Facilitated spin models: recent and new results

N. Cancrini[1], F. Martinelli[2], C. Roberto[3], and C. Toninelli[4]

[1] Dip. Matematica Univ.l'Aquila, 1-67100 L'Aquila, Italy
nicoletta.cancrini@roma1.infn.it
[2] Dip.Matematica, Univ. Roma Tre, Largo S.L.Murialdo 00146, Roma,Italy
martin@mat.uniroma3.it
[3] L.A.M.A., Univ Marne-la-Vallée, 5 bd Descartes 77454 Marne-la-Vallée France
cyril.roberto@univ-mlv.fr
[4] Laboratoire de Probabilités et Modèles Alèatoires CNRS-UMR 7599 Universités Paris VI-VII 4, Place Jussieu F-75252 Paris Cedex 05 France
Cristina.Toninelli@lpt.ens.fr

Summary. Facilitated or kinetically constrained spin models are a class of interacting particle systems, reversible w.r.t. to a simple product measure, in which each dynamical variable (spin) is re-sampled from its equilibrium distribution only if the surrounding configuration fulfills a simple local constraint which *does not involve* the chosen dynamical variable itself. Such models have been quite popular in the glass community as simple models of glassy dynamics showing the phenomenon of dynamical arrest. Due to the fact that the jumps rates of the Markov process can be zero, the whole analysis of the long time behavior becomes quite delicate and, until recently, such models have escaped a rigorous analysis with the notable exception of the East model. In these notes we will mainly review several recent mathematical results which, besides being applicable to a wide class of interesting models, have contributed to settle some debated questions arising in numerical simulations made by physicists. We will also provide some interesting new extensions. In particular we will show how to deal with interacting models reversible w.r.t. to a high temperature Gibbs measure and we will provide a detailed analysis of the so called one spin facilitated model on a general connected graph. [5]

Key words: Glauber dynamics, spectral gap, constrained models, dynamical phase transition, glass transition.

1 Introduction and motivations

Consider the following simple interacting particle system. At each site of the lattice $\mathbb{Z}$ there is a dynamical variable σ_x, called in the sequel "spin", taking

[5] The material presented here is an expanded version of a series of lectures delivered by F.Martinelli at the Prague summer school 2006 on *Mathematical Statistical Mechanics*.

values in $\{0,1\}$. With rate one each spin attempts to change its current value by tossing a coin which lands head with probability $p \in (0,1)$ and setting the new value to 1 if head and 0 if tail. However the whole operation is performed only if the current value on its right neighbor is 0. Such a model is known under the name of the East model [] and it is easily checked to have the product Bernoulli(p) measure, denote it by μ, as a stationary reversible measure. A characteristic feature of the East model is that, when $p \approx 1$ *i.e.* $q := 1-p \approx 0$ the relaxation to the reversible measure μ is extremely slow namely [11]

$$T_{\rm relax} \approx (1/q)^{\frac{1}{2}\log_2(1/q)}$$

where $T_{\rm relax}$ is the inverse spectral gap in the spectrum of the generator $\mathcal{L}$ of the process. Notice that if one writes $p = \frac{e^\beta}{1+e^\beta}$ then $T_{\rm relax} \approx e^{c\beta^2}$ as $\beta \to \infty$, a behavior that is refereed to as a *super-Arrhenius law* in the physics literature.

The East model is one of the simplest example of a general class of interacting particles models in which each dynamical variable, taking values in a finite set S, waits an exponential time of mean one and then, if the surrounding current configuration satisfies a simple local constraint, is refreshed by sampling according to some apriori specified measure ν a new value from S. These systems have been introduced in the physical literature [17, 18] to model the liquid/glass transition and more generally the slow "glassy" dynamics which occurs in different systems (see [30, 9] for recent review). In particular, they were devised to mimic the fact that the motion of a molecule in a dense liquid can be inhibited by the presence of too many surrounding molecules. That explains why, in all physical models, $S = \{0,1\}$ and the constraints specify the maximal number of particles on certain sites around a given one in order to allow creation/destruction on the latter. As a consequence, the dynamics becomes increasingly slow as p is increased. Moreover there usually exist *blocked configurations*, namely configurations with all creation/destruction rates identically equal to zero. This implies the existence of several invariant measures (see [24] for a somewhat detailed discussion of this issue in the context of the North-East model), the occurrence of unusually long mixing times compared to standard high-temperature stochastic Ising models and may induce the presence of ergodicity breaking transitions without any counterpart at the level of the reversible measure.

Because of the presence of the constraints a mathematical analysis of these models have been missing for a long time with the notable exception of the East model [3], until a first recent breackthrough [11].

In this work we partly review the results and the techniques of [11] but we also extend them in two directions. Firstly we show that the main technique can be adapted to deal with a weak interaction among the variables obtained by replacing the reversible product measure with a general high-temperature Gibbs measure. Secondly, motivated by some unpublished considerations of D. Aldous [2], we analyze a special model, the so called *FA-1f* model, on a general connected graph and relate its relaxation time to that of the East model.

2 The models

2.1 Setting and notation

The models considered here are defined on a locally finite, bounded degree, connected graph $\mathcal{G} = (V, E)$ with vertex set V and edge set E. The associated graph distance will be denoted by $d(\cdot, \cdot)$ and the degree of a vertex x by Δ_x. The set of neighbors of x, *i.e.* $y \in V$ such that $d(y, x) = 1$, will be denoted by $\mathcal{N}_x$. For every subset $V' \subset V$ we denote by $\partial V'$ the set of vertices in $V \setminus V'$ with one neighbor in V'. In most cases the graph G will either be the d-dimensional lattice $\mathbb{Z}^d$ or a finite portion of it and in both cases we need some additional notation that we fix now. For any vertex $x \in \mathbb{Z}^d$ we define the $*$, thw oriented and the $*$-oriented neighborhood of x as

$$\begin{aligned} \mathcal{N}_x^* &= \{y \in \mathbb{Z}^d : \ y = x + \textstyle\sum_{i=1}^d \alpha_i \mathbf{e}_i, \ \alpha_i = \pm 1, 0 \text{ and} \sum_i \alpha_i^2 \neq 0\} \\ \mathcal{K}_x &= \{y \in \mathcal{N}_x : \ y = x + \textstyle\sum_{i=1}^d \alpha_i \mathbf{e}_i, \ \alpha_i \geq 0\} \\ \mathcal{K}_x^* &= \{y \in \mathcal{N}_x^* : \ y = x + \textstyle\sum_{i=1}^d \alpha_i \mathbf{e}_i, \ \alpha_i = 1, 0\} \end{aligned}$$

where $\mathbf{e}_i$ are the basis vactors of $\mathbb{Z}^d$. Accordingly, the oriented and *-oriented neighborhoods $\partial_+ \Lambda, \ \partial_+^* \Lambda$ of a finite subset $\Lambda \subset \mathbb{Z}^d$ are defined as $\partial_+ \Lambda := \{\cup_{x \in \Lambda} \mathcal{K}_x\} \setminus \Lambda, \ \partial_+^* \Lambda := \{\cup_{x \in \Lambda} \mathcal{K}_x^*\} \setminus \Lambda$. A rectangle R will be a set of sites of the form

$$R := [a_1, b_1] \times \cdots \times [a_d, b_d]$$

while the collection of finite subsets of $\mathbb{Z}^d$ will be denoted by $\mathbb{F}$.

2.2 The probability space

Let (S, ν) be a finite probability space with $\nu(s) > 0$ for any $s \in S$. $G \subset S$ will denote a distinguished event in S, often referred to as the set of "good states", and $q \equiv \nu(G)$ its probability.

Given (S, ν) we will consider the configuration space $\Omega \equiv \Omega_V = S^V$ whose elements will be denoted by Greek letters $(\omega, \eta \dots)$. If $\mathcal{G}' = (V', E')$ is a subgraph of $\mathcal{G}$ and $\omega \in \Omega_V$ we will write $\omega_{V'}$ for its restriction to V'. We will also say that a vertex x is good for the configuration ω if $\omega_x \in G$.

On Ω equipped with the natural σ-algebra we will consider the product measure $\mu := \prod_{x \in V} \nu_x, \ \nu_x \equiv \nu$. If $\mathcal{G}' = (V', E')$ is a subgraph of $\mathcal{G}$ we will write $\mu_{V'}$ or $\mu_{\mathcal{G}'}$ for the restriction of μ to $\Omega_{V'}$. Finally, for any $f \in L^1(\mu)$, we will use the shorthand notation $\mu(f)$ to denote its expected value and $\mathrm{Var}(f)$ for its variance (when it exists).

2.3 The Markov process

The general interacting particle models that will be studied here are Glauber type Markov processes in Ω, reversible w.r.t. the measure μ and characterized by a finite collection of *influence classes* $\{\mathcal{C}_x\}_{x\in V}$, where $\mathcal{C}_x$ is just a collection of subsets of V (often of the neighbors of the vertex x) satisfying the following general hypothesis:

Hp1 For all $x \in V$ and all $A \in \mathcal{C}_x$ the vertex x does not belong to A.
Hp2 $r := \sup_x \sup_{A\in\mathcal{C}_x} d(x, A) < +\infty$.

In turn the influence classes together with the good event G are the key ingredients to define the constraints of each model.

Definition 1. *Given a vertex $x \in V$ and a configuration ω, we will say that the constraint at x is satisfied by ω if the indicator*

$$c_x(\omega) = \begin{cases} 1 & \text{if there exists a set } A \in \mathcal{C}_x \text{ such that } \omega_y \in G \text{ for all } y \in A \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

is equal to one.

Remark 1. The two general hypotheses above tell us that in order to check whether the constraint is satisfied at a given vertex we do not need to check the current state of the *vertex itself* and we only need to check *locally* around the vertex. This last requirement can actually be weakened and indeed, in order to analyze certain spin exchange kinetically constrained models [10], a very efficient tool is to consider *long range* constraints !

The process that will be studied in the sequel can then be informally described as follows. Each vertex x waits an independent mean one exponential time and then, provided that the current configuration ω satisfies the constraint at x, the value ω_x is refreshed with a new value in S sampled from ν and the whole procedure starts again.

The generator $\mathcal{L}$ of the process can be constructed in a standard way (see e.g. [25, 24]) and it is a non-positive self-adjoint operator on $L^2(\Omega, \mu)$ with domain $Dom(\mathcal{L})$ and Dirichlet form given by

$$\mathcal{D}(f) = \sum_{x\in V} \mu\left(c_x \operatorname{Var}_x(f)\right), \quad f \in Dom(\mathcal{L})$$

Here $\operatorname{Var}_x(f) \equiv \int d\nu(\omega_x) f^2(\omega) - \left(\int d\nu(\omega_x) f(\omega)\right)^2$ denotes the local variance with respect to the variable ω_x computed while the other variables are held fixed. To the generator $\mathcal{L}$ we can associate the Markov semigroup $P_t := e^{t\mathcal{L}}$ with reversible invariant measure μ.

Notice that the constraints $c_x(\omega)$ are increasing functions w.r.t the partial order in Ω for which $\omega \le \omega'$ iff $\omega'_x \in G$ whenever $\omega_x \in G$. However that does not imply in general that the process generated by $\mathcal{L}$ is attractive in the sense of Liggett [25].

Due to the fact that in general the jump rates are not bounded away from zero, irreducibility of the process is not guaranteed and the reversible measure μ is usually not the only invariant measure (typically there exist initial configurations that are blocked forever). An interesting question when $\mathcal{G}$ is infinite is therefore whether μ is ergodic/mixing for the Markov process and whether there exist other ergodic stationary measures. To this purpose it is useful to recall the following well known result (see e.g. Theorem 4.13 in [25]).

Theorem 2.1 *The following are equivalent,*

(a) $\lim_{t\to\infty} P_t f = \mu(f)$ *in* $L^2(\mu)$ *for all* $f \in L^2(\mu)$.
(b) 0 *is a simple eigenvalue for* $\mathcal{L}$.

Clearly (a) implies that $\lim_{t\to\infty} \mu\left(fP_t g\right) = \mu(f)\mu(g)$ for any $f, g \in L^2(\mu)$, *i.e.* μ is mixing.

Remark 2. Even if μ is mixing there will exist in general infinitely many stationary measures, *i.e.* probability measures $\tilde{\mu}$ satisfying $\tilde{\mu}P_t = \tilde{\mu}$ for all $t \geq 0$. As an example, assume c_x not identically equal to one and take an arbitrary probability measure $\tilde{\mu}$ such that $\tilde{\mu}\big(\{S \setminus G\}^V\big) = 1$. An interesting problem is therefore to classify all the stationary ergodic measures $\tilde{\mu}$ of $\{P_t\}_{t\geq 0}$, where ergodicity means that $P_t f = f$ ($\tilde{\mu}$ a.e.) for all $t \geq 0$ implies that f is constant ($\tilde{\mu}$ a.e.). As we will see later, when $\mathcal{G} = \mathbb{Z}^2$ and for a specific choice of the constraint known as the North-East model, a rather detailed answer is now available [24].

When $\mathcal{G}$ is finite connected subgraph of an infinite graph $\mathcal{G}_\infty = (V_\infty, E_\infty)$, the ergodicity issue of the resulting continuous time Markov chain can be attacked in two ways.

The first one is to analyze the chain restricted to a suitably defined ergodic component. Although such an approach is feasible and natural in some cases (see section 6 for an example), the whole analysis becomes quite cumbersome.

Another possibility, which has several technical advantages over the first one, is to unblock certain special vertices of $\mathcal{G}$ by relaxing their constraints and restore irreducibility of the chain. A natural way to do that is to imagine to extend the configuration ω, apriori defined only in V, to the vertices in $V_\infty \setminus V$ and to keep it there frozen and equal to some reference configuration τ that will be referred to as the *boundary condition.* If enough vertices in $V_\infty \setminus V$ are good for τ, then enough vertices of $\mathcal{G}$ will become unblocked and the whole chain ergodic.

More precisely we can define the finite volume constraints with boundary condition τ as

$$c^\tau_{x,V}(\omega) := c_x(\omega \cdot \tau) \tag{2}$$

where c_x are the constraints for $\mathcal{G}_\infty$ defined in 1 and $\omega \cdot \tau \in \Omega$ denotes the configuration equal to ω inside V and equal to τ in $V_\infty \setminus V$. Notice that, for any $x \in V$, the rate $c^\tau_{x,V}(\omega)$ (2) depends on τ only through the indicators

$\{\mathbb{1}_{\tau_z \in G}\}_{z \in \mathcal{B}}$, where $\mathcal{B}$ is the *boundary set* $\mathcal{B} := (V_\infty \setminus V) \cap (\cup_{z \in V} \mathcal{C}_z)$. Therefore, instead of fixing τ, it is enough to choose a subset $\mathcal{M} \subset \mathcal{B}$, called the *good boundary set*, and define

$$c^{\mathcal{M}}_{x,V}(\omega) := c^{\tau}_{x,V}(\omega) \tag{3}$$

where τ is *any* configuration satisfying $\tau_z \in G$ for all $z \in \mathcal{M}$ and $\tau_z \notin G$ for $z \in \mathcal{B} \backslash \mathcal{M}$. We will say that a choice of $\mathcal{M}$ is *minimal* if the corresponding chain in $\mathcal{G}$ with the rates (3) is irreducible and it is non-irreducible for any other choice $\mathcal{M}' \subset \mathcal{M}$. The choice $\mathcal{M} = \mathcal{B}$ will be called *maximal*. For convenience we will write $\mathcal{L}^{\max}_{\Lambda}$ ($\mathcal{L}^{\min}_{\Lambda}$) for the corresponding generators.

Remark 3. Without any other specification for the influence classes of the model it may very well be the case that there exists no boundary conditions for which the chain is irreducible and/or their existence may depend on the choice of the finite subgraph $\mathcal{G}$. However, as we will see later, for all the interesting models discussed in the literature all these issues will have a rather simple solution.

We will now describe some of the basic models and solve the problem of boundary conditions for each one of them.

2.4 0-1 Kinetically constrained spin models

In most models considered in the physical literature the finite probability space (S, ν) is a simple $\{0, 1\}$ Bernoulli space and the good set G is conventionally chosen as the empty (vacant) state $\{0\}$. Any model with these features will be called in the sequel a "0-1 KCSM" (kinetically constrained spin model). Although in most cases the underlying graph $\mathcal{G}$ is a regular lattice like $\mathbb{Z}^d$, whenever is possible we will try to work in full generality.

Given a 0-1 KCSM, the parameter $q = \nu(0)$ can be varied in $[0, 1]$ while keeping fixed the basic structure of the model (*i.e.* the notion of the good set and the constraints c_x) and it is natural to define a critical value q_c as

$$q_c = \inf\{q \in [0, 1] : \text{ 0 is a simple eigenvalue of } \mathcal{L}\}$$

As we will prove below q_c coincides with the *bootstrap percolation threshold* q_{bp} of the model defined as follows [32] [6]. For any $\eta \in \Omega$ define the bootstrap map $T : \Omega \mapsto \Omega$ as

$$(T\eta)_x = 0 \quad \text{if either} \quad \eta_x = 0 \quad \text{or} \quad c_x(\eta) = 1. \tag{4}$$

Denote by $\mu^{(n)}$ the probability measure on Ω obtained by iterating n-times the above mapping starting from μ. As $n \to \infty$ $\mu^{(n)}$ converges to a limiting measure $\mu^{(\infty)}$ [32] and it is natural to define the critical value q_{bp} as

[6] In most of the bootstrap percolation literature the role of the 0's and the 1's is inverted

$$q_{bp} = \inf\{q \in [0,1] : \mu^{\infty} = \delta_0\}$$

where δ_0 is the probability measure assigning unit mass to the constant configuration identically equal to zero. In other words q_{bp} is the infimum of the values q such that, with probability one, the graph $\mathcal{G}$ can be entirely emptied. Using the fact that the c_x's are increasing function of η it is easy to check that $\mu^{(\infty)} = \delta_0$ for any $q > q_{bp}$.

Proposition 2.2 ([11]) *$q_c = q_{bp}$ and for any $q > q_c$ 0 is a simple eigenvalue for $\mathcal{L}$.*

Remark 4. In [11] the proposition has been proved in the special case $\mathcal{G} = \mathbb{Z}^d$ but actually the same arguments apply to any bounded degree connected graph.

Having defined the bootstrap percolation it is natural to divide the 0-1 KCSM into two distinct classes.

Definition 2. *We will say that a 0-1 KCSM is* non cooperative *if there exists a finite set $\mathcal{B} \subset V$ such that any configuration η which is empty in all the sites of $\mathcal{B}$ reaches the empty configuration (all 0's) under iteration of the bootstrap mapping. Otherwise the model will be called* cooperative.

Remark 5. Notice that for a non-cooperative model the critical value q_c is obviously zero since with μ-probability one a configuration will contain the required finite set $\mathcal{B}$ of zeros.

We will now illustrate some of the most studied models.

[1] Frederickson-Andersen (FA-jf) facilitated models [17, 18]. In the facilitated models the constraint at x requires that at least $j \leq \Delta_x$ neighbors are vacant. More formally

$$\mathcal{C}_x = \{A \subset \mathcal{N}_x : |A| \geq j\}$$

When $j = 1$ the model is non-cooperative for any connected graph $\mathcal{G}$ and ergodicity of the Markov chain is clearly guaranteed by the presence of at least one unblocked vertex. When $j > 1$ ergodicity on a general graph is more delicate and we restrict ourselves to finite rectangles R in $\mathbb{Z}^d$. In that case and for the most constrained cooperative case $j = d$ among the irreducible ones, irreducibility is guaranteed if we assume a boundary configuration identically empty on $\partial_+ R$. Quite remarkably, using results from bootstrap percolation [32] combined with proposition 2.2, when $\mathcal{G} = \mathbb{Z}^d$ and $2 \leq j \leq d$ the ergodicity threshold q_c always vanishes.

[2] Spiral model [8, 7] This model is defined on $\mathbb{Z}^2$ with the following choice for the influence classes

$$\mathcal{C}_x = \{NE_x \cup SE_x;\ SE_x \cup SW_x;\ SW_x \cup NW_x;\ NW_x \cup NE_x\}$$

where $NE_x = (x + \mathbf{e}_2, x + \mathbf{e}_1 + \mathbf{e}_2)$, $SE_x = (x + \mathbf{e}_1, x + \mathbf{e}_1 - \mathbf{e}_2)$, $SW_x = (x - \mathbf{e}_2, x - \mathbf{e}_2 - \mathbf{e}_1)$ and $NW_x = (x - \mathbf{e}_1; x - \mathbf{e}_1 + \mathbf{e}_2)$. In other words the vertex x can flip iff either its North-East (NE_x) or its South-West (SW_x) neighbours (or both of them) are empty *and* either its North-West (NW_x) or its South-East (SE_x) neighbours (or both of them) are empty too. The model is clearly cooperative and in [7] it has been proven that its critical point q_c coincides with $1-p_c^o$, where p_c^o is the critical treshold for oriented percolation. The interest of this model lies on the fact that its bootstrap percolation is expected to display a peculiar mixed discontinuous/critical character which makes it relevant as a model for the liquid glass and more general jamming transitions [8, 7].

[3] Oriented models. Oriented models are similar to the facilitated models but the neighbors of a given vertex x that must be vacant in order for x to become free to flip, are chosen according to some orientation of the graph. Instead of trying to describe a very general setting we present three important examples.

Example 1. The first and best known example is the so called East model [16]. Here $\mathcal{G} = \mathbb{Z}$ and for every $x \in \mathbb{Z}$ the influence class $\mathcal{C}_x$ consists of the vertex $x + 1$. In other words any vertex can flip iff its right neighbor is empty. The minimal boundary conditions in a finite interval which ensure irreducibility of the chain are of course empty right boundary, *i.e.* the rightmost vertex is always unconstrained. The model is clearly cooperative but $q_c = 0$ since in order to empty $\mathbb{Z}$ it is enough to start from a configuration for which any site x has some empty vertex to its right. One could easily generalize the model to the case when $\mathcal{G}$ is a rooted tree (see section 6). In that case any vertex different from the root can be updated iff its ancestor is empty. The root itself is unconstrained.

Example 2. The second example is the North-East model in $\mathbb{Z}^2$ [23]. Here one chooses $\mathcal{C}_x$ as the North and East neighbor of x. The model is clearly cooperative and its critical point q_c coincides with $1 - p_c^o$, where p_c^o is the critical threshold for oriented percolation in $\mathbb{Z}^2$ [32]. For such a model much more can be said about the stationary ergodic measures of the Markov semigroup P_t.

Theorem 2.3 ([24]) *If $q < q_c$ the trivial measure δ_1 that assigns unit mass to the configuration identically equal to 1 is the only translation invariant, ergodic, stationary measure for the system. If $q \geq q_c$ the reversible measure μ is the unique, non trivial, ergodic, translation invariant, stationary measure.*

Example 3. The third model was suggested in [3] and it is defined on a rooted (finite or infinite) binary tree $\mathcal{T}$. Here a vertex x can flip iff its two children are vacant. If the tree is finite then ergodicity requires that all the leaves of $\mathcal{T}$ are unconstrained. It is easy to check that the critical threshold satisfies $q_c = 1/2$, the site percolation threshold on the binary tree.

3 Quantities of interest and related problems

Back to the general model we now define two main quantities that are of mathematical and physical interest.

The first one is the spectral gap of the generator $\mathcal{L}$, defined as

$$\text{gap}(\mathcal{L}) := \inf_{f \neq \text{const}} \frac{\mathcal{D}(f)}{\text{Var}(f)} \tag{5}$$

A positive spectral gap implies that the reversible measure μ is mixing for the semigroup P_t with exponentially decaying correlations:

$$\text{Var}\,(P_t f) \leq e^{-2t\,\text{gap}(\mathcal{L})}\,\text{Var}(f), \qquad \forall\, f \in L^2(\mu).$$

Remark 6. In the sequel the time scale $T_{\text{rel}} := \text{gap}^{-1}$ which is naturally fixed by the spectral gap will be refered to as the *relaxation time* of the process.

For a 0-1 KCSM, two natural questions arise.

1. Define the new critical point $q_c' := \inf\{q \in [0,1] : \ \text{gap}(\mathcal{L}) > 0\}$. Obviously $q_c' \geq q_c$. Is it the case that equality holds ?
2. If $q_c' = q_c$ what is the behaviour of $\text{gap}(\mathcal{L})$ as $q \downarrow q_c$?

As we will see later for most of the relevant models it is possible to answer in rather detailed way to both questions.

The second quantity of interest is the so called *persistence function* (see e.g. [21, 34]) defined by

$$F(t) := \int d\mu(\eta)\; \mathbb{P}(\sigma_0^\eta(s) = \eta_0,\ \forall s \leq t) \tag{6}$$

where $\{\sigma_s^\eta\}_{s\geq 0}$ denotes the process started from the configuration η. In some sense the persistence function, a more accessible quantity to numerical simulation than the spectral gap, provides a measure of the "mobility" of the system. Here the main questions are:

1. What is the behavior of $F(t)$ for large time scales ?
2. For a 0-1 KCSM is it the case that $F(t)$ decays exponentially fast as $t \to \infty$ for any $q > q_c'$?
3. If the answer to the previous question is positive, is the decay rate related to the spectral gap in a simple way or the decay rate of $F(t)$ requires a deeper knowledge of the spectral density of $\mathcal{L}$?
4. Is it possible to exhibit examples of 0-1 KCSM in which the persistence function shows a crossover between a stretched and a pure exponential decay ?

Unfortunately the above questions are still mostly unanswered except for the first two.

3.1 Some useful observations to bound the spectral gap

It is important to observe the following kind of monotonicity that can be exploited in order to bound the spectral gap of one model with the spectral gap of another one.

Definition 3. *Suppose that we are given two influence classes $\mathcal{C}_0$ and $\mathcal{C}_0'$, denote by $c_x(\omega)$ and $c_x'(\omega)$ the corresponding rates and by $\mathcal{L}$ and $\mathcal{L}'$ the associated generators on $L^2(\mu)$. If, for all $\omega \in \Omega$ and all $x \in V$, $c_x'(\omega) \le c_x(\omega)$, we say that $\mathcal{L}$ is dominated by $\mathcal{L}'$.*

Remark 7. The term domination here has the same meaning it has in the context of bootstrap percolation. It means that the KCSM associated to $\mathcal{L}'$ is *more* constrained than the one associated to $\mathcal{L}$.

Clearly, if $\mathcal{L}$ is dominated by $\mathcal{L}'$, $\mathcal{D}'(f) \le \mathcal{D}(f)$ and therefore $\text{gap}(\mathcal{L}') \le \text{gap}(\mathcal{L})$.

Example 4. Assume that the graph $\mathcal{G}$ has n vertices and contains a Hamilton path $\Gamma = \{x_1, x_2, \dots, x_n\}$, *i.e.* $d(x_{i+1}, x_i) = 1$ for all $1 \le i \le n-1$ and $x_i \neq x_j$ for all $i \neq j$. Consider the FA-1f model on $\mathcal{G}$ with one special vertex, e.g. x_n, unconstrained ($c_{x_n} \equiv 1$). Then, if we replace $\mathcal{G}$ by Γ equipped with its natural graph structure and we denote by $\mathcal{L}$ and $\mathcal{L}'$ the respective generators, we get that $\text{gap}(\mathcal{L}) \ge \text{gap}(\mathcal{L}')$. Clearly $\mathcal{L}'$ describes the FA-1f model on the finite interval $[1, \dots, n] \subset \mathbb{Z}$ with the last vertex free to flip. This in turn is dominated by $\mathcal{L}_{\text{East}}$, the generator of the East model on $[1, \dots, n]$, which is known to have a positive [3, 11] spectral gap uniformly in n. Therefore the latter result holds also for $\text{gap}(\mathcal{L}')$ and $\text{gap}(\mathcal{L})$.

Example 5. Along the lines of the previous example we could lower bound the spectral gap of the FA-2f model in $\mathbb{Z}^d$, $d \ge 2$, with that in $\mathbb{Z}^2$, by restricting the sets $A \in \mathcal{C}_0$ to e.g. the $(\mathbf{e}_1, \mathbf{e}_2)$-plane.

For a last and more detailed example of the comparison technique we refer the reader to section 6.

Although the comparison technique can be quite effective in proving positivity of the spectral gap, one should keep in mind that, in general, it provides quite poor bounds, particularly in the limiting case $q \downarrow q_c$.

The second observation we make consists in relating $\text{gap}(\mathcal{L})$ when the underlying graph is infinite to its finite graph analogue. Fix $r \in V$ and let $\mathcal{G}_{n,r} \subset \mathcal{G}$ be the connected ball centered at r of radius n. Suppose that $\inf_n \text{gap}(\mathcal{L}^{\max}_{\mathcal{G}_{n,r}}) > 0$. It is then easy to conclude that $\text{gap}(\mathcal{L}) > 0$.

Indeed, following Liggett Ch.4 [25], for any $f \in Dom(\mathcal{L})$ with $\text{Var}(f) > 0$ pick $f_n \in L^2(\Omega, \mu)$ depending only on finitely many spins so that $f_n \to f$ and $\mathcal{L} f_n \to \mathcal{L} f$ in L^2. Then $\text{Var}(f_n) \to \text{Var}(f)$ and $\mathcal{D}(f_n) \to \mathcal{D}(f)$. But since f_n depends on finitely many spins

$$\text{Var}(f_n) = \text{Var}_{\mathcal{G}_{m,r}}(f_n) \quad \text{and} \quad \mathcal{D}(f_n) = \mathcal{D}_{\mathcal{G}_{m,r}}(f_n)$$

provided that m is a large enough square (depending on f_n). Therefore

$$\frac{\mathcal{D}(f)}{\mathrm{Var}(f)} \geq \inf_n \mathrm{gap}(\mathcal{L}_{\mathcal{G}_{n,r}}) > 0.$$

and $\mathrm{gap}(\mathcal{L}) \geq \inf_n \mathrm{gap}(\mathcal{L}_{\mathcal{G}_{n,r}}) > 0$.

4 Main results for 0-1 KCSM on regular lattices

In this section we state some of the main results for a general 0-1 KCSM on $\mathbb{Z}^d$ which have been obtained in [11].

Fix an integer length scale ℓ larger than the range of the constraints and let $\mathbb{Z}^d(\ell) \equiv \ell\,\mathbb{Z}^d$. Consider a partition of $\mathbb{Z}^d$ into disjoint rectangles $\Lambda_z := \Lambda_0 + z$, $z \in \mathbb{Z}^d(\ell)$, where $\Lambda_0 = \{x \in \mathbb{Z}^d : \ 0 \leq x_i \leq \ell - 1, \ i = 1, .., d\}$.

Definition 4. *Given $\epsilon \in (0,1)$ we say that $G_\ell \subset \{0,1\}^{\Lambda_0}$ is a ϵ-good set of configurations on scale ℓ if the following two conditions are satisfied:*

(a) $\mu(G_\ell) \geq 1 - \epsilon$.
(b) For any collection $\{\xi^{(x)}\}_{x \in \mathcal{K}_0^*}$ of spin configurations such that $\xi^{(x)} \in G_\ell$ for all $x \in \mathcal{K}_0^*$, the following holds. For any $\xi \in \Omega$ which coincides with $\xi^{(x)}$ in $\cup_{x \in \mathcal{K}_0^*} \Lambda_{\ell x}$, there exists a sequence of legal moves inside $\cup_{x \in \mathcal{K}_0^*} \Lambda_{\ell x}$ (i.e. single spin moves compatible with the constraints) which transforms ξ into a new configuration $\tau \in \Omega$ such that the Markov chain in Λ_0 with boundary conditions τ is ergodic.

Remark 8. In general the transformed configuration τ will be identically equal to zero on $\partial_+^* \Lambda_0$. It is also clear that assumption (b) has been made having in mind models like the East, the FA-jf or the N-E which, modulo rotations, are dominated by a model with influence class $\tilde{\mathcal{C}}_x$ *entirely contained* in the sector $\{y : y = x + \sum_{i=1}^d \alpha_i \mathbf{e}_i, \ \alpha_i \geq 0\}$. Here we deal only with these models and we refer to [12] for the analysis of models which do not have the above property, such as the Spiral Model introduced in [8, 7]. In this case one should use a non rectangular geometry for the tiles of the partition of $\mathbb{Z}^d$, adapted to the choice of the influence classes $\{\mathcal{C}_x\}_{x \in \mathbb{Z}^d}$.

With the above notation the first main result of [11] can be formulated as follows.

Theorem 4.1 *There exists a universal constant $\epsilon_0 \in (0,1)$ such that, if there exists ℓ and a ϵ_0-good set G_ℓ on scale ℓ, then* $\inf_{\Lambda \in \mathbb{F}} \mathrm{gap}(\mathcal{L}_\Lambda^{\max}) > 0$. *In particular* $\mathrm{gap}(\mathcal{L}) > 0$.

In several examples, e.g. the FA-jf models, the natural candidate for the event G_ℓ is the event that the tile Λ_0 is "internally spanned", a notion borrowed from bootstrap percolation [1, 32, 13, 22, 14]:

Definition 5. *We say that a finite set $\Gamma \subset \mathbb{Z}^d$ is* internally spanned *by a configuration $\eta \in \Omega$ if, starting from the configuration η^Γ equal to one outside Γ and equal to η inside Γ, there exists a sequence of legal moves inside Γ which connects η^Γ to the configuration identically equal to zero inside Γ and identically equal to one outside Γ.*

Of course whether or not the set Λ_0 is internally spanned for η depends only on the restriction of η to Λ_0. One of the major results in bootstrap percolation problems has been the exact evaluation of the μ-probability that the box Λ_0 is internally spanned as a function of the length scale ℓ and the parameter q [22, 32, 13, 14, 1]. For non-cooperative models it is obvious that for any $q > 0$ such probability tends very rapidly (exponentially fast) to one as $\ell \to \infty$, since the existence of at least one completely empty finite set $\mathcal{B} + x \subset \Lambda_0$ (see definition 2), allows to empty all Λ_0. For some cooperative systems like e.g. the FA-2f and Modified Basic model in $\mathbb{Z}^2$, it has been shown that for any $q > 0$ such probability tends very rapidly (exponentially fast) to one as $\ell \to \infty$ and that it abruptly jumps from being very small to being close to one as ℓ crosses a critical scale $\ell_c(q)$. In most cases the critical length $\ell_c(q)$ diverges very rapidly as $q \downarrow 0$. Therefore, for such models and $\ell > \ell_c(q)$, one could safely take G_ℓ as the collection of configurations η such that Λ_0 is internally spanned for η. We now formalize what we just said.

Corollary 4.2 *Assume that* $\lim_{\ell \to \infty} \mu(\Lambda_0$ *is internally spanned* $) = 1$ *and that the Markov chain in Λ_0 with zero boundary conditions on $\cup_{x \in \mathcal{K}_0^*} \Lambda_{\ell x}$ is ergodic. Then* $\text{gap}(\mathcal{L}) > 0$.

The second main result concerns the long time behavior of the persistence function $F(t)$ defined in (6).

Theorem 4.3 *Assume that* $\text{gap}(\mathcal{L}) > 0$. *Then there exists a constant $c = c(q) > 0$ such that $F(t) \leq e^{-ct}$. For small values of q the constant c can be taken proportional to $q \times$* $\text{gap}(\mathcal{L})$.

Remark 9. The above theorems disprove some conjectures which appeared in the physics literature [19, 21, 4, 5], based on numerical simulations and approximate analytical treatments, on the existence of a second critical point $q_c' > q_c$ at which the spectral gap vanishes and/or below which $F(t)$ would decay in a stretched exponential form $\simeq \exp(-t/\tau)^\beta$ with $\beta < 1$.

Theorem 4.3 also indicates that one can obtain *upper bounds* on the spectral gap by proving *lower bounds* on the persistence function. Concretely a lower bound on the persistence function can be obtained by restricting the μ-average to those initial configurations η for which the origin is blocked with high probability for all times $s \leq t$. Unfortunately in most models such a strategy leads to lower bound on $F(t)$ which are usually quite far from the above upper bound and it is an interesting open problem to find an exact asymptotic as $t \to \infty$ of $F(t)$.

Finally we observe that for the North-East model on $\mathbb{Z}^2$ at the critical value $q = q_c$ the spectral gap vanishes and the persistence function satisfies $\int_0^\infty dt\, F(\sqrt{t}) = \infty$ (see Theorem 6.17 and Corollary 6.18 in [11]).

4.1 Some ideas of the strategy for proving theorems 4.1, 4.3

The main idea behind the proof of theorem 4.1 goes as follows. First of all one covers the lattice with non overlapping cubic blocks $\{\Lambda_{\ell x}\}_{x\in\mathbb{Z}^d}$ and, on the rescaled lattice $\mathbb{Z}^d(\ell) := \ell\mathbb{Z}^d$, one considers the new model with single spin space $S = \{0,1\}^{\ell^d}$, good event $G := G_\ell$, single site measure the restriction of μ to S and renormalized constraints $\{c_x^{ren}\}_{x\in\mathbb{Z}^d(\ell)}$ which are a strengthening of the North-East ones namely

$$c_x^{ren}(\eta) = 1 \quad \text{iff } \eta_y \in G \text{ for all } y \in \mathcal{K}_x^*.$$

Such a model is referred to in [11] as the **-general model*. By assumption the probability of G can be made arbitrarily close to one by taking ℓ large enough and therefore, by the so called *Bisection-Constrained approach* which is detailed in the next section for the case when μ is a high temperature Gibbs measure, the spectral gap of the *-general model is positive. Next one observes that assumption (b) of the theorem is there exactly to allow one to reconstruct any *legal* move of the *-general model, *i.e.* a full update of an entire block of spins, by means of a finite (depending only on ℓ) sequence of *legal* moves for the original 0-1 KCMS. It is then an easy step, using standard path techniques for comparing two different Markov chains (see e.g. [31]), to go from the Poincaré inequality for the *-general model to the Poincaré inequality for the original model.

The proof of Theorem 4.3 is instead based on rather standard large deviation considerations. Namely, one first observe that $F(t) = F_1(t) + F_0(t)$ where

$$F_1(t) = \int d\mu(\eta)\, \mathbb{P}(\sigma_0^\eta(s) = 1 \text{ for all } s \le t)$$

and similarly for $F_0(t)$. Consider now $F_1(t)$, the case of $F_0(t)$ being similar. The exponential Chebychev inequality gives

$$F_1(t) = \int d\mu(\eta)\, \mathbb{P}\Big(\int_0^t ds\, \sigma_0^\eta(s) = t\Big) \le e^{-\lambda t}\ \mathbb{E}_\mu\big(e^{\lambda \int_0^t ds\, \sigma_0^\eta(s)}\big)$$

where $\mathbb{E}_\mu$ denotes the expectation over the process started from the equilibrium distribution μ. On $L^2(\mu)$ consider the self-adjoint operator $H_\lambda := \mathcal{L}+\lambda V$, where V is the multiplication operator by σ_0. By the very definition of the scalar product $< f, g >$ in $L^2(\mu)$ and the Feynman–Kac formula, one can rewrite $\mathbb{E}_\mu(e^{\lambda \int_0^t \sigma_0(s)})$ as $< 1, e^{tH_\lambda} 1 >$. Thanks to the fact that $\text{gap}(\mathcal{L}) > 0$, standard perturbation theory now shows that there exists $\lambda > 0$ such that $\lambda \times$ (supremum of the spectrum of H_λ) < 1 and the proof is complete.

4.2 Asymptotics of the spectral gap near the ergodicity threshold.

An important question, particularly in connection with numerical simulations or non-rigorous approaches, is the behavior near the ergodicity threshold q_c of the spectral gap for each specific model. Here is a set of results proven in [11].

East Model.

$$\lim_{q\to 0} \log(1/\operatorname{gap})/(\log(1/q))^2 = (2\log 2)^{-1} \tag{7}$$

FA-1f. For any $d \geq 1$, there exists a constant $C = C(d)$ such that for any $q \in (0,1)$, the spectral gap on $\mathbb{Z}^d$ satisfies:

$$\begin{aligned} C^{-1}q^3 \leq \operatorname{gap}(\mathcal{L}) &\leq Cq^3 && \text{for } d=1,\\ C^{-1}q^2/\log(1/q) \leq \operatorname{gap}(\mathcal{L}) &\leq Cq^2 && \text{for } d=2,\\ C^{-1}q^2 \leq \operatorname{gap}(\mathcal{L}) &\leq Cq^{1+\frac{2}{d}} && \text{for } d\geq 3. \end{aligned}$$

FA-df in $\mathbb{Z}^d$. Fix $\epsilon > 0$. Then there exists $c = c(d)$ such that

$$\Big[\exp^{d-1}(c/q^2)\Big]^{-1} \leq \operatorname{gap}(\mathcal{L}) \leq \Big[\exp^{d-1}\big(\frac{\lambda_1-\epsilon}{q}\big)\Big]^{-1} \qquad d\geq 3 \tag{8}$$

$$\exp(-c/q^5) \leq \operatorname{gap}(\mathcal{L}) \leq \exp\big(-\frac{(\lambda_1-\epsilon)}{q}\big) \qquad d=2 \tag{9}$$

as $q \downarrow 0$, where $\exp^{d-1}$ denote the $(d-1)^{\text{th}}$-iterate of the exponential function and $\lambda_1 = \pi^2/18$.

The proof of the lower bounds is a rather delicate combination of the renormalization scheme described above together with paths techniques as described in [31]. The upper bounds are proved instead either by a careful choice of a test function in the variational characterization of the spectral gap or by a lower bound on the persistence function $F(t)$ combined with the upper bound given in Theorem 4.3.

Remark 10. Again some of the above findings disprove previous claims for the East model [34] and for the FA-1f model in $d = 2, 3$ [5]. The result for the East model actually came out as a surprise. In [34] the model was considered "essentially" solved and the result for the spectral gap was gap $\approx q^{\log_2(q)}$ as $q \downarrow 0$ to be compared to the correct scaling $q^{\log_2(q)/2}$. In [3] the above solution was proved to be a *lower* bound and an upper bound of the form $q^{\log_2(q)/2}$ was rigorously established but considered poor because off by a power 1/2 from the supposedly correct behavior.

The scaling indicated in [34] is based in part on the following consideration. Fix $q \ll 1$ and consider the East model on the interval $\Lambda_q := [0, \ldots, 1/q]$ with the last site free to flip (*i.e.* zero boundary conditions). Notice that $1/q$ is the average distance between the zeros. Start from the configuration identically equal to one and let T be the (random) time at which the origin is able to flip. Energy barriers consideration (see [2, 3, 15]) suggest that $\mathbb{E}(T)$ should scale as $q^{\log_2(q)}$ and that is what was assumed in [34]. However it is not difficult to prove that the scaling of $\mathbb{E}(T)$ is bounded above by $(q\,\mathrm{gap})^{-1}$. Indeed we can write for any $t \geq 0$

$$\exp(-cq\,\mathrm{gap}(\mathcal{L}_{\Lambda_q})t) \geq \tilde{F}(t) \geq \mu(\text{all ones})\mathbb{P}(T \geq t) \geq e^{-2}\mathbb{P}(T \geq t)$$

where $\tilde{F}(t)$ is the finite volume persistence function. Integrating over t and using the monoticity of the gap (see [11, Lemma 2.11]) give $\mathbb{E}(T) \leq e^2 c(q\,\mathrm{gap}(\mathcal{L}_{\Lambda_q})))^{-1} \leq e^2 c(q\,\mathrm{gap}(\mathcal{L}))^{-1}$. This, in view of Theorem 4.1, is incompatible with the assumed scaling $q^{\log_2(q)}$.

Moreover one can obtain a lower bound on $\mathbb{E}(T)$ as follows. Let λ be such that $\mathbb{P}(T \geq \lambda) = e^{-1}$ then clearly $\mathbb{P}(T \geq t) \leq e^{-\lfloor t/\lambda \rfloor}$ and $\mathbb{E}(T) \geq e^{-1}\lambda$. We can always couple in the natural way two copies of the process, one started from all ones and the other from any other configuration η, and conclude that

$$\mathbb{P}(\text{the two copies have not coupled at time } t) \leq \mathbb{P}(T \geq t) \leq e^{1-\lambda t}.$$

Standard arguments give immediately that $\mathrm{gap}^{-1} \leq \lambda$ *i.e.* $\mathbb{E}(T) \geq e^{-1}\,\mathrm{gap}^{-1}$. In conclusion

$$e^{-1}(\mathrm{gap}(\mathcal{L}_{\Lambda_q}))^{-1} \leq \mathbb{E}(T) \leq e^2 c(q\,\mathrm{gap}(\mathcal{L}_{\Lambda_q})))^{-1}$$

5 Extension to interacting models

In this section we show how to extend the results on the positivity of the spectral for 0-1 KCSM on a regular lattice $\mathbb{Z}^d$ to the case in which a weak interaction is present among the spins. We begin by defining what we mean by an *interaction*.

Definition 6. *A finite range interaction Φ is a collection $\Phi := \{\Phi_A\}_{A\in\mathbb{F}}$ where*

i) $\Phi_A : \Omega_A \mapsto \mathbb{R}$ *for every* $A \in \mathbb{F}$*;*
ii) $\Phi_A = 0$ *if* $\mathrm{diam}(A) \geq r$ *for some finite* $r = r(\Phi)$ *called the range of the interaction;*
iii) $\|\Phi\| \equiv \sup_{x\in\mathbb{Z}^d} \sum_{A\ni x} \|\Phi_A\|_\infty < \infty$*;*

We will say that $\Phi \in \mathcal{B}_{M,r}$ *if* $r(\Phi) \leq r$ *and* $\|\Phi\| \leq M$*.*

Given an interaction $\Phi \in \mathcal{B}_{r,M}$ and $\Lambda \in \mathbb{F}$, we define the energy in Λ of a spin configuration $\sigma \in \Omega$ by

$$H_\Lambda(\sigma) = \sum_{A\cap\Lambda\neq\emptyset} \Phi_A(\sigma)$$

For $\sigma \in \Omega_\Lambda$ and $\tau \in \Omega_{\Lambda^c}$ we also let $H_\Lambda^\tau(\sigma) := H_\Lambda(\sigma\cdot\tau)$ where $\sigma\cdot\tau$ denotes the configuration equal to σ inside Λ and to τ outside it. Finally, for any

$\Lambda \in \mathbb{F}$ and $\tau \in \Omega_{\Lambda^c}$, we define the finite volume Gibbs measure on Ω_Λ with boundary conditions τ and apriori single spin measure ν by the formula

$$\mu_\Lambda^{\Phi,\tau}(\sigma) := \frac{1}{Z_\Lambda^{\Phi,\tau}} e^{-H_\Lambda^\tau(\sigma)} \prod_{x\in\Lambda} \nu(\sigma_x)$$

where $Z_\Lambda^{\Phi,\tau}$ is a normalization constant.

The key property of Gibbs measures is that, for any $V \subset \Lambda$ and any ξ in $\Lambda \setminus V$, the conditional Gibbs measure in Λ with boundary conditions τ given ξ coincides with the Gibbs measure in V with boundary condition $\tau_{\Lambda^c} \cdot \xi$. More formally

$$\mu_\Lambda^{\Phi,\tau}(\cdot \mid \sigma_{V^c} = \xi) = \mu_V^{\Phi,\tau_{\Lambda^c}\cdot\xi}(\cdot)$$

Clearly averages w.r.t. $\mu_\Lambda^{\Phi,\tau}(\cdot \mid \sigma_{V^c} = \xi)$ are function of ξ and, whenever confusion does not arise, we will systematically drop ξ from our notation.

As it is well known (see e.g. [33]), for any $r < \infty$ there exists $M_0 > 0$ such that for any $0 < M < M_0$ the following holds. For any $\Phi \in \mathcal{B}_{r,M}$ there exists a unique probability measure μ^Φ on Ω, called the unique Gibbs measure associated to the interaction Φ with apriori measure ν, such that, for any τ,

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \mu_\Lambda^{\Phi,\tau} = \mu^\Phi$$

where the limit is to be understood as a weak limit. Moreover the limit is reached "exponentially fast" in the strongest possible sense. Namely, for any $\Delta \subset \Lambda \in \mathbb{F}$ and any two boundary conditions τ, τ',

$$\max_{\sigma_\Delta} \left| \frac{\mu_\Lambda^{\Phi,\tau'}(\sigma_\Delta)}{\mu_\Lambda^{\Phi,\tau}(\sigma_\Delta)} - 1 \right| \le K|D_r(\tau,\tau')|\, e^{-md\left(\Delta,\, D_r(\tau,\tau')\right)} \tag{10}$$

where $D_r(\tau,\tau') = \{y :\ 0 < d(y,\Lambda) \le r,\ \tau_y \neq \tau'_y\}$ and the constants m, K depend only on M, r, d. Moreover $m \uparrow +\infty$ as $M \downarrow 0$. When $d = 1$ the threshold M_0 can be taken equal to $+\infty$. In all what follows we will always assume that $\Phi \in \mathcal{B}_{r,M}$ for some apriori given r, M and that $M < M_0$.

Remark 11. In general the constant M_0 does not coincide with any "critical point" for the model. It is only a sort of "high temperature threshold" (see [27, 28, 29] for more details about this issue).

Having described the notion of the unique Gibbs measure corresponding to Φ, we can define the generator $\mathcal{L}^\Phi$ of a 0-1 KCSM with interaction Φ and constraints c_x given by (1), as the unique self-adjoint operator on $L^2(\Omega, \mu^\Phi)$ with quadratic form

$$\mathcal{D}^\Phi(f) = \sum_x \mu^\Phi\left(c_x \operatorname{Var}_x^\Phi(f)\right), \qquad f \text{ local}$$

where now the local variance $\operatorname{Var}_x^\Phi(f)$ is computed with the conditional Gibbs measure given all the spins outside x. The construction of the generator in a finite volume Λ with boundary conditions τ is exactly the same as in the non-interacting case and we skip it.

5.1 Spectral gap for a weakly interacting North-East model

Instead of trying to prove a very general result on the spectral gap of a weakly interacting KCSM, we will explain how to deal with the interaction in the concrete case of the North-East model introduced in section 2.4. Moreover, in order not to obscure the discussion with renormalization or block constructions, we will make the unnecessary assumption that the basic parameter q of the reference measure ν is very close to one.

Theorem 5.1 *Let $\{c_x\}_{x\in\mathbb{Z}^2}$ be those of the North-East model. There exists $q_0 \in (0,1)$ and for any $r < \infty$ there exists M_1 such that, for any $M < \min(M_0, M_1)$ and $q \geq q_0$,*

$$\inf_{\Phi\in\mathcal{B}_{r,M}} \operatorname{gap}(\mathcal{L}^{\Phi}) > 0$$

Remark 12. As we will see in the proof of the theorem, the restriction on strength of the interaction comes from two different requirements. The first one is that the finite volume Gibbs measure has the very strong mixing property uniformly in the boundary conditions given in (10). That, as we pointed out previously, is guaranteed as long as $M < M_0$. The second one requires that the zeros, which certainly percolate in a robust way w.r.t. the unperturbed measure ν because of the assumption $q \approx 1$, continue to do so even when we switch on the interaction. It is worthwhile to observe that for the one dimensional East model, the first requirement is satisfied for any $M < \infty$ and that the second one is simply not necessary. Therefore for the East model the above theorem should be reformulated as follows.

Theorem 5.2 *Let $\{c_x\}_{x\in\mathbb{Z}}$ be those of the East model. For any finite pair (r, M)*

$$\inf_{\Phi\in\mathcal{B}_{r,M}} \operatorname{gap}(\mathcal{L}^{\Phi}) > 0$$

Proof (of Theorem 5.1). We will follow the pattern of the proof for the non interacting case given in [11] and we will establish the stronger result

$$\sup_{\Lambda\in\mathbb{F}} \gamma(\Lambda) < +\infty, \quad \text{where} \quad \gamma(\Lambda) := \Big(\inf_{\Phi\in\mathcal{B}_{r,M}} \inf_{\tau\in\mathrm{Max}_\Lambda} \operatorname{gap}(\mathcal{L}^{\Phi,\tau}_{\Lambda})\Big)^{-1} \qquad (11)$$

provided that $q > q_0$ is large and M is taken sufficiently small. Above Max_Λ denotes the set of configurations in Ω_{Λ^c} which are identically equal to zero on $\partial^*_+\Lambda$.

As in [11] the first step consists in proving a certain monotonicity property of $\gamma(\Lambda)$.

Lemma 5.3 *For any $V \subset \Lambda \in \mathbb{F}$,*

$$0 < \gamma(V) \leq \gamma(\Lambda) < \infty$$

Proof (Proof of the Lemma). Fix $\Phi \in \mathcal{B}_{r,M}$ and, for any $\xi \in \text{Max}_V$, define the new interaction Φ^ξ as follows:

$$\Phi^\xi_A(\sigma_A) = \begin{cases} 0 & \text{if } A \cap V^c \neq \emptyset \\ \sum_{A' :\, A' \cap V = A} \Phi_{A'}(\sigma_A \cdot \xi_{A' \setminus A}) & \text{if } A \subset V \end{cases}$$

Notice that, by construction,

$$r(\Phi^\xi) \le r(\Phi) \quad \text{and} \quad \sup_x \sum_{A \ni x} \|\Phi^\xi_A\|_\infty \le \|\Phi\|_\infty$$

so that $\Phi^\xi \in \mathcal{B}_{r,M}$. Next observe that the Gibbs measure on Λ with interaction Φ^ξ is simply the product measure

$$\mu^{\Phi^\xi}_\Lambda(\sigma_\Lambda) := \mu^{\Phi,\xi}_V(\sigma_V) \otimes \nu_{\Lambda \setminus V}(\sigma_{\Lambda \setminus V}) \quad \text{on} \quad \Omega_\Lambda = \Omega_V \otimes \Omega_{\Lambda \setminus V}$$

Thus, for any $f \in L^2(\Omega_V, \mu^{\Phi,\xi}_V)$ and $\tau \in \text{Max}_\Lambda$, we can write ($\text{Var}^{\Phi,\tau}_\Lambda \equiv \text{Var}_{\mu^{\Phi,\tau}_\Lambda}$)

$$\begin{aligned} \text{Var}^{\Phi,\xi}_V(f) &= \text{Var}^{\Phi^\xi,\tau}_\Lambda(f) \\ &\le \gamma(\Lambda)\, \mathcal{D}^{\Phi^\xi,\tau}_\Lambda(f) \\ &\le \gamma(\Lambda)\, \mathcal{D}^{\Phi,\xi}_V(f) \end{aligned}$$

where, in the last inequality, we used the fact that, for any $x \in V$ and any $\omega \in \Omega_\Lambda$, $c_{x,\Lambda}(\omega) \le c_{x,V}(\omega)$ because $\xi \in \text{Max}_V$, together with

$$\text{Var}^{\Phi^\xi,\tau}_\Lambda(f \mid \{\sigma_y\}_{y \neq x}) = \text{Var}^{\Phi,\xi}_V(f \mid \{\sigma_y\}_{y \neq x}).$$

□

Thanks to Lemma 5.3 we need to prove (11) only when Λ runs through all possible rectangles. For this purpose our main ingredient will be the bisection technique of [26] which, in its essence, consists in proving a suitable recursion relation between spectral gap on scale $2L$ with that on scale L, combined with the novel idea of considering an accelerated block dynamics which is itself constrained. Such an approach is referred to in [11] as the *Bisection-Constrained* or *B-C approach*.

In order to present it we first need to recall some simple facts from two dimensional percolation.

A *path* is a collection $\{x_0, x_1, \dots, x_n\}$ of distinct points in $\mathbb{Z}^2$ such that $d(x_i, x_{i+1}) = 1$ for all i. A $*$-*path* is a collection $\{x_0, x_1, \dots, x_n\}$ of distinct points in $\mathbb{Z}^2$ such that $x_{i+1} \in \mathcal{N}^*_{x_i}$ for all i. Given a rectangle Λ and a direction $\mathbf{e}_i$, we will say that a path $\{x_0, \dots, x_n\}$ traverses Λ in the i^{th}-direction if $\{x_0, \dots, x_n\} \subset \Lambda$ and x_0, x_n lay on the two opposite sides of Λ orthogonal to $\mathbf{e}_i$.

Definition 7. *Given a rectangle Λ and a configuration $\omega \in \Omega_\Lambda$, a path $\{x_0, \dots, x_n\}$ is called a* top-bottom crossing *(*left-right crossing*) if it traverses Λ in the vertical (horizontal) direction and $\omega_{x_i} = 0$ for all $i = 0, \dots, n$. The rightmost (lower-most) such crossings (see [20] page 317) will be denoted by Π_ω*

Remark 13. Given a rectangle Λ and a path Γ traversing Λ in e.g. the vertical direction, let Λ_Γ consists of all the sites in Λ which are in Γ or to the right of it. Then, as remarked in [20], the event $\{\omega : \ \Pi_\omega = \Gamma\}$ depends only on the variables ω_x with $x \in \Lambda_\Gamma$.

We are now ready to start the actual proof of the theorem. At the beginning the method requires a simple geometric result (see [6]) which we now describe.

Let $l_k := (3/2)^{k/2}$, and let $\mathbb{F}_k$ be the set of all rectangles $\Lambda \subset \mathbb{Z}^2$ which, modulo translations and permutations of the coordinates, are contained in $[0, l_{k+1}] \times [0, l_{k+2}]$. The main property of $\mathbb{F}_k$ is that each rectangle in $\mathbb{F}_k \setminus \mathbb{F}_{k-1}$ can be obtained as a "slightly overlapping union" of two rectangles in $\mathbb{F}_{k-1}$.

Lemma 5.4 *For all $k \in \mathbb{Z}_+$, for all $\Lambda \in \mathbb{F}_k \setminus \mathbb{F}_{k-1}$ there exists a finite sequence $\{\Lambda_1^{(i)}, \Lambda_2^{(i)}\}_{i=1}^{s_k}$ in $\mathbb{F}_{k-1}$, where $s_k := \lfloor l_k^{1/3} \rfloor$, such that, letting $\delta_k := \frac{1}{8}\sqrt{l_k} - 2$,*

(i) $\Lambda = \Lambda_1^{(i)} \cup \Lambda_2^{(i)}$,
(ii) $d(\Lambda \setminus \Lambda_1^{(i)}, \Lambda \setminus \Lambda_2^{(i)}) \geq \delta_k$,
(iii) $\left(\Lambda_1^{(i)} \cap \Lambda_2^{(i)}\right) \cap \left(\Lambda_1^{(j)} \cap \Lambda_2^{(j)}\right) = \emptyset$, *if* $i \neq j$.

The *B-C approach* then establishes a simple recursive inequality between the quantity $\gamma_k := \sup_{\Lambda \in \mathbb{F}_k} \gamma(\Lambda)$ on scale k and the same quantity on scale $k-1$ as follows.

Fix $\Lambda \in \mathbb{F}_k \setminus \mathbb{F}_{k-1}$ and write it as $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_1, \Lambda_2 \in \mathbb{F}_{k-1}$ satisfying the properties described in Lemma 5.4 above. Without loss of generality we can assume that all the horizontal faces of Λ_1 and of Λ_2 lay on the horizontal faces of Λ except for the face orthogonal to the first direction $\mathbf{e}_1$ and that, along that direction, Λ_1 comes before Λ_2. Set $\Delta \equiv \Lambda_1 \cap \Lambda_2$ and write, for definiteness, $\Delta = [a_1, b_1] \times [a_2, b_2]$. Lemma 5.4 implies that the width of Δ in the first direction, $b_1 - a_1$, is at least δ_k. Set also

$$I \equiv [a_1 + (b_1 - a_1)/2, \ b_1] \times [a_2, b_2]$$

and let $\partial_r I = \{b_1\} \times [a_2, b_2]$ be the right face of I along the first direction.

Definition 8. *Given a configuration $\omega \in \Omega$ we will say that ω is I-good iff there exists a top-bottom crossing of I.*

Given $\tau \in \mathrm{Max}_\Lambda$, we run the following constrained "block dynamics" on Ω_Λ (in what follows, for simplicity, we suppress the index i) with boundary conditions τ and blocks $B_1 := \Lambda_1 \setminus I$, $B_2 := \Lambda_2$. The block B_2 waits a mean one exponential random time and then the current configuration inside it is refreshed with a new one sampled from the Gibbs measure of the block given the previous configuration outside it (and τ outside Λ). The block B_1 does the same but now the configuration is refreshed only if the current configuration ω in B is I-good (see Figure 5.1).

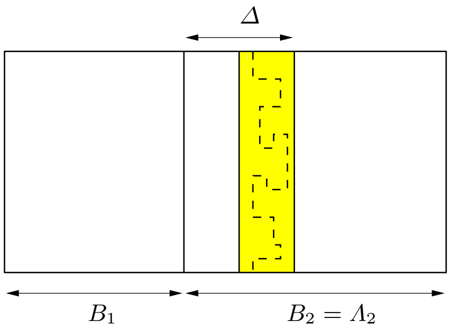

Fig. 1. The rectangle Λ divided into two blocks B_1 and B_2. The grey region is the strip I with a top-bottom crossing.

The generator of the block dynamics applied to f can be written as

$$\mathcal{L}_{\text{block}} f = c_1(\mu_{B_1}^{\Phi,\tau}(f) - f) + \mu_{B_2}^{\Phi,\tau}(f) - f \tag{12}$$

and the associated Dirichlet form is

$$\mathcal{D}_{\text{block}}^{\Phi,\tau}(f) = \mu_{\Lambda}^{\Phi,\tau}\left(c_1 \operatorname{Var}_{B_1}^{\Phi,\tau}(f) + \operatorname{Var}_{B_2}^{\Phi,\tau}(f)\right)$$

where $c_1(\omega)$ is just the indicator of the event that ω is I-good and $\operatorname{Var}_{B_1}^{\Phi,\tau}(f)$, $\operatorname{Var}_{B_2}^{\Phi,\tau}(f)$ depend of course on $\omega_{B_1^c}$ and $\omega_{B_2^c}$ respectively. In order to study the mixing property of the chain we need the following two lemmas.

Lemma 5.5 ([26]) *Fix (r, M) with $M < M_0$. Then, for any $\Phi \in \mathcal{B}_{r,M}$,*

$$\|\mu_{B_1}^{\Phi}(g) - \mu_{\Lambda}^{\Phi,\tau}(g)\|_\infty \leq \lambda_k \|g\|_\infty \quad \forall g : \Omega_{B_2^c} \mapsto \mathbb{R} \tag{13}$$

$$\|\mu_{B_2}^{\Phi}(g) - \mu_{\Lambda}^{\Phi,\tau}(g)\|_\infty \leq \lambda_k \|g\|_\infty \quad \forall g : \Omega_{B_1^c} \mapsto \mathbb{R} \tag{14}$$

where $\lambda_k := Krl_{k+1}e^{-m\delta_k/2}$ and the constants K, m are given in (10).

Lemma 5.6 *There exists $q_0 \in (0, 1)$ and for any $r < \infty$ there exists M_1 such that, for any $M < \min(M_0, M_1)$ and $q \geq q_0$,*

$$\varepsilon_k := \max_{\Phi \in \mathcal{B}_{r,M}} \max_{\tau} \mu_{B_2}^{\Phi,\tau}(\omega \text{ is not } I\text{-good }) \leq e^{-\delta_k}.$$

Proof. It follows immediately from standard percolation arguments together with

$$\sup_{\Phi\in\mathcal{B}_{r,M}}\sup_{\tau}\mu^{\Phi,\tau}_{\{x\}}(\sigma_x=1)\le(1-q)e^{2M}$$

□

We can now state the main consequence of Lemma 5.5, 5.6.

Proposition 5.7 *There exists $q_0\in(0,1)$ and for any $r<\infty$ there exists M_1 such that, for any $M<\min(M_0,M_1)$ and $q\ge q_0$,*

$$\gamma^{(k)}_{\text{block}}:=\sup_{\Phi\in\mathcal{B}_{r,M}}\sup_{\tau\in\text{Max}_\Lambda}\Big(\text{gap}(\mathcal{L}^{\Phi,\tau}_{\text{block}})\Big)^{-1}\le\Big(1-\sqrt{\lambda_k+5\varepsilon_k}\Big)^{-1}\tag{15}$$

for all k so large that the r.h.s. of (15) *is smaller than* 2*.*

Proof (Proof of the proposition). Fix r,M and Φ as prescribed, let $\tau\in\text{Max}_\Lambda$ and, in order to simplify the notation, drop all the superscripts Φ,τ. Let $f:\Omega_\Lambda\mapsto\mathbb{R}$ be a mean zero function, the eigenvalue equation associated to the generator (12) is

$$c_1(\mu_{B_1}(f)-f)+\mu_{B_2}(f)-f=\lambda f\tag{16}$$

By construction $\lambda\ge-2$.

Assume that $\lambda>-1+\sqrt{\lambda_k}$ since otherwise there is nothing to be proved. By applying μ_{B_1} to both sides of (16) and using (13) we obtain

$$(1+\lambda)\mu_{B_1}f=\mu_{B_1}\big(\mu_{B_2}(f)\big)\quad\Rightarrow\quad\|\mu_{B_1}(f)\|_\infty\le\sqrt{\lambda_k}\,\|\mu_{B_2}(f)\|_\infty\tag{17}$$

If we rewrite (16) as

$$f=\frac{1}{1+\lambda+c_1}\mu_{B_2}(f)+\frac{c_1}{1+\lambda+c_1}\mu_{B_1}(f)$$

and apply μ_{B_2} to both sides, by using (17) together with the assumption $\lambda>-1+\sqrt{\lambda_k}$, we get

$$\|\mu_{B_2}(f)\|_\infty\le\|\mu_{B_2}(f)\|_\infty\,\|\mu_{B_2}(\frac{1}{1+\lambda+c_1})\|_\infty\tag{18}$$

$$+\lambda_k\|\frac{c_1}{1+\lambda+c_1}\|_\infty\|\mu_{B_1}(f)\|_\infty\tag{19}$$

$$\le\|\mu_{B_2}(f)\|_\infty\left(\|\mu_{B_2}(\frac{1}{1+\lambda+c_1})\|_\infty+\sqrt{\lambda_k}\right)\tag{20}$$

which is possible only if

$$\|\mu_{B_2}(\frac{1}{1+\lambda+c_1})\|_\infty\ge1-\sqrt{\lambda_k}$$

i.e.

$$\lambda \le -1 + \sqrt{\lambda_k + 5\varepsilon_k}$$

In conclusion we have shown that

$$1 + \lambda \le \sqrt{\lambda_k + 5\varepsilon_k}$$

and the proof is complete. □

By writing down the standard Poincaré inequality for the block auxiliary chain, we get that for any f

$$\operatorname{Var}_\Lambda^{\Phi,\tau}(f) \le \gamma_{\text{block}}^{(k)} \, \mu_\Lambda^{\Phi,\tau}\Big(c_1 \operatorname{Var}_{B_1}^{\Phi}(f) + \operatorname{Var}_{B_2}^{\Phi}(f)\Big) \tag{21}$$

Remark 14. The reader should keep in mind that the e.g. the notation

$$\mu_\Lambda^{\Phi,\tau}\big(c_1 \operatorname{Var}_{B_1}^{\Phi}(f)\big)$$

stands for $\sum_\xi \mu_\Lambda^{\Phi,\tau}(\xi) c_1(\xi) \operatorname{Var}_{B_1}^{\Phi,\xi}(f)$ and that one can imagine the sum restricted to those configurations outside B_1 that coicide with τ outside Λ since otherwise their probability $\mu_\Lambda^{\Phi,\tau}(\xi)$ is zero.

The second term in the r.h.s. of (21), using the definition of γ_k and the fact that $B_2 = \Lambda_2 \in \mathbb{F}_{k-1}$ is bounded from above by

$$\mu_\Lambda^{\Phi,\tau}\Big(\operatorname{Var}_{B_2}^{\Phi}(f)\Big) \le \gamma_{k-1} \sum_{x \in B_2} \mu_\Lambda^{\Phi,\tau}\big(c_{x,B_2} \operatorname{Var}_x^{\Phi}(f)\big) \tag{22}$$

Notice that, by construction, for all $x \in B_2$ and all ω, $c_{x,B_2}(\omega) = c_{x,\Lambda}(\omega)$. Therefore the term $\sum_{x \in B_2} \mu_\Lambda^{\Phi,\tau}\big(c_{x,B_2} \operatorname{Var}_x^{\Phi}(f)\big)$ is nothing but the contribution carried by the set B_2 to the full Dirichlet form $\mathcal{D}_\Lambda^{\Phi,\tau}(f)$.

Next we examine the more complicate term $\mu_\Lambda^{\Phi,\tau}\Big(c_1 \operatorname{Var}_{B_1}^{\Phi}(f)\Big)$. For any ω such that there exists a rightmost crossing Π_ω in I denote by Λ_ω the set of all sites in Λ which are to the *left* of Π_ω. Since $\operatorname{Var}_{B_1}^{\Phi}(f)$ depends only on $\omega_{\Lambda \setminus B_1}$ and, for any top-bottom crossing Γ of I, $\mathbb{1}_{\{\Pi_\omega = \Gamma\}}$ does not depend on the variables ω's to the left of Γ, we can write

$$\mu_\Lambda^{\Phi,\tau}\Big(c_1 \operatorname{Var}_{B_1}^{\Phi}(f)\Big) = \mu_\Lambda^{\Phi,\tau}\Big(\mathbb{1}_{\{\exists\, \Pi_\omega \text{ in } I\}} \mu_{\Lambda_\omega}^{\Phi}\big(\operatorname{Var}_{B_1}^{\Phi}(f)\big)\Big) \tag{23}$$

The convexity of the variance implies that

$$\mu_{\Lambda_\omega}^{\Phi}\big(\operatorname{Var}_{B_1}^{\Phi}(f)\big) \le \operatorname{Var}_{\Lambda_\omega}^{\Phi}(f)$$

where it is understood that the r.h.s. depends on the variables in Π_ω and to the right of it. The key observation at this stage, which explains the role and the need of the event $\{\exists\, \Pi_\omega \text{ in } I\}$, is the following. For any ω such that Π_ω exists the variance $\operatorname{Var}_{\Lambda_\omega}^{\Phi}(f)$ is computed with boundary conditions (τ outside

Λ and $\omega_{\Lambda\setminus\Lambda_\omega}$) which belong to $\mathrm{Max}_{\Lambda_\omega}$. Therefore we can bound it from above using the Poincaré inequality by

$$\mathrm{Var}^{\Phi}_{\Lambda_\omega}(f) \le \gamma(\Lambda_\omega)\mathcal{D}^{\Phi}_{\Lambda_\omega}(f) \le \gamma(B_1\cup I)\mathcal{D}^{\Phi}_{\Lambda_\omega}(f)$$

where we used Lemma 5.3 together with the observation that $\Lambda_\omega \subset B_1 \cup I = \Lambda_1$. In conclusion

$$\begin{aligned}
&\mu^{\Phi,\tau}_{\Lambda}\Big(\mathbb{1}_{\{\exists\,\Pi_\omega \text{ in } I\}}\mu^{\Phi}_{\Lambda_\omega}\big(\mathrm{Var}^{\Phi}_{B_1}(f)\big)\Big)\\
&\le \gamma(\Lambda_1)\mu^{\Phi,\tau}_{\Lambda}\Big(\mathbb{1}_{\{\exists\,\Pi_\omega \text{ in } I\}}\mathcal{D}^{\Phi}_{\Lambda_\omega}(f)\Big)\\
&\le \gamma(\Lambda_1)\mu^{\Phi,\tau}_{\Lambda}\Big(\mathbb{1}_{\{\exists\,\Pi_\omega \text{ in } I\}}\sum_{x\in\Lambda_\omega} c_{x,\Lambda_\omega}\,\mathrm{Var}^{\Phi}_{x}(f)\Big)\\
&\le \gamma(\Lambda_1)\mu^{\Phi,\tau}_{\Lambda}\Big(\sum_{x\in\Lambda_1} c_{x,\Lambda}\,\mathrm{Var}^{\Phi}_{x}(f)\Big)
\end{aligned}\tag{24}$$

because, by construction,

$$c_{x,\Lambda_\omega}(\omega) = c_{x,\Lambda}(\omega) \quad \forall x\in\Lambda_\omega\,. \tag{25}$$

If we finally plug (24) into the r.h.s. of (23) and recall that $\Lambda_1 \in \mathcal{F}_{k-1}$, we obtain

$$\mu^{\Phi,\tau}_{\Lambda}\Big(c_1\,\mathrm{Var}^{\Phi}_{B_1}(f)\Big) \le \gamma_{k-1}\,\mu^{\Phi,\tau}_{\Lambda}\big(\sum_{x\in\Lambda_1} c_{x,\Lambda}\,\mathrm{Var}^{\Phi}_{x}(f)\big) \tag{26}$$

In conclusion we have shown that

$$\mathrm{Var}_{\Lambda}(f) \le \gamma^{(k)}_{\text{block}}\gamma_{k-1}\Big(\mathcal{D}_{\Lambda}(f) + \sum_{x\in\Delta}\mu^{\Phi,\tau}_{\Lambda}\big(c_{x,\Lambda}\,\mathrm{Var}_{x}(f)\big)\Big) \tag{27}$$

Averaging over the $s_k = \lfloor l_k^{1/3}\rfloor$ possible choices of the sets Λ_1, Λ_2 gives

$$\mathrm{Var}_{\Lambda}(f) \le \gamma^{(k)}_{\text{block}}\gamma_{k-1}(1+\frac{1}{s_k})\mathcal{D}_{\Lambda}(f) \tag{28}$$

which implies that

$$\begin{aligned}
\gamma_k &\le (1+\frac{1}{s_k})\gamma^{(k)}_{\text{block}}\gamma_{k-1} &(29)\\
&\le \gamma_{k_0}\prod_{j=k_0}^{k}(1+\frac{1}{s_j})\gamma^{(j)}_{\text{block}} &(30)
\end{aligned}$$

where k_0 is the smallest integer such that $\gamma^{(k_0)}_{\text{block}} < 2$. If we now recall the expression (15) for $\gamma^{(j)}_{\text{block}}$ together with Lemma 5.5 and 5.6, we immediately conclude that the product $\prod_{j=k_0}^{\infty}\gamma^{(j)}_{\text{block}}(1+\frac{1}{s_j})$ is bounded. □

6 One spin facilitated model on a general graph

In this section we prove our second set of new results by examining the *one spin facilitated* model (FA-1f in short) on a general connected graph $\mathcal{G} = (V, E)$. Our motivation comes from some unpublished speculation by D. Aldous [2] that, in this general setting, the FA-1f may serve as an algorithm for information storage in dynamic graphs.

We begin by discussing the finite setting. Let r be one of the vertices and $\mathcal{T}$ be a rooted spanning tree of $\mathcal{G}$ with root r. On $\Omega = \{0,1\}^V$ consider the FA-1f constraints:

$$\begin{cases} c_{x,\mathcal{G}}(\omega) = 1 & \text{if } \omega_y = 0 \text{ for some neighbor } y \text{ of } x \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

and let $\hat{c}_{x,\mathcal{G}} = c_{x,\mathcal{G}}$ if $x \neq r$ and $\hat{c}_{r,\mathcal{G}} \equiv 1$. Let $\hat{\mathcal{L}}$ be the corresponding Markov generator and notice that associated Markov chain is ergodic since the vertex r is unconstrained. For shortness we will refer in the sequel to $\hat{\mathcal{L}}$ as the ($\mathcal{G}$, r, FA-1f) model. Our first result reads as follows.

Theorem 6.1

$$\text{gap}(\mathcal{G}, r, \text{FA-1f}) \geq \text{gap}(\mathbb{Z}, \text{East})$$

Proof. By monotonicity $\hat{c}_{x,\mathcal{G}}(\omega) \geq \hat{c}_{x,\mathcal{T}}(\omega)$ and therefore $\text{gap}(\mathcal{G}, r, \text{FA-1f}) \geq \text{gap}(\mathcal{T}, r, \text{FA-1f})$. We can push the monotonicity argument a bit further and consider the following $(\mathcal{T}, r, \text{East})$ model:

$$\tilde{c}_{x,\mathcal{T}}(\omega) = \begin{cases} 1 & \text{if either } x = r \text{ or } \omega_y = 0, \text{ where } y \text{ is the ancestor (in } \mathcal{T}) \text{ of } x \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

Clearly $\hat{c}_{x,\mathcal{T}}(\omega) \geq \tilde{c}_{x,\mathcal{T}}(\omega)$ and therefore $\text{gap}(\mathcal{G}, r, \text{FA-1f}) \geq \text{gap}(\mathcal{T}, r, \text{East})$. We will now proceed to show that

$$\text{gap}(\mathcal{T}, r, \text{East}) \geq \text{gap}(\mathbb{Z}, \text{East}) \tag{33}$$

If all the vertices of $\mathcal{T}$ have degree 2 with the exception of the root and the leaves, *i.e.* if $\mathcal{T} \subset \mathbb{Z}$, then (33) follows from [11, Lemma 2.11]. Thus let us assume that there exists $x \in \mathcal{T}$ with $\Delta_x \geq 3$ and let us order the vertices of $\mathcal{T}$ by first assigning some arbitrary order to all vertices belonging to any given layer ($\equiv$ same distance from the root) and then declaring $x < y$ iff either $d(x, r) < d(y, r)$ or $d(x, r) = d(y, r)$ and x comes before y in the order assigned to their layer. Let v be equal to the root if $\Delta_r \geq 2$ or equal to the first descendant of r with degree $\Delta_v \geq 3$ otherwise and let $\Gamma_v = \{r, v_1, \ldots, v_k, v\}$ be the path in $\mathcal{T}$ leading from r to v. Let a be a child of v and let $\mathcal{T}_a = (V_a, E_a)$ be the subtree of $\mathcal{T}$ rooted in a. Finally we denote by A and B the two subgraphs of $\mathcal{T}$: $A := \Gamma_v \cup \mathcal{T}_a$, $B := \mathcal{T} \setminus \mathcal{T}_a$. (see Fig 6).

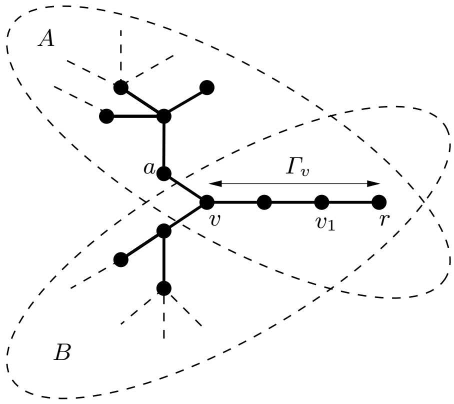

Fig. 2. The subtrees A and B

Lemma 6.2

$$\text{gap}(\mathcal{T}, r, \text{East}) \geq \min\left(\text{gap}(A, r, \text{East}), \text{gap}(B, r, \text{East})\right) \tag{34}$$

By recursively applying the above result to A and B separately, we immediately reduce ourselves to the case of a tree $\mathcal{T}' \subset \mathbb{Z}$ and the proof of the theorem is complete.

□

Proof (of Lemma 6.2). In $L^2(\Omega, \mu)$ consider the set $\mathcal{H}_B$ of functions f that do not depend on $\omega_x,\ x \in \mathcal{T}_a$. Because of the choice of the constraints $\tilde{c}_{x,\mathcal{T}}(\omega)$, $\mathcal{H}_B$ is an invariant subspace for the generator of the $(\mathcal{T}, r, \text{East})$ model and

$$\inf_{\substack{f \in \mathcal{H}_B \\ \mu(f)=0}} \frac{\tilde{\mathcal{D}}(f)}{\text{Var}(f)} = \text{gap}(B, r, \text{East}) \tag{35}$$

Let us now consider the orthogonal subspace $\mathcal{H}_B^{\perp}$. Any zero mean element $f \in \mathcal{H}_B^{\perp}$ satisfies $\mu_{\mathcal{T}_a}(f) = 0$ and therefore we can write

$$\begin{aligned} \text{Var}(f) &= \mu\left(\text{Var}_{\mathcal{T}_a}(f)\right) \leq \mu\left(\text{Var}_A(f)\right) \\ &\leq \text{gap}(A, r, \text{East})^{-1} \sum_{x \in A} \mu\left(\tilde{c}_{x,A} \, \text{Var}_x(f)\right) \\ &\leq \text{gap}(A, r, \text{East})^{-1} \tilde{\mathcal{D}}(f) \end{aligned} \tag{36}$$

where the first inequality follows from convexity of the variance and the second one is nothing but the Poincaré inequality for the East model in A. The proof of the Lemma follows at once from (35), (36). □

Theorem 6.1 has two consequences that will be the content of the following Theorems. The first one deals with the case of an infinite graph. The second one deals with the FA-1f model on general graph $\mathcal{G}$ without the special unblocked vertex r but with the Markov chain restricted to a suitable ergodic component.

Theorem 6.3 *Let $\mathcal{G}_\infty$ be an infinite connected graph of bounded degree and let $\mathcal{L}$ be the generator of the FA-1f model on $\mathcal{G}_\infty$ with constraints $\{c_{x,\mathcal{G}_\infty},\ x \in V_\infty\}$, i.e. no apriori unblocked vertex. Then*

$$\operatorname{gap}(\mathcal{G}_\infty, \text{FA-1f}) \ge \operatorname{gap}(\mathbb{Z}, \text{East})$$

Proof. The proof combines Theorem 6.1 together with the finite subgraph approximation described in section 3. □

Theorem 6.4 *Let $\mathcal{G}$ be as in Theorem 6.1 and let $\mathcal{L}^+$ be the FA-1f generator with constraints $\{c_{x,\mathcal{G}}\}_{x\in V}$ on the restricted configuration space $\Omega^+ := \{\eta \in \Omega :\ \sum_{x\in V}(1-\eta_x) \ge 1\}$ equipped with the reversible measure $\mu^+ := \mu(\cdot \mid \Omega^+)$. Then*

$$\operatorname{gap}(\mathcal{L}^+) \ge \frac{1}{2}\operatorname{gap}(\mathbb{Z}, \text{East})\mu(\Omega^+)$$

Proof. As in the proof of Theorem 6.1 we can safely assume that $\mathcal{G}$ is a tree $\mathcal{T}$ with root $r \in V$. We extend any $f : \Omega^+ \mapsto \mathbb{R}$ to a function $\tilde f$ on Ω by setting $\tilde f(\eta_y = 1\ \forall y) \equiv f(\eta_y = 1\ \forall y \neq r,\ \eta_r = 0)$. Using Theorem 6.1, we then write

$$\begin{aligned} \operatorname{Var}^+(f) &= \operatorname{Var}^+(\tilde f) \le \big(\mu(\Omega^+)\big)^{-2}\operatorname{Var}(\tilde f) \\ &\le \big(\mu(\Omega^+)\big)^{-2}\operatorname{gap}(\mathcal{T}, r, \text{East})^{-1}\sum_x \mu\left(\hat c_{x,\mathcal{T}}\operatorname{Var}_x(\tilde f)\right) \end{aligned} \tag{37}$$

where the constraints $\{\hat c_{x,\mathcal{T}}\}_{x\in\mathcal{T}}$ have been defined right after (31).

Let us examine a generic term $\mu\left(\hat c_{x,\mathcal{T}}\operatorname{Var}_x(\tilde f)\right)$ with $x \neq r$. Remember that $\hat c_{x,\mathcal{T}} = c_{x,\mathcal{T}}$ and moreover $c_{x,\mathcal{T}}(\eta) = 0$ if $\eta_y = 1$ for all $y \neq x$. Furthermore, for any η such that there exists $y \neq x$ with $\eta_y = 0$, $\mu^+(\eta_x = 1 \mid \{\eta_y\}_{y\neq x}) = p$. In conclusion we have shown that

$$\mu\left(\hat c_{x,\mathcal{T}}\operatorname{Var}_x(\tilde f)\right) = \mu(\Omega^+)\mu^+\left(c_{x,\mathcal{T}}\operatorname{Var}^+_x(f)\right) \qquad \forall x \neq r \tag{38}$$

We now examine the dangerous term $\mu\left(\hat c_{r,\mathcal{T}}\operatorname{Var}_r(\tilde f)\right) = \mu\left(\operatorname{Var}_r(\tilde f)\right)$. Because of the definition of $\tilde f$ we can safely rewrite it as

$$\mu\left(\operatorname{Var}_r(\tilde f)\right) = \mu\left(\chi_{\{\exists\, y\neq r:\ \eta_y=0\}}\operatorname{Var}_r(f)\right)$$

Let us order the vertices of the tree $\mathcal{T}$ starting from the *furthermost ones* by first assigning some arbitrary order to all vertices belonging to any given layer ($\equiv$ same distance from the root) and then declaring $x < y$ iff either $d(x,r) > d(y,r)$ or $d(x,r) = d(y,r)$ and x comes before y in the order assigned

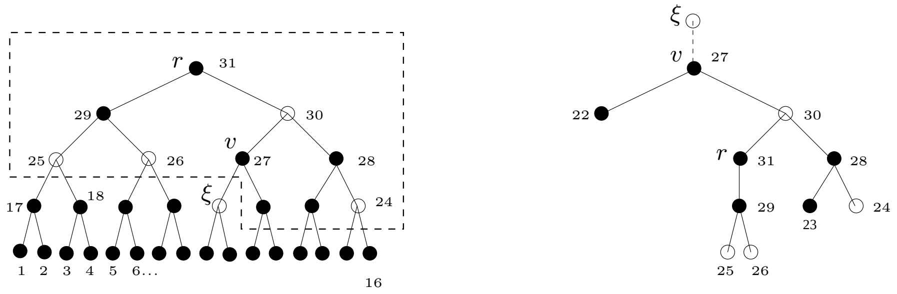

Fig. 3. An example of a tree $\mathcal{T}$ on the left, with a choice of an ordering. The dotted line delimitates the subtree $\mathcal{T}_\xi$ that we reproduce on the right, with root v.

to their layer. Next, for any η such that $\eta_y = 0$ for some $y \neq r$, define $\xi = \min\{y : \ \eta_y = 0\}$ and let $\mathcal{T}_\xi := \{z \in \mathcal{T} : \ z > \xi\}$ (see Fig 3).

Notice that the subgraph $\mathcal{T}_\xi$ is again a tree and we define its root to be the ancestor v of ξ in $\mathcal{T}$. Then, using convexity of the variance, we can write

$$\begin{aligned}\mu\left(\chi_{\{\exists\, y\neq r:\, \eta_y=0\}} \operatorname{Var}_r(f)\right) &= \mu\left(\chi_{\xi\neq r}\mu\left(\operatorname{Var}_r(f)\,|\,\xi\right)\right)\\ &\le \mu\left(\chi_{\xi\neq r} \operatorname{Var}_{\mathcal{T}_\xi}(f)\right)\end{aligned}$$

In order to bound from above $\operatorname{Var}_{\mathcal{T}_\xi}(f)$ we apply the Poincaré inequality in $\mathcal{T}_\xi$ with constraints $\{\hat{c}_{z,\mathcal{T}_\xi}\}$ and root v together with Theorem 6.1:

$$\operatorname{Var}_{\mathcal{T}_\xi}(f) \le \operatorname{gap}(\mathbb{Z},\mathrm{East})^{-1} \sum_{z\in\mathcal{T}_\xi} \mu_{\mathcal{T}_\xi}\left(\hat{c}_{z,\mathcal{T}_\xi} \operatorname{Var}_z(f)\right)$$

Notice that, by construction, $\hat{c}_{z,\mathcal{T}_\xi}(\eta) = c_{z,\mathcal{T}}(\eta)$ for any $z \in \mathcal{T}_\xi$, including the root v of $\mathcal{T}_\xi$ where $\hat{c}_{v,\mathcal{T}_\xi}(\eta) = 1$ by definition and $c_{v,\mathcal{T}}(\eta) = 1$ because $\eta_\xi = 0$. Putting all together we conclude that

$$\begin{aligned}\mu\left(\chi_{\{\exists\, y\neq r:\, \eta_y=0\}} \operatorname{Var}_r(f)\right) &\le \operatorname{gap}(\mathbb{Z},\mathrm{East})^{-1} \sum_{x\in\mathcal{T}} \mu\left(\chi_{\{\exists\, y\neq r:\, \eta_y=0\}} c_{x,\mathcal{T}} \operatorname{Var}_x(f)\right)\\ &\le \operatorname{gap}(\mathbb{Z},\mathrm{East})^{-1}\mu(\Omega^+) \sum_{x\in\mathcal{T}} \mu^+\left(c_{x,\mathcal{T}} \operatorname{Var}^+_x(f)\right) \qquad (39)\end{aligned}$$

where we have used once more the observation before (38) to write

$$c_{x,\mathcal{T}} \operatorname{Var}_x(f) = c_{x,\mathcal{T}} \operatorname{Var}^+_x(f).$$

If we now combine (37), (38) and (39) together we get

$$\operatorname{Var}^+(f) \le 2\left(\operatorname{gap}(\mathbb{Z},\mathrm{East})\mu(\Omega^+)\right)^{-1} \sum_{x\in\mathcal{T}} \mu^+\left(c_{x,\mathcal{T}} \operatorname{Var}^+_x(f)\right)$$

and the proof is complete. □

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References

1. M. Aizenman and J. L. Lebowitz, *Metastability effects in bootstrap percolation*, J. Phys. A **21** (1988), no. 19, 3801–3813. MR MR968311 (90e:82047)
2. D. Aldous, slides available at http://www.math.ucsd.edu/ williams/diaconis/aldous-ising.pdf.
3. D. Aldous and P. Diaconis, *The asymmetric one-dimensional constrained Ising model: rigorous results*, J. Statist. Phys. **107** (2002), no. 5-6, 945–975. MR MR1901508 (2003e:82046)
4. L. Berthier and J. P. Garrahan, *Non-topographic description of inherent structure dynamics in glass formers*, J.Chem.Phys. **119** (2003), 4367–4371.
5. L. Berthier, J. P. Garrahan, and S. Whitelam, *Dynamic criticality in glass forming liquids*, Phys.Rev.Lett. (2004), no. 92, 185705–185709.
6. L. Bertini, N. Cancrini, and F. Cesi, *The spectral gap for a Glauber-type dynamics in a continuous gas*, Ann. Inst. H. Poincaré Probab. Statist. **38** (2002), no. 1, 91–108. MR MR1899231 (2003d:82073)
7. G. Biroli, D. S. Fisher, and C. Toninelli, *On the universality of jamming percolation: reply to the comment of jeng and schwarz*, Phys.Rev.Lett. 2007 in press (preprint cond-mat/0612485).
8. G. Biroli and C. Toninelli, *A new class of cellular automata with a discontinuous transition*, preprint.
9. ———, *Jamming percolation and glassy dynamics*, J.Stat.Phys. (2006) in press, preprint cond-mat/0603860, 2006.
10. N. Cancrini, F. Martinelli, C. Roberto, and C. Toninelli, *Kinetically constrained lattice gases*, in preparation.
11. ———, *Kinetically constrained spin models*, Probability Theory and Related Fields in press.
12. ———, *On the relaxation time for the spiral model*, in preparation.
13. R. Cerf and E. N. M. Cirillo, *Finite size scaling in three-dimensional bootstrap percolation*, Ann. Probab. **27** (1999), no. 4, 1837–1850. MR MR1742890 (2001b:82047)
14. R. Cerf and F. Manzo, *The threshold regime of finite bootstrap percolation*, Stochastic Process. Appl. **101** (2002), no. 1, 69–82.
15. Fan Chung, Persi Diaconis, and Ronald Graham, *Combinatorics for the East model*, Adv. in Appl. Math. **27** (2001), no. 1, 192–206. MR MR1835679 (2002g:82035)
16. S. Eisinger and J. Jackle, *A hierarchically constrained kinetic ising model*, Z.Phys.B **84** (1991), 115–124.
17. G. H. Fredrickson and H. C. Andersen, *Kinetic ising model of the glass transition*, Phys. Rev. Lett. **53** (1984), 1244–1247.

18. ———, *Facilitated kinetic ising models and the glass transition*, J.Chem.Phys. **83** (1985), 5822–5831.
19. I. S. Graham, M. Grant, and L. Piché, *Model for dynamics of structural glasses*, Phys.Rev.E **55** (1997), 2132–2144.
20. G. Grimmett, *Percolation*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 321, Springer-Verlag, Berlin, 1999. MR MR1707339 (2001a:60114)
21. P. Harrowell, *Visualizing the collective motion responsible for α and β relaxations in a model glass*, Phys.Rev.E **48** (1993), 4359–4363.
22. A. E. Holroyd, *Sharp metastability threshold for two-dimensional bootstrap percolation*, Probab. Theory Related Fields **125** (2003), no. 2, 195–224. MR MR1961342 (2003k:60257)
23. J. Jackle, F. Mauch, and J. Reiter, *Blocking transitions in lattice spin models with directed kinetic constraints*, Physica A **184** (1992), no. 3-4, 458–476.
24. G. Kordzakhia and S. Lalley, *Ergodicity and mixing properties of the northeast models*, preprint, 2006.
25. T. M. Liggett, *Interacting particle systems*, Springer-Verlag, New York, 1985. MR 86e:60089
26. F. Martinelli, *Lectures on Glauber dynamics for discrete spin models*, Lectures on probability theory and statistics (Saint-Flour, 1997), Springer, Berlin, 1999, pp. 93–191. MR 2002a:60163
27. F. Martinelli and E. Olivieri, *Approach to equilibrium of Glauber dynamics in the one phase region. I. The attractive case*, Comm. Math. Phys. **161** (1994), no. 3, 447–486. MR MR1269387 (96c:82040)
28. ———, *Approach to equilibrium of Glauber dynamics in the one phase region. II. The general case*, Comm. Math. Phys. **161** (1994), no. 3, 487–514. MR MR1269388 (96c:82041)
29. F. Martinelli, E. Olivieri, and R. H. Schonmann, *For 2-D lattice spin systems weak mixing implies strong mixing*, Comm. Math. Phys. **165** (1994), no. 1, 33–47. MR MR1298940 (95j:82015)
30. F. Ritort and P. Sollich, *Glassy dynamics of kinetically constrained models*, Advances in Physics **52** (2003), no. 4, 219–342.
31. L. Saloff-Coste, *Lectures on finite Markov chains*, Lectures on probability theory and statistics (Saint-Flour, 1996), Springer, Berlin, 1997, pp. 301–413. MR 99b:60119
32. R. H. Schonmann, *On the behavior of some cellular automata related to bootstrap percolation*, Ann. Probab. **20** (1992), no. 1, 174–193. MR MR1143417 (93b:60231)
33. Barry Simon, *The statistical mechanics of lattice gases. Vol. I*, Princeton Series in Physics, Princeton University Press, Princeton, NJ, 1993. MR MR1239893 (95a:82001)
34. P. Sollich and M. R. Evans, *Glassy time-scale divergence and anomalous coarsening in a kinetically constrained spin chain*, Phys.Rev.Lett **83** (1999), 3238–3241.