weaker form of Lemma 12 was necessary.

4.6. A block dynamics bound

In this section, we give a bound on the entropy that play an important role in our recursive approach. For simplicity we discuss the next estimates in two dimensions and at the end we explain how to generalize it to higher dimensions.

Assume that the number of layers $j_{\max}$ is greater than 4, fix j_0 such that $3 \leqslant j_0 \leqslant j_{\max}$ and let $A = \bigcup_{j=1}^{j_0} \Lambda_j$, $B = \bigcup_{j_0-1}^{j_{\max}} \Lambda_j$ and $S = \Lambda_{j_0-1} \cup \Lambda_{j_0}$. Define also $v_X(\cdot) := v(\cdot \,|\, \mathscr{F}_{X^c})$ for $X = A, B, S$. Notice that v-almost surely N_A, N_B and N_S are constant.

Proposition 14. *Assume that $J \in \Theta_{\text{good}}(L, M, \varepsilon)$. Then, there exist $L_0 = L_0(M, \varepsilon)$ such that, for any $L \geqslant L_0$ and any function f,*

$$\mathrm{Ent}_v(f^2) \leqslant (1+\varepsilon) v(\mathrm{Ent}_{v_A}(f^2) + \mathrm{Ent}_{v_B}(f^2)). \tag{19}$$

Remark 15. Remark that for an arbitrary product measure $\nu = \nu_1 \otimes \nu_2$, we have the general tensorization result (see Ané et al., 2000, for instance):

$$\mathrm{Ent}_\nu(f^2) \leqslant \nu(\mathrm{Ent}_{\nu_1}(f^2) + \mathrm{Ent}_{\nu_2}(f^2)).$$

In some sense, the extra factor takes into account the difference between ν and the product measure $\nu_A \otimes \nu_B$ due to the presence of the overlapping strip S.

Proof. With the result of Lemma 12 we can apply Proposition 2.1 of Cesi (2001). □

Remark 16. It is clear from the proof that the key input for the result is Lemma 12. Therefore, one can easily formulate the proposition in dimension greater than two simply by assuming that the set S consists of a sufficiently large number of layers (just three in $d = 3$) as it was explained already in Remark 13 after the proof of Lemma 12.

4.7. On the distribution of the particle number

The goal of this section is to recall from Cancrini et al. (2002a) some general result on the logarithmic Sobolev constant of the distribution γ of the number of particles inside one particular block of Λ.

Pick $j \in [1, \dots, j_{\max}]$ and divide Λ_j into two disjoint subsets V and W that we assume to be also element of $\mathscr{R}_L^\varepsilon$. Then, we denote by $\bar{n} = \mu(N_V)$ the average of the number of particles in V according to μ and we let $\gamma(n) = \nu(N_V = n)$ be the distribution of the number of particles inside V. Then, we have the following result.

Theorem 17. *Assume that $J \in \Theta_{\mathrm{good}}(M, \varepsilon, L)$. Then, there exists $k = k(M, \varepsilon, \beta)$ such that for all $f : \Omega_\Lambda \mapsto \mathbb{R}$ that depend only on the number of particles $N_V(\sigma)$, the following logarithmic Sobolev inequality holds*:

$$\mathrm{Ent}_\gamma(f) \leqslant k\bar{n} \sum_n (\gamma(n) \wedge \gamma(n-1))\,[f(n) - f(n-1)]^2.$$

Proof. Thank to the assumption $J \in \Theta_{\mathrm{good}}(M, \varepsilon, L)$, the proof is almost identical to that one of Proposition 3.17 of Cancrini et al. (2002a) with the difference that the properties coming from the mixing condition are substituted by Proposition 7. □

4.8. A general upper bound on the logarithmic Sobolev constant in a finite subgraph

In this section we obtain a rough upper bound for the logarithmic Sobolev constant of the dynamics in a finite subgraph Λ of $\mathbb{Z}^d$ which depends on the size of Λ and on the size of the largest cluster inside Λ. As a corollary we get that, if $J \in \Theta_{\mathrm{good}}(M, \varepsilon, L)$ and $\Lambda \in \mathscr{R}_L^\varepsilon$, the logarithmic Sobolev constant in Λ is not greater than $|\Lambda|^b$ for a suitable b independent of ε.

Theorem 18. *Let Λ be a finite subgraph of $\mathbb{Z}^d$ and let $\Gamma_J(\Lambda) := \max_{x\in\Lambda_v} |C_x \cap \Lambda|^{(d-1)/d}$. Then there exist a positive constant c, depending only on J and on d, and a numerical constant $b > 9$ such that*

$$c^{\tau}_{\Lambda,N} \leqslant |\Lambda_v|^b \exp(c\Gamma_J(\Lambda)) \quad \forall N, \tau.$$

Proof. First, from Cancrini and Martinelli (2001, Theorem 5.1) we learn that for some constant $b > 9$,

$$\operatorname{gap}(L^{\tau}_{\Lambda,N})^{-1} \leqslant |\Lambda_v|^{b-1} \exp(c\Gamma_J(\Lambda))$$

for all N and all τ. Here $\operatorname{gap}(L^{\tau}_{\Lambda,N})$ is the spectral gap of the generator $L^{\tau}_{\Lambda,N}$, defined in (9). On the other hand, we learn in Martinelli (1999, Proposition 3.9) that

$$c^{\tau}_{\Lambda,N} \leqslant |\Lambda| \operatorname{gap}(L^{\tau}_{\Lambda,N})^{-1}.$$

The proof is complete. □

Corollary 19. *Let $\Lambda \in \mathscr{R}^{\varepsilon}_L$ and assume $J \in \Theta_{\text{good}}(M, \varepsilon, L)$. Then there exists a positive numerical constant $b > 9$, independent of L, such that*

$$c^{\tau}_{\Lambda,N} \leqslant |\Lambda|^b \quad \forall\ N, \tau.$$

Proof. It follows immediately from Theorem 18 and the fact that $\max_{x\in\Lambda} |C_x| \leqslant M \log L$ for any $J \in \Theta_{\text{good}}(M, \varepsilon, L)$. □

4.9. A special path bound

In this section, we recall a useful result from Yau (1996).

Fix $\varepsilon \in (0,1)$ and $l \in [2L^{\varepsilon}, L]$. We then consider a volume $\Lambda \in \mathscr{R}^{\varepsilon}_L(l)$ such that $\Lambda = \bigcup_{j=1}^{j_{\max}} \Lambda_j$, where $\Lambda_j \in \mathscr{R}^{\varepsilon}_L(l)$ and $|\Lambda_j|/|\Lambda| \geqslant \varepsilon$ for $j = 1, \ldots, j_{\max}$. Let $\mathbf{N} := \{N_j\}_{j=1}^{j_{\max}}$ be a set of possible values of $\mathbf{N}_\Lambda := \{N_{\Lambda_j}\}_{j=1}^{j_{\max}}$ and define $\rho_j = N_j/|\Lambda_j|$. Recall the definition of $\mu := \mu^{\tau,\lambda}_{\Lambda}$ and $v := v^{\tau}_{\Lambda,\mathbf{N}}$.

Finally, let also $V, W \subset \Lambda$ such that $V \cap W = \emptyset$ and $V, W \in \mathscr{R}^{\varepsilon}_L(l)$.

Lemma 20. *There exists a suitable constant $k = k(\varepsilon, d, j_{\max})$ such that*

$$\sum_{x,z\in V\times W} v((\nabla_{xz} f)^2) \leqslant k l^{d+2} \mathscr{E}_v(f,f).$$

Proof. See Definition 3.9 and inequality (3.15) of Cancrini and Martinelli (2001). □

5. The recursive approach

In this section, we prove the main result of the paper, Theorem 2, via a recursive analysis on the behavior of the logarithmic Sobolev constant, when the linear size of the volume under consideration is doubled, developed in Cancrini and Martinelli (2001) for the high-temperature case.

For simplicity we carry out our analysis in two dimensions but the extension to higher dimension is straightforward (see Remark 13).

Let

$$c_s(l,L) := c_s(J,l,L,\varepsilon) = \max_{R\in\bar{\mathscr{R}}_L^\varepsilon(l)} \max_{N,\tau} c_{R,N}^\tau,$$

where $c_{R,N}^\tau$ is the logarithmic Sobolev constant in R, with boundary condition τ and N particles, defined in (12).

Notice that by definition of $\bar{\mathscr{R}}_L^\varepsilon(l)$, necessarily, $l \geqslant L^\varepsilon$.

Let also $\varepsilon' = \varepsilon/d(3b+2)$ where d is the dimension and b is the constant appearing in Corollary 19. With the above notation we will prove the following recursive bound.

Theorem 21. *Assume $J \in \Theta_{\text{good}}(M,\varepsilon',L)$. Then, there exist $L_0(\varepsilon',M)$ and $k(d,\beta,M,\varepsilon')$ such that, for $L \geqslant L_0$,*

$$c_s(l,L) \leqslant \frac{3}{2}\, c_s\left(\frac{l}{2},L\right) + kl^2L^\varepsilon \quad \text{for any } l\in[2L^\varepsilon,L].$$

In particular

$$\max_{N,\tau} c_s(L_{Q_L,N}^{\tau,J}) \leqslant L^{2+\varepsilon}[L^{(1-\varepsilon)\log_2(3/8)-3\varepsilon} c_s(L^\varepsilon,L) + 2k].$$

Proof. The fact that

$$\max_{N,\tau} c_s(L_{Q_L,N}^{\tau,J}) \leqslant L^{2+\varepsilon}[L^{(1-\varepsilon)\log_2(3/8)-3\varepsilon} c_s(L^\varepsilon,L) + 2k]$$

is a direct consequence of the recursive bound, by induction.

Fix now $l\in[2L^\varepsilon,L]$ and consider a rectangle $\Lambda\in\bar{\mathscr{R}}_L^\varepsilon(l)$. Without loss of generality, we can write

$$\Lambda = \{(x_1,x_2)\colon\, 0\leqslant x_1 < l_1,\ 0\leqslant x_2 < l_2\}$$

with $l_1 \leqslant l_2$.

If $l_2 \leqslant l/2$, then $\max_{N,\tau} c_{\Lambda,N}^\tau \leqslant c_s(l/2,L)$ simply because of the definition of $c_s(l,L)$. Thus, we can assume that $l/2 < l_2 \leqslant l$.

Let $d := \lfloor \varepsilon l\rfloor$ and pick an integer $i\in[1,\lfloor 1/10\varepsilon\rfloor - 1]$. We partition Λ into four disjoint sub-rectangles $\{\Lambda_j\}_{j=1}^4$ as follows:

$$\begin{aligned}
\Lambda_1 &= \{x\in\Lambda;\ 0\leqslant x_2 \leqslant l_2/2+(i-1)d\},\\
\Lambda_2 &= \{x\in\Lambda;\ l_2/2+(i-1)d < x_2 \leqslant l_2/2+id\},\\
\Lambda_3 &= \{x\in\Lambda;\ l_2/2+id < x_2 \leqslant l_2/2+(i+1)d\},\\
\Lambda_4 &= \{x\in\Lambda;\ l_2/2+(i+1)d < x_2\} \qquad (20)
\end{aligned}$$

and we set $A = \Lambda_1\cup\Lambda_2\cup\Lambda_3$, $B=\Lambda_2\cup\Lambda_3\cup\Lambda_4$ and $S=\Lambda_2\cup\Lambda_3$.

Notice that each Λ_i, $i=1,\ldots,4$, belongs to $\bar{\mathscr{R}}_L^{\varepsilon'}$, therefore, since $J \in \Theta_{\rm good}(M,\varepsilon',L)$, we are allowed to use the results of Section 4.

Fix now a boundary condition τ outside Λ, a number of particles $N \in [0,\ldots,|\Lambda|]$ and let $v := v^{\tau}_{\Lambda,N}$. We then use three times the conditional formula on the entropy: for any sub σ-algebra $\mathscr{F}_0$, any f,

$$\mathrm{Ent}_v(f^2) = v(\mathrm{Ent}_v(f^2|\mathscr{F}_0)) + \mathrm{Ent}_v(v(f^2|\mathscr{F}_0))$$

to write

$$\begin{aligned}
\mathrm{Ent}_v(f^2) &= v(\mathrm{Ent}_v(f^2|N_A)) + \mathrm{Ent}_v(v[f^2|N_A]) \\
&= v(\mathrm{Ent}_v(f^2|N_A,N_S)) + v(\mathrm{Ent}_v(v[f^2|N_A,N_S]|N_A)) + \mathrm{Ent}_v(v[f^2|N_A]) \\
&= v(\mathrm{Ent}_v(f^2|N_{\Lambda_1},N_{\Lambda_2},N_{\Lambda_3})) + v(\mathrm{Ent}_v(v[f^2|N_{\Lambda_1},N_{\Lambda_2},N_{\Lambda_3}]|N_A,N_S)) \\
&\quad + v(\mathrm{Ent}_v(v[f^2|N_A,N_S]|N_A)) + \mathrm{Ent}_v(v[f^2|N_A]).
\end{aligned} \tag{21}$$

We recall that N_V denotes the number of particles in the region V. The previous formula is the basic starting point of our recursive approach. We will now examine separately each term in the r.h.s. of (21).

As usual, in what follows, k will denote a generic constant depending on M,β, the dimensions of the lattice and on ε, whose value may vary from line to line.

5.1. Analysis of the first term in the r.h.s. of (21)

By Proposition 14 we have for L large enough

$$v(\mathrm{Ent}_v(f^2|N_{\Lambda_1},N_{\Lambda_2},N_{\Lambda_3})) \leqslant (1+\varepsilon)v(\mathrm{Ent}_{v_A}(f^2) + \mathrm{Ent}_{v_B}(f^2)).$$

As in Cancrini and Martinelli (2001, Section 4.1), let us examine the geometry of the bottom rectangle A, the reasoning being similar for the top one.

There are two cases to analyze:

(a) $l_1 \leqslant \frac{3}{4}l$. In this case one easily verifies that $\Lambda_1 \in \bar{\mathscr{R}}_L^{\varepsilon}(\frac{3}{4}l)$.

(b) $l_1 > \frac{3}{4}L$. In this case $\Lambda_1 \in \mathscr{R}_L^{\varepsilon}$ but now the *longest* side is l_1 and the *shortest* one is smaller than $l_2/2 + l/10 \leqslant \frac{3}{5}l$ since $l_2 \leqslant l$.

Therefore, $\max_{N,\tau} c^{\tau}_{A,N} \leqslant \max\{c_s(\frac{3}{4}l,L), \hat{c}_s(l,L)\}$ where

$$\hat{c}_s(l,L) = \max_{\substack{R\in\bar{\mathscr{R}}_L^{\varepsilon}(l)\\ l_1<\frac{3}{5}l,\, l_2\geqslant\frac{3}{4}l}} \max_{\tau,N} c^{\tau}_{R,N}.$$

In other words,

$$v(\mathrm{Ent}_{v_A}(f^2)) \leqslant \max\{c_s(\tfrac{3}{4}l,L), \hat{c}_s(l,L)\}\mathscr{E}_v(f,f)$$

and similarly for B.

In conclusion, we obtain that

$$\nu(\mathrm{Ent}_\nu(f^2|N_{A_1},N_{A_2},N_{A_3})) \leqslant (1+\varepsilon)\max\left\{c_s\left(\frac{3}{4}l,L\right),\hat{c}_s(l,L)\right\} \times \left[\mathscr{E}_\nu(f,f)+\frac{1}{2}\sum_{[x,y]\in\mathscr{E}_S}\nu[c_{xy}(\nabla_{xy}f)^2]\right] \quad (22)$$

uniformly in $i\in[1,\lfloor 1/10\varepsilon\rfloor - 1]$. Notice that the "spurious" term $\frac{1}{2}\sum_{[x,y]\in\mathscr{E}_S}\nu[c_{xy}(\nabla_{xy}f)^2]$ comes from the fact that $A\cap B=S$.

5.2. Analysis of the remaining terms in the r.h.s. of (21)

Here we bound from above the other three terms in (21). The necessary steps are almost identical for all of them and therefore, for shortness, we treat only the second one. Later on we will state without further comments the analogous result for the first and third one.

For a given value N_A of the number of particles in A, let $\rho_A := N_A/|A|$ and assume, without loss of generality, that $\rho_A\leqslant\frac{1}{2}$. Let $\hat{\nu}(\cdot):=\nu(\cdot|N_A)$ be the associated multi-canonical measure and let $\hat{\mu}$ be the corresponding (multi)-grand canonical measure. Let also $\bar{n}_S=\hat{\mu}(N_S)$, and let $\gamma(n):=\hat{\nu}(N_S=n)$ be the distribution of the number of particles inside S. Let finally $c_n=n(|A\setminus S|-N_A+n)$, that is (number of particles in S) $\times$ (number of holes in $A\setminus S$), similarly $c'_n=n(|S|-N_A+n)$, and let $u=\lfloor\rho_A|S|\rfloor$.

Then, using Theorem 17 and Corollary 3.7 of Cancrini et al. (2002a), we can write

$$\begin{aligned}
&\mathrm{Ent}_\nu(\nu[f^2|N_A,N_S]|N_A)\\
&\quad=\mathrm{Ent}_{\hat{\nu}}(\hat{\nu}[f^2|N_S]=\mathrm{Ent}_\gamma(\hat{\nu}[f^2|N_S])\\
&\quad\leqslant k\bar{n}_S\sum_n\gamma(n)\wedge\gamma(n-1)(\sqrt{\hat{\nu}(f^2|N_S=n)}-\sqrt{\hat{\nu}(f^2|N_S=n-1)})^2\\
&\quad\leqslant k\bar{n}_S\sum_n\gamma_f(n)(A(n)^2+B(n)^2),
\end{aligned}\quad (23)$$

where

$$A(n)=\begin{cases}\dfrac{1}{c_n}\dfrac{\gamma(n-1)}{\gamma(n)}\displaystyle\sum_{\substack{x\in S\\ z\in A\setminus S}}\hat{\nu}((\nabla_{zx}f^2)\mathbb{1}_{E_{zx}}\mathrm{e}^{-\nabla_{xz}H_A}|N_S=n-1) & \text{if } n\leqslant u,\\[2ex] \dfrac{1}{c'_{N_A-n+1}}\dfrac{\gamma(n)}{\gamma(n-1)}\displaystyle\sum_{\substack{x\in S\\ z\in A\setminus S}}\hat{\nu}((\nabla_{xz}f^2)\mathbb{1}_{E_{xz}}\mathrm{e}^{-\nabla_{xz}H_A}|N_S=n) & \text{otherwise,}\end{cases}$$

$$B(n) = \begin{cases} \dfrac{1}{c_n} \dfrac{\gamma(n-1)}{\gamma(n)} \displaystyle\sum_{\substack{x \in S \\ z \in \Lambda \setminus S}} \hat{v}((\mathrm{e}^{-\nabla_{xz} H_\Lambda} - 1) \mathbb{1}_{E_{zx}}, f^2 | N_S = n-1) & \text{if } n \leqslant u, \\ \dfrac{1}{c'_{N_\Lambda - n+1}} \dfrac{\gamma(n)}{\gamma(n-1)} \displaystyle\sum_{\substack{x \in S \\ z \in \Lambda \setminus S}} \hat{v}((\mathrm{e}^{-\nabla_{xz} H_\Lambda} - 1) \mathbb{1}_{E_{xz}}, f^2 | N_S = n) & \text{otherwise,} \end{cases}$$

$$E_{xz} = \{\sigma \in \Omega\colon \sigma(x) = 1,\ \sigma(z) = 0\},$$

and

$$\gamma_f(n) := \frac{\gamma(n) \wedge \gamma(n-1)}{\hat{v}(f^2 | N_S = n) \vee \hat{v}(f^2 | N_S = n-1)}.$$

5.2.1. Bound on $\bar{n}_S \sum_n \gamma_f(n) A(n)^2$

Exactly as in Proposition 3.8 of Cancrini and Martinelli (2001), we have

$$\bar{n}_S \sum_n \gamma_f(n) A(n)^2 \leqslant k l^2 \mathscr{E}_{\hat{v}}(f, f).$$

5.2.2. Bound on $\bar{n}_S \sum_n \gamma_f(n) B(n)^2$

This bound is one of the technical part of this work. It is at this point the term L^ε appears. Actually, using the same proof as Proposition 3.10 of Cancrini and Martinelli (2001) and thanks to the assumption $J \in \Theta_{\mathrm{good}}(M, \varepsilon', L)$ and Proposition 32, we obtain that for any $\zeta > 0$, there exist C_ζ independent of ρ_Λ such that

$$\bar{n}_S \sum_n \gamma_f(n) B(n)^2 \leqslant C_\zeta \hat{v}(f^2) + C_\zeta l^2 L^\varepsilon \mathscr{E}_{\hat{v}}(f, f) + \zeta \,\mathrm{Ent}_{\hat{v}}(f^2). \tag{24}$$

5.2.3. Bound on $v(\mathrm{Ent}_v(v[f^2 | N_\Lambda, N_S] | N_\Lambda))$

If we average w.r.t. the canonical measure v inequality (23) and use the simple inequality $v(\mathrm{Ent}_v(f^2 | \mathscr{F}_0)) \leqslant \mathrm{Ent}_v(f^2)$ for any $\mathscr{F}_0$, we get

$$v(\mathrm{Ent}_v(v[f^2 | N_\Lambda, N_S] | N_\Lambda)) \leqslant C_\zeta v(f^2) + C_\zeta l^2 L^\varepsilon \mathscr{E}_v(f, f) + \zeta \,\mathrm{Ent}_v(f^2). \tag{25}$$

Similar bounds hold also for the first and third term in the r.h.s. of (21).

5.3. The recursion completed

We are finally in a position to complete the proof of Theorem 21. If we put together (22) and (25) we get that, for any ζ small enough

$$\begin{aligned} \text{r.h.s. of (21)} \leqslant{} & (1+\varepsilon) \max\left\{ c_s\left(\frac{3}{4} l, L\right), \hat{c}_s(l, L) \right\} \\ & \times \left[\mathscr{E}_v(f, f) + \frac{1}{2} \sum_{[x,y] \in \mathscr{E}_S} v[c_{xy} (\nabla_{xy} f)^2] \right] \\ & + C_\zeta v(f^2) + C_\zeta l^2 L^\varepsilon \mathscr{E}_v(f, f) + \zeta \,\mathrm{Ent}_v(f^2) \end{aligned}$$

that is

$$\mathrm{Ent}_\nu(f^2) \leqslant (1+\varepsilon)\left(\frac{1}{1-\zeta}\right)\max\left\{c_s\left(\frac{3}{4}l,L\right),\hat{c}_s(l,L)\right\}$$

$$\times\left[\mathscr{E}_\nu(f,f)+\frac{1}{2}\sum_{[x,y]\in\mathscr{E}_S}\nu[c_{xy}(\nabla_{xy}f)^2]\right]$$

$$+C_\zeta\nu(f^2)+C_\zeta l^2L^\varepsilon\,\mathscr{E}_\nu(f,f)$$

for a suitable constant C_ζ.

Finally, following Martinelli (1999), we average the above inequality w.r.t. to the integer i (see (20)) and use the observation that, as i varies in $[1,\lfloor 1/10\varepsilon\rfloor-1]$, the strips $S\equiv S_i$ are disjoint. In particular

$$\frac{1}{2}\sum_{i\in[1,\lfloor\frac{1}{10\varepsilon}\rfloor-1]}\sum_{[x,y]\in\mathscr{E}_{S_i}}\nu[c_{xy}(\nabla_{xy}f)^2]\leqslant\mathscr{E}_\nu(f,f)$$

so that

$$\mathrm{Ent}_\nu(f^2)\leqslant\left(\frac{1}{1-\zeta}\right)(1+20\varepsilon)^2\max\left\{c_s\left(\frac{3}{4}l,L\right),\hat{c}_s(l,L)\right\}\mathscr{E}_\nu(f,f)$$

$$+C_\zeta\nu(f^2)+C_\zeta l^2L^\varepsilon\mathscr{E}_\nu(f,f)$$

for ε small enough. Notice that if we write $f=[f-\nu(f)]+\nu(f)$ and we use the Poincaré bound $\mathrm{Var}_\nu(f)\leqslant kl^2\mathscr{E}_\nu(f,f)$ we get

$$\mathrm{Ent}_\nu(f^2)\leqslant\mathrm{Ent}_\nu([f-\nu(f)]^2)+2\,\mathrm{Var}_\nu(f)$$

$$\leqslant\left(\frac{1}{1-\zeta}\right)(1+20\varepsilon)^2\max\left\{c_s\left(\frac{3}{4}\,l,L\right),\hat{c}_s(l,L)\right\}\mathscr{E}_\nu(f,f)$$

$$+kl^2L^\varepsilon\mathscr{E}_\nu(f,f),$$

where in the first line we have used the Rothaus inequality $\mathrm{Ent}_\nu(f^2)\leqslant 2\,\mathrm{Var}_\nu(f)+\mathrm{Ent}_\nu([f-\nu(f)]^2)$ (see Rothaus, 1985; Ané et al., 2000). In other words

$$c^{\tau}_{\Lambda,N}\leqslant\left(\frac{1}{1-\zeta}\right)(1+20\varepsilon)^2\max\left\{c_s\left(\frac{3}{4}l,L\right),\hat{c}_s(l,L)\right\}+kl^2L^\varepsilon. \tag{26}$$

Notice that if the original rectangle Λ was chosen in the subclass of $\bar{\mathscr{R}}^{\varepsilon}_L(l)$ entering in the definition of $\hat{c}_s(l,L)$, i.e. $l_1\leqslant l_2/2+l/10$, then we would have obtained the inequality (26) with the factor $\max\{c_s(\frac{3}{4}l,L),\hat{c}_s(l,L)\}$ replaced by $c_s(\frac{3}{4}l,L)$ simply because, for any $i\in[1,\lfloor\frac{1}{10\varepsilon}\rfloor-1]$, Λ_1, Λ_2 and Λ_3 would belong to $\bar{\mathscr{R}}^{\varepsilon}_L(\frac{3}{4}l)$. Thus,

$$\hat{c}_s(l,L)\leqslant\left(\frac{1}{1-\zeta}\right)(1+20\varepsilon)^2c_s\left(\frac{3}{4}l,L\right)+kl^2L^\varepsilon. \tag{27}$$

If we combine (26) with (27) we finally get

$$c^{\tau}_{\Lambda,N} \leqslant \left(\frac{1}{1-\zeta}\right)^2 (1+20\varepsilon)^4 c_s\left(\frac{3}{4}l, L\right) + kl^2 L^{\varepsilon}.$$

Thus,

$$c_s(l,L) \leqslant \left(\frac{1}{1-\zeta}\right)^4 (1+20\varepsilon)^2 c_s\left(\frac{3}{4}l, L\right) + kl^2 L^{\varepsilon}$$

and two more iterations prove the recursive inequality of the theorem provided that the two parameters ε, ζ were chosen small enough. □

5.4. Proof of Theorem 2

The lower bound is a direct consequence of the general comparison

$$\mathrm{gap}(L^{\tau,J}_{Q_L,N})^{-1} \leqslant c_s(L^{\tau,J}_{Q_L,N}),$$

between the spectral gap and the logarithmic Sobolev constant (see Ané et al., 2000), and the result of Cancrini and Martinelli (2001) on the spectral gap of the dilute Ising model, $\mathrm{gap}(L^{\tau,J}_{Q_L,N})^{-1} \geqslant c_1 L^2$.

We turn to the upper bound. Let $\varepsilon_0 = 1/(10d+1)$ and let us fix M large enough and $0 < \varepsilon < \min\{\log_2 \frac{8}{3}/(\log_2 \frac{8}{3} - 3 + db), \varepsilon_0\}$, where b is the constant defined in Corollary 19. Let also $\varepsilon' = \varepsilon/(d(3b+2))$.

Then, by Proposition 6 there exists a set $\Theta_0 \subset \Theta$ with $\mathbb{P}(\Theta_0) = 1$ such that for any $J \in \Theta_0$, there exists $L(J) < \infty$ with the property that $J \in \Theta_{\mathrm{good}}(L, M, \varepsilon')$ for any $L \geqslant L(J)$.

Without loss of generality we can assume that $L(J)$ is larger that some large constant $L_0 = L_0(M, \varepsilon')$.

By Theorem 21 we have

$$\max_{N,\tau} c_s(L^{\tau,J}_{Q_L,N}) \leqslant L^{2+\varepsilon}[L^{(1-\varepsilon)\log_2(3/8)-3\varepsilon} c_s(L^{\varepsilon}, L) + 2k]$$

provided that $L \geqslant L(J)$. But thanks to Corollary 19,

$$c_s(L^{\varepsilon}, L) \leqslant L^{db\varepsilon}.$$

So that,

$$\max_{N,\tau} c_s(L^{\tau,J}_{Q_L,N}) \leqslant L^{2+\varepsilon}[1+2k].$$

The proof is complete. □

Remark 22. In the particular case of very high temperature ($\beta \ll 1$), a proof similar (but much more simple) to that one given in Section 5.2.2 leads to the following inequality:

$$\bar{n}_S \sum_n \gamma_f(n) B(n)^2 \leqslant C_\zeta \hat{v}(f^2) + \zeta\, \mathrm{Ent}_{\hat{v}}(f^2).$$

And thus, by the previous recursion procedure, we get that the logarithmic Sobolev constant, in that particular case, grows like L^2.

We obtain the same conclusion when there are a finite number (independent of $|\Lambda|$ of cluster inside Λ, see the appendix).

6. On the grand canonical Laplace transform

In this section we seek Gaussian bounds on quantities of the form $\mu(e^{tf})$ where μ is the grand canonical Gibbs measure on some finite set and f is a mean zero function, namely bounds of the type

$$\mu(e^{tf}) \leqslant e^{t^2 K_f}$$

Once bounds like the one above are proved, then we can transfer them to the *canonical* Laplace transform by means of Proposition 11.

We first recall a technical result established in Cancrini et al. (2002a) and then we apply it to a practical case.

Let Λ be a finite set, for a given boundary configuration τ and (possible vector) chemical potential $\underline{\lambda}$, let $\mu := \mu_\Lambda^{\tau,\underline{\lambda}}$. Let $\{V_\alpha\}_{\alpha\in I}$ be a collection of subsets of Λ such that $\mathrm{dist}(V_\alpha, V_{\alpha'}) \geqslant r+1$ for $\alpha \neq \alpha'$, r being the range of the interaction, and let $V = \bigcup_\alpha V_\alpha$. Let also $f : \Omega_\Lambda \mapsto \mathbb{R}$ be such that $\mu(f) = 0$.

Proposition 23. *Fix $t_0 > 0$. Then, for all $t \in [0, t_0]$*

$$\mu(e^{tf}) \leqslant e^{t^2 K_f},$$

where $K_f = e^{2\beta} \sum_{x\in\Lambda} \|\nabla_x f\|_\infty^2 c_\mu$ and c_μ is the logarithmic Sobolev constant of μ w.r.t. to the Heat Bath rates.

Proof. See Proposition 5.1 in Cancrini et al. (2002a). □

Now, we discuss an application of our bounds to a concrete case that will be important in the next section. We fix $\varepsilon \in (0,1)$ and $l \in [2L^\varepsilon, L]$. We then consider a volume $\Lambda \in \mathscr{R}_L^\varepsilon(l)$ such that $\Lambda = \bigcup_{j=1}^{j_{\max}} \Lambda_j$, where $\Lambda_j \in \mathscr{R}_L^\varepsilon(l)$ and $|\Lambda_j|/|\Lambda| \geqslant \varepsilon$ for $j = 1, \ldots, j_{\max}$. Let $\mathbf{N} := \{N_j\}_{j=1}^{j_{\max}}$ be a set of possible values of $\mathbf{N}_\Lambda := \{N_{\Lambda_j}\}_{j=1}^{j_{\max}}$ and define $\rho_j = N_j/|\Lambda_j|$.

We denote by ν the multi-canonical measure on Λ as in the standard multi-canonical setting of Section 4.

We define $\{Q_\alpha\}_{\alpha\in I}$ to be a collection of cubes in Λ_j of side $l_0 = L^{\varepsilon'}$, where $0 < \varepsilon' < \varepsilon$ (ε' will be defined more precisely in the next section), such that for any $\alpha \neq \beta$ $\mathrm{dist}(Q_\alpha, Q_\beta) \geqslant 3M \log L$, $\mathrm{dist}(Q_\alpha, \partial\Lambda) \geqslant 3M \log L$ and $|\Lambda \setminus Q| \leqslant |\Lambda| \log L / l_0$, where $Q := \bigcup_{\alpha\in I} Q_\alpha$. Clearly such collection exists. Let $Q_\alpha^{\mathrm{int}} := \{x \in Q_\alpha \colon d(x, Q_\alpha^c) \geqslant 3M \log L\}$, $Q^{\mathrm{int}} = \bigcup_\alpha Q_\alpha^{\mathrm{int}}$ and denote by $n_c = |I|$ the number of such cubes.

For any $\alpha \in I$, let $\{Q_{\alpha\beta}\}_{\beta\in I_\alpha}$ be a chessboard-like partition of Q_α into cubes of side $l_1 = l_0^{\varepsilon'}$. It is easy to see that we can partition the indexes I_α into $2d$ subsets $\{I_i\}_{i=1}^{2d}$ in such a way that $\min_{\beta\neq\beta' \in I_i} d(Q_{\alpha\beta}, Q_{\alpha\beta'}) \geqslant l_1$. Then, for each α, we arrange in regular order the collection of cubes $\{Q_{\alpha\beta}\}_\beta$ with respect to β in such a way that we can write $\sum_\alpha \sum_\beta = \sum_\beta \sum_\alpha$.

Given $\eta \in \Omega_\Lambda$ and $s \in [0, l_0^d]$, write $\mu_{Q_\alpha}^{\eta,\lambda(\eta,s)}(\cdot)$ to denote the grand canonical Gibbs measure on Q_α with boundary condition η and constant chemical potential $\lambda(\eta,s)$ such that $\mu_{Q_\alpha}^{\eta,\lambda(\eta,s)}(N_{Q_\alpha}) = s$. Whenever s is also an integer, say $s = n \in [0, 1, \dots, \lfloor l_0^d \rfloor]$, we will use the standard notation $v_{Q_\alpha,s}^\eta$ for the corresponding canonical Gibbs measure. In the same way, we define $\mu_{Q_{\alpha\beta}}^{\eta,\lambda(\eta,s)}$ and $v_{Q_{\alpha\beta},s}^\eta$. Then, for any $\alpha \in I$ and $\beta \in I_i$, $n_\alpha(\eta) := N_{Q_\alpha}(\eta)$ and $n_{\alpha\beta}(\eta) := N_{Q_{\alpha\beta}}(\eta)$ stand for the number of particles inside Q_α and $Q_{\alpha\beta}$, respectively, and we define $\bar{n}_\alpha := \mu(N_{Q_\alpha})$, $\bar{n}_{\alpha\beta} := \mu(N_{Q_{\alpha\beta}})$.

For notation convenience, we define for any α, β, $F_{\alpha\beta} := n_{\alpha\beta} - \bar{n}_{\alpha\beta}$ and for all β,

$$F_\beta := \sum_{\alpha \in I} \sum_{\substack{\beta' \in I_i \\ \beta' \neq \beta}} \frac{1}{\bar{n}_\alpha}((n_{\alpha\beta} - \bar{n}_{\alpha\beta})(n_{\alpha\beta'} - \bar{n}_{\alpha\beta'})) = \sum_{\alpha \in I} \sum_{\substack{\beta' \in I_i \\ \beta' \neq \beta}} \frac{1}{\bar{n}_\alpha} F_{\alpha\beta} F_{\alpha\beta'}.$$

Consider a local function g with support Λ_g containing the origin and of diameter smaller than $2r$. Then we define g_x the translated of g by x and write $g^\delta(\sigma) := g_x(\sigma) - \delta\,\sigma(x)$, δ being a constant independent of x. Next we write

$$\xi_\alpha^\delta(\eta, s) := \sum_{x \in Q_\alpha^{\text{int}}} \mu_\alpha^{\eta,\lambda(\eta,s)}(g_x^\delta), \quad \xi_\alpha^\delta(\eta) := \xi_\alpha^\delta(\eta, n_\alpha(\eta)), \quad \xi_\alpha^\delta(s) := \xi_\alpha^\delta(0, s),$$

$$g_\alpha^\delta(\eta, s) := \sum_{x \in Q_\alpha^{\text{int}}} v_{Q_\alpha,s}^\eta(g_x^\delta), \quad g_\alpha^\delta(\eta) := g_\alpha^\delta(\eta, n_\alpha(\eta)). \tag{28}$$

We introduce the set of constants (w.r.t. μ and v) $\{\delta_\alpha\}_{\alpha \in I}$ where

$$\delta_\alpha := \frac{\mu_{Q_\alpha}^{0,\lambda(0,\bar{n}_\alpha)}\left(\sum_{x \in Q_\alpha^{\text{int}}} g_x, N_{Q_\alpha}\right)}{\mu_{Q_\alpha}^{0,\lambda(0,\bar{n}_\alpha)}(N_{Q_\alpha^{\text{int}}}, N_{Q_\alpha})} \tag{29}$$

and substituting the constant δ with δ_α in each cube Q_α we have $\xi_\alpha^{\delta_\alpha}(s)$.

Finally we define

$$G_v(\eta) := \sum_{\alpha \in I} [g_\alpha^\delta(\eta) - \mu(g_\alpha^\delta)], \quad G^{\text{ext}}(\eta) := \sum_{x \in \Lambda_j \setminus Q^{\text{int}}} [g_x^\delta(\eta) - \mu(g_x^\delta)], \tag{30}$$

$$G_\mu(\eta) := \sum_{\alpha \in I} [\xi_\alpha^\delta(\eta) - \mu(\xi_\alpha^\delta)], \quad G_\mu^0(\eta) := \sum_{\alpha \in I} [\xi_\alpha^{\delta,0}(n_\alpha(\eta)) - \mu(\xi_\alpha^{\delta,0})],$$

$$\hat{G}_\mu^0(\eta) := \sum_{\alpha \in I} [\, \xi_\alpha^{\delta_\alpha}(n_\alpha(\eta)) - \mu(\xi_\alpha^{\delta_\alpha})\,]\,.$$

Let $\{D_\alpha\}_{\alpha \in I}$ be a set of functions constant w.r.t. μ and v, and define $D := \sup_{\alpha \in I} D_\alpha$. Let furthermore Γ_Q be a function whose support is $Q = \bigcup_{\alpha \in I} Q_\alpha$ and define $\Gamma := \|\Gamma_Q\|_\infty$, $\nabla\Gamma := \sup_{x \in Q} \|\nabla_x \Gamma_Q\|_\infty$. We have the following proposition.

Proposition 24. *Assume* $J \in \Theta_{\rm good}(L, M, \varepsilon')$ *and fix* $t_0 > 0$. *Then there exists a constant* A *depending on* $M, \varepsilon', \beta, r, d, t_0, \|g\|_\infty, \rho_j$, *such that*

(i) $\mu(\mathrm{e}^{tG^{\rm ext}}) \leqslant \mathrm{e}^{t^2|\Lambda_j|A \log L c(L) l_0^{-1}} \quad \forall t,$

(ii) $\mu(\mathrm{e}^{t(G_v - G_\mu)}) \leqslant \mathrm{e}^{t^2|\Lambda_j|A l_0^{-d}} \quad \forall t \in [0, t_0],$

(iii) $\mu(\mathrm{e}^{t(G_\mu - G_\mu^0)}) \leqslant \mathrm{e}^{t^2}|\Lambda_j|\, A\, (\log L)^2\, c(L)\, l_0^{-1} \quad \forall t,$

(iv) $\mu(\mathrm{e}^{t(\hat G_\mu^0)}) \leqslant \mathrm{e}^{t^2 A|\Lambda_j| l_0^{-d}} \quad \forall t \in [0, t_0 l_0^{-d}],$

(v) $\mu\left(\exp\left\{t \sum_{\alpha\in I, \beta\in I_i} D_\alpha F_{\alpha\beta}\right\}\right) \leqslant \mathrm{e}^{t^2|\Lambda_j|AD^2} \quad \forall t \in [0, t_0/D l_1^d],$

(vi) $\mu\left(\exp\left\{t \sum_{\alpha\in I, \beta\in I_i} \frac{1}{\bar n_\alpha}(F_{\alpha\beta}^2 - \mu(F_{\alpha\beta}^2))\right\}\right) \leqslant \mathrm{e}^{t^2|\Lambda_j|\, A l_1^d l_0^{-d}} \quad \forall t \in [0, t_0],$

(vii) $\prod_{\beta\in I_i} \mu(\mathrm{e}^{tF_\beta}) \leqslant \mathrm{e}^{t^2|\Lambda_j|\, A l_0^{-d}} \quad \forall t \in [0, t_0(l_0^d/l_1^{2d})],$

(viii) $\mu\left(\exp\left\{t\Gamma_Q \sum_{x\in\Lambda_j\setminus Q} [\mu(\sigma(x)) - \sigma(x)]\right\}\right) \leqslant \mathrm{e}^{t^2 c(L)[a(L)^2|\Lambda_j|\nabla\Gamma^2 + a(L)\Gamma^2]} \quad \forall t,$

where $c(L) := \mathrm{e}^{A(\log L)^{(d-1)/d}}$, $a(L) := |\Lambda_j \setminus Q| \leqslant A|\Lambda_j| l_0^{-1} \log L$ *and* d *is the dimension.*

If μ *is replaced by the multi-canonical measure* v, *the same bounds hold, but with an extra factor* $B = B(M, \varepsilon', \beta, r, d, t_0, \|g\|_\infty, \rho_j)$ *in front of the exponential.*

Proof. In what follows k will always denote a generic numerical constant depending only on $\beta, M, \varepsilon', d, t_0, r, d, \|g\|_\infty, \rho_j$ and whose value may vary in different estimates.

We will use the following bound on the global logarithmic Sobolev constant (see Martinelli, 1999)

$$c_\mu \leqslant c\mathrm{e}^{(\log L)^{(d-1)/d}} \tag{31}$$

where $c = c(M, \beta) > 0$.

(i) We simply apply Proposition 23 to the function $G^{\rm ext}$. Indeed, since

$$\sum_{x\in\Lambda_j} \|\nabla_x G^{\rm ext}\|_\infty^2 = \sum_{x\in\Lambda_j\setminus Q^{\rm int}} \|\nabla_x G^{\rm ext}\|_\infty^2 \leqslant k|\Lambda_j|\, \frac{\log L}{l_0}.$$

Proposition 23 and (31) yield at once the result.

(ii) First observe that for any $\alpha \in I$, by point (a) of Proposition 9 together with, as $J \in \Theta_{\rm good}(L, M, \varepsilon')$, Assumption 1, as $J \in \Theta_{\rm good}(L, M, \varepsilon')$, we have

$$\sup_\alpha \|g_\alpha^\delta - \xi_\alpha^\delta\|_\infty \leqslant \sup_\alpha \sum_{x\in Q_\alpha^{\rm int}} \|v_{Q_\alpha}(g_x^\delta) - \mu_{Q_\alpha}(g_x^\delta)\|_\infty \leqslant \sup_\alpha k \sum_{x\in Q_\alpha^{\rm int}} \frac{|C_x|^3}{|Q_\alpha|} \leqslant k.$$

Thus, for any $\alpha \in I$

$$\mu_{\bar Q_\alpha}(g_\alpha^\delta - \xi_\alpha^\delta, g_\alpha^\delta - \xi_\alpha^\delta) \leqslant k. \tag{32}$$

Next, we recall that the measure μ factorizes over the collection of sets $\{\bar Q_\alpha\}_{\alpha\in I}$, thus we have, by a simple Taylor expansion,

$$\mu\left(\mathrm{e}^{t(G_v - G_\mu)}\right) = \prod_{\alpha\in I} \mu_{\bar Q_\alpha}(\mathrm{e}^{t(g_\alpha^\delta - \xi_\alpha^\delta)}) \leqslant \prod_{\alpha\in I} (1 + kt^2 \mu_{\bar Q_\alpha}(g_\alpha^\delta - \xi_\alpha^\delta, g_\alpha^\delta - \xi_\alpha^\delta)) \leqslant \mathrm{e}^{t^2 A|\Lambda_j|/l_0^d},$$

where we used (32) and $|I| \leqslant k|\Lambda_j|/l_0^d$ in the last inequality.

(iii) Notice that

$$\|\nabla_x(\xi^{\delta}_{\alpha}(\eta)-\xi^{\delta,0}_{\alpha}(\eta))\|_\infty \leqslant \begin{cases} k|C_x| & \text{if } x\in\partial^+_r Q_\alpha,\\ k l_0^{-1}\log L & \text{if } x\in Q_\alpha \end{cases}$$

because of Lemma 8 point (i) of (1) and (2) and Assumption 1. Thus, by point (i) of Proposition 23, Assumption 1 and (31) the result is obtained.

(iv) By the definition of δ_α (29) we have that $d\xi^{\delta_\alpha}_\alpha(s)/\mathrm{d}s|_{s=\bar n_\alpha}=0$ so that

$$|\xi^{\delta_\alpha}_\alpha(n_\alpha)-\xi^{\delta_\alpha}_\alpha(\bar n_\alpha)| = \left|\int_{\bar n_\alpha}^{n_\alpha}\mathrm{d}s\int_{\bar n_\alpha}^{s}\mathrm{d}t\,\frac{d^2}{\mathrm{d}t^2}\,\xi^{\delta_\alpha}_\alpha(t)\right| \leqslant k\,\frac{(n_\alpha-\bar n_\alpha)^2}{\bar n_\alpha},$$

where we used Lemma 8 in the last inequality. Then, by adding and subtracting $\xi^{\delta_\alpha}_\alpha(\bar n_\alpha)$, we have

$$\mu_{\bar Q_\alpha}(\xi^{\delta_\alpha}_\alpha(n_\alpha),\xi^{\delta_\alpha}_\alpha(n_\alpha)) \leqslant 4\mu_{\bar Q_\alpha}((\xi^{\delta_\alpha}_\alpha(n_\alpha)-\xi^{\delta_\alpha}_\alpha(\bar n_\alpha))^2) \leqslant \mu_{\bar Q_\alpha}\left(\frac{(n_\alpha-\bar n_\alpha)^4}{\bar n_\alpha^2}\right) \leqslant k, \quad (33)$$

where we used points (d) and (f) of Proposition 7 and, as $J\in\Theta_{\text{good}}(L,M,\varepsilon')$, Assumption 1. The measure μ factorizes over the collection of sets $\{\bar Q_\alpha\}_{\alpha\in I}$, so we have, using a simple Taylor expansion and that $t\leqslant t_0 l_0^{-d}$,

$$\begin{aligned}\mu(\mathrm{e}^{t\sum_{\alpha\in I}[\xi^{\delta_\alpha}_\alpha(n_\alpha)-\mu(\xi^{\delta_\alpha}_\alpha(n_\alpha))]}) &= \prod_{\alpha\in I}\mu_{\bar Q_\alpha}(\mathrm{e}^{t[\xi^{\delta_\alpha}_\alpha(n_\alpha)-\mu_{\bar Q_\alpha}(\xi^{\delta_\alpha}_\alpha(n_\alpha))]})\\ &\leqslant \prod_{\alpha\in I}(1+k\,t^2\mu_{\bar Q_\alpha}(\xi^{\delta_\alpha}_\alpha(n_\alpha),\xi^{\delta_\alpha}_\alpha(n_\alpha))) \leqslant \mathrm{e}^{t^2A|\Lambda_j|/l_0^d},\end{aligned}$$

where we used (33) and $|I|\leqslant k|\Lambda_j|/l_0^d$ to obtain the last inequality.

(v) Consider the collections of sets $\{\bar Q_\alpha\}_{\alpha\in I}$ and $\{\bar Q_{\alpha\beta}\}_{\beta\in I_i}$, as the measure μ factorizes over the clusters we have

$$\mu\left(\exp\left\{t\sum_{\alpha\in I,\beta\in I_i}D_\alpha F_{\alpha\beta}\right\}\right) = \prod_{\alpha\in I,\beta\in I_i}\mu_{\bar Q_{\alpha\beta}}(\mathrm{e}^{tD_\alpha F_{\alpha\beta}}). \quad (34)$$

By a Taylor expansion up to the first order and using the fact that for $t\in[0,1/Dl_1^d]$ there exists a positive constant k such that $\|tD_\alpha F_{\alpha\beta}\|_\infty\leqslant k$ and

$$\mu_{\bar Q_{\alpha\beta}}(\mathrm{e}^{tD_\alpha F_{\alpha\beta}}) \leqslant 1+kt^2D_\alpha^2\mu_{\bar Q_{\alpha\beta}}(n_{\alpha\beta},n_{\alpha\beta}) \leqslant 1+kt^2\sum_{x\in Q_{\alpha\beta}}|C_x\cap Q_{\alpha\beta}|,$$

where we used point (c) of Proposition 7 in the last inequality; thus the r.h.s. of (34), as $J\in\Theta_{\text{good}}(M,L,\varepsilon')$, is bounded by

$$\exp\left\{t^2\sum_{\alpha\in I,\beta\in I_i}D_\alpha^2\sum_{x\in Q_{\alpha\beta}}|C_x\cap Q_{\alpha\beta}|\right\} \leqslant \mathrm{e}^{t^2AD^2|\Lambda_j|}.$$

(vi) We omit the proof because it is similar to that one of point (v).

(vii) For simplicity for every $\beta,\beta'\in I_i$ we write $\mu_{\alpha\beta\beta'}(\cdot)=\mu_{\bar{Q}_\beta\cup\bar{Q}_{\beta'}}(\cdot)$. Fix $\beta\in I_i$ then,

$$\mu(\mathrm{e}^{t(F_\beta-\mu(F_\beta))})=\prod_{\alpha\in I}\prod_{\beta'\in I_i,\beta'\neq\beta}\mu_{\alpha\beta\beta'}(\mathrm{e}^{t(1/\bar{n}_\alpha)F_{\alpha\beta}F_{\alpha\beta'}}). \tag{35}$$

Now, as $\|1/\bar{n}_\alpha F_{\alpha\beta}F_{\alpha\beta'}\|_\infty\leqslant kl_1^{2d}/l_0^d$, a simple Taylor expansion with $t\leqslant t_0(l_0^d/l_1^{2d})$ and the fact that the measure μ produces over the clusters, gives

$$\begin{aligned}\mu_{\alpha\beta\beta'}(\mathrm{e}^{t(1/\bar{n}_\alpha)F_{\alpha\beta}F_{\alpha\beta'}})&\leqslant 1+kt^2\frac{1}{\bar{n}_\alpha^2}\mu_{\bar{Q}_{\alpha\beta}}(F_{\alpha\beta}^2)\mu_{\bar{Q}_{\alpha\beta'}}(F_{\alpha\beta'}^2)\\&\leqslant 1+kt^2\frac{1}{\bar{n}_\alpha^2}\sum_{\substack{x\in Q_{\alpha\beta}\\x'\in Q_{\alpha\beta'}}}|C_x\cap Q_{\alpha\beta}||C_{x'}\cap Q_{\alpha\beta'}|,\end{aligned}$$

where we used point (c) of Proposition 7 in the last inequality. Thus, as $J\in\Theta_{\text{good}}(M,L,\varepsilon')$ we can use Assumption 1 and obtain the result.

(viii) We have

$$\sum_{x\in\Lambda_j}\left\|\nabla_x\left(\Gamma_Q\sum_{y\in\Lambda_j\setminus Q}[\mu(\sigma(y))-\sigma(y)]\right)\right\|_\infty^2\leqslant|\Lambda_j||\Lambda_j\setminus Q|^2\nabla\Gamma^2+|\Lambda_j\setminus Q|\Gamma^2$$

so that by Proposition 23 and (31) the result follows.

In order to prove the analogous bound for the multi-canonical measure v, we observe that e.g. the function G_μ can be written as $G_\mu=G_\mu^1+G_\mu^2$, where G_μ^1 and G_μ^2 have the same expression of G_μ but with the sum over x restricted to two halves of the set Λ_j. Then

$$v(\mathrm{e}^{tG_\mu})\leqslant v(\mathrm{e}^{2tG_\mu^1})^{1/2}v(\mathrm{e}^{2tG_\mu^2})^{1/2}$$

and we can apply to each terms the bound (see Proposition 11)

$$v(\mathrm{e}^{2tG_\mu^i})\leqslant B\mu(\mathrm{e}^{2tG_\mu^i}),\quad i=1,2.$$

The final result follows at once from the bound on the grand canonical expectation. □

7. On the covariance of f^2 with sums of local functions

Fix $\varepsilon\in(0,1)$ and $l\in[2L^\varepsilon,L]$. We then consider a volume $\Lambda\in\mathscr{R}_L^\varepsilon(l)$ such that $\Lambda=\bigcup_{j=1}^{j_{\max}}\Lambda_j$, where $\Lambda_j\in\mathscr{R}_L^\varepsilon(l)$ and $|\Lambda_j|/|\Lambda|\geqslant\varepsilon$ for $j=1,\dots,j_{\max}$. Let $\mathbf{N}:=\{N_j\}_{j=1}^{j_{\max}}$ be a set of possible values of $\mathbf{N}_\Lambda:=\{N_{\Lambda_j}\}_{j=1}^{j_{\max}}$ and define $\rho_j=N_j/|\Lambda_j|$. We define $\mu:=\mu_\Lambda^{\tau,\lambda}$ and $v:=v_{\Lambda,\mathbf{N}}^\tau$.

In this section we discuss some important bounds on covariances of the form $v(f^2,G)^2$ where f an arbitrary function with $v(f^2)=1$ and $G=\sum_{x\in\Lambda_j}g_x$ for some $1\leqslant j\leqslant j_{\max}$, where $g_x=\bar{g}_x$ or $g_x=\tilde{g}_x$ where $\bar{g}_x$ and $\tilde{g}_x$ (defined later in 61) are local functions.

In the recursion approach (developed in Section 5), we need to bound $v(f^2,G)^2$ in terms of the only quantities that enter in the logarithmic Sobolev inequality, namely the entropy $\mathrm{Ent}_\nu(f^2)$ and the Dirichlet form $\mathscr{E}_\nu(f,f)$. But we want to keep track of the right dependence on the volume and in particular, $\mathrm{Ent}_\nu(f^2)$ must appear multiplied by a very small constant times the volume. More precisely, we expect a bound of the form

$$\nu(f^2,G)^2 \leqslant |\Lambda_j|(C_\zeta l^2 \mathscr{E}_\nu(f,f) + \zeta\,\mathrm{Ent}_\nu(f^2)), \tag{36}$$

where ζ is a small number. In order to appreciate the difficulty of the problem, we notice that, since f enters as f^2, one of the natural tool to bound covariances, namely Schwarz inequality, becomes useless since no L^p-norm of f, $p>2$, enters into the logarithmic Sobolev inequality. This is precisely one of the main technical difference and new challenge between the Poincaré inequality (where f appears linearly) and the logarithmic Sobolev inequality for conservative stochastic dynamics.

A natural counterpart to Schwarz inequality in this context is the so-called *entropy inequality* that can be stated as follows.

Lemma 25. *Let $(\Omega,\mathscr{F},\mu)$ be a probability space. Then, for any $t>0$ and any real-valued functions f,G on Ω with $\mu(f^2)=1$,*

$$\mu(f^2G) \leqslant \frac{1}{t}\ln(\mu(\mathrm{e}^{tG})) + \frac{1}{t}\mathrm{Ent}_\mu(f^2). \tag{37}$$

Proof. It is an immediate consequence of Jensen inequality. □

As $\nu(f^2,G)$ is a covariance, we can assume that G is of zero mean w.r.t. to μ and at the light of Section 6, it is natural to expect a Gaussian bound of the Laplace transform of the form $\nu(\mathrm{e}^{tG}) \leqslant \mathrm{e}^{t^2 C|\Lambda_j|}$ for some constant C. The entropy inequality (37) gives

$$\nu(f^2G) \leqslant tC|\Lambda_j| + \frac{1}{t}\,\mathrm{Ent}_\nu(f^2).$$

Optimizing over the free parameter t, namely $t^2 = 1/C|\Lambda_j|\,\mathrm{Ent}_\nu(f^2)$, we get

$$\nu(f^2G)^2 \leqslant 4C|\Lambda_j|\mathrm{Ent}_\nu(f^2). \tag{38}$$

Let us explain more precisely how we plan to use the results of Section 6 and the entropy inequality. First observe that Section 6 gives bounds on the Laplace transform of G under the assumption that $\mu(G)=0$. Moreover, the resulting estimates are distorted Gaussian bounds because of the presence of an extra constant B in front of the exponential. Thus, in reality, (38) is slightly more complicated. We have the following lemma.

Lemma 26. *Let f with $\nu(f^2)=1$ and K a constant. Then,*

(i) *If $t_*^2 = (1/|\Lambda_j|)(1 \vee K\,\mathrm{Ent}_\nu(f^2))$ and $\nu(\mathrm{e}^{t_* G}) \leqslant B\mathrm{e}^{t_*^2 C|\Lambda_j|}$ for some constants B and C,*

$$\nu(f^2G)^2 \leqslant k|\Lambda_j|\left[C^2 + 1 + \left(KC^2 + \frac{1}{K}\right)\mathrm{Ent}_\nu(f^2)\right].$$

(ii) *If* $t_*^2 = (K/|\Lambda_j|)(\mathrm{Ent}_\nu(f^2)) \geqslant 1/|\Lambda_j|$ *and* $\nu(\mathrm{e}^{t_* G}) \leqslant B\mathrm{e}^{t_*^2 C|\Lambda_j|}$ *for some constants* B *and* C,

$$\nu(f^2G)^2 \leqslant k|\Lambda_j| \left[(\log B)^2 + \left(KC^2 + \frac{1}{K}\right)\mathrm{Ent}_\nu(f^2)\right]$$

for a suitable constant k *depending only on* B (*and in particular independent of* L).

Proof. The proofs of (i) and (ii) are similar. We only deal with point (i). By the entropy inequality (Lemma 25) applied to $t = t_*$, we get

$$\begin{aligned}\nu(f^2G)^2 &\leqslant \left[\frac{1}{t_*}\log(\nu(\mathrm{e}^{t_*G})) + \frac{1}{t_*}\mathrm{Ent}_\nu(f^2)\right]^2\\ &\leqslant 3\frac{1}{t_*^2}(\log B)^2 + 3t_*^2C^2|\Lambda_j|^2 + 3\frac{1}{t_*^2}\mathrm{Ent}_\mu^2(f^2).\end{aligned}$$

Let us examine the three terms separately.

By definition of t_* we have $t_*^2 \geqslant 1/|\Lambda_j|$, thus $3(\log B)^2/t_*^2 \leqslant k|\Lambda_j|$.

Next, $3t_*^2C^2|\Lambda_j|^2 \leqslant k|\Lambda_j|[C^2 + KC^2\,\mathrm{Ent}_\mu(f^2)]$.

Finally, if $t_*^2 = 1/|\Lambda_j|$ then $K\,\mathrm{Ent}_\mu(f^2) \leqslant 1$, hence, in this case

$$\frac{1}{t_*^2}\mathrm{Ent}_\mu^2(f^2) = |\Lambda_j|\mathrm{Ent}_\mu^2(f^2) \leqslant |\Lambda_j|\frac{\mathrm{Ent}_\mu(f^2)}{K}.$$

The same result holds obviously if $t_*^2 = (1/|\Lambda_j|)K\,\mathrm{Ent}_\nu(f^2)$. Putting all together point (i) is proved. □

Let us continue our informal discussion. Suppose that $t_* \geqslant K$ where t_* and K are defined in Lemma 26. We certainly have $t_*^2 = (K/|\Lambda_j|)\mathrm{Ent}_\nu(f^2)$ for $|\Lambda_j|$ large enough, thus $|\Lambda_j| \leqslant (1/K)\mathrm{Ent}_\nu(f^2)$. Then, trivially,

$$\nu(f^2G)^2 \leqslant \|G\|_\infty^2 \leqslant \|g\|_\infty^2|\Lambda_j|^2 \leqslant \|g\|_\infty^2\frac{1}{K}|\Lambda_j|\mathrm{Ent}_\nu(f^2) \tag{39}$$

which is like (38) but with a smaller constant in front of $|\Lambda_j|\mathrm{Ent}_\nu(f^2)$ if K is large enough. In other words, if $t_* \geqslant K$ a simple L_∞ estimate gives a better result than the entropy bound.

In order to understand this point we remark that, on the basis of the central limit theorem and for "normal" values of the particle density in Λ_j, one expects the distribution $\mathrm{d}P(G)$ of the random variable $\sum_{x\in\Lambda_j} g_x$ to be close to a centered Gaussian with variance proportional to $|\Lambda_j|$. If this is the case, then, for t large enough, the distorted distribution $\mathrm{d}P_t(G) \propto \mathrm{e}^{tG}\,\mathrm{d}P(G)$ becomes concentrated on the largest value of G and the Gaussian bound $\nu(\mathrm{e}^{tG}) \leqslant \mathrm{e}^{t^2K|\Lambda_j|}$ becomes unnatural and certainly worst than the trivial one $\nu(\mathrm{e}^{tG}) \leqslant \mathrm{e}^{t\|G\|_\infty}$. On the contrary, for "moderate" values of t, the distortion only moves the center of the Gaussian and in this case the entropy inequality will perform better.

Thus, in what follows, our strategy will always be, roughly speaking, the following. Depending on the ratio between the entropy and the volume, we will either apply the entropy bound with the optimal t and appeal several times to the results of Section 6 and Lemma 26 or we will apply the trivial L_∞ bound. It remains to explain how we get in both cases a small constant in front of the entropy. For large values of the entropy it will follow quite easily from the results of Section 6. In the other cases we will have to appeal to a partial average argument, almost identical to the one used in Lu and Yau (1993), Cancrini and Martinelli (2000b) and Cancrini et al. (2002a) under the name of "two–blocks estimates", in order to reduce the fluctuations of the function G. It is at this point that appears the "spurious" ε in our estimate $L^{2+\varepsilon}$ of the logarithmic Sobolev constant.

Our proof follows essentially the same lines of the one given in Cancrini et al. (2002a) for the translation invariant interaction under a mixing condition, but with some important difference due to the presence of clusters where the particle variables are strongly interacting. In particular, the main difficulty and challenging point with respect to the proof given in Cancrini et al. (2002a) is the lack of translation invariance property.

We now explain more precisely our results.

7.1. Low-density case

Here we discuss our first result in the low-density regime.

Proposition 27. *Assume there exists a constant* $k = k(\beta, M, \varepsilon) > 0$ *such that*

$$\mu(|g_x|) \leqslant k\rho_j^2 \|g\|_\infty \quad \forall x \in \Lambda_j \text{ such that } \operatorname{dist}(x, \Lambda_j^c) \geqslant 2r,$$

$$\mu(|g_x|) \leqslant k\rho_j \|g\|_\infty \quad \forall x \in \Lambda_j \text{ such that } \operatorname{dist}(x, \Lambda_j^c) \leqslant 2r.$$

$$\|G\|_\infty \leqslant kN_j$$

Then, for any $\zeta > 0$, *there exists* C_ζ *and* ρ_0 *such that for all* $\rho_j < \rho_0$, *for all function* f *with* $\nu(f^2) = 1$,

$$\nu(f^2, G)^2 \leqslant N_j(C_\zeta + \zeta \operatorname{Ent}_\nu(f^2))$$

for L large enough.

Proof. For every β, there exists $\rho_0(\beta)$ such that the standard low activity expansion hold (see Simon, 1993) so that the proof is the same of that one of Proposition 6.2 of Cancrini et al. (2002a). □

7.2. Normal density

Here we treat instead the case of "normal" density ρ_j, namely we assume that $\rho_j \geqslant \rho_0$ for some constant ρ_0 independent of L. Let $\varepsilon' = \varepsilon/[d(3b+2)]$ where b is defined in Corollary 19.

Proposition 28. *Assume that* $J \in \Theta_{\rm good}(L, M, \varepsilon')$. *Then, for any* $\zeta > 0$, *there exists a positive constant* C_ζ *and* $L_0(M, \varepsilon', \zeta)$ *such that for all* $\rho_0 \leqslant \rho_j \leqslant \frac{1}{2}$,

$$\nu(f^2, G)^2 \leqslant |\Lambda_j|[C_\zeta + C_\zeta l^2 L^\varepsilon \mathscr{E}_\nu(f, f) + \zeta\,\mathrm{Ent}_\nu(f^2)],$$

for all $L \geqslant L_0$.

Remark 29. We explain briefly here why in the estimate of the inverse of the spectral gap (ISG) in Cancrini and Martinelli (2001) the factor L^ε does not appear. Studying the ISG one has to estimate $\nu(f, G)^2$ in terms of the Dirichlet form and the variance instead of $\nu(f^2, G)^2$ in terms of the Dirichlet form and the entropy. As explained at the beginning of this section since f enters as f^2, a natural tool to bound covariances, namely Schwarz inequality, becomes useless since no L^p-norm of f, $p > 2$, enters into the logarithmic Sobolev inequality and a natural counterpart is the entropy inequality (37). As a consequence, already in the standard Ising model with a high temperature mixing condition, a factor L^2 appears in front of the Dirichlet form (compare Cancrini and Martinelli (2000b, Proposition A.1) to Cancrini et al. (2002a, Proposition 6.3) or see Cancrini et al. (2002b)). Furthermore, roughly speaking, when the dilute Ising model under the percolation threshold is considered, the fact that the homogeneity properties hold only from scale L to scale L^ε produces in any case a factor L^ε in front of the Dirichlet form. This does not affect the diffusive estimate of the ISG.

Proof. Fix f with $\nu(f^2) = 1$, together with $\zeta > 0$, $\rho_0 > 0$ and K large enough (how large will be specified later on).

As usual, in what follows, k will always denote a generic positive numerical constant depending only on $M, \varepsilon', \beta, r, d, \|g\|_\infty, \rho_0$ and whose value may vary in different estimates.

Define

$$t_*^2 := \frac{1}{|\Lambda_j|}(1 \vee K\,\mathrm{Ent}_\nu(f^2)). \tag{40}$$

If $t_* \geqslant K$, then, by definition of G and t_*, according to the general discussion above, we can safely apply an L_∞ bound to get

$$\nu(f^2, G)^2 \leqslant 2\|G\|_\infty^2 \leqslant k|\Lambda_j|^2 \leqslant k|\Lambda_j| \left(\frac{\mathrm{Ent}_\nu(f^2)}{K} \vee 1 \right).$$

And the proof is finished provided that K was taken large enough. □

Let us now examine the much more complicate case of $t_* \leqslant K$.

As in Section 6, let $\{Q_\alpha\}_{\alpha \in I}$ be a collection of cubes of side $l_0 = L^{\varepsilon'}$ inside Λ_j, such that for any $\alpha \neq \alpha'$, $d(Q_\alpha, Q_{\alpha'}) \geqslant 3M \log L$, for any $\alpha \in I$, $d(Q_\alpha, \partial\Lambda_j) \geqslant 3M \log L$, $Q := \bigcup_\alpha Q_\alpha$ and $|\Lambda_j \setminus Q| \leqslant (\log L / l_0)|\Lambda|$. Clearly such a construction is always possible. We define also $Q_\alpha^{\rm int} = \{x \in Q_\alpha : d(x, Q_\alpha^c) \geqslant 3M \log L\}$ and $Q^{\rm int} = \bigcup_\alpha Q_\alpha^{\rm int}$.

Next we observe that, without loss of generality, we can replace g_x by $g_x - \delta\sigma(x)$, δ being an arbitrary constant independent of x, because $\sum_{x \in \Lambda_j} \sigma(x) = N_j$ ν-almost surely. Our choice of δ will be made later.

Finally we set $g_x^\delta(\sigma) := g_x(\sigma) - \delta\sigma(x)$ and

$$G^{\text{ext}} = \sum_{x\in\Lambda_j\setminus Q^{\text{int}}} [g_x^\delta - \mu(g_x^\delta)],$$

$$G^{\text{int}} = G - G^{\text{ext}} = \sum_{x\in Q^{\text{int}}} [g_x^\delta - \mu(g_x^\delta)].$$

Then we write

$$\begin{aligned} v(f^2, G^{\text{ext}})^2 &\leqslant 2v(f^2 G^{\text{ext}})^2 + 2v(G^{\text{ext}})^2 \leqslant k|\Lambda_j| \left[\left(\frac{c(L)\log L}{l_0}\right)^2 + \frac{(\log L)^2}{l_0} + 1 \right. \\ &\quad \left. + \left[K\left(\frac{c(L)\log L}{l_0}\right)^2 + \frac{1}{K}\right] \text{Ent}_v(f^2) \right] \\ &\leqslant |\Lambda_j|(C_\zeta + \zeta\,\text{Ent}_v(f^2)). \end{aligned}$$

Here we have applied Lemma 26 together with point (i) of Proposition 24 to bound the first term and point (a) of Proposition 9, Assumption 1 and $|\Lambda_j\setminus Q| \leqslant (\log L/l_0)|\Lambda|$ to bound the second one. The last inequality holds if we choose first K and then L sufficiently large.

We now turn to the relevant term $v(f^2, G^{\text{int}})^2$.

Let $\mathscr{F}_0$ be the σ-algebra generated by the random variables $\{\sigma(x)\}_{x\in\Lambda\setminus\bigcup_\alpha Q_\alpha}, \{N_\alpha\}_{\alpha\in I}$, where $N_\alpha(\sigma) := \sum_{x\in Q_\alpha}\sigma(x)$. Then, by the formula for the conditional covariance, we get

$$v(f^2, G_\delta^{\text{int}})^2 \leqslant 2v(v(f^2, G^{\text{int}}|\mathscr{F}_0))^2 + 2v(f^2, v(G^{\text{int}}|\mathscr{F}_0))^2. \tag{41}$$

For simplicity let $v_0(\cdot) := v(\cdot|\mathscr{F}_0)$, $f_0^2 := f^2/v_0(f^2)$. Notice that v_0 is the product of the standard canonical measures on each cube Q_α with a certain number of particles and boundary conditions. To bound the first term in the r.h.s. of (41) we use the entropy inequality (37)

$$v_0(f_0^2, G^{\text{int}}) \leqslant \frac{1}{t}\ln\left(v_0(\mathrm{e}^{t(G^{\text{int}} - v_0(G^{\text{int}}))})\right) + \frac{1}{t}\text{Ent}_{v_0}(f_0^2). \tag{42}$$

To estimate the argument of the logarithm we need some result. Using the classical tensorization property of the entropy for a product measure (see for instance, Ané et al., 2000, Chapter 1) we have

$$\text{Ent}_{v_0}(\mathrm{e}^{t(G^{\text{int}} - v_0(G^{\text{int}}))}) \leqslant \sum_{\alpha\in I} v_0(\mathrm{e}^{t\sum_{\beta\neq\alpha}(G_\beta^{\text{int}} - v_\beta(G_\beta^{\text{int}}))}\,\text{Ent}_{v_\alpha}(\mathrm{e}^{t(G_\alpha^{\text{int}} - v_\alpha(G_\alpha^{\text{int}}))})),$$

where for any $\alpha\in I$, $G_\alpha^{\text{int}} := \sum_{x\in Q_\alpha^{\text{int}}}[g_x^\delta - \mu(g_x^\delta)]$.

Now, for each cube, ν_α satisfies a logarithmic Sobolev inequality. More precisely by Corollary 19 we have

$$\begin{aligned}\mathrm{Ent}_{\nu_\alpha}(\mathrm{e}^{t(G_\alpha^{\mathrm{int}}-\nu_\alpha(G_\alpha^{\mathrm{int}}))}) &\leqslant kl_0^{db}\,\mathscr{E}_{\nu_\alpha}(\mathrm{e}^{t/2}(G_\alpha^{\mathrm{int}}-\nu_\alpha(G_\alpha^{\mathrm{int}})),\mathrm{e}^{t/2}(G_\alpha^{\mathrm{int}}-\nu_\alpha(G_\alpha^{\mathrm{int}})))\\ &= k\,l_0^{db}\sum_{[x,y]\in Q_a}\nu_\alpha([\nabla_{xy}\mathrm{e}^{t/2}(G_\alpha^{\mathrm{int}}-\nu_\alpha(G_\alpha^{\mathrm{int}}))]^2)\\ &\leqslant kl_0^{d(b+2)}t^2\nu_\alpha(\mathrm{e}^{t(G_\alpha^{\mathrm{int}}-\nu_\alpha(G_\alpha^{\mathrm{int}}))})\end{aligned}$$

so that

$$\mathrm{Ent}_{\nu_0}(\mathrm{e}^{t(G^{\mathrm{int}}-\nu_0(G^{\mathrm{int}}))}) \leqslant kl_0^{d(b+2)}\frac{|\Lambda_j|}{l_0^d}t^2\,\nu_0(\mathrm{e}^{t(G^{\mathrm{int}}-\nu_0(G^{\mathrm{int}}))}). \tag{43}$$

We can now use the following lemma.

Lemma 30. *Let $(\Omega,\mathscr{F},\mu)$ be a finite probability space and f a function on Ω. Assume that there exists $K>0$ such that for all $t\in[0,t_0]$,*

$$\mathrm{Ent}_\mu(\mathrm{e}^{tf})\leqslant Kt^2\mu(\mathrm{e}^{tf}).$$

Then, for all $t\in[0,t_0]$,

$$\mu(\mathrm{e}^{tf})\leqslant \mathrm{e}^{t\mu(f)+Kt^2}.$$

This property is known as the Herbst argument. See Cancrini et al. (2002a, Lemma 5.2) for the proof (see also Ané et al., 2000, Chapter 7).

By this result applied to (43), we have that the argument of the logarithm in (42) can be bounded by

$$\nu_0(\mathrm{e}^{t(G^{\mathrm{int}}-\nu_0(G^{\mathrm{int}}))})\leqslant \mathrm{e}^{t^2kl_0^{d(b+1)}|\Lambda_j|}.$$

We now choose $t=s_*$ where $s_*^2:=K/|\Lambda_j|\mathrm{Ent}_{\nu_0}(f_0^2)$ and obtain

$$\nu_0(f_0^2,G^{\mathrm{int}})^2\leqslant k|\Lambda_j|l_0^{2d(b+1)}\left(K+\frac{1}{K}\right)\mathrm{Ent}_{\nu_0}(f_0^2).$$

Using once more Corollary 19, we get that for any α, ν_α satisfies the logarithmic Sobolev inequality $\mathrm{Ent}_{\nu_\alpha}(f_0^2)\leqslant kl_0^{db}\mathscr{E}_{\nu_\alpha}(f_0,f_0)$. As ν_0 is a product measure, we get $\mathrm{Ent}_{\nu_0}(f_0^2)\leqslant kl_0^{db}\mathscr{E}_{\nu_0}(f_0,f_0)$ (see Ané et al., 2000, Chapter 3), and we can conclude that

$$\nu(\nu(f^2,G^{\mathrm{int}}|\mathscr{F}_0))^2\leqslant kl_0^{d(3b+2)}|\Lambda_j|\mathscr{E}_\nu(f,f).$$

Finally, by our choice of ε', we certainly have $l_0^{d(3b+2)}\leqslant L^\varepsilon$. This complete the proof for the term $\nu(\nu(f^2,G^{\mathrm{int}}|\mathscr{F}_0))^2$.

The second term in the r.h.s. of (41) needs some more reductions. We recall first some definitions introduced in Section 4.

Given $\eta \in \Omega_{\Lambda_j}$ and $s \in [0,\dots,l_0^d]$, write $\mu_{Q_\alpha}^{\eta,\lambda(\eta,s)}(\cdot)$ to denote the grand canonical Gibbs measure on Q_α with boundary condition η and constant chemical potential $\lambda(\eta,s)$ such that $\mu_{Q_\alpha}^{\eta,\lambda(\eta,s)}(N_{Q_\alpha}) = s$. We will use the standard notation $v_{Q_\alpha,s}^{\eta}$ for the corresponding canonical Gibbs measure. With this notation we define (see (28)),

$$\xi_\alpha^\delta(\eta,n) = \sum_{x\in Q_\alpha^{\rm int}} \mu_{Q_\alpha}^{\lambda(\eta,n)}(g_x - \delta\sigma(x)); \quad \xi_\alpha^\delta(\eta) := \xi_\alpha^\delta(\eta, N_{Q_\alpha}(\eta)),$$

$$g_\alpha^\delta(\eta,n) = \sum_{x\in Q_\alpha^{\rm int}} v_{Q_\alpha,n}^{\eta}(g_x - \delta\sigma(x)); \quad g_\alpha^\delta(\eta) := g_\alpha^\delta(\eta, N_{Q_\alpha}(\eta)).$$

Next we define

$$G_v(\eta) := \sum_\alpha [g_\alpha^\delta(\eta) - \mu(g_\alpha^\delta)],$$

$$G_\mu(\eta) := \sum_\alpha [\xi_\alpha^\delta(\eta) - \mu(\xi_\alpha^\delta)].$$

Then we write

$$\begin{aligned} v(f^2, v(G^{\rm int}|\mathscr{F}_0))^2 &= v(f^2, G_v)^2 \\ &\leqslant 4v(f^2(G_v - G_\mu))^2 + 4(v(G_v - G_\mu))^2 + 2v(f^2, G_\mu)^2. \end{aligned} \quad (44)$$

Let us examine the three terms separately. Using Lemma 26 combined with point (ii) of Proposition 24, we can bound the first term by

$$4v(f^2(G_v - G_\mu))^2 \leqslant k|\Lambda_j| \left[\frac{A^2}{l_0^{2d}} + 1 + \left(K\frac{A^2}{l_0^{2d}} + \frac{1}{K} \right) {\rm Ent}_v(f^2) \right], \quad (45)$$

where A is the constant appearing in Proposition 24.

Because of point (a) of Proposition 9, as $J \in \Theta_{\rm good}(L,M,\varepsilon')$ Assumption 1 and the above definitions we have

$$\begin{aligned} |v(G_v - G_\mu)| &\leqslant \sum_{\alpha\in I} |v(g_\alpha^\delta) - \mu(g_\alpha^\delta)| + |v(\xi_\alpha^\delta) - \mu(\xi_\alpha^\delta)| \\ &\leqslant k \sum_{\alpha\in I} l_0^d \frac{\sum_{x\in\bar{Q}_\alpha} |C_x \cap \bar{Q}_\alpha|^3}{|\Lambda_j|} \leqslant k l_0^d. \end{aligned} \quad (46)$$

Therefore, $(v(G_v - G_\mu))^2 \leqslant k|\Lambda_j|$. In conclusion, by a suitable choice of K and L large enough we get

$$4v(f^2(G_v - G_\mu))^2 + 4(v(G_v - G_\mu))^2 \leqslant |\Lambda_j|(C_\zeta + \zeta\, {\rm Ent}_v f^2) \quad (47)$$

for a suitable constant C_ζ.

So, it remains to bound the last term in the r.h.s. of (44). We define

$$\xi_\alpha^\delta(s) := \xi_\alpha^\delta(0,s) \quad \text{and} \quad G_\mu^0(\eta) := \sum_{\alpha\in I} \xi_\alpha^\delta(N_{Q_\alpha}(\eta)) - \mu(\xi_\alpha^\delta).$$

We can thus write

$$\nu(f^2, G_\mu)^2 \leqslant 2\nu(f^2, G_\mu - G_\mu^0)^2 + 2\nu(f^2, G_\mu^0)^2. \tag{48}$$

By point (i) of Lemma 26 and point (iii) of Proposition 24 together with $|\nu(G_\mu - G_\mu^0)| \leqslant k l_0^d$ (see (46)), for the first term in the r.h.s. of (48) we have

$$\begin{aligned}\nu(f^2, G_\mu - G_\mu^0)^2 &\leqslant 2\nu(f^2(G_\mu - G_\mu^0))^2 + 2\nu(G_\mu - G_\mu^0)^2 \\ &\leqslant k|\Lambda_j|\left[\frac{(\log L)^4\, c(L)}{l_0^2} + 1 + \left(K\frac{(\log L)^4\, c(L)}{l_0^2} + \frac{1}{K}\right)\mathrm{Ent}_\nu(f^2)\right] \\ &\leqslant |\Lambda_j|(C_\zeta + \zeta\,\mathrm{Ent}_\nu\, f^2)\end{aligned}$$

taking first K and then L large enough.

We are now left with the bound of the second term in the r.h.s. of (48). It is at this point that the subtraction with the free parameter δ made at the beginning becomes important. We define

$$\delta := \frac{\mathbb{E}[|C_0|^{-1}\mu_{C_0}^{\lambda_0}(\sum_{x\in C_0} g_x, N_{C_0})]}{\mathbb{E}[C_0|^{-1}\mu_{C_0}^{\lambda_0}(N_{C_0}, N_{C_0})]} \quad \text{and} \quad \delta_\alpha := \frac{\mu_{Q_\alpha}^{0,\lambda(0,\bar{n}_\alpha)}(\sum_{x\in Q_\alpha^{\mathrm{int}}} g_x, N_{Q_\alpha})}{\mu_{Q_\alpha}^{0,\lambda(0,\bar{n}_\alpha)}(N_{Q_\alpha^{\mathrm{int}}}, N_{Q_\alpha})}, \tag{49}$$

where $\lambda_0 = \lambda_0(\Lambda_j, N_j)$ is the chemical potential such that $\mathbb{E}(|C_0|^{-1}\mu_{C_0}^{\eta,\lambda_0}(N_{C_0})) = N_j/|\Lambda_j|$ and C_0 is a fixed cluster of Λ_j, and $\bar{n}_\alpha := \mu(N_{Q_\alpha})$. By definition and the fact that $\mathbb{E}(|C_0|^n) \leqslant k$ for $p < p_c$ (see e.g. Grimmett, 1999), we have $\delta \leqslant k$ (uniformly in L). We define $\xi_\alpha^{\delta_\alpha}$ by replacing δ with δ_α, then by the above definitions we have

$$\xi_\alpha^\delta(n_\alpha) - \xi_\alpha^\delta(\bar{n}_\alpha) = (\delta - \delta_\alpha)(n_\alpha - \bar{n}_\alpha) + \xi_\alpha^{\delta_\alpha}(n_\alpha) - \xi_\alpha^{\delta_\alpha}(\bar{n}_\alpha)$$

so that the second term in the r.h.s of (48) can be bounded by

$$\begin{aligned}&2\,\nu\left(f^2, \sum_{x\in I}(\delta - \delta_\alpha)(n_\alpha - \bar{n}_\alpha)\right)^2 + 2\nu\left(f^2, \sum_{\alpha\in I}[\xi_\alpha^{\delta_\alpha}(n_\alpha) - \xi_\alpha^{\delta_\alpha}(\bar{n}_\alpha)]\right)^2 \\ &\quad\leqslant 4\nu\left(f^2\sum_{\alpha\in I}(\delta - \delta_\alpha)(n_\alpha - \bar{n}_\alpha)\right)^2 + 4\nu\left(\sum_{\alpha\in I}(\delta - \delta_\alpha)(n_\alpha - \bar{n}_\alpha)\right)^2 \\ &\qquad + 2\nu\left(f^2, \sum_{\alpha\in I}[\xi_\alpha^{\delta_\alpha}(n_\alpha) - \xi_\alpha^{\delta_\alpha}(\bar{n}_\alpha)]\right)^2.\end{aligned} \tag{50}$$

Using the fact that $J \in \Theta_{\mathrm{good}}(M, L, \varepsilon')$ it is easy to prove that

$$|\delta - \delta_\alpha| \leqslant \frac{k}{l_0^{d\varepsilon'}}. \tag{51}$$

For any $\alpha \in I$, let $\{Q_{\alpha\beta}\}_{\beta\in I_\alpha}$ be a chessboard-like partition of Q_α into cubes of side $l_1 = l_0^{\varepsilon'}$. It is easy to see that we can partition the indexes I_α into $2d$ subsets $\{I_{\alpha,i}\}_{i=1}^{2d}$ in such a way that $\min_{\beta\neq\beta'\in I_{\alpha,i}} d(Q_{\alpha\beta}, Q_{\alpha\beta'}) \geqslant l_1$. Then, for each α, we arrange in regular order the collection of cubes $\{Q_{\alpha\beta}\}_\beta$ with respect to β in such a way that we can write $\sum_\alpha \sum_\beta = \sum_\beta \sum_\alpha$. We define $n_{\alpha\beta} := N_{Q_{\alpha\beta}}$ and $\bar{n}_{\alpha\beta} := \mu(N_{Q_{\alpha\beta}})$, $D_\alpha := \delta - \delta_a$ and $F_{\alpha\beta} := n_{\alpha\beta} - \bar{n}_{\alpha\beta}$. The first term in the r.h.s. of (50) can be written as

$$\sum_{i=1}^{2d} \nu\left(f^2 \sum_{\substack{\alpha\in I\\ \beta\in I_i}} D_\alpha F_{\alpha\beta} \right) \leqslant k|\Lambda_j| \left[\frac{A^2}{l_1^{4d}} + 1 + \left(\frac{KA^2}{l_1^{4d}} + \frac{1}{K} \right) \mathrm{Ent}_\nu(f^2) \right]$$
$$\leqslant |\Lambda_j|(C_\zeta + \zeta\, \mathrm{Ent}_\nu f^2), \tag{52}$$

where we used point (v) of Proposition 24 and the fact that by (51) and the above definition of l_1 we have $D := \sup_{\alpha\in I} D_\alpha \leqslant k l_1^{-d}$ to apply Lemma 26 and obtain the first inequality and we took first K and then L large enough to obtain the second one.

Using point (a) of Proposition 9 the second term in (50) is bounded by

$$kD^2 \left(\sum_{\alpha\in I} \frac{\sum_{x\in Q_\alpha} |C_x|}{|\Lambda_j|} \right) \leqslant \frac{k}{l_1^{2d}}.$$

To bound the third term in (50) we are (unfortunately) forced to distinguish between two sub-cases.

(a) $t_* \leqslant K/l_0^d$.
We have

$$\nu\left(f^2, \sum_{\alpha\in I} [\xi_\alpha^{\delta_\alpha}(n_\alpha) - \xi_\alpha^{\delta_\alpha}(\bar{n}_\alpha)] \right)^2$$
$$\leqslant 2\nu\left(f^2 \sum_{\alpha\in I} [\xi_\alpha^{\delta_\alpha}(n_\alpha) - \mu(\xi_\alpha^{\delta_\alpha}(n_\alpha))] \right)^2 + 2\left[\sum_{\alpha\in I} \left(\nu(\xi_\alpha^{\delta_\alpha}(n_\alpha)) - \mu(\xi_\alpha^{\delta_\alpha}(n_\alpha)) \right) \right]^2$$
$$\leqslant 2\nu\left(f^2 \sum_{\alpha\in I} [\xi_\alpha^{\delta_\alpha}(n_\alpha) - \mu(\xi_\alpha^{\delta_\alpha}(n_\alpha))] \right)^2 + k l_0^{2d}, \tag{53}$$

where we used point (a) of Proposition 9 to obtain the second inequality. By point (iv) of Proposition 24 we can apply Lemma 26 and obtain that (53) can be bounded by

$$k|\Lambda_j| \left[\frac{A^2}{l_0^{2d}} + 1 + \left(\frac{KA^2}{l_0^{2d}} + \frac{1}{K} \right) \mathrm{Ent}_\nu(f^2) \right] + k l_0^{2d} \leqslant |\Lambda_j|(C_\zeta + \zeta\, \mathrm{Ent}_\nu(f^2))$$

taking K and then L large enough.

(b) $K/l_0^d \leqslant t_* \leqslant K$.

In this case we can assume, without loss of generality, that $|\Lambda_j|$ is large enough so that $t_*^2 = (1/|\Lambda_j|) K \,\mathrm{Ent}_\nu(f^2)$. By the definition of δ_α (49) we have that $\mathrm{d}\xi_\alpha^{\delta_\alpha,0}/\mathrm{d}s(s)|_{s=\bar n_\alpha} = 0$, then

$$|\xi_\alpha^{\delta_\alpha}(n_\alpha) - \xi_\alpha^{\delta_\alpha}(\bar n_\alpha)| = \left| \int_{\bar n_\alpha}^{n_\alpha} \mathrm{d}s \int_{\bar n_\alpha}^{s} \mathrm{d}t \, \frac{\mathrm{d}^2}{\mathrm{d}t^2} \xi_\alpha^{\delta_\alpha}(t) \right| \leqslant k \, \frac{(n_\alpha - \bar n_\alpha)^2}{\bar n_\alpha},$$

where we used Proposition 8 to obtain the last inequality. Here $n_\alpha(\eta) := N_{Q_\alpha}(\eta)$ is the number of particles inside Q_α. We thus have

$$\begin{aligned} &\nu\left(f^2, \sum_{\alpha \in I} [\xi_\alpha^{\delta_\alpha}(n_\alpha) - \xi_\alpha^{\delta_\alpha}(\bar n_\alpha)] \right)^2 \\ &\quad \leqslant 2\nu\left(f^2 \sum_{\alpha \in I} \left(\frac{(n_\alpha - \bar n_\alpha)^2}{\bar n_\alpha} \right)^2 \right)^2 + 2\left[\sum_{\alpha \in I} \nu\left(\frac{(n_\alpha - \bar n_\alpha)^2}{\bar n_\alpha} \right) \right]^2 \\ &\quad \leqslant 2\nu\left(f^2 \sum_{\alpha \in I} \frac{(n_\alpha - \bar n_\alpha)^2}{\bar n_\alpha} \right)^2 + k\left(\frac{|\Lambda_j|}{l_0^d} \right)^2, \end{aligned} \tag{54}$$

where we used point (a) of Proposition 9 and point (c) of Proposition 7 to obtain the last inequality. Then, as $t_*^2 \geqslant K/l_0^d$, it follows that $|\Lambda_j|/l_0^{2d} \leqslant \mathrm{Ent}_\nu f^2/K$. We are thus left with the first term in r.h.s. of (54).

Lemma 31. *Assume that $J \in \Theta_{\mathrm{good}}(L, M, \varepsilon')$. Then, for any $\zeta > 0$, there exists a positive constant C_ζ and $L_0(M, \varepsilon', \zeta)$ such that for all $\rho_0 \leqslant \rho_j \leqslant \frac{1}{2}$,*

$$\nu\left(f^2 \sum_{\alpha \in I} \frac{(n_\alpha - \bar n_\alpha)^2}{\bar n_\alpha} \right)^2 \leqslant |\Lambda_j| [C_\zeta + C_\zeta l^2 L^\varepsilon \mathscr{E}_\nu(f,f) + \zeta \,\mathrm{Ent}_\nu(f^2)].$$

Proof. The main difficulties here with respect to the proof given in Cancrini et al. (2002a) are the lack of invariance by translation and the presence of clusters of volume $\log L$, that is, dependent of L. Indeed, we cannot reduce the problem until cubes of side l_0 independent of L simply because our Assumptions 1–3 (see Section 4.2) should not hold in this case. Hence, for technical reasons, we must take care very carefully to all the resulting terms that appear in our reduction. We are forced once more to distinguish between two cases.

(c) $K/l_0^d \leqslant t_* \leqslant K/l_1^d$.

For any $\alpha \in I$ we introduce a chessboard-like partition of Q_α into cubes of side $l_1 = l_0^{\varepsilon'}$ as we did to obtain (52).

$$\begin{aligned} &\nu\left(f^2 \sum_{\alpha} \frac{(n_\alpha - \bar n_\alpha)^2}{\bar n_\alpha} \right)^2 = \nu\left(f^2 \sum_{\alpha \in I} \frac{1}{\bar n_\alpha} \left[\sum_{i=1}^{2d} \sum_{\beta \in I_i} F_{\alpha\beta} \right]^2 \right)^2 \\ &\quad \leqslant 2d \sum_{i=1}^{2d} \left\{ \nu\left(f^2 \sum_{\alpha \in I} \frac{1}{\bar n_\alpha} \left[\sum_{\beta \in I_i} F_{\alpha\beta} \right]^2 - \mu\left(\left[\sum_{\beta \in I_i} F_{\alpha\beta} \right]^2 \right) \right) \right)^2 \end{aligned}$$

$$+\left[\sum_{\alpha\in I}\frac{1}{\bar n_\alpha}\,\mu\left(\left[\sum_{\beta\in I_i}F_{\alpha\beta}\right]^2\right)\right]^2\Bigg\}$$

$$\leqslant 2d\sum_{i=1}^{2d}v\left(f^2\sum_{\alpha\in I}\frac{1}{\bar n_\alpha}\left[\sum_{\beta\in I_i}F_{\alpha\beta}\right]^2-\mu\left(\left[\sum_{\beta\in I_i}F_{\alpha\beta}\right]^2\right)\right)^2+k\left(\frac{|\Lambda_j|}{l_0^d}\right)^2, \quad (55)$$

where we used point (a) of Proposition 7, as $J\in\Theta_{\rm good}(L,M\varepsilon')$ Assumption 1 to obtain the last inequality. Again as $t_*^2\geqslant K/l_0^d$, it follows that $|\Lambda_j|/l_0^{2d}\leqslant \mathrm{Ent}_v f^2/K$. We are thus left with the first term in r.h.s. of (55). We have, for a fixed i,

$$v\left(f^2\sum_{\alpha\in I}\frac{1}{\bar n_\alpha}\left[\sum_{\beta\in I_i}F_{\alpha\beta}\right]^2-\mu\left(\left[\sum_{\beta\in I_i}F_{\alpha\beta}\right]^2\right)\right)^2$$

$$\leqslant 2v\left(f^2\sum_{\alpha\in I}\frac{1}{\bar n_\alpha}\sum_{\beta\in I_i}(F_{\alpha\beta}^2-\mu(F_{\alpha\beta}^2))\right)^2$$

$$+2v\left(f^2\sum_{\alpha\in I}\frac{1}{\bar n_\alpha}\sum_{\substack{\beta,\beta'\in I_i\\ \beta\neq\beta'}}(F_{\alpha\beta}F_{\alpha\beta'}-\mu(F_{\alpha\beta}F_{\alpha\beta'}))\right)^2. \quad (56)$$

By point (v) and (vi) of Proposition 24 and we can apply Lemma 26 and the first term on the r.h.s of (56) can be bounded by

$$k|\Lambda_j|\left[(\log B)^2+\left(\frac{KA^2l_1^{2d}}{l_0^{2d}}+\frac{1}{K}\right)\mathrm{Ent}_v(f^2)\right]\leqslant|\Lambda_j|[C_\zeta+\zeta\,\mathrm{Ent}_v(f^2)],$$

where we took K and then L large enough in the last inequality.

Now, we recall that for each α we arranged in regular order the collection $\{Q_{\alpha\beta}\}_\beta$ with respect to β in such a way that we can write

$$v\left(f^2\sum_{\alpha\in I}\frac{1}{\bar n_\alpha}\sum_{\substack{\beta,\beta'\in I_i\\ \beta\neq\beta'}}(F_{\alpha\beta}F_{\alpha\beta'}-\mu(F_{\alpha\beta}F_{\alpha\beta'}))\right)^2$$

$$=\left[\sum_{\beta\in I_i}v\left(f^2\sum_{\alpha}\frac{1}{\bar n_\alpha}\sum_{\substack{\beta'\in I_i\\ \beta'\neq\beta}}(F_{\alpha\beta}F_{\alpha\beta'}-\mu(F_{\alpha\beta}F_{\alpha\beta'}))\right)\right]^2$$

$$\leqslant \frac{1}{s^2}\left[\sum_{\beta\in I_i}\mathrm{Ent}_\nu(f^2)+\log\prod_{\beta\in I_i}\nu\left(\exp\left\{s\sum_{\alpha\in I}\sum_{\substack{\beta'\in I_i\\ \beta'\neq\beta}}(F_{\alpha\beta}F_{\alpha\beta'}-\mu(F_{\alpha\beta}F_{\alpha\beta'}))\right\}\right)\right]^2$$

$$\leqslant\left[\frac{1}{t_*}(\log B+\mathrm{Ent}_\nu(f^2))+t_*\frac{A|\Lambda_j|}{l_1^d}\right]^2\leqslant|\Lambda_j|\,\left[C_\zeta+\zeta\,\mathrm{Ent}_\nu(f^2)\right],$$

where we used the entropy inequality (37) to obtain the first inequality, we chose $s=t_*l_0^d/l_1^d\leqslant Kl_0^d/l_1^{2d}$ and used point (vii) of Proposition 24 to obtain the second one, and we used that $t_*^2=(K/|\Lambda_j|)\mathrm{Ent}_\nu(f^2)$ together with $t_*\geqslant K/l_0^d$ and took K and then L large enough to obtain the last one.

(d) $K/l_1^d\leqslant t_*\leqslant K$.

By point (c) of Proposition 7 together with Assumption 1 and $t_*^2=(K/|\Lambda_j|)\mathrm{Ent}_\nu(f^2)\geqslant K/l_0^d$ we have

$$\sum_{\alpha\in I}\mu\left(\frac{(n_\alpha-\bar n_\alpha)^2}{\bar n_\alpha}\right)\leqslant k\left(\frac{|\Lambda_j|}{l_0^d}\right)^2\leqslant\frac{|\Lambda_j|}{K}\,\mathrm{Ent}_\nu(f^2)$$

and

$$\nu\left(f^2\sum_{\alpha\in I}\frac{(n_\alpha-\bar n_\alpha)^2}{\bar n_\alpha}\right)^2\leqslant 2\,\nu\left(f^2\sum_{\alpha\in I}\left[\frac{(n_\alpha-\bar n_\alpha)^2}{\bar n_\alpha}-\mu\left(\frac{(n_\alpha-\bar n_\alpha)^2}{\bar n_\alpha}\right)\right]\right)^2$$
$$+k\,\frac{|\Lambda_j|}{K}\,\mathrm{Ent}_\nu(f^2). \tag{57}$$

To bound the first term in the r.h.s. of (57) we need some reductions. Let $\langle\cdot\rangle$ denote the average (normalized sum) over the cubes $\{Q_\alpha\}_{\alpha\in I}$ so that $\langle n_{\alpha'}\rangle:=(1/|I|)\sum_{\alpha'\in I}n_{\alpha'}$ where $|I|$ is the number of cubes in Λ_j.

$$\frac{(n_\alpha-\bar n_\alpha)^2}{\bar n_\alpha}=\frac{n_\alpha}{\bar n_\alpha}(n_\alpha-\langle n_{\alpha'}\rangle+\langle n_{\alpha'}\rangle-\langle\bar n_{\alpha'}\rangle+\langle\bar n_{\alpha'}\rangle-\bar n_\alpha)+n_\alpha-\bar n_\alpha.$$

Thus, the first term in the r.h.s. of (57) can be bounded by

$$\frac{4}{|I|^2}\left[\sum_{\alpha,\alpha'\in I}\nu\left(f^2\frac{n_\alpha}{\bar n_\alpha}(n_\alpha-n_{\alpha'})-\mu\left(\frac{n_\alpha}{\bar n_\alpha}(n_\alpha-n_{\alpha'})\right)\right)\right]^2$$
$$+4\nu\left(f^2\sum_{\alpha'\in I}\left\langle\frac{n_\alpha}{\bar n_\alpha}\right\rangle(n_{\alpha'}-\bar n_{\alpha'})-\mu\left(\left\langle\frac{n_\alpha}{\bar n_\alpha}\right\rangle(n_{\alpha'}-\bar n_{\alpha'})\right)\right)^2.$$
$$+4\nu\left(f^2\sum_{\alpha\in I}\frac{\langle\bar n_{\alpha'}\rangle-\bar n_\alpha}{\bar n_\alpha}(n_\alpha-\bar n_\alpha)\right)^2+4\nu\left(f^2\sum_{\alpha\in I}(n_\alpha-\bar n_\alpha)\right)^2. \tag{58}$$

To apply Lemma 26 to the second and fourth term in (58) we need to bound

$$\nu(\mathrm{e}^{t[\Gamma_Q \sum_{\alpha\in I}(n_\alpha-\bar{n}_\alpha)-\mu(\Gamma_Q\sum_{\alpha\in I}(n_\alpha-\bar{n}_\alpha))}),$$

where $\Gamma_Q=\langle n_\alpha/\bar{n}_\alpha\rangle$ or $\Gamma_Q=1$; as ν-a.s. $\sum_{x\in\Lambda_j}\sigma(x)=N_j=\sum_{x\in\Lambda_j}\mu(\sigma(x))$ we certainly have $\sum_{\alpha\in I}(n_\alpha-\bar{n}_\alpha)=\sum_{x\in\Lambda_j\setminus Q}(\mu(\sigma(x))-\sigma(x))$. Thus, in both cases,

$$\nu(\mathrm{e}^{t(\Gamma_Q \sum_{\alpha\in I}(n_\alpha-\bar{n}_\alpha)-\mu(\Gamma_Q\sum_{\alpha\in I}(n_\alpha-\bar{n}_\alpha))}) \leqslant B\,\mathrm{e}^{t^2 A|\Lambda_j|c(L)\log L/l_0},$$

where we used point (viii) of Proposition 24 to obtain the last inequality. Thus, by Lemma 26 the second and fourth term can be bounded by

$$k|\Lambda_j|\left[(\log B)^2+\left(K\left(\frac{Ac(L)\log L}{l_0}\right)^2+\frac{1}{K}\right)\mathrm{Ent}_\nu(f^2)\right]\leqslant|\Lambda_j|\left[C_\zeta+\zeta\,\mathrm{Ent}_\nu(f^2)\right],$$

where we used that $c(L)=\mathrm{e}^{A(\log L)^{(d-1)/d}}$ and we took K and then L large enough.

For the third term in (58) we can proceed as for the first term in (50) (see (52)) with $D_\alpha:=(\langle\bar{n}_{\alpha'}\rangle-\bar{n}_\alpha)/\bar{n}_\alpha$.

Therefore, we can focus our attention on the first term of the r.h.s. of (58). First we observe that by point (a) of Proposition 9 and Assumption 1, we can safely write

$$\frac{1}{|I|^2}\left[\sum_{\alpha,\alpha'\in I}\nu\left(f^2\frac{n_\alpha}{\bar{n}_\alpha}(n_\alpha-n_{\alpha'})-\mu\left(\frac{n_\alpha}{\bar{n}_\alpha}(n_\alpha-n_{\alpha'})\right)\right)\right]^2$$
$$\leqslant\frac{1}{|I|^2}\left[\sum_{\alpha,\alpha'\in I}\nu\left(f^2,\frac{n_\alpha}{\bar{n}_\alpha}(n_\alpha-n_{\alpha'})\right)\right]^2+kl_0^{2d}.$$

At this stage we cannot appeal to the same old argument based on the entropy inequality and we must proceed differently. Following Yau (1996) we introduce $\mathscr{F}_{\alpha,\alpha'}$, the σ-algebra generated by the random variables $\{\sigma(x)\}_{x\in\Lambda\setminus(Q_\alpha\cup Q_{\alpha'})}$, and we write

$$\left|\nu\left(f^2,\frac{n_\alpha}{\bar{n}_\alpha}(n_\alpha-n_{\alpha'})\right)\right|\leqslant\left|\nu\left(\nu\left(f^2,\frac{n_\alpha}{\bar{n}_\alpha}(n_\alpha-n_{\alpha'})|\mathscr{F}_{\alpha,\alpha'}\right)\right)\right|$$
$$+\left|\nu\left(f^2,\nu\left(\frac{n_\alpha}{\bar{n}_\alpha}(n_\alpha-n_{\alpha'})|\mathscr{F}_{\alpha,\alpha'}\right)\right)\right|. \tag{59}$$

Define $\nu_{\alpha\alpha'}(\cdot):=\nu(\cdot|\mathscr{F}_{\alpha\alpha'})$. We have

$$\left|\nu_{\alpha\alpha'}\left(\frac{n_\alpha}{\bar{n}_\alpha}(n_\alpha-n_{\alpha'})\right)\right|\leqslant\frac{1}{\bar{n}_\alpha}\nu_{\alpha\alpha'}(n_\alpha^2)^{1/2}\nu_{\alpha\alpha'}((n_\alpha-n_{\alpha'})^2)^{1/2}\leqslant kl_0^{d(1-\varepsilon')},$$

where as $J\in\Theta_{\mathrm{good}}(M,L,\varepsilon')$ we used Assumption 2 to obtain the last inequality. Thus, the second term in the r.h.s. of (59) can be bounded by

$$k\left(\frac{|\Lambda_j|}{l_1^d}\right)^2\leqslant|\Lambda_j|\frac{k}{K}\,\mathrm{Ent}_\nu(f^2),$$

where we used $t_*^2=K/|\Lambda_j|\mathrm{Ent}_\nu(f^2)\geqslant(K/l_1^d)^2$ in the last inequality.

We now consider the first term in the r.h.s. of (59). Define

$$\mathscr{E}_{\alpha\alpha'}(f,f) := \sum_{x,y\in Q_\alpha\cup Q_{\alpha'}} v((\nabla_{xy} f)^2|\mathscr{F}_{\alpha\alpha'}).$$

By Corollary 19,

$$v(f,f|\mathscr{F}_{\alpha\alpha'}) \leqslant k l_0^b \mathscr{E}_{\alpha\alpha'}(f,f) \quad \forall f.$$

Thus,

$$\begin{aligned}\left| v\left(f^2, \frac{n_\alpha}{\bar{n}_\alpha}(n_\alpha - n_{\alpha'})|\mathscr{F}_{\alpha\alpha'}\right)\right| &\leqslant 2\left\|\frac{n_\alpha}{\bar{n}_\alpha}(n_\alpha - n_{\alpha'})\right\|_\infty (v(f,f|\mathscr{F}_{\alpha\alpha'}))^{1/2}(v(f^2|\mathscr{F}_{\alpha\alpha'}))^{1/2} \\ &\leqslant k l_0^{d+\frac{b}{2}} (\mathscr{E}_{\alpha\alpha'}(f,f))^{1/2}(v(f^2|\mathscr{F}_{\alpha\alpha'}))^{1/2}.\end{aligned}$$

Schwarz inequality yields

$$v\left(\left|v\left(f^2, \frac{n_\alpha}{\bar{n}_\alpha}(n_\alpha - n_{\alpha'})|\mathscr{F}_{\alpha\alpha'}\right)\right|\right) \leqslant k l_0^{d+b/2} v(\mathscr{E}_{\alpha\alpha'}(f,f))^{1/2}. \tag{60}$$

Finally, from (60), we get that the first term in the r.h.s. of (58) is bounded from above by

$$\begin{aligned}&\frac{1}{|I|^2}\left[\sum_{\alpha,\alpha'\in I} v\left(f^2, \frac{n_\alpha}{\bar{n}_\alpha}(n_\alpha - n_{\alpha'})\right)\right]^2 \\ &\quad \leqslant k l_0^{2d+b} \sum_{\alpha,\alpha'\in I} v(\mathscr{E}_{\alpha\alpha'}(f,f)) + k\left(\frac{|\Lambda_j|}{l_0^{d\varepsilon}}\right)^2 \\ &\quad \leqslant k l_0^{2d+b} \sum_{\alpha,\alpha'\in I} v\left(\sum_{x,y\in Q_\alpha} v_{\alpha\alpha'}([\nabla_{x,y} f]^2) + \sum_{x,y\in Q_{\alpha'}} v_{\alpha\alpha'}([\nabla_{x,y} f]^2)\right. \\ &\qquad \left. + \sum_{\substack{x\in Q_\alpha \\ y\in Q_{\alpha'}}} v_{\alpha\alpha'}([\nabla_{x,y} f]^2)\right) + \frac{k}{K}|\Lambda_j| \,\mathrm{Ent}_v(f^2) \\ &\quad \leqslant k l_0^{2d+b} |\Lambda_j| (l_0^2 + l^2) \mathscr{E}(f,f) + \frac{k}{K}|\Lambda_j| \mathrm{Ent}_v(f^2)\end{aligned}$$

since $t_* \geqslant K/l_1^d = K/l_0^{d\varepsilon'}$. Above we have used twice the "path" bound $\sum_{x,y\in\Lambda_j} v(\nabla_{xy} f)^2 \leqslant k l^{d+2} \mathscr{E}_v(f,f)$ given in Section 4.9. Then, by our choice of ε', $l_0^{2d+b} \leqslant L^\varepsilon$, by choosing K large enough we get the sought result also in this case. □

7.3. Applications

Here we discuss the applications of the above results which are relevant for the proof of the main theorem (see Section 5.2.2).

Fix $i, j \in \{1, \ldots, j_{\max}\}$ with $i \neq j$ and let $\rho = (N_j + N_j)/(|\Lambda_j| + |\Lambda_j|)$. Clearly $(\delta_0/2)(\rho_i + \rho_j) \leqslant \rho \leqslant \rho_i + \rho_j$ where ρ_i and ρ_j are the densities in Λ_i and Λ_j and δ_0 is defined in the general setting of Section 4. Without loss of generality we assume that $\rho_i \leqslant \rho$ which implies $\rho_j \geqslant \rho_i$. Let also

$$\tilde{g}_x(\sigma) = [\mathrm{e}^{-\nabla_x H^\tau(\sigma)} - 1]\sigma(x), \quad \tilde{G} := \sum_{x \in \Lambda_i} \tilde{g}_x, \tag{61}$$

$$\bar{g}_x(\sigma) = [\mathrm{e}^{-\nabla_x H^\tau(\sigma)} - 1](1 - \sigma(x)), \quad \bar{G} := \sum_{z \in \Lambda_j} \bar{g}_z.$$

Notice that $\tilde{g}_x$ satisfies the hypotheses of Proposition 27 simply because $\tilde{g}_x = 0$ if there are less than two particles (spins equal to 1) inside its support. Similarly $\bar{g}_z = 0$ if there is less than one particle inside its support. In particular $\|\bar{G}\|_\infty \leqslant kN_j$.

We recall that $\varepsilon' = \varepsilon/d(3b + 2)$ where b is defined in Corollary 19.

Proposition 32. *Assume that* $J \in \Theta_{\mathrm{good}}(L, M, \varepsilon')$. *Then, for any* $\zeta > 0$, *there exists a positive constant* C_ζ *and* $L_0(M, \varepsilon', \zeta)$ *such that for any* f *with* $\nu(f^2) = 1$

(i) *if* $\rho \leqslant \rho_0$

$$\nu\left(f^2, \sum_{\substack{x \in \Lambda_i \\ z \in \Lambda_j}} \tilde{g}_x \bar{g}_z\right)^2 \leqslant \rho |\Lambda|^3 \, [C_\zeta + \zeta \, \mathrm{Ent}_\nu(f^2)].$$

(ii) *if* $\rho > \rho_0$

$$\nu\left(f^2, \sum_{\substack{x \in \Lambda_i \\ z \in \Lambda_j}} \tilde{g}_x \bar{g}_z\right)^2 \leqslant |\Lambda|^3 [C_\zeta + C_\zeta l^2 L^\varepsilon \mathscr{E}_\nu(f, f) + \zeta \, \mathrm{Ent}_\nu(f^2)]$$

provided that $|\Lambda|$ *is large enough.*

Proof. The proof is identical of that one of Cancrini et al. (2002a, Proposition 6.5) using Propositions 27 and 28 instead of Propositions 6.2 and 6.3 of Cancrini et al. (2002a). □

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Appendix

In this appendix we give a direct and simple bound of the l.h.s. of (24) in the particular case where there are a finite number of clusters of size greater than one inside Λ. We refer the reader to Section 5.2 for notations. We assume that $\rho_\Lambda > \rho_0$ for a fixed ρ_0.

As mentioned in Section 5.2, to obtain inequality (24), we must bound

$$\frac{1}{|\Lambda|^4}\hat{v}_{n-1}\left(f^2, \sum_{\substack{x\in S\\ z\in\Lambda\setminus S}} (\mathrm{e}^{-\nabla_{xz}H_\Lambda}-1)\mathbb{1}_{E_{zx}}\right)^2.$$

Let I_S (resp. $I_{\Lambda\setminus S}$, I_Λ) be the set of clusters of size greater than one that intersect S (resp. $\Lambda\setminus S$, Λ). In particular, for all $i\in I_\Lambda$, $|C_i|\geqslant 2$ and $I_\Lambda = I_{\Lambda\setminus S}\cup I_S$.

Then, by definition of H_Λ, $\mathrm{e}^{-\nabla_{xz}H_\Lambda}-1=0$ unless there exists $i\in I_\Lambda$ such that $\{x,z\}\cap C_i\neq\emptyset$. Thus,

$$\sum_{\substack{x\in S\\ z\in\Lambda\setminus S}} (\mathrm{e}^{-\nabla_{xz}H_\Lambda}-1) = \sum_{x\in S}\sum_{i\in I_{\Lambda\setminus S}}\sum_{z\in C_i}(\mathrm{e}^{-\nabla_{xz}H_\Lambda}-1) + \sum_{z\in\Lambda\setminus S}\sum_{i\in I_S}\sum_{x\in C_i}(\mathrm{e}^{-\nabla_{xz}H_\Lambda}-1).$$

Now, a simple L^∞ bound gives

$$\hat{v}_{n-1}\left(f^2, \sum_{\substack{x\in S\\ z\in\Lambda\setminus S}} (\mathrm{e}^{-\nabla_{xz}H_\Lambda}-1)\mathbb{1}_{E_{zx}}\right)$$

$$\leqslant k\left[|S||I_{\Lambda\setminus S}|\sup_{i\in I_{\Lambda\setminus S}}|C_i| + |\Lambda\setminus S||I_S|\sup_{i\in I_S}|C_i|\right]\hat{v}_{n-1}(f^2).$$

Hence, by Assumption 1,

$$B(n)^2 \leqslant \frac{k}{|\Lambda|^4}[|\Lambda|^2(\log L)^2|I_\Lambda|^2\hat{v}_{n-1}(f^2)^2].$$

Then, it comes that

$$\bar{n}_S\sum_n \gamma_f(n)B(n)^2 \leqslant k\,\frac{(\log L)^2}{|\Lambda|}\,|I_\Lambda|^2\hat{v}(f^2). \tag{A.1}$$

Finally, if $|I_\Lambda|$ is independent of L, using inequality (A.1) instead of inequality (24) the diffusive scaling of the LSC follows.

Remark 33. In general, inequality (A.1) is not sharp enough. Indeed we learn in Grimmett (1999) that $|I_\Lambda| = \mathrm{O}(|\Lambda|)$. Thus inequality (A.1) gives

$$\bar{n}_S \sum_n \gamma_f(n) B(n)^2 \leqslant k|\Lambda|(\log L)^2 \hat{v}(f^2).$$

The constant $k|\Lambda|(\log L)^2$ is not sharp enough in order to use it in the recursive proof (we need not more than constant!).

On the other hand, if one takes care of the size of the clusters, using the results of Grimmett (1999), one can prove that there are $|\Lambda|\mathrm{e}^{-km}$ clusters of size m (where k is a suitable constant), and thus,

$$\bar{n}_S \sum_n \gamma_f(n) B(n)^2 \leqslant k|\Lambda| \hat{v}(f^2).$$

The improving bound is no more useful.

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