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Kinetically constrained spin models

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Abstract We analyze the density and size dependence of the relaxation time for kinetically constrained spin models (KCSM) intensively studied in the physics literature as simple models sharing some of the features of the glass transition. KCSM are interacting particle systems on $\mathbb{Z}^d$ with Glauber-like dynamics, reversible w.r.t. a simple product i.i.d Bernoulli(p) measure. The essential feature of a KCSM is that the creation/destruction of a particle at a given site can occur only if the current configuration around it satisfies certain constraints which completely define each specific model. No other interaction is present in the model. From the mathematical point of view, the basic issues concerning positivity of the spectral gap inside the ergodicity region and its scaling with the particle density p remained open for most KCSM (with the notably exception of the East model in $d = 1$; Aldous and Diaconis in J Stat Phys 107(5–6):945–975, 2002). Here for the first time we: (i) identify the ergodicity region by establishing a connection with an associated bootstrap percolation model;

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(ii) develop a novel multi-scale approach which proves positivity of the spectral gap in the whole ergodic region; (iii) establish, sometimes optimal, bounds on the behavior of the spectral gap near the boundary of the ergodicity region and (iv) establish pure exponential decay at equilibrium for the persistence function, i.e. the probability that the occupation variable at the origin does not change before time t. Our techniques are flexible enough to allow a variety of constraints and our findings disprove certain conjectures which appeared in the physical literature on the basis of numerical simulations.

Keywords Glauber dynamics · Spectral gap · Constrained models · Dynamical phase transition · Glass transition

Mathematics Subject Classification (2000) 60K35 · 60K40 · 82B20

1 Introduction

Kinetically constrained spin models (KCSM) are interacting particle systems on the integer lattice $\mathbb{Z}^d$. A configuration is defined by assigning to each site x its occupation variable $\eta_x \in \{0, 1\}$. The evolution is given by a simple Markovian stochastic dynamics of Glauber type with generator $\mathcal{L}$. Each site waits an independent, mean one, exponential time and then, provided that the current configuration around it satisfies an a priori specified constraint which does not involve η_x, it refreshes its state by declaring it to be occupied with probability p and empty with probability $q = 1 - p$. Detailed balance w.r.t. Bernoulli(p) product measure μ is easily verified and μ is therefore an invariant reversible measure for the process.

These models have been introduced in physical literature [17,18] to model liquid/glass transition and more generally the slow "glassy" dynamics which occurs in different systems (see [31,37] for recent review). In particular, they were devised to mimic the fact that the motion of a molecule in a dense liquid can be inhibited by the presence of too many surrounding molecules. That explains why, in all physical models, the constraints specify the maximal number of particles on certain sites around a given one in order to allow creation/destruction on the latter. As a consequence, the dynamics of KCSM becomes increasingly slow as p is increased. Moreover, there usually exist *blocked configurations*, namely configurations with all creation/destruction rates identically equal to zero. This implies the existence of several invariant measures (see [26] for a somewhat detailed discussion of this issue in the context of the North-East model), the occurrence of unusually long mixing times compared to standard high-temperature stochastic Ising models (see Sect. 7.1 below) and may induce the presence of ergodicity breaking transitions. Finally we observe that a KCSM model is in general not attractive so that the usual coupling arguments valid for e.g. ferromagnetic stochastic Ising models cannot be applied.

The above little discussion explains why the basic issues concerning the large time behavior of the process, even if started from the equilibrium reversible measure μ, are non-trivial and justifies why they remained open for most of the interesting models, with the only exception of the East model [3]. This is a one-dimensional model for which creation/destruction at a given site can occur only if the nearest neighbor to its right is empty. In [3] it has been proved that the spectral gap of the generator $\mathcal{L}$ for the East model is positive for all $q > 0$ and, for $q \downarrow 0$, shrinks faster than any polynomial in q (see Sect. 6 for more details). However, the method in [3] uses quite heavily the specifics of the model and its extension to higher dimensions or to other models introduced in physical literature seems to be non trivial. Among the latter we just recall the North-East model (N-E) [25] in $\mathbb{Z}^2$ and the Fredrickson Andersen $j \leq d$ spin facilitated (FA-jf) [17] models in $\mathbb{Z}^d$. For the first, destruction/creation at a given site can occur only if its North and East neighbors are empty, while for the FA-jf model the constraint requires that at least j among the nearest neighbors are empty.

The main results of this paper can be described as follows. In Sect. 2.3, given a generic KCSM with constraints satisfying few rather mild conditions, we identify the critical value of the density of vacancies defined as $q_c = \inf\{q : 0 \text{ is a simple eigenvalue of } \mathcal{L}\}$ with the critical value of a naturally related bootstrap percolation model (Proposition 2.5). Notice that a general result on Markov semigroups (see Theorem 2.3 below) implies that for any $q > q_c$ the reversible measure μ is mixing for the process generated by $\mathcal{L}$. Next, in Sect. 3, we identify a natural general condition on the associated bootstrap percolation model which implies the positivity of the spectral gap of $\mathcal{L}$ (Theorem 3.3) and therefore exponential decay of correlations. In its simplest form the condition requires that the probability that a large cube is internally spanned (i.e. the block does not contain blocked configurations, see Definition 3.4 below) is close to one. For all the models discussed in Sect. 6 our condition is satisfied for all q strictly larger than q_c. Our findings disprove some conjectures in the physics literature [19,21], based on numerical simulations and approximate analytical treatments, on the existence of a second critical point $q_c' > 0$ at which the spectral gap of FAjf ($j \geqslant 2$) vanishes and below which correlation functions would decay in a stretched exponential form $\simeq \exp(-t/\tau)^\beta$ with $\beta < 1$.

There are two main strategies in our proofs. The first one is the *Bisection-Constrained* or *B-C* approach (see Sect. 4), which combines the bisection technique of [28] with the novel idea of considering auxiliary constrained models on large length scales with scale dependent constraints. This allows to prove positivity of the spectral gap provided constraints are satisfied with high probability. The second one is the *Renormalization-Constrained* or *R-C* approach (see Sect. 5), which consists in a renormalization procedure which allows to map all different models at any density into a basic KCSM, the so called **-general model*. Inside the ergodicity region and with a proper choice of the size of the blocks, the R-C procedure leads to the **-general model* in a regime in which its constraints are satisfied with high probability and therefore susceptible to be analyzed via the B-C method.

At the end of the Sect. 3 we also analyze the so called persistence function $F(t)$ which represents the probability for the equilibrium process that the occupation variable at the origin does not change before time t. In Theorem 3.6 we prove that, whenever the spectral gap is strictly positive, $F(t)$ must decay exponentially, with characteristic time $\tau_F < 1/(q\,\mathrm{gap}(\mathcal{L}))$. Such a universal upper bound suggests a dynamical way (alternative to the use of test functions) to obtain upper bounds on the spectral gap, see Remark 3.7 and the proof of Theorem 6.9 (i). The fact that τ_F is finite whenever the spectral gap is positive disproves previous conjectures of a stretched exponential decay $F(t) \simeq \exp(-t/\tau)^\beta$ with $\beta < 1$ for FA1f in $d = 1$ [5,6] and for FA2f in $d = 2$ [21]. For the North-East model at the critical point where gap $= 0$ we show instead (see corollary 6.18) that $\int_0^\infty dt\, F(\sqrt{t}) = \infty$, a signature of a slow polynomial decay.

After establishing the positivity of the spectral gap, in Sect. 6 we analyze its behavior as $q \downarrow q_c$ for some of the models discussed in Sect. 2.3. For the East model ($q_c = 0$) we significantly improve the lower bound on the spectral gap proved in [3] and claimed to provide the leading behavior in [34]. Our lower bound, in leading order, coincides with the upper bound of [3], yielding that the gap shrinks as $q^{\log_2(q)/2}$ for small values of q.

For the FA-1f model ($q_c = 0$) we show that for $q \approx 0$, the spectral gap is $O(q^3)$ in $d = 1$, $O(q^2)$ in $d = 2$ apart from logarithmic corrections and between $O(q^{1+2/d})$ and $O(q^2)$ in $d \geq 3$. Again these findings disprove previous claims in $d = 2, 3$ [6] .

For the FA-2f model ($q_c = 0$) in e.g. $d = 2$ we get instead

$$\exp(-c/q^5) \leq \mathrm{gap}(\mathcal{L}) \leq \exp\Big(-\frac{\pi^2}{18q}\big(1 + o(1)\big)\Big) \tag{1.1}$$

as $q \downarrow 0$. Notice that the r.h.s. of (1.1) represents the inverse of the critical length for bootstrap percolation [22], i.e. the smallest length scale above which a region of the lattice becomes mobile or unjammed under the FA-2f dynamics, and its square has been conjectured [30,38] to provide the leading behavior of the spectral gap for small values of q.

As explained above, the techniques developed in this paper are flexible enough to deal with a variety of KCSM, i.e. with different choices of the constraints, even with long range. Furthermore, they can be extend to cover models with some additional weak static interaction between the occupation variables [11] and to *Kinetically Constrained Lattice Gases (KCLG)* [10], namely KCSM models with a spin-exchange (*i.e.* conservative) Kawasaki dynamics replacing Glauber dynamics (see [31,37] for review).

2 The models

2.1 Setting and notation

The models considered here are defined on the integer lattice $\mathbb{Z}^d$ with sites $x = (x_1, \ldots, x_d)$ and basis vectors $\vec{e}_1 = (1, \ldots, 0), \vec{e}_2 = (0, 1, \ldots, 0), \ldots, \vec{e}_d = (0, \ldots, 1)$. On $\mathbb{Z}^d$ we will consider the Euclidean norm $\|x\|$, the ℓ^1 (or graph

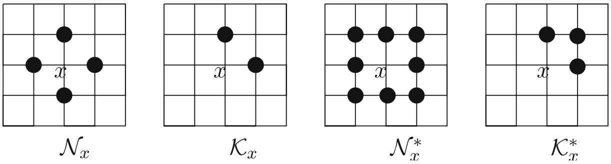

Fig. 1 The various neighborhoods of a vertex x in two dimensions

theoretic) norm $\|x\|_1$ and the sup-norm $\|x\|_\infty$. The associated distances will be denoted by $d(\cdot,\cdot)$, $d_1(\cdot,\cdot)$ and $d_\infty(\cdot,\cdot)$, respectively. For any vertex x we let

$$\begin{aligned}
&\mathcal{N}_x = \{y \in \mathbb{Z}^d :\ d_1(x,y) = 1\},\\
&\mathcal{K}_x = \{y \in \mathcal{N}_x :\ y = x + \textstyle\sum_{i=1}^d \alpha_i \vec{e}_i,\ \alpha_i \geq 0\}\\
&\mathcal{N}^*_x = \{y \in \mathbb{Z}^d :\ y = x + \textstyle\sum_{i=1}^d \alpha_i \vec{e}_i,\ \alpha_i = \pm 1, 0 \text{ and } \textstyle\sum_i \alpha_i^2 \neq 0\}\\
&\mathcal{K}^*_x = \{y \in \mathcal{N}^*_x :\ y = x + \textstyle\sum_{i=1}^d \alpha_i \vec{e}_i,\ \alpha_i = 1, 0\}
\end{aligned}$$

and write $x \sim y$ if $y \in \mathcal{N}^*_x$ (Fig. 1).
The neighborhood, the *-neighborhood, the oriented and *-oriented neighborhoods $\partial\Lambda,\ \partial^*\Lambda, \partial_+\Lambda,\ \partial^*_+\Lambda$ of a finite subset $\Lambda \subset \mathbb{Z}^d$ are defined accordingly as $\partial\Lambda := \{\cup_{x\in\Lambda}\mathcal{N}_x\} \setminus \Lambda,\ \partial^*\Lambda := \{\cup_{x\in\Lambda}\mathcal{N}^*_x\} \setminus \Lambda,\ \partial_+\Lambda := \{\cup_{x\in\Lambda}\mathcal{K}_x\} \setminus \Lambda,\ \partial^*_+\Lambda := \left\{\cup_{x\in\Lambda}\mathcal{K}^*_x\right\} \setminus \Lambda$. A rectangle R will be a set of sites of the form

$$R := [a_1, b_1] \times \cdots \times [a_d, b_d]$$

while the collection of finite subsets of $\mathbb{Z}^d$ will be denoted by $\mathbb{F}$.

The pair (S,ν) will denote a finite probability space with $\nu(s) > 0$ for any $s \in S$. $G \subset S$ will denote a distinguished event in S, often referred to as the set of "good states", and $q \equiv \nu(G)$ its probability.

Given (S,ν) we will consider the configuration space $\Omega = S^{\mathbb{Z}^d}$ equipped with the product measure $\mu := \prod_{x\in\mathbb{Z}^d} \nu_x$, $\nu_x \equiv \nu$. Similarly we define Ω_Λ and μ_Λ for any subset $\Lambda \subset \mathbb{Z}^d$. Elements of Ω (Ω_Λ) will be denoted by Greek letters ω, η, etc. while the value of the configuration at site x will be denoted by η_x, ω_x. We will use the shorthand notation $\mu(f)$, $\mathrm{Var}(f)$ ($\mu_\Lambda(f), \mathrm{Var}_\Lambda(f)$) to denote the expected value and variance of any $f \in L^2$. A function $f : \Omega \mapsto \mathbb{R}$ that depends only on finitely many variables $\{\omega_x\}_{x\in\mathbb{Z}^d}$ will be called *local*. Finally, if the set S coincides with the two state-space $\{0,1\}$, we denote by η^x the configuration η flipped at x namely

$$\eta^x_y = \begin{cases} \eta_y & \text{if } y \neq x \\ 1 - \eta_x & \text{if } y = x \end{cases}$$

2.2 The Markov process

Kinetically constrained spin models (KCSM) are Glauber type Markov processes in Ω (Ω_Λ), reversible w.r.t. the product measure μ (μ_Λ). In the following, we give a general definition which covers the KCSM which have been more extensively studied in the physical literature in the context of liquid/glass transition (see Sect. 2.3 for specific definitions). It also includes the so called Spiral Model, which have been introduced more recently [36,39] and which we will analyze in [12]. We also remark that our definition is formulated in order to include models with more general occupation variables than the 0–1 variables which are usually considered in the physics literature. The reason for this more general setting will be clarified in Sects. 4 and 5. In the sequel ergodicity issues force us to carefully distinguish between the process defined directly on the infinite lattice $\mathbb{Z}^d$ and the one defined on a finite subset $\Lambda \subset \mathbb{Z}^d$.

KCSM on the infinite lattice $\mathbb{Z}^d$

Each specific model is characterized by a collection $\{\mathcal{C}_x\}_{x\in\mathbb{Z}^d}$ of families of subsets of $\mathbb{Z}^d$ called *influence classes* and by the choice of the good event $G \subset S$. The influence classes will satisfy the following basic hypothesis:

(a) *independence of x:* for all $x \in \mathbb{Z}^d$ and all $A \in \mathcal{C}_x$ $x \notin A$;
(b) *translation invariance:* $\mathcal{C}_x = \mathcal{C}_0 + x$ for all x;
(c) *finite range interaction:* there exists $r < \infty$ such that any element of $\mathcal{C}_x$ is contained in $\cup_{j=1}^r \{y : \ d_1(x,y) = j\}$

The pair $(\mathcal{C}_x, G)$ defines the local constraint at x via the following definition.

Definition 2.1 Given a vertex $x \in \mathbb{Z}^d$ we will say that the constraint at x is satisfied by the configuration ω if the indicator

$$c_x(\omega) = \begin{cases} 1 & \text{if there exists a set } A \in \mathcal{C}_x \text{ such that } \omega_y \in G \text{ for all } y \in A \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

is equal to one.[1]

Remark 2.2 The constraints $c_x(\omega)$ are increasing functions w.r.t the partial order in Ω for which $\omega \le \omega'$ iff $\omega'_x \in G$ whenever $\omega_x \in G$. However, this does not imply in general that the process described below is attractive in the sense of Liggett [27].

The process that will be studied in the sequel can be informally described as follows. Each vertex x waits an independent mean one exponential time and then, provided that the current configuration ω satisfies the constraint at x, the value ω_x is refreshed with a new value in S sampled from ν and the all procedure starts again. More formally, we will consider the Markov process associated to

[1] In other words the presence of good sites around x facilitates the possibility for ω_x to fulfill its constraint. There exist other interesting models (see e.g. [35]) in which the constraints requires a more complicate configuration around a given vertex than just "all good" and our techniques and results do not apply to them.

the self-adjoint Markov semigroup $P_t := e^{t\mathcal{L}}$ on $L^2(\mu)$, where the generator $\mathcal{L}$, a non-positive self-adjoint operator with domain $\mathrm{Dom}(\mathcal{L})$, can be constructed in a standard way (see e.g. [26,27]) starting from its action on local functions:

$$\mathcal{L}f(\omega) = \sum_{x\in\mathbb{Z}^d} c_x(\omega)\left[\mu_x(f) - f(\omega)\right] \tag{2.2}$$

where $\mu_x(f) \equiv \int d\nu(\omega_x)f(\omega)$ is a function of all ω_y with $y \in \mathbb{Z}^d \setminus \{x\}$ and corresponds to the local mean with respect to the variable ω_x computed while the other variables are held fixed. The Dirichlet form corresponding to $\mathcal{L}$, $\mathcal{D}(f) = -\mu(f \cdot \mathcal{L}f)$, can be rewritten by using (2.2) as

$$\mathcal{D}(f) = \sum_{x\in\mathbb{Z}^d} \mu\left(c_x \operatorname{Var}_x(f)\right), \quad f \in Dom(\mathcal{L})$$

where $\operatorname{Var}_x(f) \equiv \int d\nu(\omega_x)f^2(\omega) - \left(\int d\nu(\omega_x)f(\omega)\right)^2$. Due to the fact that the jump rates are not bounded away from zero, the reversible measure μ is certainly not the only invariant measure: there exist initial configurations that are blocked forever (all creation/destruction rates are zero) and any measure concentrated on them is invariant. An interesting question is therefore whether μ is ergodic or mixing for the Markov process and whether there exist other translation invariant, ergodic stationary measures. To this purpose it is useful to recall the following well known result (see e.g. Theorem 4.13 in [27]).

Theorem 2.3 *The following are equivalent,*

(a) $\lim_{t\to\infty} P_t f = \mu(f)$ *in* $L^2(\mu)$ *for all* $f \in L^2(\mu)$.
(b) 0 *is a simple eigenvalue for* $\mathcal{L}$.

Clearly (a) above implies that $\lim_{t\to\infty} \mu\left(fP_t g\right) = \mu(f)\mu(g)$ for any $f, g \in L^2(\mu)$, i.e. μ is mixing. As we show in the following section in a certain generality, using condition (b) we are able to prove (see Proposition 2.5 below) that whenever μ is ergodic it is also mixing. However, the following remark should be kept in mind.

Remark 2.4 Even if μ is mixing there will exist in general infinitely many stationary measures, i.e. probability measures $\tilde{\mu}$ satisfying $\tilde{\mu}P_t = \tilde{\mu}$ for all $t \geq 0$. As an example take an arbitrary probability measure $\tilde{\mu}$ such that $\tilde{\mu}\left(\{S \setminus G\}^{\mathbb{Z}^d}\right) = 1$, namely a measure concentrated on a configuration which is blocked forever. Note that, by taking proper superposition of blocked configurations, it is possible to construct stationary measures which are also translational invariant. We refer the interested reader to [26] for a discussion of this point in the context of the North-East model.

KCSM on a finite subset $\Lambda \subset \mathbb{Z}^d$

In a finite region $\Lambda \subset \mathbb{Z}^d$ the process, a continuous time Markov chain, is characterized by the action of the generator $\mathcal{L}_\Lambda$ which takes the form

$$\mathcal{L}_\Lambda f(\omega) = \sum_{x\in\Lambda} c_{x,\Lambda}(\omega)\left[\mu_x(f) - f(\omega)\right] \tag{2.3}$$

where $c_{x,\Lambda}$ describe now the finite volume constraints. As in infinite volume, we denote by $\mathcal{D}_\Lambda$ the associated Dirichlet form. For a given choice of the infinite volume influence classes $\{\mathcal{C}_x\}_{x\in\mathbb{Z}^d}$ and of the good event G, the finite volume constraints $c_{x,\Lambda}$ could again be defined as in (2.1). However, some care has to be taken for those $x \in \Lambda$ with an influence class $\mathcal{C}_x$ not entirely contained inside Λ.

One possibility is to modify in a Λ-dependent way the definition of the influence classes by setting

$$\mathcal{C}_{x,\Lambda} := \{A \cap \Lambda :\ A \in \mathcal{C}_x\} \tag{2.4}$$

and consequently define $c_{x,\Lambda}$ as in (2.1) with $\mathcal{C}_x$ replaced by $\mathcal{C}_{x,\Lambda}$. Although such an approach is feasible and natural, at least for some of the models discussed below, an important drawback is a loss of ergodicity of the chain. Consider for example the East model on $\mathbb{Z}$ with influence class $\mathcal{C}_x = \{x+1\}$ (see Sect. 2.3). If we consider the model on $\Lambda = \{1,\ldots,L\}$, ergodicity is certainly lost since $c_{L,\Lambda}(\omega) = 0\ \forall\omega$. One is then forced to consider the chain restricted to an ergodic component making the whole analysis more cumbersome (see Sect. 7 for this approach in the case of the FA-1f model in one dimension).

Another alternative is to imagine that the configuration ω is defined also over the sites outside Λ where it is frozen and equal to some reference configuration τ, that will be referred to as the *boundary condition*. Then we can define the finite volume constraints with boundary condition τ as

$$c^\tau_{x,\Lambda}(\omega) := c_x(\omega\cdot\tau) \tag{2.5}$$

where c_x are the infinite volume rates (2.1) and $\omega\cdot\tau \in \Omega$ denotes the configuration equal to ω inside Λ and equal to τ in $\mathbb{Z}\setminus\Lambda$. Notice that, for any $x \in \Lambda$, the rate $c^\tau_{x,\Lambda}(\omega)$ (2.5) depends on τ only through the indicators $\{\mathbb{1}_{\tau_z\in G}\}_{z\in\mathcal{B}}$, where $\mathcal{B}$ is the *boundary set* $\mathcal{B} := \left(\mathbb{Z}^d \setminus \Lambda\right) \cap (\cup_{z\in\Lambda}\mathcal{C}_z)$. Therefore, instead of fixing τ, it is enough to choose a subset $\mathcal{M} \subset \mathcal{B}$, called the *good boundary set*, and define

$$c^{\mathcal{M}}_{x,\Lambda}(\omega) := c^\tau_{x,\Lambda}(\omega) \tag{2.6}$$

where τ is *any* configuration satisfying $\tau_z \in G$ for all $z \in \mathcal{M}$ and $\tau_z \notin G$ for $z \in \mathcal{B}\setminus\mathcal{M}$. We will say that a choice of $\mathcal{M}$ is *minimal* if $\mathcal{L}_\Lambda$ with the rates (2.6) is ergodic and non-ergodic for any other choice $\mathcal{M}' \subset \mathcal{M}$. The choice $\mathcal{M} = \mathcal{B}$ will be called *maximal*. For convenience we will write $\mathcal{L}_\Lambda^{\max}$ ($\mathcal{L}_\Lambda^{\min}$) for the generators with maximal (minimal) choice of the boundary conditions.

2.3 0–1 Kinetically constrained spin models

In most models considered in the physical literature the finite probability space (S, ν) is chosen with S being the two-state configuration space, $S = \{0, 1\}$ and the good set G is conventionally chosen as the empty state $\{0\}$. Any model with these features will be called a "0–1 KCSM" (kinetically constrained spin model).

Given a 0–1 KCSM, the parameter $q = \mu(\eta_0 = 0)$ can be varied in $[0, 1]$ while keeping fixed the basic structure of the model (i.e. the notion of the good set and the functions c_x's expressing the constraints) and it is natural to define a critical value q_c as

$$q_c = \inf\{q \in [0,1] : \text{ 0 is a simple eigenvalue of } \mathcal{L}\}$$

As we will prove below, $\forall q > q_c$ the value 0 is simple eigenvalue of $\mathcal{L}$ and q_c coincides with the *bootstrap percolation threshold* q_{bp} of the model defined as follows [33]. For any $\eta \in \Omega$ define the bootstrap map $T : \Omega \mapsto \Omega$ by[2]

$$T(\eta)_x = 0 \quad \text{if either} \quad \eta_x = 0 \quad \text{or} \quad c_x(\eta) = 1 \tag{2.7}$$

Denote by $\mu^{(n)}$ the probability measure on Ω obtained by iterating n-times the above mapping starting from μ. As $n \to \infty$ $\mu^{(n)}$ converges to a limiting measure $\mu^{(\infty)}$ [33] and it is natural to define the critical value q_{bp} as

$$q_{bp} = \inf\{q \in [0,1] : \mu^{(\infty)}(\eta_0 = 0) = 1\}$$

i.e. the infimum of the values q such that, with probability one, the lattice can be entirely emptied. Using the fact that the c_x's are increasing function of η it is easy to check that $\mu^{(\infty)}(\eta_0 = 0) = 1$ for any $q > q_{bp}$.

Proposition 2.5 *0 is a simple eigenvalue for $\mathcal{L}$ if and only if $\mu^{(\infty)}(\eta_0 = 0) = 1$. Therefore $q_c = q_{bp}$.*

Proof Assume $q < q_{bp}$ and call f the indicator of the event that the origin cannot be emptied by any finite number of iterations of the bootstrap map T (2.7). By construction $\mathrm{Var}(f) \neq 0$ and $\mathcal{L}f = 0$ a.s. (μ). Therefore 0 is not a simple eigenvalue of $\mathcal{L}$ and $q \leq q_c$.

Suppose now that $q > q_{bp}$ and that $f \in Dom(\mathcal{L})$ satisfies $\mathcal{L}f = 0$ or, what is the same, $\mathcal{D}(f) = 0$. We want to conclude that $f = $ const. a.e. (μ). For this purpose we will show that $\mathcal{D}(f) = 0$ implies that the unconstrained Glauber

[2] In most of the bootstrap percolation literature the role of the 0's and the 1's is inverted.

Dirichlet form $\sum_x \mu\,(\mathrm{Var}_x(f))$ is zero which makes the sought conclusion obvious since $\mathrm{Var}(f) \le \sum_x \mu\,(\mathrm{Var}_x(f))$.

Given $x \in \mathbb{Z}^d$, let $A_n \equiv A_{n,x} = \{\eta : \; T^n(\eta)_x = 0\}$. Since $q > q_{bp}$, clearly $\mu\,(\cup_n A_n) = 1$. Write

$$\mu\,(\mathrm{Var}_x(f)) = q \sum_{n=1} \int_{A_n \setminus A_{n-1}} d\mu(\eta)[f(\eta^x) - f(\eta)]^2 \eta_x$$

where we recall from Sect. 2.1 that η^x denotes the flipped configuration at x. For any $\eta \in A_n$ it is easy to convince oneself that it is possible to find a collection of vertexes $x^{(1)}, \dots, x^{(k)}$, with k and $d(x, x^{(j)})$ bounded by a constant depending only on n, and a collection of configurations $\eta^{(1)}, \eta^{(2)}, \dots, \eta^{(k)}$ such that $\eta^{(1)} = \eta$, $\eta^{(k)} = \eta^x$, $\eta^{(j+1)} = (\eta^{(j)})^{x^{(j)}}$ and $c_{x^{(j)}}(\eta^{(j)}) = 1$. We can then write $[f(\eta^x) - f(\eta)]$ as a telescopic sum of terms like $[f(\eta^{(j+1)}) - f(\eta^{(j)})]$ and apply Schwartz inequality to get

$$\int_{A_n \setminus A_{n-1}} d\mu(\eta)[f(\eta^x) - f(\eta)]^2 \le C(n) \sum_{y:\, d(y,x) \le C'(n)} \int d\mu(\sigma) c_y(\sigma)[f(\sigma^y) - f(\sigma)]^2$$

where $C(n)$ takes care of the relative density $\sup_{\eta \in A_n} \mu(\eta)/\mu(\eta^{(j)})$ and of the number of possible choice of the vertexes $\{x^{(j)}\}_{j=1}^k$.

By assumption $\mathcal{D}(f) = 0$ i.e. $\int d\mu(\sigma) c_y(\sigma)[f(\sigma^y) - f(\sigma)]^2 = 0$ for any y and the above inequality implies that the unconstrained Dirichlet form is also zero. Therefore, $f = const.$ a.e. (μ) and the proof is complete. □

Having defined the bootstrap percolation it is natural to divide the 0–1 KCSM into two distinct classes.

Definition 2.6 We will say that a 0–1 KCSM is *non-cooperative* if there exists a finite set $\mathcal{V} \subset \mathbb{Z}^d$ such that any configuration η which is empty in all the sites of $\mathcal{V}$ reaches the empty configurations (all 0's) under iteration of the bootstrap mapping. Otherwise the model will be called *cooperative*.

Remark 2.7 Because of the translation invariance of the constraints it is obvious that any configuration η identically equal to zero in $\mathcal{V} + x$, $x \in \mathbb{Z}^d$, will reach the empty configuration under iterations of T. It is also obvious that q_{bp} and therefore q_c are zero for all non-cooperative models.

In what follows we will now illustrate some of the most studied models.

[1] The East model [16]. The model is one-dimensional, $\Omega = \{0,1\}^{\mathbb{Z}}$, with influence class $\mathcal{C}_x = \{x+1\}$: a vertex can flip iff its right neighbor is empty. On a finite volume $\Lambda \subset \mathbb{Z}$ the boundary set is given by the site to the right of the rightmost $x \in \Lambda$, namely $\mathcal{B} = \partial_+ \Lambda$. The minimal boundary condition is

τ empty on the right boundary, i.e. $\mathcal{M} = \mathcal{B}$. The model is clearly cooperative and $q_c = 0$ since in order to empty the whole lattice it is enough to start from a configuration for which any site x has a vacancy to its right.

[2] Frederickson–Andersen (FA-jf) models [17,18]. The model is d-dimensional, $\Omega = \{0,1\}^{\mathbb{Z}^d}$. Take $1 \le j \le d$, the influence class of FA-jf model are

$$\mathcal{C}_x = \{A \subset \mathcal{N}_x : |A| \ge j\}$$

In words a vertex can be updated iff at least j of its neighbors are 0's. For all choices of j, the set of boundary sites is $\mathcal{B} = \partial\Lambda$. When $j = 1$ the minimal boundary conditions is exactly one 0 on $\partial\Lambda$, namely $\mathcal{M} = \{z\}$, $z \in \partial\Lambda$. If instead $1 < j \le d$, a boundary condition which guarantees ergodicity is $\mathcal{M} = \partial_+\Lambda$. This condition is minimal for rectangles when $j = d$. If $j = 1$ the model is non-cooperative with $\mathcal{V} = \{0\}$ while for $j \ge 2$ it is cooperative. In any case $q_c = 0$ [33].

[3] The Modified Basic (MB) model. The model is d-dimensional, $\Omega = \{0,1\}^{\mathbb{Z}^d}$, and the influence classes are

$$\mathcal{C}_x = \{A \subset \mathcal{N}_x : A \cap \{-\vec{e}_i, \vec{e}_i\} \neq \emptyset, \text{ for all } i = 1, \dots, d\}$$

i.e. a move at x can occur iff in each direction there is a 0. The boundary set is $\mathcal{B} = \partial\Lambda$, i.e. coincides with the one of FA-jf models for the same value of d. Once again a minimal boundary condition on a rectangle is $\mathcal{M} = \partial_+\Lambda$. The model is cooperative and $q_c = 0$ [33].

[4] The North-East (N-E) model [25]. The model is two-dimensional, $\Omega = \{0,1\}^{\mathbb{Z}^2}$, and its influence classes are

$$\mathcal{C}_x = \{\mathcal{K}_x\}$$

i.e. a move at x can occur iff both in the north and in the east direction there is a 0. In this case $\mathcal{B} = \partial_+\Lambda$ and the only possible (and therefore minimal) choice of boundary conditions which guarantee ergodicity is $\mathcal{M} = \partial_+\Lambda$. The model is cooperative and the critical point q_c coincides with $1 - p_c^o$ where p_c^o is the critical threshold for oriented percolation in $\mathbb{Z}^2$ [33].

2.4 Quantities of interest

We now define the main quantities that will be studied in the sequel.

The first object of mathematical and physical interest is the spectral gap (or inverse of the relaxation time) of the generator $\mathcal{L}$, defined as

$$\text{gap}(\mathcal{L}) := \inf_{\substack{f \in Dom(\mathcal{L}) \\ f \neq \text{const}}} \frac{\mathcal{D}(f)}{\text{Var}(f)} \tag{2.8}$$

and similarly for the finite volume version of the process. A positive spectral gap implies that the reversible measure μ is mixing for the semigroup P_t with exponentially decaying correlations,

$$\operatorname{Var}(P_t f) \leqslant \exp(-t\,\operatorname{gap}(\mathcal{L}))\operatorname{Var}(f), \quad f \in L^2(\mu).$$

It is important to observe the following kind of monotonicity that can be exploited in order to bound the spectral gap of one model with the spectral gap of another one.

Definition 2.8 Suppose that we are given two influence classes $\mathcal{C}_0$ and $\mathcal{C}'_0$, denote by $c_x(\omega)$ and $c'_x(\omega)$ the corresponding rates and by $\mathcal{L}$ and $\mathcal{L}'$ the associated generators on $L^2(\mu)$. If, for all $\omega \in \Omega$ and all $x \in \mathbb{Z}^d$, $c'_x(\omega) \leq c_x(\omega)$, we say that $\mathcal{L}$ is dominated by $\mathcal{L}'$.

Remark 2.9 The term domination here has the same meaning it has in the context of bootstrap percolation. It means that the KCSM associated to $\mathcal{L}'$ is *more* constrained than the one associated to $\mathcal{L}$.

Clearly, if $\mathcal{L}$ is dominated by $\mathcal{L}'$, $\mathcal{D}'(f) \leqslant \mathcal{D}(f)$ and therefore $\operatorname{gap}(\mathcal{L}') \leq \operatorname{gap}(\mathcal{L})$.

As an example we can consider the FA-1f model in $\mathbb{Z}^d$, where $\mathcal{C}_0$ is the collection of non-empty subsets A of $\mathcal{N}_0$ (see above). If instead we consider $\mathcal{C}'_0$ with the extra constraint that A must contain at least one vertex between $\pm\vec{e}_1$, we get that the spectral gap of the FA-1f model in $\mathbb{Z}^d$ is bounded from below by the spectral gap of the FA-1f model in $\mathbb{Z}$. This in turn is bounded from below by the spectral gap of the East model which is known to be positive [3]. Similarly we could lower bound the spectral gap of the FA-2f model in $\mathbb{Z}^d$, $d \geq 2$, with that in $\mathbb{Z}^2$, by restricting the sets $A \in \mathcal{C}_0$ to e.g. the $(\vec{e}_1, \vec{e}_2)$-plane.

In finite volume the comparison argument is a bit more delicate since it heavily depends on the boundary conditions. For example, if we consider the FA-1f model in a rectangle with minimal boundary conditions, i.e. a single 0 in a site belonging to $\partial\Lambda$, the argument discussed above would lead to a comparison with a non-ergodic Markov chain whose spectral gap is zero.

Remark 2.10 The comparison technique can be quite effective in proving positivity of the spectral gap but the resulting bounds are in general quite poor, particularly in the limiting case $q \approx q_c$.

A second, quite effective, observation concerns the monotonicity of the spectral gap for a given model when considered on different finite volumes.

Lemma 2.11 *For any finite $\Lambda \subset \mathbb{Z}^d$ and $V \subset \Lambda$,*

$$0 < \operatorname{gap}(\mathcal{L}_\Lambda^{\max}) \leq \operatorname{gap}(\mathcal{L}_V^{\max})$$

Proof For any $f \in L^2(\Omega_V, \mu_V)$ we have $\operatorname{Var}_V(f) = \operatorname{Var}_\Lambda(f)$ because of the product structure of the measure μ_Λ and $\mathcal{D}_\Lambda(f) \leq \mathcal{D}_V(f)$ because, for any $x \in V$ and any $\omega \in \Omega_\Lambda$, $c_{x,\Lambda}(\omega) \leq c_{x,V}(\omega)$ since $\mathcal{B}_V \subset \mathcal{B}_\Lambda \cup \Lambda$. The proof follows at once from the variational characterization of the spectral gap. □

Remark 2.12 The above monotonicity is in general not true if we replace the maximal choice of the boundary conditions with the minimal one.

The third observation we make consists in relating $\mathrm{gap}(\mathcal{L})$ to its finite volume analogue with maximal boundary conditions.

Proposition 2.13 *Assume that* $\inf_{\Lambda\in\mathbb{F}} \mathrm{gap}(\mathcal{L}_\Lambda^{\max}) > 0$. *Then* $\mathrm{gap}(\mathcal{L}) > 0$.

Proof Following Liggett Chap. 4 [27], for any $f \in Dom(\mathcal{L})$ with $\mathrm{Var}(f) > 0$ pick a sequence of local functions $f_n \in L^2(\Omega, \mu)$ so that $f_n \to f$ and $\mathcal{L}f_n \to \mathcal{L}f$ in L^2. Then $\mathrm{Var}(f_n) \to \mathrm{Var}(f)$ and $\mathcal{D}(f_n) \to \mathcal{D}(f)$. But since f_n is local

$$\mathrm{Var}(f_n) = \mathrm{Var}_\Lambda(f_n) \quad \text{and} \quad \mathcal{D}(f_n) = \mathcal{D}_\Lambda^{\max}(f_n)$$

provided that Λ is a large enough square (depending on f_n) centered at the origin. Therefore

$$\frac{\mathcal{D}(f)}{\mathrm{Var}(f)} \geq \inf_{\Lambda\in\mathbb{F}} \mathrm{gap}(\mathcal{L}_\Lambda^{\max}) > 0.$$

and $\mathrm{gap}(\mathcal{L}) \geq \inf_{\Lambda\in\mathbb{F}} \mathrm{gap}(\mathcal{L}_\Lambda) > 0$. □

The second quantity of interest is the so called *persistence function* (see e.g. [21,34]) defined by

$$F(t) := \int d\mu(\eta)\, \mathbb{P}(\sigma_0^\eta(s) = \eta_0,\ \forall s \leq t) \tag{2.9}$$

where $\{\sigma^\eta(s)\}_{s\geq 0}$ denotes the process at time s started from the configuration η at time 0. In some sense the persistence function provides a measure of the "mobility" of the system.

3 Main results for 0–1 KCSM

In this section, we state our main results for a general 0–1 KCSM with generator $\mathcal{L}$.

Fix an integer length scale ℓ larger than the range r and let $\mathbb{Z}^d(\ell) \equiv \ell\,\mathbb{Z}^d$. Consider a partition of $\mathbb{Z}^d$ into disjoint rectangles $\Lambda_z := \Lambda_0 + z$, $z \in \mathbb{Z}^d(\ell)$, where $\Lambda_0 = \{x \in \mathbb{Z}^d :\ 0 \leq x_i \leq \ell - 1,\ i = 1, \ldots, d\}$.

Definition 3.1 Given $\epsilon \in (0,1)$ we say that $G_\ell \subset \{0,1\}^{\Lambda_0}$ is a ϵ-good set of configurations on scale ℓ if the following two conditions are satisfied:

(a) $\mu(G_\ell) \geq 1 - \epsilon$.
(b) For any collection $\{\xi^{(x)}\}_{x\in\mathcal{K}_0^*}$ of spin configurations such that $\xi^{(x)} \in G_\ell$ for all $x \in \mathcal{K}_0^*$, the following holds. For any $\xi \in \Omega$ which coincides with $\xi^{(x)}$ in $\cup_{x\in\mathcal{K}_0^*}\Lambda_{\ell x}$, there exists a sequence of legal moves inside $\cup_{x\in\mathcal{K}_0^*}\Lambda_{\ell x}$ (i.e. single spin moves compatible with the constraints) which transforms

ξ into a new configuration $\tau \in \Omega$ such that the Markov chain in Λ_0 with boundary conditions τ is ergodic.

Remark 3.2 In general the transformed configuration τ will be identically equal to zero on $\partial_+^* \Lambda_0$. As will be clear fom Corollary 3.5 below assumption (b) has been made having in mind models like the East, FA-jf, M-B or N-E which, modulo rotations, are dominated by a model with influence class $\tilde{\mathcal{C}}_x$ entirely contained in the sector $\{y : y = x + \sum_{i=1}^d \alpha_i \vec{e}_i,\ \alpha_i \geq 0\}$. Here we deal only with these models and we refer to [12] for the analysis of models which do not have the above property, such as the Spiral Model introduced in [36,39]. In this case a non rectangular geometry for the tiles in the partition of $\mathbb{Z}^d$ adapted to the choice of the influence classes $\{\mathcal{C}_x\}_{x\in\mathbb{Z}^d}$ should be used.

With the above notation our first main result, whose proof can be found in Sect. 5, can be formulated as follows.

Theorem 3.3 *There exists a universal constant* $\epsilon_0 \in (0,1)$ *such that if there exists* ℓ *and a* ϵ_0*-good set* G_ℓ *on scale* ℓ *then* $\inf_{\Lambda\in\mathbb{F}}(\text{gap}(\mathcal{L}_\Lambda^{\max}) > 0$. *In particular* $\text{gap}(\mathcal{L}) > 0$.

In several examples, e.g. the FA-jf and Modified Basic models, the natural candidate for the event G_ℓ is the event that the tile Λ_0 is "internally spanned", a notion borrowed from bootstrap percolation [2,13,14,22,33]:

Definition 3.4 We say that a finite set $\Gamma \subset \mathbb{Z}^d$ is *internally spanned* by a configuration $\eta \in \Omega$ if, starting from the configuration η^Γ equal to one outside Γ and equal to η inside Γ, there exists a sequence of legal moves inside Γ which connects η^Γ to the configuration identically equal to zero inside Γ and identically equal to one outside Γ.

Of course whether or not the set Λ_0 is internally spanned for η depends only on the restriction of η to Λ_0. One of the major results in bootstrap percolation problems has been the exact evaluation of the μ-probability that the box Λ_0 is internally spanned as a function of the length scale ℓ and the parameter q [2,13,14,22,33]. For non-cooperative models it is obvious that for any $q > 0$ such probability tends very rapidly (exponentially fast) to one as $\ell \to \infty$, since the existence of at least one completely empty finite set $\mathcal{V} + x$, $x \in \Lambda_0$, allows to empty all Λ_0 (see Definition 2.6). For some cooperative systems like e.g. the FA-2f and Modified Basic model in $\mathbb{Z}^2$, it has been shown that for any $q > 0$ such probability tends very rapidly (exponentially fast) to one as $\ell \to \infty$ and that it abruptly jumps from being very small to being close to one as ℓ crosses a critical scale $\ell_c(q)$. In most cases the critical length $\ell_c(q)$ diverges very rapidly as $q \downarrow 0$. Therefore, for such models and $\ell > \ell_c(q)$, one could safely take G_ℓ as the collection of configurations η such that Λ_0 is internally spanned for η. We now formalize what we just said.

Corollary 3.5 *Assume that* $\lim_{\ell\to\infty} \mu(\Lambda_0$ *is internally spanned*$) = 1$ *and that the Markov chain in* Λ_0 *with zero boundary conditions on* $\cup_{x\in\mathcal{K}_0^*} \Lambda_{\ell x}$ *is ergodic. Then* $\text{gap}(\mathcal{L}) > 0$.

The second main result concerns the long time behavior of the persistence function $F(t)$ defined in (2.9).

Theorem 3.6 *Assume that* $\mathrm{gap}(\mathcal{L}) \geq 1/\gamma > 0$. *Then there exists a constant $c > 0$ such that $F(t) \leq e^{-ct}$. For small values of q the constant c can be taken proportional to q/γ.*

Proof Clearly $F(t) = F_1(t) + F_0(t)$ where

$$F_1(t) = \int d\mu(\eta)\, \mathbb{P}(\sigma_0^\eta(s) = 1 \text{ for all } s \leq t)$$

and similarly for $F_0(t)$. We will prove the exponential decay of $F_1(t)$; the case of $F_0(t)$ can be treated in a similar way.

For any $\lambda > 0$ the exponential Chebychev inequality gives

$$F_1(t) = \int d\mu(\eta)\, \mathbb{P}\Big(\int_0^t ds\, \sigma_0^\eta(s) = t\Big) \leq e^{-\lambda t}\, \mathbb{E}_\mu\big(e^{\lambda \int_0^t ds\, \sigma_0^\eta(s)}\big)$$

where $\mathbb{E}_\mu$ denotes the expectation over the process started from the equilibrium distribution μ. On $L^2(\mu)$ consider the self-adjoint operator $H_\lambda := \mathcal{L} + \lambda V$, where V is the multiplication operator by σ_0. By the very definition of the scalar product $< f, g >$ in $L^2(\mu)$ and the Feynman–Kac formula, we can rewrite $\mathbb{E}_\mu(e^{\lambda \int_0^t \sigma_0(s)})$ as $< 1, e^{tH_\lambda} 1 >$. Thus, if β_λ denotes the supremum of the spectrum of H_λ,

$$\mathbb{E}_\mu(e^{\lambda \int_0^t \sigma_0(s)}) \leqslant e^{t\beta_\lambda}.$$

In order to complete the proof we need to show that for suitable positive λ the constant β_λ/λ is strictly smaller than one.

For any norm one function f in the domain of H_λ (which coincides with $\mathrm{Dom}(\mathcal{L})$) write $f = \alpha\mathbf{1} + g$ with $< 1, g > = 0$. Thus

$$\begin{aligned} < f, H_\lambda f > &= < g, \mathcal{L}g > + \alpha^2\lambda < 1, V1 > + \lambda < g, Vg > + 2\lambda\alpha < 1, Vg > \\ &\leq (\lambda - 1/\gamma) < g, g > + \alpha^2\lambda p + 2\lambda|\alpha|\big(< g, g > pq\big)^{1/2} \end{aligned} \tag{3.1}$$

Since $\alpha^2 + < g, g > = 1$

$$\beta_\lambda/\lambda \leq \sup_{0\leq\alpha\leq 1} \Big\{(1 - 1/(\gamma\lambda))(1-\alpha^2) + p\alpha^2 + 2\alpha\big((1-\alpha^2)pq\big)^{1/2}\Big\} \tag{3.2}$$

If we choose $\lambda = 1/(2\gamma)$ the r.h.s. of (3.2) becomes

$$\sup_{0\le\alpha\le 1}(1+p)\alpha^2 - 1 + 2\alpha\big((1-\alpha^2)pq\big)^{1/2}$$
$$\le \sup_{0\le\alpha\le 1}(1+p)\alpha^2 - 1 + 2\big((1-\alpha^2)pq\big)^{1/2} = \frac{pq}{1+p} + p < 1$$

since $p \neq 1$. Thus $F_1(t)$ satisfies

$$F_1(t) \le e^{-t\frac{1}{2\gamma}\frac{q}{1+p}}.$$

A similar computation shows that $F_0(t) \le e^{-t/\gamma c}$ with c independent of q. □

Remark 3.7 The above result indicates that one can obtain *upper bounds* on the spectral gap by proving *lower bounds* on the persistence function. Concretely a lower bound on the persistence function can be obtained by restricting the μ-average to those initial configurations η for which the origin is blocked with high probability for all times $s \le t$. In Sect. 6, we will see few examples of this strategy.

4 Analysis of a general auxiliary model

Consider the model characterized by the influence classes $\mathcal{C}_x = \{\mathcal{K}^*_x\}$, $x \in \mathbb{Z}^d$, an arbitrary finite probability space (S, ν) and a choice of the good event $G \subset S$ with $q := \nu(G)$. For concreteness we will call it the *-*general model* and denote its generator by $\mathcal{L}$. In a finite region $\Lambda \in \mathbb{F}$ and with the above choice for the influence classes, it is immediate to verify that the boundary set is $\mathcal{B} = \partial^*_+\Lambda$. Moreover, due to the oriented character of the influence classes, the maximal choice for the good boundary set, $\mathcal{M} = \mathcal{B}$, is the only possible choice which guarantees ergodicity[3] in finite volume and therefore it is also the minimal one. The generator of the *-general model in Λ, denoted by $\mathcal{L}_\Lambda$, is given by (2.3) with rates (2.6) and $\mathcal{M} = \mathcal{B}$.

The Proof of Theorem 3.3 is based on the positivity of the spectral gap of $\mathcal{L}$ for q large enough, a result which we will prove in Theorem 4.1 below.

In turn, the main ingredient in the Proof of Theorem 4.1 is the bisection technique of [28], combined with the novel idea of considering an accelerated block dynamics which is itself constrained. We will refer to this strategy with the name *Bisection-Constrained* or *B-C approach*.

We now state our main theorem concerning the *-general model.

[3] The fact that with this choice of $\mathcal{M}$ the chain is ergodic follows once we observe that, starting from the sites in Λ whose *-oriented neighborhood is entirely contained in Λ^c and whose existence is proved by induction, we can reach any good configuration $\omega' \in G^\Lambda$ and from there any other configuration $\tilde{\omega}$.

Theorem 4.1 *There exists $q_0 < 1$ independent of S, ν such that for any $q > q_0$*

$$\inf_{\Lambda \in \mathbb{F}} \text{gap}(\mathcal{L}_\Lambda) > 1/2$$

and in particular $\text{gap}(\mathcal{L}) > 0$.

Proof Since we are choosing maximal good boundary conditions, thanks to Lemma 2.11 on the monotonicity of the spectral gap in nested volumes, we need to prove the result only for rectangles. Our approach is based on the "bisection method" introduced in [28,29] and which, in its essence, consists in proving a suitable recursion relation between the spectral gap on scale $2L$ with that on scale L. At the beginning the method requires a simple geometric result (see [8]) which we now describe.

Let $l_k := (3/2)^{k/2}$, and let $\mathbb{F}_k$ be the set of all rectangles $\Lambda \subset \mathbb{Z}^d$ which, modulo translations and permutations of the coordinates, are contained in

$$[0, l_{k+1}] \times \cdots \times [0, l_{k+d}]$$

The main property of $\mathbb{F}_k$ is that each rectangle in $\mathbb{F}_k \backslash \mathbb{F}_{k-1}$ can be obtained as a "slightly overlapping union" of two rectangles in $\mathbb{F}_{k-1}$. More precisely we have:

Lemma 4.2 *For all $k \in \mathbb{Z}_+$, for all $\Lambda \in \mathbb{F}_k \backslash \mathbb{F}_{k-1}$ there exists a finite sequence $\{\Lambda_1^{(i)}, \Lambda_2^{(i)}\}_{i=1}^{s_k}$ in $\mathbb{F}_{k-1}$, where $s_k := \lfloor l_k^{1/3} \rfloor$, such that, letting $\delta_k := \frac{1}{8}\sqrt{l_k} - 2$,*

(i) $\Lambda = \Lambda_1^{(i)} \cup \Lambda_2^{(i)}$,
(ii) $d(\Lambda \backslash \Lambda_1^{(i)}, \Lambda \backslash \Lambda_2^{(i)}) \geq \delta_k$,
(iii) $\left(\Lambda_1^{(i)} \cap \Lambda_2^{(i)}\right) \cap \left(\Lambda_1^{(j)} \cap \Lambda_2^{(j)}\right) = \emptyset, \quad \textit{if } i \neq j$

Proof Proof is given in [8], Proposition 3.2. □

The bisection method then establishes a simple recursive inequality between the quantity $\gamma_k := \sup_{\Lambda \in \mathbb{F}_k} \text{gap}(\mathcal{L}_\Lambda)^{-1}$ on scale k and the same quantity on scale $k-1$ as follows.

Fix $\Lambda \in \mathbb{F}_k \backslash \mathbb{F}_{k-1}$ and write it as $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_1, \Lambda_2 \in \mathbb{F}_{k-1}$ satisfying the properties described in Lemma 4.2 above. Without loss of generality we can assume that all the faces of Λ_1 and of Λ_2 lay on the faces of Λ except for one face orthogonal to the first direction $\vec{e}_1$ and that, along that direction, Λ_1 comes before Λ_2. Set $I \equiv \Lambda_1 \cap \Lambda_2$ and write, for concreteness, $I = [a_1, b_1] \times \cdots \times [a_d, b_d]$. Lemma 4.2 implies that the width of I in the first direction, $b_1 - a_1$, is at least δ_k. Let also $\partial_r I = \{b_1\} \times \cdots \times [a_d, b_d]$ be the right face of I along the first direction.

In what follows, for simplicity, we suppress the index i and we set $B_1 := \Lambda \setminus \Lambda_2$ and $B_2 := \Lambda_2$ (see Fig. 2).

Next, for any $x, y \in \Lambda_1$ and any $\omega \in \Omega$, we write $x \stackrel{\omega}{\to} y$ if there exists a sequence $(x^{(1)}, \ldots, x^{(n)})$ in Λ_1, starting at x and ending at y, such that, for any $j = 1, \ldots, n-1$, $x^{(j)} \sim x^{(j+1)}$ and $\omega_{x^{(j)}} \notin G$, where $\sim$ has been defined in Sect. 2.1. With this notation we finally define the *bad cluster* of x as the

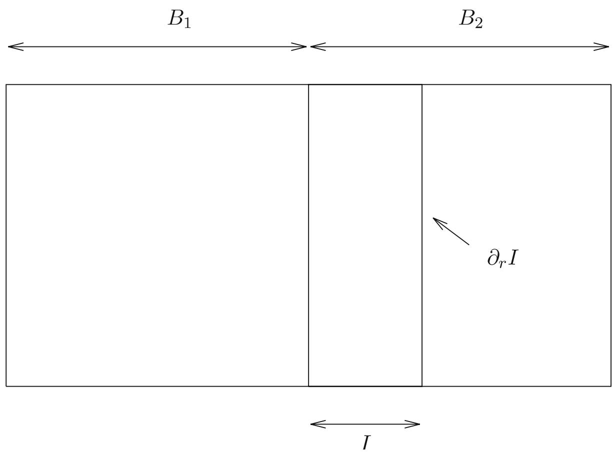

Fig. 2 The two blocks and the strip I

set $A_x(\omega) = \{y \in \Lambda_1; x \overset{\omega}{\to} y\}$. Notice that, by construction, $\omega_z \in G$ for any $z \in \partial^* A_x(\omega)$.

Definition 4.3 We will say that ω is I-good iff, for all $x \in \partial_r I$, the set $A_x(\omega) \cup \partial^* A_x(\omega)$ is contained in B_2.

With the help of the above decomposition we now run the following constrained "block dynamics" on Ω_Λ. The block B_2 waits a mean one exponential random time and then the current configuration inside it is refreshed with a new one sampled from μ_{Λ_2}. The block B_1 does the same but now the configuration is refreshed only if the current configuration ω is I-good.

The Dirichlet form of this auxiliary chain is simply

$$\mathcal{D}_{\text{block}}(f) = \mu_\Lambda \left(c_1 \operatorname{Var}_{B_1}(f) + \operatorname{Var}_{B_2}(f)\right)$$

where $c_1(\omega)$ is just the indicator of the event that ω is I-good and $\operatorname{Var}_{B_1}(f)$, $\operatorname{Var}_{B_2}(f)$ depend on $\omega_{B_1^c}$ and $\omega_{B_2^c}$ respectively.

Denote by $\gamma_{\text{block}}(\Lambda)$ the inverse spectral gap of this auxiliary chain. The following bound, whose proof is postponed for clarity of the exposition, is not difficult to prove.

Proposition 4.4 *Let* $\varepsilon_k \equiv \max_I \mathbb{P}(\omega$ *is not I-good*) *where the* $\max_I$ *is taken over the* s_k *possible choices of the pair* (Λ_1, Λ_2). *Then*

$$\gamma_{\text{block}}(\Lambda) \le \frac{1}{1 - \sqrt{\varepsilon_k}}$$

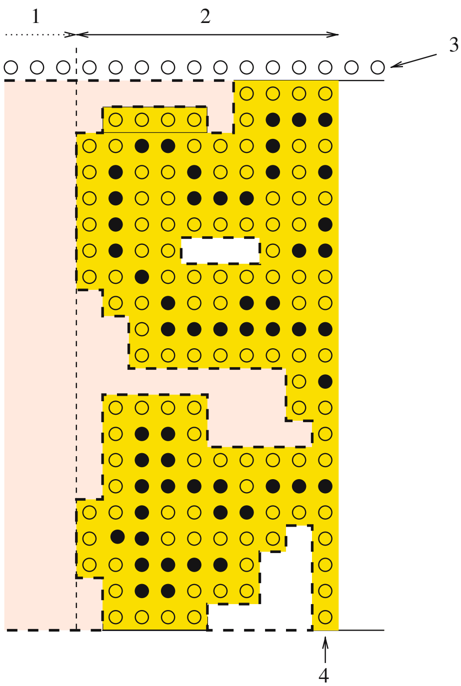

Fig. 3 An example of an I-good configuration ω: empty sites are good and filled ones are not good. The *grey* region is the set $\Pi_\omega \cup \partial^* \Pi_\omega \cup \partial I_r \cap I$. The *dotted lines* mark the connected components of $B_1 \cup I \setminus (\Pi_\omega \cup \partial^* \Pi_\omega \cup \partial I_r)$. The connected component containing B_1 is the *shaded* one

In conclusion, by writing down the standard Poincaré inequality for the block auxiliary chain, we get that for any f

$$\mathrm{Var}_\Lambda(f) \le \left(\frac{1}{1-\sqrt{\varepsilon_k}}\right) \mu_\Lambda\Big(c_1 \,\mathrm{Var}_{B_1}(f) + \mathrm{Var}_{B_2}(f)\Big) \tag{4.1}$$

The second term, using the definition of γ_k and the fact that $B_2 \in \mathbb{F}_{k-1}$ is bounded from above by

$$\mu_\Lambda\Big(\mathrm{Var}_{B_2}(f)\Big) \le \gamma_{k-1} \sum_{x\in B_2} \mu_\Lambda\big(c_{x,B_2}\,\mathrm{Var}_x(f)\big) \tag{4.2}$$

Notice that, by construction, for all $x \in B_2$ and all ω, $c_{x,B_2}(\omega) = c_{x,\Lambda}(\omega)$. Therefore the term $\sum_{x\in B_2} \mu_\Lambda\big(c_{x,B_2}\,\mathrm{Var}_x(f)\big)$ is nothing but the contribution carried by the set B_2 to the full Dirichlet form $\mathcal{D}_\Lambda(f)$.

Next we examine the more complicate term $\mu_\Lambda\Big(c_1\,\mathrm{Var}_{B_1}(f)\Big)$ with the goal in mind to bound it with the missing term of the full Dirichlet form $\mathcal{D}_\Lambda(f)$.

For any I-good ω, let $\Pi_\omega = \cup_{x\in\partial_r I} A_x(\omega)$ and let B_ω be the connected (w.r.t. the graph structure induced by the $\sim$ relationship) component of $B_1 \cup I \setminus (\Pi_\omega \cup \partial^*\Pi_\omega \cup \partial_r I)$ which contains B_1 (see Fig. 3).

A first key observation is now the following.

Claim 4.5 *For any $z \in \partial^*_+ B_\omega$ it holds true that $\omega_z \in G$.*

Proof of the claim To prove the claim suppose the opposite and let $z \in \partial_+^* B_\omega$ be such that $\omega_z \notin G$ and let $x \in B_\omega$ be such that $\mathcal{K}_x^* \ni z$. Necessarily $z \in \Pi_\omega$ because of the good boundary conditions in $\partial_+^*\Lambda$ and the fact that $\omega_y \in G$ for all $y \in \partial^*\Pi_\omega \cup (\partial_r I \setminus \Pi_\omega)$. However $z \in \Pi_\omega$ is impossible because in that case $z \in A_y(\omega)$ for some $y \in \partial_r I$ and therefore $x \in A_y(\omega) \cup \partial^* A_y(\omega)$ i.e. $x \in \Pi_\omega \cup \partial^*\Pi_\omega$, a contradiction. □

The second observation is the following.

Claim 4.6 *For any* $\Gamma \in \Omega_\Pi := \{\Pi_{\omega,\omega} I - \text{good}\}$, *the event* $\{\omega : \ \Pi_\omega = \Gamma\}$ *does not depend on the values of* ω *in* B_Γ, *the connected component* (*w.r.t.* $\sim$) *of* $B_1 \cup I \setminus \Gamma \cup \partial^*\Gamma \cup \partial_r I$ *which contains* B_1.

Proof of the claim Fix $\Gamma \in \Omega_\Pi$. The event $\Pi_\omega = \Gamma$ is equivalent to:

(i) $\omega_z \in G$ for any $z \in \partial_r I \setminus \Gamma$;
(ii) $\omega_z \in G$ for any $z \in \partial^*\Gamma \cap I$;
(iii) $\omega_z \notin G$ for all $z \in \Gamma$.

In fact trivially $\Pi_\omega = \Gamma$ implies (i),(ii) and (iii). To prove the other direction we first observe that (i) and (iii) imply that $\Pi_\omega \supset \Gamma$. If $\Pi_\omega \neq \Gamma$ there exists $z \in \Pi_\omega \setminus \Gamma$ which is in $\partial^*\Gamma \cap I$ and such that $\omega_z \notin G$. That is clearly impossible because of (ii). □

If we observe that $\mathrm{Var}_{B_1}(f)$ depends only on ω_{B_2}, we can write (we omit the subscript Λ for simplicity)

$$\begin{aligned}
\mu\Big(c_1 \,\mathrm{Var}_{B_1}(f)\Big) &= \sum_{\Gamma\in\Omega_\Pi} \mu\big(\mathbb{1}_{\{\Pi_\omega=\Gamma\}}\,\mathrm{Var}_{B_1}(f)\big) \\
&= \sum_{\Gamma\in\Omega_\Pi}\sum_{\omega_{B_2\setminus I}} \mu(\omega_{B_2\setminus I}) \sum_{\omega_I} \mu(\omega_I)\mathbb{1}_{\{\Pi_\omega=\Gamma\}}\,\mathrm{Var}_{B_1}(f) \\
&= \sum_{\Gamma\in\Omega_\Pi}\sum_{\omega_{B_2\setminus I}} \mu(\omega_{B_2\setminus I}) \sum_{\omega_{I\setminus I_\Gamma}} \mu(\omega_{I\setminus I_\Gamma})\mathbb{1}_{\{\Pi_\omega=\Gamma\}} \sum_{\omega_{I_\Gamma}} \mu(\omega_{I_\Gamma})\,\mathrm{Var}_{B_1}(f) \qquad (4.3)
\end{aligned}$$

where $I_\Gamma = B_\Gamma \cap I$ and we used the independence of $\mathbb{1}_{\{\Pi_\omega=\Gamma\}}$ from ω_{I_Γ}.

The convexity of the variance implies that

$$\sum_{\omega_{I_\Gamma}} \mu(\omega_{I_\Gamma})\,\mathrm{Var}_{B_1}(f) \le \mathrm{Var}_{B_\Gamma}(f) \qquad (4.4)$$

The Poincaré inequality together with Lemma 2.11 finally gives

$$\begin{aligned}
\mathrm{Var}_{B_\Gamma}(f) &\le \mathrm{gap}(\mathcal{L}_{B_\Gamma})^{-1} \sum_{x\in B_\Gamma} \mu_{B_\Gamma}\big(c_{x,B_\Gamma}\,\mathrm{Var}_x(f)\big) \\
&\le \mathrm{gap}(\mathcal{L}_{B_1\cup I})^{-1} \sum_{x\in B_\Gamma} \mu_{B_\Gamma}\big(c_{x,B_\Gamma}\,\mathrm{Var}_x(f)\big) \qquad (4.5)
\end{aligned}$$

The role of the event $\{\Pi_\omega = \Gamma\}$ should at this point be clear: it guarantees (see Claim 4.5) that for any $\omega \in \Omega_\Lambda$ such that $\Pi_\omega = \Gamma$, ω_z is in the good event G for all sites z in the boundary set of the restricted volume B_Γ, which implies

$$c_{x,\Lambda}(\omega) = c_{x,B_\Gamma}(\omega_{B_\Gamma}) \quad \forall x \in B_\Gamma \tag{4.6}$$

if $\mathbb{1}_{\{\Pi_\omega=\Gamma\}} = 1$, where ω_{B_Γ} is the restriction of ω to the set B_Γ. Indeed if $\mathcal{C}_x \subset B_\Gamma \subset \Lambda$, then $c_{x,\Lambda}$ and c_{x,B_Γ} depend only on the configuration in $\mathcal{C}_x \subset B_\Gamma$, where ω and ω_{B_Γ} coincide. Otherwise, if $\mathcal{C}_x \not\subset B_\Gamma$ and $z \in \mathcal{C}_x$ with $z \notin B_\Gamma$, then c_{x,B_Γ} is defined by fixing good configuration on z, which is also true for $c_{x,\Lambda}(\omega)$ even if $z \notin \partial_+^* \Lambda$ thanks to the result of Claim 4.5 and to the constraint forced by $\mathbb{1}_{\{\Pi_\omega=\Gamma\}} = 1$.

If we finally plug (4.4), (4.5) and (4.6) in the r.h.s. of (4.3) and recall that $B_1 \cup I = \Lambda_1 \in \mathcal{F}_{k-1}$, we obtain

$$\begin{aligned} \mu_\Lambda\Big(c_1 \operatorname{Var}_{B_1}(f)\Big) &\le \operatorname{gap}(\mathcal{L}_{\Lambda_1})^{-1} \mu_\Lambda\big(c_1 \sum_{x \in B_{\Pi_\omega}} c_{x,\Lambda} \operatorname{Var}_x(f)\big) \\ &\le \gamma_{k-1}\, \mu_\Lambda\big(\sum_{x\in\Lambda_1} c_{x,\Lambda} \operatorname{Var}_x(f)\big) \end{aligned} \tag{4.7}$$

This, together with (4.1) and (4.2), yields

$$\operatorname{Var}_\Lambda(f) \le \Big(\frac{1}{1-\sqrt{\varepsilon_k}}\Big)\gamma_{k-1}\Big(\mathcal{D}_\Lambda(f) + \sum_{x\in\Lambda_1\cap\Lambda_2} \mu_\Lambda\big(c_{x,\Lambda} \operatorname{Var}_x(f)\big)\Big) \tag{4.8}$$

Averaging over the $s_k = \lfloor l_k^{1/3} \rfloor$ possible choices of the sets Λ_1, Λ_2 gives

$$\operatorname{Var}_\Lambda(f) \le \Big(\frac{1}{1-\sqrt{\varepsilon_k}}\Big)\gamma_{k-1}\Big(1+\frac{1}{s_k}\Big)\mathcal{D}_\Lambda(f) \tag{4.9}$$

which implies that

$$\gamma_k \le \Big(\frac{1}{1-\sqrt{\varepsilon_k}}\Big)\Big(1+\frac{1}{s_k}\Big)\gamma_{k-1} \tag{4.10}$$

$$\le \gamma_{k_0} \prod_{j=k_0}^{k} \Big(\frac{1}{1-\sqrt{\varepsilon_j}}\Big)\Big(1+\frac{1}{s_j}\Big) \tag{4.11}$$

where k_0 is the smallest integer such that $\delta_{k_0} > 1$ and we recall (see Proposition 4.4) that ε_j is the probability that a configuration is not I-good, maximized over all the s_j possible choices for the stripe I.

It is at this stage (and only here) that we need a restriction on the probability q of the good set G. If q is taken large enough (but uniformly in the cardinality of S), elementary percolation arguments imply that the quantity ε_j becomes exponentially small in $\delta_j = \frac{1}{8}\sqrt{l_j} - 2$ (the minimum width of the intersection between the rectangles Λ_1, Λ_2 on scale l_j) with a large constant rate and the convergence to 1 of the infinite product $\prod_{j=k_0}^{\infty}\big(\frac{1}{1-\sqrt{\varepsilon_j}}\big)(1+\frac{1}{s_j})$ follows. Furthermore, since an elementary coupling argument gives $\gamma_{k_0} \leqslant (l/q)^{\alpha_{k_0}}$ for a suitable constant α_{k_0}, we conclude that there exists a universal constant $q_0 < 1$ (i.e. independent on the choice of S, ν which characterize each *-general model) such that for $q > q_0$ the product $\gamma_{k_0}\ \prod_{j=k_0}^{k}\big(\frac{1}{1-\sqrt{\varepsilon_j}}\big)(1+\frac{1}{s_j})$ is smaller than 2. □

Proof of Proposition 4.4 For any mean zero function $f \in L^2(\Omega_\Lambda, \mu_\Lambda)$ let

$$\pi_1 f := \mu_{B_2}(f), \quad \pi_2 f := \mu_{B_1}(f)$$

be the natural projections onto $L^2(\Omega_{B_i}, \mu_{B_i})$, $i = 1, 2$. Obviously $\pi_1\pi_2 f = \pi_2\pi_1 f = 0$. The generator of the block dynamics can then be written as:

$$\mathcal{L}_{\text{block}} f = c_1\big(\pi_2 f - f\big) + \pi_1 f - f$$

and the associated eigenvalue equation as

$$c_1\big(\pi_2 f - f\big) + \pi_1 f - f = \lambda f. \tag{4.12}$$

By taking $f(\sigma_\Lambda) = g(\sigma_{B_2})$ we see that $\lambda = -1$ is an eigenvalue. Moreover, since $c_1 \leq 1$, $\lambda \geq -2$. Assume now $0 > \lambda > -1$ and apply π_2 to both sides of (4.12) to obtain (recall that $c_1 = c_1(\sigma_{B_2})$)

$$-\pi_2 f = \lambda \pi_2 f \Rightarrow \pi_2 f = 0 \tag{4.13}$$

For any f with $\pi_2 f = 0$ the eigenvalue equation becomes

$$f = \frac{\pi_1 f}{1 + \lambda + c_1} \tag{4.14}$$

and that is possible only if

$$\mu_{B_2}\Big(\frac{1}{1+\lambda+c_1}\Big) = 1.$$

We can solve the equation to get

$$\lambda = -1 + \sqrt{1 - \mu_{B_2}(c_1)} \leq -1 + \sqrt{\varepsilon_k}.$$

□

5 Proof of Theorem 3.3

If we consider the *-general model analyzed in the previous section with $S = \{0, 1\}$ and $G = \{0\}$, we immediately realized that all the 0–1 KCSM defined in Sect. 2.3 are dominated by it. Therefore Theorem 4.1 together with the comparison technique explained in Sect. 2.4 imply positivity of the spectral gap for all the models in Sect. 2.3 provided that the corresponding parameter q satisfies $q > q_0$. However in Theorem 3.3 a much stronger result is claimed, namely positivity of the spectral gap in the whole ergodic region $q > q_c$. It is therefore not surprising that we need some renormalization or block analysis to go beyond the perturbative regime. Such an approach is what we call the *Renormalization-Constrained* or *R-C approach* and it basically allows us to map all models at $q > q_c$ into a *-general model with triple (S, ν, G) depending on the original model but with $\nu(G) \geq q_0$.

Let us now provide the technical details. For the relevant notation we refer the reader to Sect. 3.

Define $\epsilon_0 = 1 - q_0$ where q_0 is the threshold appearing in Theorem 4.1 and assume that ℓ is such that there exists a ϵ_0-good event G_ℓ on scale ℓ. Consider the *-general model on $\mathbb{Z}^d(\ell)$ with $S = \{0, 1\}^{\Lambda_0}$, $\nu = \mu_{\Lambda_0}$ and good event G_ℓ. Obviously the two probability spaces $\Omega = \left(\{0, 1\}^{\mathbb{Z}^d}, \mu\right)$ and $\Omega(\ell) = \left(S^{\mathbb{Z}^d(\ell)}, \prod_{x\in\mathbb{Z}^d(\ell)} \nu_x\right)$ coincide. Thanks to condition (a) on G_ℓ we can use theorem 4.1 to get that for any $f \in Dom(\mathcal{L})$

$$\operatorname{Var}(f) \leq 2 \sum_{x\in\mathbb{Z}^d(\ell)} \mu\left(\tilde{c}_x \operatorname{Var}_{\Lambda_x}(f)\right) \tag{5.1}$$

where the (renormalized) rate $\tilde{c}_x(\sigma)$ is simply the indicator function of the event that for any $y \in \mathcal{K}^*_{\{x/\ell\}}$ the restriction of σ to the rectangle $\Lambda_{\ell y}$ belongs to the good set G_ℓ on scale ℓ.

In the sequel we will often refer to (5.1) as the *renormalized-Poincaré inequality* with parameters (ℓ, G_ℓ).

Let us examine a generic term $\mu\left(\tilde{c}_x(\xi) \operatorname{Var}_{\Lambda_x}(f)\right)$ which we write as

$$\frac{1}{2} \int d\mu(\xi)\tilde{c}_x(\xi) \int\int d\mu_{\Lambda_x}(\sigma) d\mu_{\Lambda_x}(\eta)\left[f(\sigma\cdot\xi) - f(\eta\cdot\xi)\right]^2 \tag{5.2}$$

By assumption, if $\tilde{c}_x(\xi) = 1$ necessarily there exists τ and a sequence of configurations $(\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(n)})$, $n \leq 3\ell^d$, with the following properties:

(i) $\xi^{(0)} = \xi$ and $\xi^{(n)} = \tau$;
(ii) the chain in Λ_x with boundary conditions τ is ergodic;
(iii) $\xi^{(i+1)}$ is obtained from $\xi^{(i)}$ by changing exactly only one spin at a suitable site $x^{(i)} \in \cup_{y\in\mathcal{K}^*_{\{x/\ell\}}} \Lambda_{\ell y}$;
(iv) the move at $x^{(i)}$ leading from $\xi^{(i)}$ to $\xi^{(i+1)}$ is permitted i.e. $c_{x^{(i)}}(\xi^{(i)}) = 1$ for every $i = 0, \ldots, n$.

Remark 5.1 Notice that for any $i = 0, \dots, n$, the intermediate configuration $\xi^{(i)}$ coincides with ξ outside $\cup_{y \in \mathcal{K}^*_{\{x/\ell\}}} \Lambda_{\ell y}$. Therefore, given $\xi^{(i)} = \eta$, the number of starting configurations $\xi = \xi^{(0)}$ compatible with η is bounded from above by $2^{(2^d-1)\ell^d}$ and the relative probability $\mu(\xi)/\mu(\eta)$ by $\max\big(\frac{p}{q}, \frac{q}{p}\big)^{(2^d-1)\ell^d}$.

By adding and subtracting the terms $f(\sigma \cdot \tau), f(\eta \cdot \tau)$ inside $\big[f(\sigma \cdot \xi) - f(\eta \cdot \xi)\big]^2$ and by writing $f(\sigma \cdot \tau) - f(\sigma \cdot \xi)$ as a telescopic sum $\sum_{i=1}^{n-1}\big[f(\sigma \cdot \xi_{i+1}) - f(\sigma \cdot \xi_i)\big]$ we get

$$\begin{aligned} &\big[f(\sigma \cdot \xi) - f(\eta \cdot \xi)\big]^2 \le 3\big[f(\sigma \cdot \tau) - f(\eta \cdot \tau)\big]^2 \\ &\quad + 3n \sum_{i=1}^{n-1}\big[f(\sigma \cdot \xi^{(i+1)}) - f(\sigma \cdot \xi^{(i)})\big]^2 + 3n \sum_{i=1}^{n-1}\big[f(\eta \cdot \xi^{(i+1)}) - f(\eta \cdot \xi^{(i)})\big]^2 \end{aligned} \tag{5.3}$$

If we plug (5.3) inside the r.h.s. of (5.2) and use properties (i),…,(iv) of the intermediate configurations $\{\xi^{(i)}\}_{i=1}^n$ together with the Remark 5.1 and the fact that the inverse spectral gap in Λ_x with ergodic boundary conditions τ is bounded from above by a constant depending only on (q, ℓ), we get that there exists a finite constant $c := c(q, \ell)$ such that

$$\mu\big(\tilde{c}_x(\xi)\,\mathrm{Var}_{\Lambda_x}(f)\big) \le c \sum_{y \in \Lambda_x \cup_{y \in \mathcal{K}^*_{\{x/\ell\}}} \Lambda_{\ell y}} \mu\big(c_y\,\mathrm{Var}_y(f)\big)$$

and therefore, thanks to (5.1), the proof of the positivity of the spectral gap is complete.

6 Specific models

In this section, we analyze the specific models that have been introduced in Sect. 2 and for each of them we prove positivity of the spectral gap for $q > q_c$ together with upper and lower bounds as $q \downarrow q_c$.

6.1 The East model

As a first application of the Bisection-Constrained method explained in Sect. 5 in the context of the *-general model, we reprove the result contained in [3] on the positivity of the spectral gap, but we sharpen (by a power of 2) their lower bound.

Theorem 6.1 *For any $q \in (0, 1)$ the spectral gap of the East model is positive. Moreover, for any $\delta \in (0, 1)$ there exists $C_\delta > 0$ such that*

$$\mathrm{gap}(\mathcal{L}) \ge C_\delta q^{\log_2(1/q)/(2-\delta)} \tag{6.1}$$

In particular

$$\lim_{q\to 0} \log(1/\operatorname{gap})/(\log(1/q))^2 = (2\log 2)^{-1} \tag{6.2}$$

Remark 6.2 Notice that (6.2) disproves the asymptotic behavior of the spectral gap suggested in [34].

Proof The limiting result (6.2) follows at once from the lower bound together with the analogous upper bound proved in [3].

In order to get the lower bound (6.1) we want to apply directly the bisection method used in the proof of Theorem 4.1 but we need to choose the length scales l_k a little bit more carefully.

Fix $\delta \in (0,1)$ and define $l_k = 2^k$, $\delta_k = \lfloor l_k^{1-\delta/2} \rfloor, s_k := \lfloor l_k^{\delta/6} \rfloor$. Let also $\mathbb{F}_k$ be the set of intervals which, modulo translations, have the form $[0,\ell]$ with $\ell \in [l_k, l_k + l_k^{1-\delta/6}]$ and define γ_k as the worst case over the elements $\Lambda \in \mathbb{F}_k$ of the inverse spectral gap in Λ with empty boundary condition at the right boundary of Λ. Thanks to Lemma 2.11 the worst case is attained for the interval $\Lambda_k = [0, l_k + l_k^{1-\delta/6}]$. With these notation there exists k_δ independent of q such that the same result of Lemma 4.2 holds true as long as $k \geq k_\delta$. We can then repeat exactly the same analysis done in the proof of Theorem 4.1 to get that

$$\gamma_k \leq \gamma_{k_\delta} \prod_{j=k_\delta}^{\infty} \left(\frac{1}{1-\sqrt{\varepsilon_j}}\right) \prod_{j=k_\delta}^{\infty} \left(1 + \frac{1}{s_j}\right) \tag{6.3}$$

Here the quantity ε_k is just the probability that an interval of width δ_k is fully occupied (see Proposition 4.4) i.e. $\varepsilon_k = p^{\delta_k}$. The convergence of the product in (6.3) is thus guaranteed and the positivity of the spectral gap follows.

Let us now discuss the asymptotic behavior of the gap as $q \downarrow 0$.

We first observe that $\gamma_{k_\delta} < (1/q)^{\alpha_\delta}$ for some finite α_δ. That follows e.g. from a coupling argument. In a time lag one and with probability larger than q^{α_δ} for suitable α_δ, any configuration in Λ_{k_δ} can reach the empty configuration by just flipping one after another the spins starting from the right boundary. In other words, under the maximal coupling, two arbitrary configurations will couple in a time lag one with probability larger than q^{α_δ} i.e. $\gamma_{k_\delta} < (1/q)^{\alpha_\delta}$. We now analyze the two infinite products in (6.3). The second one, due to the exponential growth of the scales, is bounded by a constant independent of q.

To bound the first factor, we define

$$j_* = \min\{j : \varepsilon_j \leq e^{-1}\} \approx \log_2(1/q)/(1-\delta/2)$$

and we obtain

$$\prod_{j=k_\delta}^{\infty} \left(\frac{1}{1-\sqrt{\varepsilon_j}}\right) \leq \prod_{j=1}^{j_*} \left(\frac{1+\sqrt{\varepsilon_j}}{1-\varepsilon_j}\right) \prod_{j>j_*}^{\infty} \left(\frac{1}{1-\sqrt{\varepsilon_j}}\right) \leq e^C\, 2^{j_*} \prod_{j=1}^{j_*} \left(\frac{1}{1-\varepsilon_j}\right) \tag{6.4}$$

where we used the bound $1/(1-\sqrt{\varepsilon_i}) \le 1 + (e/(e-1))\sqrt{\varepsilon_j}$ valid for any $j \ge j_*$ together with

$$\sum_{j>j_*}^{\infty} \log\left(1 + \frac{e}{e-1}\sqrt{\varepsilon_j}\right) \le \frac{e}{e-1}\sum_{j>j_*}^{\infty}\sqrt{\varepsilon_j}$$
$$\le \frac{e}{e-1}\int_{j_*-1}^{\infty} dx\ \exp(-q(2^{x(1-\delta/2)})/2) = A_\delta \int_{2^{(j_*-1)(1-\delta/2)}}^{\infty} dz\ \exp(-qz/2)/z$$
$$\le 2A_\delta 2^{-(j_*-1)(1-\delta/2)} q^{-1} \exp(-q2^{(j_*-1)(1-\delta/2)}/2) \le C$$

for some constant C independent of q.

Observe now that $1-\varepsilon_j \ge 1-e^{-q\delta_j} \ge Aq\delta_j$ for any $j \le j_*$ and some constant $A \approx e^{-1}$. Thus the r.h.s. of (6.4) is bounded from above by

$$e^C\,(\frac{2}{Aq})^{j_*}\prod_{j=1}^{j_*}\delta_j^{-1} \le \frac{1}{q^a}\,(1/q)^{j_*}\,2^{-(1-\delta/2)j_*^2/2} \approx \frac{1}{q^a}\,(1/q)^{\log_2(1/q)/(2-\delta)}$$

as $q \downarrow 0$ for some constant a.

6.2 FA-1f model

In this section, we deal with the FA-1f model. Our main result are the followings

Theorem 6.3 *For any $q \in (0,1)$ the spectral gap of the FA-1f model is positive.*

Theorem 6.4 *For any $d \ge 1$, there exists a constant $C = C(d)$ such that for any $q \in (0,1)$, the spectral gap* $\mathrm{gap}(\mathcal{L})$ *satisfies the following bounds.*

$$\begin{aligned} C^{-1}q^3 &\leqslant \mathrm{gap}(\mathcal{L}) \leqslant Cq^3 && \text{for } d=1,\\ C^{-1}q^2/\log(1/q) &\leqslant \mathrm{gap}(\mathcal{L}) \leqslant Cq^2 && \text{for } d=2,\\ C^{-1}q^2 &\leqslant \mathrm{gap}(\mathcal{L}) \leqslant Cq^{1+\frac{2}{d}} && \text{for } d\geqslant 3.\end{aligned}$$

Proof of Theorem 6.3 The proof follows at once from Corollary 3.5 because the probability that the rectangle Λ_0 of side ℓ is internally spanned is equal to the probability that Λ_0 is not fully occupied which is equal to $1-(1-q)^{\ell^d} \uparrow 1$ as $\ell \to \infty$. □

In the next result we discuss the asymptotic of the spectral gap for $q \downarrow 0$. Such a problem has been discussed at length in the physical literature with varying results based on numerical simulations and/or analytical work [6,7,24]. As a preparation for our bounds we observe that on average the vacancies are at distance $O(q^{-1/d})$ and each one of them roughly performs a random walk with jump rate proportional to q. Therefore a possible guess is that

$$\text{gap}(\mathcal{L}) = O(q \times \text{gap of a simple RW in a box of side } O(q^{-1/d})\ ) = O(q^{1+2/d})$$

However, a recent work in the physics community [24], which uses the mapping of FA-1f into a diffusion limited aggregation model and its treatment by renormalization theory, leads to the conjecture gap $\approx q^2$ for any $d \geq 2$. For $d = 2$ the two above conjectures coincide and we prove that indeed $\text{gap}(\mathcal{L}) = O(q^2)$ up to log corrections. On the other hand, for $d \geq 3$ we are not able to prove or disprove any of the two conjectures: our bounds are consistent with both of them.

Proof of Theorem 6.4 We begin by proving the upper bounds via a careful choice of a test function to plug into the variational characterization for the spectral gap. Before doing that, we notice that we could alternatively use the strategy outlined in Remark 3.7, which uses the result in Theorem 3.6 relating the persistence function to the spectral gap. However this strategy, which will be used in the next sections to obtain the upper bounds for MB, FA-jf and for N-E models, provides a very poor upper bound for FA-1f. Indeed, the use of the persistence result together with the fact that a region has probability $\simeq 1/2$ to be internally spanned if $l \simeq 1/q^{1/d}$ (see proof of Theorem 6.3), would imply $\text{gap}(\mathcal{L}) \leqslant q^{1/d-1}$. Let us now describe our choice for the test functions.

Fix $d \geq 1$ and assume, without loss of generality, $q \ll 1$. Let also $\ell_q = \left(\frac{\log(1-q_0)}{\log(1-q)}\right)^{1/d} \approx \lambda_0 q^{-1/d}$ with $\lambda_0 = |\log(1-q_0)|^{1/d}$, where q_0 is as in Theorem 4.1. Let g be a smooth function on $[0,1]$ with support in $[1/4, 3/4]$ and such that

$$\int_0^1 \alpha^{d-1} e^{-\alpha^d} g(\alpha) d\alpha = 0 \quad \text{and} \quad \int_0^1 \alpha^{d-1} e^{-\alpha^d} g^2(\alpha) d\alpha = 1. \tag{6.5}$$

Set (see Fig. 4)

$$\xi(\sigma) := \sup \{\ell : \sigma(x) = 1 \text{ for all } x \text{ such that } \|x\|_\infty < \ell\}$$

and notice that for any $k = 0, \dots, \ell_q$,

$$\mu(\xi = k) = p^{k^d} - p^{(k+1)^d} \approx qdk^{d-1} e^{-qk^d} \tag{6.6}$$

Having defined the r.v. ξ the test function we will use is $f = g(\xi/\ell_q)$. Using (6.6) together with (6.5) one can check that

$$\text{Var}(f) \approx \frac{1}{\ell_q} \approx q^{1/d}. \tag{6.7}$$

On the other hand, by writing T_x for the spin-flip operator in x, *i.e.* $T_x(\sigma) = \sigma^x$ using reversibility we have

$$\begin{aligned}\mathcal{D}(f) &= \sum_{x\in\mathbb{Z}^d} \mu\left[c_x\left[g(\frac{\xi\circ T_x}{\ell_q}) - g(\frac{\xi}{\ell_q})\right]^2\right] \\ &= \sum_{x\in\mathbb{Z}^d}\sum_{k=0}^{\ell_q} \mu\left[c_x\left[g(\frac{\xi\circ T_x}{\ell_q}) - g(\frac{\xi}{\ell_q})\right]^2 \mathbb{1}_{\xi=k}\right] \\ &= 2\sum_{k=\lfloor\frac{1}{4}\ell_q-1\rfloor}^{\lfloor\frac{3}{4}\ell_q\rfloor} \left(g(\frac{k+1}{\ell_q}) - g(\frac{k}{\ell_q})\right)^2 \sum_{\substack{x\\ \|x\|_\infty=k+1}} \mu\left(c_x \mathbb{1}_{\xi\circ T_x=k+1}\mathbb{1}_{\xi=k}\right).\end{aligned} \tag{6.8}$$

Notice that for any k, any x such that $\|x\|_\infty = k+1$,

$$\begin{aligned}&\mu\left(c_x \mathbb{1}_{\xi\circ T_x=k+1}\mathbb{1}_{\xi=k}\right) \\ &\quad= \mu\left(c_x \,|\, \xi\circ T_x = k+1, \xi = k\right)\mu\left(\xi\circ T_x = k+1 \,|\, \xi = k\right)\mu\left(\xi = k\right) \\ &\quad\leqslant c\,\frac{q}{k^{d-1}}\mu\left(\xi = k\right)\end{aligned}$$

for some constant c depending only on d. The factor q above comes from the fact that, given that $\xi = k$ and $\xi\circ T_x = k+1$, x is necessarily the only empty site in the $(k+1)$th-layer. Therefore, the flip at x can occur only if the nearest neighbor of x in the next layer is empty (see Fig. 4). Moreover, given $\xi = k$, the conditional probability of having zero at x and the rest of the layer completely filled is of order $1/k^{d-1}$. It follows that

$$\sum_{x:\|x\|_\infty=k+1} \mu\left(c_x \mathbb{1}_{\xi\circ T_x=k+1}\mathbb{1}_{\xi=k}\right) \leqslant c' q\mu\left(\xi = k\right).$$

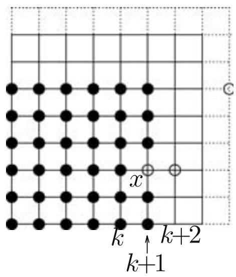

Fig. 4 In dimension 2, a configuration σ where $\xi(\sigma) = k$ and $\xi(\sigma^x) = k+1$

In conclusion, using (6.6) and writing $\alpha := k/\ell_q$,

$$\begin{aligned}\mathcal{D}(f) &\leqslant c''q \sum_{k=\lfloor\frac{1}{4}\ell_q-1\rfloor}^{\lfloor\frac{3}{4}\ell_q\rfloor} \mu\left(\xi = k\right) \Big(g\big(\frac{k+1}{\ell_q}\big) - g\big(\frac{k}{\ell_q}\big)\Big)^2 \\ &\approx \frac{q}{\ell_q^3} \int_{\frac{1}{4}}^{\frac{3}{4}} \alpha^{d-1} e^{-(\lambda_0\alpha)^d} g'\left(\alpha\right)^2 d\alpha \approx \frac{q^{1+\frac{2}{d}}}{\ell_q}.\end{aligned} \tag{6.9}$$

as $q \downarrow 0$. The upper bound on the spectral gap follows from (6.7), (6.9) and (2.8).

We now discuss the lower bound. We will not use the B-C approach used for the East model, since this would lead to very poor bounds. For example in $d = 1$ we would obtain the same bound (6.1) as for East, which is far from the claimed $gap \geqslant cq^3$. We will instead start by relating the spectral gap in infinite volume to the spectral gap in a q-dependent finite region via a variant Renormalization-Constrained approach discussed in Sect. 5. More precisely, we will start again by the renormalized-Poincaré inequality (5.1) for a proper *-general model. However, instead of writing inequalities corresponding to (5.2) and (5.3) to treat the variances inside the blocks, we use a different method which is feasible thanks to the non-cooperative character of the model.[4]

Denote by $\mathcal{L}_\Lambda^{\{z\}}$ the generator of the FA-1f model in Λ with good (and minimal) boundary set $\mathcal{M} = \{z\} \subset \partial\Lambda$. With a slight abuse of notation we denote by $c_{x,\Lambda}^{\{z\}}$ the corresponding rates.

Lemma 6.5 *Let* $\Lambda_r := \{x \in \mathbb{Z}^d : \|x\|_\infty \leqslant r-1\}$ *and define* $\operatorname{gap}(q) := \inf_{z,r} \operatorname{gap}(\mathcal{L}_{\Lambda_r}^{\{z\}})$ *where the infimum is over* $z \in \partial\Lambda_r$ *and* $r \in [\ell_q, 2\ell_q]$. *Then there exists a constant* $C = C(d)$ *such that*

$$\operatorname{gap}(\mathcal{L}) \geq C \operatorname{gap}(q)$$

Notice that, since we are dealing with minimal good boundary conditions and not with the maximal ones ($\mathcal{M} = \mathcal{B} = \partial\Lambda$), the monotonicity of the spectral gap in nested volumes does not necessarily hold (see the remark below Lemma 2.11).

Proof of the Lemma The starting point is the bound (5.1) for $\ell = \ell_q$:

$$\operatorname{Var}(f) \leq 2 \sum_{x\in\mathbb{Z}^d(\ell_q)} \mu\big(\tilde{c}_x \operatorname{Var}_{\Lambda_x}(f)\big) \tag{6.10}$$

[4] Alternatively we could follow exactly the R-C approach, as will be done in next section for FA-2f and MB. However, since we are interested in the asymptotic for $q \downarrow 0$, properly weighted paths (in the same spirit as the proof of Proposition 6.6) should be used when rewriting the second and third term of (5.3) in terms of the Dirichlet form of FA-1f.

Recall that $\tilde{c}_x(\sigma)$ is simply the indicator function of the event that for any $y \in \mathcal{K}^*_{\{x/\ell\}}$ the block $\Lambda_{\ell y}$ is internally spanned for σ, i.e. it is not completely filled. Let us examine a generic term $\mu\big(\tilde{c}_x \operatorname{Var}_{\Lambda_x}(f)\big)$. Given σ such that $\tilde{c}_x(\sigma) = 1$, let $r = r(\sigma)$ be the largest $0 \le r \le \ell_q$ such that there exists an empty site on $\partial\Lambda_{r,x}$, where $\Lambda_{r,x} = \{y : d_\infty(y, \Lambda_x) \le r\}$. Let also $\xi(\sigma)$ be the first (w.r.t a chosen order) empty site in $\partial\Lambda_{r,x}$ of σ. Exactly as in the proof of Theorem 4.1 the convexity of the variance implies that

$$\mu\big(\tilde{c}_x \operatorname{Var}_{\Lambda_x}(f)\big) \le \mu\Big(\mathbb{1}_{r\le\ell_q} \sum_{z\in\partial\Lambda_{r,x}} \mathbb{1}_{\xi=z} \operatorname{Var}_{\Lambda_{r,x}}(f)\Big) \tag{6.11}$$

Thanks to the constraint $\mathbb{1}_{\xi=z}$, the variance $\operatorname{Var}_{\Lambda_{r,x}}(f)$ is computed with at least an empty site in z and we can use the Poincaré inequality for the FA-1f model in $\Lambda_{r,x}$ with good boundary set $\mathcal{M} = \{z\}$ (i.e. corresponding to the generator $\mathcal{L}^{\{z\}}_{\Lambda_{r,x}}$) to get

$$\mu\big(\tilde{c}_x \operatorname{Var}_{\Lambda_x}(f)\big) \le \mu\Big(\operatorname{gap}(q)^{-1} \sum_{z\in\partial\Lambda_{r,x}} \mathbb{1}_{\xi=z} \sum_{y\in\Lambda_{r,x}} \mu_{\Lambda_{r,x}}\big(c^{\{z\}}_{y,\Lambda_{r,x}} \operatorname{Var}_y(f)\big)\Big) \tag{6.12}$$

Thanks again to the constraint $\mathbb{1}_{\xi=z}$, for all $y \in \Lambda_{x,r}$, $c^{\{z\}}_{y,\Lambda_{r,x}} \leqslant c_y$, where c_y are the infinite volume rates for FA-1f. If we insert this inequality in (6.12) and then we use (6.10) we finally get

$$\operatorname{Var}(f) \le \operatorname{gap}(q)^{-1} C(d) \sum_{y\in\mathbb{Z}^d} \mu\big(c_y \operatorname{Var}_y(f)\big)$$

and the lemma follows. □

The proof of the lower bound will then be complete once we prove the following result.

Proposition 6.6 *There exists a constant $C = C(d)$ such that for any $q \in (0,1)$,*

$$\operatorname{gap}(q) \ge C \begin{cases} q^3 & \text{if } d = 1 \\ q^2/\log(1/q) & \text{if } d = 2 \\ q^2 & \text{if } d = 3 \end{cases} \tag{6.13}$$

Proof of the Proposition Fix a box Λ of side $\ell_q \le \ell \le 2\ell_q$ and consider the generator in Λ with minimal choice of the good boundary conditions at $x^* \in \partial\Lambda$. We will show that the corresponding spectral gap satisfies the inequalities of the lemma uniformly in ℓ and in the choice of the site x^*.

We begin with the $d = 2$ case. The starting point is the standard Poincaré inequality for the Bernoulli product measure on Λ (see e.g. [4, Chap. 1]). For every function f

$$\operatorname{Var}_\Lambda(f) \leqslant \sum_{x\in\Lambda} \mu\left(\operatorname{Var}_x(f)\right). \tag{6.14}$$

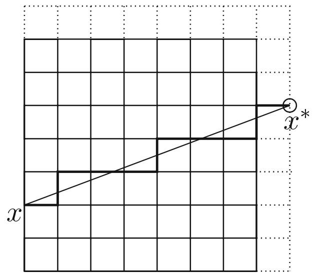

Fig. 5 An example of geodesic for the path γ_x

Our aim is to bound from above the r.h.s. of (6.14) with the Dirichlet form of the FA-1f model in Λ and minimal boundary conditions at x^*. The idea of the proof is the following. Computing the local variance $\mathrm{Var}_x(f)$ at x involves a spin–flip at site x ($\eta \to \eta^x$) which might or might not be allowed by the constraints, depending on the structure of the configuration around x. The idea is then to (see Figs. 5 and 6 for a graphical illustration):

(i) define a geometric path γ_x inside Λ connecting x to the (unique) empty site x^* at the boundary of Λ;
(ii) look for the empty site on γ_x closest to x;
(iii) move it, step by step using allowed flips, to one of the neighbors of x but keeping the configuration as close as possible to the original one;
(iv) do the spin–flip at x in the modified configuration.

In order to get an optimal result the choice of the path γ_x is not irrelevant and we will follow the strategy of [32] to analyze the simple random walk on the graph consisting of two squares grids sharing exactly one corner.

Let a pair $e = (\eta, \eta^y) \equiv (e^-, e^+)$ be an edge iff $c_{y,\Lambda}^{\{x^*\}}(\eta) = 1$ (i.e. the spin-flip at y in the configuration η is a legal one), we can rewrite the Dirichlet form as

$$\mathcal{D}(f) = \sum_e \mu(e^-) \left(f(e^+) - f(e^-)\right)^2.$$

where the sum is over the edges. To any edge $e = (\eta, \eta^y)$ we associated a weight $w(e)$ defined by $w(e) = i + 1$ if $d_1(y, x^*) = i$.

Let now, for any $x \in \Lambda$, $\gamma_x = (x^*, x^{(1)}, x^{(2)}, \ldots, x^{(n-1)}, x)$ be one of the geodesic paths from x^* to x such that, for any $y \in \gamma_x$, the Euclidean distance between y and the straight line segment $[x, x^*]$ is at most $\sqrt{2}/2$ (see Fig. 5).

Given a configuration σ we will construct a path $\Gamma_{\sigma \to \sigma^x} = \{\eta^{(0)}, \eta^{(1)}, \ldots, \eta^{(j)}\}$, $j \leq 2n$, with the properties that:

(i) $\eta^{(0)} = \sigma$ and $\eta^{(j)} = \sigma^x$;
(ii) the path is self-avoiding;
(iii) for any i the pair $(\eta^{(i-1)}, \eta^{(i)})$ forms an edge and the associated spin–flip occurs on γ_x;
(iv) for any i the configuration $\eta^{(i)}$ differs from σ in at most two sites.

We will denote by $|\Gamma_{\sigma\to\sigma^x}|_w := \sum_{e\in\Gamma_{\sigma\to\sigma^x}} \frac{1}{w(e)}$ the weighted length of the path $\Gamma_{\sigma\to\sigma^x}$. By the Cauchy–Schwartz inequality, we have

$$\begin{aligned}
\sum_{x\in\Lambda} \mu\left(\mathrm{Var}_x(f)\right) &= pq\sum_{x\in\Lambda}\mu\left(\left[f(\sigma^x)-f(\sigma)\right]^2\right)\\
&= pq\sum_{x\in\Lambda}\sum_{\sigma}\mu(\sigma)\left(\sum_{e\in\Gamma_{\sigma\to\sigma^x}}\frac{\sqrt{w(e)}(f(e^+)-f(e^-))}{\sqrt{w(e)}}\right)^2\\
&\leqslant pq\sum_{x\in\Lambda}\sum_{\sigma}\mu(\sigma)|\Gamma_{\sigma\to\sigma^x}|_w\sum_{e\in\Gamma_{\sigma\to\sigma^x}} w(e)\left(f(e^+)-f(e^-)\right)^2\\
&= pq\sum_{e}\left(f(e^+)-f(e^-)\right)^2 w(e)\sum_{\substack{x\in\Lambda,\sigma:\\ \Gamma_{\sigma\to\sigma^x}\ni e}}\mu(\sigma)|\Gamma_{\sigma\to\sigma^x}|_w\\
&\leqslant \mathcal{D}(f)\max_{e}\left\{\frac{pq\,w(e)}{\mu(e^-)}\sum_{\substack{x\in\Lambda,\sigma:\\ \Gamma_{\sigma\to\sigma^x}\ni e}}\mu(\sigma)|\Gamma_{\sigma\to\sigma^x}|_w\right\}.
\end{aligned}$$

Fix an edge $e=(\eta,\eta^y)$ with $w(e)=i+1$. Let C denotes a constant that does not depend on q and that may change from line to line. By construction, on one hand we have for any σ and x such that $\Gamma_{\sigma\to\sigma^x}\ni e$, $\frac{\mu(\sigma)}{\mu(e^-)}\leqslant C\frac{1}{q^2}$ because of property (iii) of $\Gamma_{\sigma\to\sigma^x}$. On the other hand, for any σ and x,

$$|\Gamma_{\sigma\to\sigma^x}|_w \leqslant C\sum_{i=1}^{2\ell_q}\frac{1}{i}\leqslant C\log(\ell_q).$$

And finally, by construction, one has (see [32, Sect. 3.2])

$$\#\{(x,\sigma):\Gamma_{\sigma\to\sigma^x}\ni e\}\leqslant C\#\{y:\gamma_x\ni y\}\leqslant C\frac{|\Lambda|}{i+1}.$$

Collecting these computations leads to

$$\sum_{x\in\Lambda}\mu\left(\mathrm{Var}_x(f)\right)\leqslant\frac{C}{q^2}\log(1/q)\,\mathcal{D}(f).$$

i.e. the claimed bound on gap(q).

In $d\geqslant 3$, the above strategy applies in the same way but one needs a different choice of the edge-weight $w(e)$ namely $w(e)=(i+1)^{d-2}$ (see again [32, Sect. 3.2]). In $d=1$ instead one can convince oneself that the weight function $w\equiv 1$ in the previous proof leads to the upper bound $1/q^3$, up to some constant.

It remains to discuss the construction of the path $\Gamma_{\sigma\to\sigma^x}$ with the desired properties. Given σ, x and $\gamma_x=\left(x_0=x^*,x^{(1)},x^{(2)},\ldots,x^{(n-1)},x^{(n)}=x\right)$ define

$i_0 = \max\{0 \leqslant i \leqslant n-1 : \sigma(x^{(i)}) = 0\}$. In this way for any $i \geqslant i_0+1$, $\sigma(x^{(i)}) = 1$. We will denote by $\eta^{x,y} = (\eta^x)^y$ the configuration η flipped in *x and y*.

If $i_0 = n-1$ then trivially $\Gamma_{\sigma\to\sigma^x} = \{\sigma, \sigma^x\}$. Hence assume that $i_0 \leqslant n-2$. We set

$$\Gamma_{\sigma\to\sigma^x} = \left\{\eta^{(0)} = \sigma, \eta^{(1)}, \dots, \eta^{(2(n-i_0)-1)} = \sigma^x\right\}$$

with $\eta^{(1)} = \sigma^{x^{(i_0+1)}}$ and for $k = 1, \dots, n-i_0-1$, $\eta^{(2k)} = \sigma^{x^{(i_0+k)}, x^{(i_0+k+1)}}$, $\eta^{(2k+1)} = \sigma^{x^{(i_0+k+1)}}$ (see Fig. 6). One can easily convince oneself that $\Gamma_{\sigma\to\sigma^x}$ satisfies the prescribed property $(i)-(iv)$ set above. □

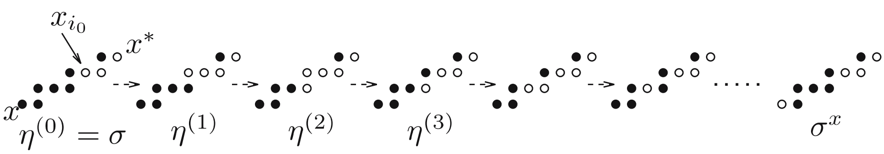

Fig. 6 The path $\Gamma_{\sigma\to\sigma^x}$

The proof of the lower bound is complete. □

6.3 FA-jf and Modified Basic model in $\mathbb{Z}^d$

Next we examine the FA-jf and Modified Basic (MB) model in $\mathbb{Z}^d$ with $d \geq 2$ and $j \leq d$.

Theorem 6.7 *For any $q \in (0,1)$ any $d \geq 2$ and $j \leq d$ the spectral gap of the FA-jf and MB models are positive.*

Proof Under the hypothesis of the theorem both models have a trivial bootstrap percolation threshold $q_{bp} = 0$ and moreover they satisfy the assumption of Corollary 3.5 (see [33]) for any $q > 0$. Therefore gap > 0 by Corollary 3.5. □

We now study the asymptotic of the spectral gap as $q \downarrow 0$ and we restrict ourselves to the most constrained case, namely either the MB model or the FA-df model. For this purpose we need to introduce some extra notation and recall some results from bootstrap percolation theory (see [23]).

Let $\delta \in \{1, \dots, d\}$. We define the δ-dimensional cube $Q^\delta(L) := \{0, \dots, L-1\}^\delta \times \{1\}^{d-\delta} \subset \mathbb{Z}^d$. By a *copy* of $Q^\delta(L)$ we mean an image of $Q^\delta(L)$ under any isometry of $\mathbb{Z}^d$.

Definition 6.8 Given a configuration η, we will say that $Q^\delta(L)$ is "δ internally spanned" if $\{1, \dots, L-1\}^\delta$ is internally spanned for the bootstrap map associated to the corresponding model restricted to $\mathbb{Z}^\delta$ (i.e. with the rules either of the FA-δf or of the MB model in $\mathbb{Z}^\delta$). Similarly for any copy of $Q^\delta(L)$.

Define now

$$I^d(L,q) := \mu\big(Q^d(L) \text{ is internally spanned}\big)$$

and let $\exp^n$ denote the n-th iterate of the exponential function. Then the following results is known to hold for both models [13,14,22,23].

There exists two positive constants $0 < \lambda_1 \le \lambda_2$ such that for any $\epsilon > 0$

$$\lim_{q\to 0} I^d\Big(\exp^{d-1}(\frac{\lambda_1 - \epsilon}{q}), q\Big) = 0 \tag{6.15}$$

$$\lim_{q\to 0} I^d\Big(\exp^{d-1}(\frac{\lambda_2 + \epsilon}{q}), q\Big) = 1 \tag{6.16}$$

Moreover there exists $c = c(d) < 1$ and $C = C(d) < \infty$ such that if ℓ is such that $I^d(\ell, q) \ge c$ then, for any $L \ge \ell$,

$$I^d(L,q) \ge 1 - Ce^{-L/\ell} \tag{6.17}$$

For the FA-2f model and for the MB model for all $d \ge 2$ the threshold is sharp in the sense that $\lambda_1 = \lambda_2 = \lambda$ with $\lambda = \pi^2/18$ for the FA-2f model and $\lambda = \pi^2/6$ for the MB model [22,23]. We are now ready to state our main result.

Theorem 6.9 *Fix $d \ge 2$ and $\epsilon > 0$. Then for both models there exists $c = c(d)$ such that*

$$\Big[\exp^{d-1}(c/q^2)\Big]^{-1} \le \mathrm{gap}(\mathcal{L}) \le \Big[\exp^{d-1}\big(\tfrac{\lambda_1-\epsilon}{q}\big)\Big]^{-1} \qquad d \ge 3 \tag{6.18}$$

$$\exp(-c/q^5) \le \quad \mathrm{gap}(\mathcal{L}) \le \exp\big(-\tfrac{(\lambda_1-\epsilon)}{q}\big) \qquad d = 2 \tag{6.19}$$

as $q \downarrow 0$.

Proof In the course of the proof we will use the following well known observation. If a configuration η is identically equal to 0 in a d-dimensional cube Q and each face F of ∂Q is "$(d-1)$ internally spanned" (by η), then $Q \cup \partial Q$ is internally spanned.

(i) We begin by proving the upper bound following the strategy outlined in Remark 3.7. Fix $\epsilon > 0$, let Λ_1 be the cube centered at the origin of side $L_1 := \exp^{d-1}\big(\frac{\lambda_1-\epsilon/2}{q}\big)$ and let $m = \exp^{d-2}(\frac{K}{q^2})$ where K is a large constant to be chosen later on. Define the two events:

$$\begin{aligned} A &= \{\eta :\ \Lambda_1 \text{ is not internally spanned}\} \\ B &= \{\eta :\ \text{any } (d-1)\text{-dimensional cube of side } m \text{ inside } \Lambda_1 \text{ is} \\ &\qquad \text{“}(d-1) \text{ internally spanned”}\}. \end{aligned} \tag{6.20}$$

Thanks to (6.16) and (6.17), $\mu(A) > 1/2$ and $\mu(B) \ge \frac{3}{4}$ if K and q are chosen large enough and small enough respectively. Therefore $\mu(A \cap B) \ge 1/4$ for small q. Pick now $\eta \in A \cap B$ and consider $\tilde{\eta}$ which is identically equal to one outside Λ_1 and equal to η inside. We begin by observing that, starting from $\tilde{\eta}$, the little square Q of side m centered at the origin cannot be completely emptied by the bootstrap map T (2.7). Assume in fact the opposite. Then, after Q has been emptied and using the fact that $\eta \in B$, we could empty ∂Q and continue layer by layer until we have emptied the whole Λ_1, a contradiction with the assumption $\eta \in A$. The above simple observation implies in particular that, if we start the Glauber dynamics from $\tilde{\eta}$, there exists a point $x \in Q$ such that $\sigma_x^{\tilde{\eta}}(s) = \eta_x$ for all $s > 0$. However, and this is the second main observation, if $t = \frac{1}{4}L_1$, by standard results on "finite speed of propagation of information" (see e.g. [28]) and the basic coupling between the process started from η and the process started from $\tilde{\eta}$,

$$\mathbb{P}\left(\exists x \in Q : \sigma_x^{\tilde{\eta}}(s) \neq \sigma_x^{\eta}(s) \text{ for some } s \leq t\right) \ll 1$$

Therefore

$$\mathbb{P}(\exists x \in Q : \ \sigma_x^{\eta}(s) = \eta_x \ \forall s \leq \frac{1}{4}L_1) \geq \frac{1}{2}$$

for all sufficiently small q.

We are finally in a position to prove the r.h.s. of (6.18). Using theorem 3.6 combined with the above discussion we can write

$$\begin{aligned} e^{-t\frac{q\,\mathrm{gap}}{2(1+p)}} &\geq F(t) \\ &\geq \frac{1}{|Q|}\int_{A\cap B} d\mu(\eta)\mathbb{P}(\exists x \in Q : \ \sigma_x^{\eta}(s) = \eta_x \ \forall s \leq t) \geq \frac{1}{8|Q|} \end{aligned}$$

that is gap $\leq c\log\big(|Q|\big)/qt$ for some constant c, i.e. the sought upper bound for q small, given our choice of t.

(ii) We now turn to the proof of the lower bound in (6.18).

It is enough to consider only the MB model since, as explained in Sect. 2.4, MB is dominated by the FA-df model and therefore has a smaller spectral gap.

Fix $\epsilon \in (0, 1)$, let $\ell = \exp^{d-1}\big((\lambda + 5\epsilon)/q\big)$, $\lambda = \pi^2/6$, and let $m = \exp^{d-2}(1/q^2)$ if $d \geq 3$ and $m = K/q^2$ if $d = 2$, where K is a large constant to be fixed later on. Let E_1 be the event that $Q^d(\ell)$ contains some copy of $Q^d(m)$ which is internally spanned and let E_2 be the event that for each $\delta \in [1, \ldots, d-1]$, every copy of $Q^\delta(m)$ in $Q^d(\ell)$ is "δ internally spanned". Then it is possible to show (see Sect. 2 of [23] for the case $d \geq 3$ and Sect. 4 of [22] for the case $d = 2$) that both $\mu(E_1)$ and $\mu(E_2)$ tend to one as $q \to 0$ if K is chosen large enough.

The first step is to relate the infinite volume spectral gap to the spectral gap in the cube $\Lambda_0 \equiv Q^d(\ell)$ with good boundary set $\mathcal{M} = \partial_+^* \Lambda_0$ and we denote by c_{x,Λ_0} the corresponding rates. To this purpose we will proceed via the Renormalization-Constrained strategy introduced to prove Theorem 3.3 in Sect. 5.

Proposition 6.10 *There exists a constant $c = c(d)$ such that, for any q small enough,*

$$\mathrm{gap}(\mathcal{L}) \geq e^{-cm^d}\,\mathrm{gap}(\mathcal{L}_{\Lambda_0})$$

Proof As in the case of the FA-1f model, our starting point is the renormalized Poincaré inequality (5.1) on scale ℓ and ϵ_0-good event $G_\ell := E_1 \cap E_2$. Thanks to (5.1) we can write

$$\mathrm{Var}(f) \leq 2 \sum_{x\in\mathbb{Z}(\ell)} \mu\left(\tilde{c}_x\,\mathrm{Var}_{\Lambda_x}(f)\right)$$

where the $\tilde{c}_x$'s are as in (5.1). Without loss of generality we now examine the term $\mu\left(\tilde{c}_0\,\mathrm{Var}_{\Lambda_0}(f)\right)$.

Lemma 6.11 *There exists a constant $c = c(d)$ such that, for any q small enough,*

$$\mu\left(\tilde{c}_0\,\mathrm{Var}_{\Lambda_0}(f)\right) \leq e^{cm^d}\,\mathrm{gap}(\mathcal{L}_{\Lambda_0})^{-1} \sum_{x\in\cup_{y\in\mathcal{K}_0^*\cup\{0\}}\Lambda_{\ell y}} \mu\left(c_x\,\mathrm{Var}_x(f)\right)$$

where the c_x's are the constraints for the MB model.

Clearly the lemma completes the proof of the proposition □

Proof of the Lemma By definition

$$\mathrm{Var}_{\Lambda_0}(f) \leq \mathrm{gap}(\mathcal{L}_{\Lambda_0})^{-1} \sum_{x\in\Lambda_0} \mu_{\Lambda_0}\left(c_{x,\Lambda_0}\,\mathrm{Var}_x(f)\right)$$

Notice that, if $\mathcal{K}_x^* \subset \Lambda_0$, then $c_{x,\Lambda_0} \leqslant c_x$. If we plug the above bound into $\mu\left(\tilde{c}_0\,\mathrm{Var}_{\Lambda_0}(f)\right)$ and use the trivial bound $\tilde{c}_0 \leq 1$, we see that it would be enough to prove

$$\mu\left(\tilde{c}_0\,c_{x,\Lambda_0}\,\mathrm{Var}_x(f)\right) \leq e^{cm^d}\mu\left(c_x\,\mathrm{Var}_x(f)\right) + \sum_{y\in\mathcal{K}_0^*}\sum_{z\in\Lambda_{\ell y}} \mu\left(c_z\,\mathrm{Var}_z(f)\right) \tag{6.21}$$

for all $x \in \Lambda_0$ such that $\mathcal{K}_x^* \nsubseteq \Lambda_0$ to conclude. For simplicity we assume that $\mathcal{K}_x^* \cap \Lambda_0^c$ consists of a unique point $z \in \Lambda_{\ell y}$ and we proceed as in the proof of Theorem 3.3. Assign some arbitrary order to all cubes of side m inside $\Lambda_{\ell y}$. Because of the constraint $\tilde{c}_0$ on the configuration ξ in $\cup_{y\in\mathcal{K}_0^*}\Lambda_{\ell y}$, for each $y \in \mathcal{K}_0^*$ there exists a sequence of configurations $(\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(n)})$, $n \leq 2m^d$, with the following properties:

(i) $\xi^{(0)} = \xi$ and $\xi^{(n)} = \xi'$, where ξ' is completely empty in the first cube $Q \subset \Lambda_{\ell y}$ of side m which was internally spanned for ξ and otherwise coincides with ξ;

(ii) $\xi^{(i+1)}$ is obtained from $\xi^{(i)}$ by changing exactly only one spin at a suitable site $x^{(i)} \in Q$;
(iii) the move at $x^{(i)}$ leading from $\xi^{(i)}$ to $\xi^{(i+1)}$ is permitted i.e. $c_{x^{(i)}}(\xi^{(i)}) = 1$ for every $i = 0, \dots, n$.

Remark 6.12 Notice that, given $\xi^{(i)} = \eta$, the number of starting configurations $\xi = \xi^{(0)}$ compatible with η is bounded from above by 2^{cm^d}, $c = c(d)$, and the relative probability $\mu(\xi)/\mu(\eta)$ by $\big(p/q\big)^{cm^d}$.

We can proceed as in (5.3) and conclude that

$$\begin{aligned}\mu\big(\tilde{c}_0\, c_{x,\Lambda_0}\operatorname{Var}_x(f)\big) &\le e^{c'm^d}\mu(\hat{c}_0\,\tilde{c}_0\, c_{x,\Lambda_0}\operatorname{Var}_x(f))\\ &\quad+\sum_{y\in\mathcal{K}_0^*}\sum_{z\in\Lambda_{\ell y}}\mu(\tilde{c}_0\, c_{x,\Lambda_0}c_z\operatorname{Var}_z(f))\end{aligned} \tag{6.22}$$

where now $\hat{c}_0$ is the indicator of the event that for each $y \in \mathcal{K}_0^*$ there exists a cube $Q \subset \Lambda_{\ell y}$ of side m which is completely empty. The second term in the above inequality arises from the above described path inside the blocks $\cup_{y\in\mathcal{K}_0^*}\Lambda_{\ell y}$ which leads from ξ to ξ' (namely it is the analogous of the second and third term of (5.3)). The above inequality together with (6.21) reduces the proof of the lemma to the proof of inequality

$$e^{c'm^d}\mu(\hat{c}_0\,\tilde{c}_0\, c_{x,\Lambda_0}\operatorname{Var}_x(f)) \leqslant e^{c''m^d}\mu\big(c_x\operatorname{Var}_x(f)\big) \tag{6.23}$$

Next we observe that for any sequence of adjacent (in e.g. the first direction) cubes $Q_1, Q_2, \dots, Q_j$ of side m inside $\Lambda_{\ell y}$, ordered from left to right, and for any configuration $\eta \in E_2$ which is identically equal to 0 in Q_1, one can construct a sequence of configurations $(\eta^{(0)}, \eta^{(1)}, \dots, \eta^{(n)})$, $n \le jm^d$, such that:

(i) $\eta^{(0)} = \eta$ and $\eta^{(n)}$ is completely empty in Q_j and otherwise coincides with η;
(ii) $\eta^{(i+1)}$ is obtained from $\eta^{(i)}$ by changing exactly only one spin at a suitable site $x^{(i)} \in \cup_{i=1}^{j} Q_i$;
(iii) the move at $x^{(i)}$ leading from $\eta^{(i)}$ to $\eta^{(i+1)}$ is permitted i.e. $c_{x^{(i)}}(\eta^{(i)}) = 1$ for every $i = 0, \dots, n$.

In other words one can move the empty square Q_1 to the position occupied by Q_j in no more than jm^d steps. The construction is very simple and it is based on the basic observation described at the beginning of the proof. Starting from Q_1 and using the fact that any copy of $Q^{d-1}(m)$ inside $\Lambda_{\ell y}$ is "$(d-1)$ internally spanned", by a sequence of legal moves one can first empty Q_2. Next one repeats the same scheme for Q_3. Once that also Q_3 has been emptied one backtracks and readjust all the spins inside Q_2 to their original value in the starting configuration η. The whole procedure is then iterated until the last square Q_j is emptied and the configuration η fully reconstructed in $\cup_{i=1}^{j-1} Q_i$.

The key observation at this point is that, given an intermediate step $\eta^{(i)}$ in the sequence, the number of starting configurations η compatible with $\eta^{(i)}$ is bounded from above by $2j \cdot 4^{m^d}$ and the relative probability $\mu(\eta^{(i)})/\mu(\eta)$ by $(p/q)^{2m^d}$. By using the path argument above and by proceeding again as in (5.3), we can finally obtain

$$e^{c'm^d}\mu(\hat{c}_0\,\tilde{c}_0\,c_{x,\Lambda_0}\,\mathrm{Var}_x(f)) \leqslant 2\ell\, e^{c''m^d}\mu\big(\tilde{c}_0\,\hat{c}_{0,x}\,c_{x,\Lambda_0}\,\mathrm{Var}_x(f)\big) \tag{6.24}$$

where $\hat{c}_{0,x}$ is the indicator of the event that there exists a cube Q of side m, laying outside Λ_0 and such that $\mathcal{K}_x^* \cap \Lambda_0^c \subset Q$, which is completely empty. This implies $\tilde{c}_0\,\hat{c}_{0,x}\,c_{x,\Lambda_0} \leq c_x$, since the sites in $\mathcal{K}_x^* \cap \Lambda_0^c$ which are considered empty for c_{x,Λ_0} due to the good boundary are also empty for the infinite volume rates c_x thanks to $\hat{c}_{0,x}$. Furthermore, since $\ell \leqslant e^{cm^d}$ for a proper c when $q \to 0$, the proof of (6.23) and therefore of the lemma is complete. □

As a second step we lower bound $\mathrm{gap}(\mathcal{L}_{\Lambda_0})$ by the spectral gap in the reduced volume $\Lambda_1 := Q^d_{\ell/2}$ (we assume here for simplicity that both ℓ and m are powers of 2). To this end we partition Λ_0 into disjoint copies of Λ_1, $\{\Lambda_1^{(i)}\}_{i=1}^{2^d}$ and, mimicking the argument of Sect. 4, we run the constrained dynamics of the $*$-general model on Λ_0 with blocks $\{\Lambda_1^{(i)}\}_{i=1}^{2^d}$ and good event the event that for each $\delta \in [1,\dots,d-1]$, every copy of $Q^\delta(m)$ in $Q^d(\ell/2)$ is "δ internally spanned". By choosing the constant K appearing in the definition of m large enough the probability of G is very close to one as $q \to 0$ and therefore the Poincaré inequality

$$\mathrm{Var}_{\Lambda_0}(f) \leq 2\sum_{i=1}^{2^d}\mu\big(c_i\,\mathrm{Var}_{\Lambda_1^{(i)}}(f)\big) \tag{6.25}$$

holds, where c_i are the constraints of the $*$-general model. At this point we can proceed exactly as in the proof of Lemma 6.11 and get that the r.h.s. of (6.25) is bounded from above by

$$e^{cm^d}\,\mathrm{gap}(\mathcal{L}_{\Lambda_1})^{-1}\mathcal{D}_{\Lambda_0}(f)$$

for some constant $c = c(d)$. We have thus proved that

$$\mathrm{gap}(\mathcal{L}_{\Lambda_0})^{-1} \leq e^{cm^d}\,\mathrm{gap}(\mathcal{L}_{\Lambda_1})^{-1}$$

If we iterate N times, where N is such that $2^{-N}\ell = m$ we finally get

$$\mathrm{gap}(\mathcal{L}_{\Lambda_0})^{-1} \leq e^{cN\,m^d}\,\mathrm{gap}(\mathcal{L}_{\Lambda_N})^{-1}$$

where $\Lambda_N = Q^d_m$. This together with Proposition 6.10 and gap $\big(\mathcal{L}_{\Lambda_N}\big)^{-1} \leqslant e^{cm^d}$ achieves the proof. □

6.4 The N-E model

The N-E model is the natural two-dimensional analogue of the one-dimensional East model. Before giving our results we need to recall some definitions of oriented percolation [15,33]. A *NE oriented path* is a sequence $\{x^{(0)}, x^{(1)}, \dots, x^{(n)}\}$ of distinct points in $\mathbb{Z}^2$ such that $x^{(i+1)} = x^{(i)} + \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2$, $\alpha_j = 0, 1$ and $\alpha_1 + \alpha_2 = 1$ for all i. Given a configuration $\eta \in \Omega$ and $x, y \in \mathbb{Z}^2$, we say that $x \to y$ if there is a NE oriented path of occupied sites starting in x and ending in y. For each site $x \in \mathbb{Z}^2$ its *NE occupied cluster* x is the random set

$$C_x(\eta) := \{y \in \mathbb{Z}^2 \ : \ x \to y\}$$

The range of $C_x(\eta)$ is the random variable

$$A_x(\eta) = \begin{cases} 0 & \text{if } C_x(\eta) = \emptyset \\ \sup\{1 + \|y - x\|_1 \ : \ y \in C_x(\eta)\} & \text{otherwise} \end{cases}$$

Remark 6.13 If $A_x(\eta) > 0$ then at least $A_x(\eta)$ legal (i.e. fulfilling the NE constraint) spin flip moves are needed to empty the site x.

Finally, we define the monotonic non decreasing function $\theta(p) := \mu(A_0 = \infty)$ and let

$$p_c^o = \inf\{p \in [0, 1] \ : \ \theta(p) > 0\}$$

It is known (see [15]) that $0 < p_c^o < 1$. In [33] it is proved that the percolation threshold and bootstrap percolation threshold of N-E model (see Sect. 2.3) are related by $p_c^o = 1 - q_{bp}$ and therefore, thanks to Proposition 2.5, $q_c = 1 - p_c^o$. The presence of a positive threshold q_c reflects a drastic change in the behavior of the NE process when $q < q_c$ due to the presence of blocked configurations (NE occupied infinite paths) with probability one. In [26] it is proved that the measure μ on the configuration space is mixing iff $q \geq q_c$, a result that also follows at once from the arguments given in the proof of our Proposition 2.5 since $\theta(p_c^o) = 0$ [9].

We now analyze the spectral gap of the N-E process above, below and at the critical point q_c.

Case $\mathbf{q} > \mathbf{q_c}$. This region is characterized by the following result of [15].

Proposition 6.14 *If $p < p_c^o$ there exists a positive constant $\varsigma = \varsigma(p) > 0$ such that*

$$\lim_{n \to \infty} -\frac{1}{n} \log \mu(A_0 \geq n) = \varsigma \tag{6.26}$$

We can now state our main theorem.

Theorem 6.15 *For any $q > q_c$ the spectral gap of the N-E model is positive.*

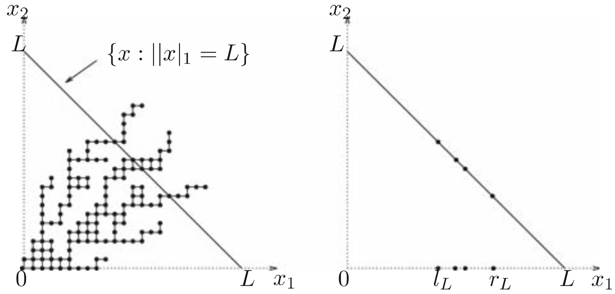

Fig. 7 An example of configuration η with the sets $C_0(\eta)$ (on the *left*) , $C_0^{(L)}(\eta)$ and $\xi_0^{(L)}(\eta)$ (on the *right*)

Proof Recall the notation of Sect. 3. Using Theorem 3.3 we need to find a set of configurations G_ℓ satisfying properties (a) and (b) of Definition 3.1. Fix $\delta \in (0, 1)$ and $\ell > 2$ and define

$$G_\ell := \{\eta \in \{0, 1\}^{\Lambda_0} : \nexists \text{ occupied oriented path in } \Lambda_0 \text{ longer than } \ell^\delta\}$$

Since $q > q_c$ we can use (6.26) to obtain that for any $\epsilon \in (0, 1)$ there exists $\ell_c(q, \varepsilon, \delta)$ such that, for any $\ell \geq \ell_c(q, \varepsilon, \delta)$, $\mu(G_\ell) \geq 1 - \varepsilon$ and property (a) follows. Property (b) also follows directly from the definition of G_ℓ. Indeed, if the restriction of a configuration η to each one of the squares $\Lambda_0 + \ell x$, $x \in \mathcal{K}_0^*$, belongs to G_ℓ, then necessarily there is no occupied oriented path in $\cup_{x \in \mathcal{K}_0^*}\{\Lambda_0 + \ell x\}$ of length greater than $3\ell^\delta$. Therefore, by a sequence of legal moves, all the $\partial_+^* \Lambda_0$ can be emptied for η and the proof is complete. ◻

Case $\mathbf{q} < \mathbf{q_c}$. Following [15] we need some extra notation. For every $L \in \mathbb{N}$ and $\eta \in \Omega$ let $C_0^{(L)}(\eta) = \{x \in C_0(\eta) : \|x\|_1 = L\}$ and let

$$\xi_0^{(L)}(\eta) := \cup_{x \in C_0^{(L)}(\eta)}\{x_1\}$$

be the projection onto the first coordinate axis of $C_0^{(L)}(\eta)$. Denote by r_L, l_L the right and left edge of $\xi_0^{(L)}(\eta)$ respectively (Fig. 7). If $p > p_c^o$ it is possible to show [15] that there exists positive constants a, ζ such that

$$\mu\left(\{\xi_0^{(L)} \neq \emptyset\} \cap \{r_L \leq aL\}\right) = \mu\left(\{\xi_0^{(L)} \neq \emptyset\} \cap \{l_L \geq aL\}\right) \leq e^{-\zeta L} \tag{6.27}$$

for any L large enough. We can now state our result for the spectral gap.

Theorem 6.16 *Let $\Lambda \subset \mathbb{Z}^2$ be a square of side $L \in \mathbb{N}$. For any $q < q_c$ there exists two positive constants c_1, c_2 such that*

$$\exp\{-c_1 L\} \leq \text{gap}(\mathcal{L}_\Lambda) \leq \exp\{-c_2 L\} \tag{6.28}$$

Proof We first discuss the upper bound by exhibiting a suitable test function f to be plugged into the variational characterization of the spectral gap. For this purpose let $B_L := \{\eta : \xi_0^{(L)} \neq \emptyset\}$ and define $f = 1\!\!1_{B_L}$. Since $q < q_c$, there exists two positive constants $0 < k_1(q) \le k_2(q) < 1$ such that $k_1 \le \mu(B_L) \le k_2$, see [15]. Thus the variance of f is bounded from below uniformly in L. On the other hand, by construction,

$$\mathcal{D}(f) = \sum_{x\in\Lambda} \mu\,(c_x \operatorname{Var}_x(f)) \le |\Lambda|\,\mu(\bar{B}_L)$$

where $\bar{B}_L := \{\eta \,:\, |\xi_0^{(L)}| = 1\} = \{\eta \,:\, r_L = l_L\}$.

Thanks to (6.27)

$$\begin{aligned}\mu(\bar{B}_L) &\le \mu(\{r_L = l_L\} \cap \{r_L > aL\}) + \mu(\{r_L = l_L\} \cap \{r_L \le aL\}) \\ &\le 2\mu(\xi_0^{(L)} \neq \emptyset\} \cap \{r_L \le aL\}) \le 2\exp\{-\zeta L\}\end{aligned}$$

and the r.h.s. of (6.28) follows.

The bound from below comes from the bisection method of Theorem 4.1 where in Proposition 4.4 ε_k is defined as the probability that there is at least one left-right NE occupied oriented path. Trivially $\varepsilon_k \le 1 - e^{-c\delta_k}$ for some constant c. If we plug such a bound bound into (4.10) and we remember that the number of steps of the iterations grows as $c \log L$, we obtain the desired result. □

The case $\mathbf{q} = \mathbf{q_c}$.

Theorem 6.17 *The spectral gap is continuous at q_c where, necessarily, it is zero.*

Proof Assume $q = q_c$ and suppose that the spectral gap is positive. Then, by Theorem 3.6, the persistence function decays exponentially fast as $t \to \infty$. We will show that such a decay necessarily implies that the all moments of the size of the oriented cluster C_0 are finite i.e. $q > q_c$, a contradiction.

Let $H(t) := \{\eta \,:\, A_0(\eta) \ge 2t\}$ and observe that, again by the "finite speed of propagation" (see Sect. 6.3), $\mathbb{P}(\sigma_0^\eta(s) = \eta_0 \text{ for all } s \le t) \ge \frac{1}{2}$ for all $\eta \in H(t)$. Using $H(t)$ we can lower bound $F(t)$ as follows.

$$\begin{aligned}F(t) &= \int d\mu(\eta)\,\mathbb{P}(\sigma_0^\eta(s) = \eta_0 \text{ for all } s \le t) \\ &\ge \int_{H(t)} d\mu(\eta)\,\mathbb{P}(\sigma_0^\eta(s) = \eta_0 \text{ for all } s \le t) \\ &\ge \frac{1}{2}\mu(A_0 \ge 2t)\end{aligned}$$

which implies,

$$\mu(A_0 \ge 2t) \le 2F(t) \le 2\,e^{-ct} \tag{6.29}$$

for a suitable constant $c > 0$. But (6.29) together with the fact that $|C_0| \leq A_0^2+1$ implies that $\mu\left(|C_0|^n\right) < \infty$ for all $n \in \mathbb{N}$, i.e. $p < p_c^o$ [1].

The same argument proves continuity at q_c.

Suppose in fact that $\limsup_{q\downarrow q_c} \text{gap} > 0$. That would imply (6.29) for any $q > q_c$ with c independent of q, i.e. $\sup_{q>q_c} \mu\left(|C_0|\right) < \infty$, again a contradiction since $\mu\left(|C_0|\right)$ is an increasing function of q which is infinite at q_c [15,20]. □

Corollary 6.18 *At $q = q_c$ the persistence function F satisfies*

$$\int_0^\infty dt\, F(\sqrt{t}) = \infty$$

Proof By (6.29)

$$\begin{aligned}\int_0^\infty dt\, F(\sqrt{t}) &\geq \frac{1}{2}\int_0^\infty dt\, \mu(A_0 \geq 2\sqrt{t})\\ &\geq \frac{1}{2}\int_0^\infty dt\, \mu\left(|C_0| \geq c' t\right) = +\infty\end{aligned}$$

because $\mu\left(|C_0|\right) = +\infty$ at q_c. □

7 Some further observations

We collect here some further comments and aside results that so far have been omitted for clarity of the exposition.

7.1 Logarithmic and modified-logarithmic Sobolev constants

A first natural question is whether it would be possible to go beyond the Poincaré inequality and prove a stronger coercive inequality for the generator $\mathcal{L}$ like the *logarithmic* or *modified-logarithmic* Sobolev inequalities [4]. As it is well known, the latter is weaker than the first one and it implies in particular that, for any non-negative mean one function f depending on finitely many variables, the entropy $\text{Ent}(P_t f) := \mu\left(P_t f \log(P_t f)\right)$ satisfies:

$$\text{Ent}(P_t f) \leq \text{Ent}(f)e^{-\alpha t} \tag{7.1}$$

for some positive α. As we briefly discuss below such a behavior is in general impossible and both the (infinite volume) logarithmic and modified logarithmic

Sobolev constants are zero.[5] For simplicity consider any of the 0–1 KCSM analyzed in Sect. 6 and choose f as the indicator function of the event that the box of side n centered at the origin is fully occupied, normalized in such a way that $\mu(f) = 1$. Denote by μ^f the probability measure whose relative density w.r.t. μ is f. If we assume (7.1) the relative entropy $\mathrm{Ent}(\mu^f P_t/\mu)$ satisfies

$$\mathrm{Ent}(\mu^f P_t/\mu) = \mathrm{Ent}(P_t f) \le Cn^d e^{-\alpha t}. \tag{7.2}$$

which implies, thanks to Pinsker inequality, that

$$\|\mu^f P_t - \mu P_t\|_{TV}^2 = \|\mu^f P_t - \mu\|_{TV}^2 \le 2\mathrm{Ent}(\mu^f P_t/\mu) \le 2Cn^d e^{-\alpha t} \tag{7.3}$$

i.e. $\|\mu^f P_t - \mu P_t\|_{TV} \le e^{-1}$ for any $t \ge O(\alpha^{-1}\log(n))$. However, the above conclusion clashes with a standard property of interacting particles systems with bounded rates known as "finite speed of propagation" (see e.g. [28]) which can be formulated as follows. Let $\tau(\eta)$ be the first time the origin is updated starting from the configuration η. Then $\int d\mu^f(\eta)\mathbb{P}(\tau(\eta) < t) \le Cn^{d-1}\mathbb{P}(Z \ge n/r)$ where Z is a Poisson variable of mean t and r is the range defined in Sect. 2.2. The above bound implies in particular that $\int d\mu^f(\eta)\mathbb{E}(\sigma_0^\eta(t)) \approx 1$ for any $t \ll n$ i.e. a contradiction with the previous reasoning.

7.2 More on the ergodicity/non ergodicity issue in finite volume

In Sect. 2.1, we mentioned that one could try to analyze a 0–1 KCSM in a finite region without inserting good boundary conditions and instead modifying the influence classes with the choice (2.4), namely by looking only to sites inside Λ to satisfy the constraints. In this case the Markov chain is in general non ergodic and a natural question is to evaluate the spectral gap of the process restricted to the different ergodic components. Although such an approach appears rather complicate for e.g. cooperative models, it is within reach for non-cooperative models.

For simplicity consider the FA-1f model in a finite interval $\Lambda = [1, \dots, L]$. We consider the generator $\mathcal{L}_\Lambda$ (2.3) with rates (2.1) and influence classes (2.4) with $\mathcal{C}_x = \{x-1, x+1\}$. In this case the configuration space Ω_L has a very simple decomposition in only two ergodic components: one contains only the completely filled configuration, the other one is given by $\Omega_\Lambda^+ := \{\eta \in \Omega_\Lambda : \sum_{x\in\Lambda} \eta_x < L\}$, i.e. contains all configurations with at least one empty site. We underline that for all other choices of the constraints different from FA-1f, more complicated decompositions with several components containing more than a single configuration will occur. Let us now consider the Markov process on Ω_Λ^+, which is ergodic and reversible w.r.t the conditional measure $\mu_\Lambda^+ := \mu_\Lambda(\cdot \mid \Omega_\Lambda^+)$.

[5] In finite volume with minimal boundary conditions it is not difficult to show that for some of the models discussed before the logarithmic Sobolev constant shrinks to zero as the inverse of the volume.

We now show how to derive that also the spectral gap of this new process stays uniformly positive as $L \to \infty$. To keep the notation simple we drop the subscript Λ from now on.

For any $\eta \in \Omega^+$, let $\xi(\eta) = \min\{x \in \Lambda : \eta_x = 0\}$ and write, for an arbitrary f,

$$\mathrm{Var}^+(f) = \mu^+\left(\mathrm{Var}^+(f \mid \xi)\right) + \mathrm{Var}^+\left(\mu^+(f \mid \xi)\right) \tag{7.4}$$

with self explanatory notation. Since $\mathrm{Var}^+(f \mid \xi)$ is computed with "good", i.e. zero, boundary condition at ξ, we get that

$$\begin{aligned}\mathrm{Var}^+(f \mid \xi) &= \mathrm{Var}(f \mid \xi)\\ &\le \text{const.} \sum_{x>\xi} \mu\left(c_x \mathrm{Var}_x(f) \mid \xi\right)\\ &= \text{const.} \sum_{x>\xi} \mu^+\left(c_x \mathrm{Var}_x(f) \mid \xi\right).\end{aligned} \tag{7.5}$$

Therefore, the first term in the r.h.s of (7.4) is bounded from above by a constant times the Dirichlet form. In order to bound the second term in the r.h.s of (7.4) we observe that ξ is a geometric random variable condition to be less or equal than L. By the classical Poincaré inequality for the geometric distribution, we can then write

$$\begin{aligned}&\mathrm{Var}^+\left(\mu^+(f \mid \xi)\right)\\ &\quad\le \text{const.} \sum_{x=1}^{L-1} \mu^+\left(b(x)\left[\mu^+(f \mid \xi = x+1) - \mu^+(f \mid \xi = x)\right]^2\right)\end{aligned} \tag{7.6}$$

where $b(x) = \mu^+(\xi = x+1)/\mu^+(\xi = x)$. A little bit of algebra now shows that

$$\begin{aligned}&\mu^+(f \mid \xi = x) - \mu^+(f \mid \xi = x+1)\\ &\quad= \mu^+\left(\eta_{x+1}(f(\eta) - f(\eta^{x+1})) \mid \xi = x\right) + \mu^+\left(f(\eta^x) - f(\eta) \mid \xi = x+1\right)\\ &\quad= \mu^+\left(\eta_{x+1}c_{x+1}(f(\eta) - f(\eta^{x+1})) \mid \xi = x\right)\\ &\qquad+ \mu^+\left(c_x(f(\eta^x) - f(\eta)) \mid \xi = x+1\right)\end{aligned} \tag{7.7}$$

In the last equality we have inserted the constraints c_{x+1} and c_x because they are identically equal to one. If we now insert (7.7) into the r.h.s. of (7.6) and use Schwartz inequality, we get that also the second term in the r.h.s of (7.4) is bounded from above by a constant times the Dirichlet form and the spectral gap stays bounded away from zero uniformly in L.

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