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Kinetically Constrained Lattice Gases*

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Abstract: Kinetically constrained lattice gases (KCLG) are interacting particle systems which show some of the key features of the liquid/glass transition and, more generally, of glassy dynamics. Their distintictive signature is the following: **i)** reversibility w.r.t. product i.i.d. Bernoulli measure at any particle density and **ii)** vanishing of the exchange rate across any edge unless the particle configuration around the edge satisfies a proper constraint besides hard core. Because of degeneracy of the exchange rates the models can show anomalous time decay in the relaxation process w.r.t. the usual high temperature lattice gas models particularly in the so-called *cooperative case*, when the vacancies have to collectively cooperate in order for the particles to move through the systems. Here we focus on the Kob-Andersen (KA) model, a cooperative example widely analyzed in the physics literature, both in a finite box with particle reservoirs at the boundary and on the infinite lattice. In two dimensions (but our techniques extend to any dimension) we prove a diffusive scaling $O(L^2)$ (apart from logarithmic corrections) of the relaxation time in a finite box of linear size L. We then use the above result to prove a diffusive decay $1/t$ (again apart from logarithmic corrections) of the density-density time autocorrelation function at *any* particle density, a result that has been sometimes questioned on the basis of numerical simulations. The techniques that we devise, based on a novel combination of renormalization and comparison with a *long-range* Glauber type constrained model, are robust enough to easily cover other choices of the kinetic constraints.

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1. Introduction

Kinetically constrained lattice gases (KCLG) are interacting particle systems on the integer lattice $\mathbb{Z}^d$ with the usual hard core exclusion. A configuration is therefore defined by assigning to each vertex $x \in \mathbb{Z}^d$ its occupation variable, $\eta(x) \in \{0, 1\}$, which represents an empty or occupied site respectively. The evolution is given by a continuous time Markov process of Kawasaki type, namely with rate $c_{x,y}(\eta)$ the occupation variables at the end points of an unoriented bond $e = (x, y)$ of $\mathbb{Z}^d$ are exchanged. The exchange rate is equal to one if the current configuration satisfies an apriori specified local constraint and zero otherwise. In the former case we say that the exchange is *legal*. A key feature of the constraint is that it does not depend on the occupation variables $\eta(x), \eta(y)$ so that any Bernoulli product measure μ_p on $\{0, 1\}^{\mathbb{Z}^d}$, where p is the particle density, is automatically an invariant reversible measure for the process.

However, at variance with the simple symmetric exclusion process (SSEP) which corresponds to the unconstrained choice $c_e(\eta) \equiv 1$ for any bond (x, y), KCLG have several other invariant measures. This is related to the fact that there exist *blocked configurations*, namely configurations for which all exchange rates are equal to zero.

KCLG have been introduced in the physics literature (see [24] for a review) to model liquid/glass transition and more generally the glassy dynamics which occurs in different systems, e.g. granular materials. In particular they were devised to mimic the fact that the motion of a molecule in a dense liquid can be inhibited by the geometrical constraints created by the surrounding molecules. The exchange rates are devised to encode this local caging mechanism and thus they typically require a minimal number of empty sites in a proper neighborhood of $e = (x, y)$ in order for the exchange at e to be legal, *i.e.* to have $c_e = 1$.

KCLG are usually classified into *cooperative* and *non-cooperative* models.

Definition 1.1. *A model is said to be non-cooperative if its rates are such that it is possible to construct a proper finite group of vacancies, the so-called mobile cluster, with the following two properties*:

(i) *for any configuration it is possible to move the mobile cluster to any other position in the lattice by a sequence of legal exchanges*;
(ii) *any exchange is legal if the mobile cluster is in a proper position in its vicinity.*

All models which are not non-cooperative are said to be cooperative.

From the point of view of the modelisation of the liquid/glass transition, cooperative models are the most relevant ones. Indeed, very roughly speaking, non-cooperative models are expected to behave like a re-scaled SSEP with the mobile cluster playing the role of a single vacancy. Therefore they are not very suitable to describe the rich behavior of glassy dynamics.

Let us start by recalling some fundamental problems which require for KCLG new ideas and techniques from those used to study SSEP or other high temperature lattice gas models.

A first basic question is whether the infinite volume process is ergodic, namely whether zero is a simple eigenvalue for the generator of the Markov process in $\mathbb{L}_2(\mu_p)$. This would in turn imply relaxation to μ_p in the $\mathbb{L}_2(\mu_p)$ sense. The constraints which are chosen in the physics literature in order to model the caging mechanism render the dynamics increasingly slow as p is increased. Therefore it is possible that the process undergoes a transition from an ergodic to a non-ergodic regime when the particle density p crosses a critical value $p_c \in (0, 1)$. This has indeed been conjectured for some cooperative models [15,17,26]. The same issue for non-cooperative models is trivially solved for any $p < 1$ because of the μ_p-almost sure existence of the mobile cluster. When ergodicity holds, the next natural issue is to establish the large time behaviour of the infinite volume process started from the reversible equilibrium measure at time zero. A typical quantity to be considered is the density-density time autocorrelation function. This problem in turn is related to the scaling with the system size of the relaxation time (*i.e.* inverse spectral gap) in a finite box. Recall that for SSEP such a scaling is diffusive, *i.e.* $O(L^2)$ if L denotes the linear size of the system, and that the induced time decay of the density-density time autocorrelation function is proportional to $t^{-d/2}$.

Numerical simulations for the Kob-Andersen model suggest the possibility of an anomalous slowing down at high density [15,21] which could correspond to an anomalous scaling of the relaxation time in finite volume.

Finally one would like to investigate the large time behaviour of a tagged particle and the evolution of macroscopic density profiles, namely the hydrodynamic limit of the process. For some cooperative models it has been conjectured that a diffusive/non-diffusive transition would occur at a finite critical density: both the self-diffusion coefficent of the tagged particle and the macroscopic diffusion coefficient of the hydrodynamic equation would be strictly positive below this critical density and zero above [15,17].

To our knowledge, the existing rigorous answers to the above questions are the following:

Non-cooperative models. In this case much more is known because the existence of mobile *finite* clusters greatly simplifies the analysis and allows the application of standard familiar techniques (e.g. paths arguments) already developed for lattice gases and exclusion models. In [6] it is proven in certain cases that both the spectral gap and the log Sobolev constant in finite volume of linear size L with boundary sources scale as $O(L^2)$. Furthermore for the same models it is established that the self-diffusion coefficient of the tagged particle is strictly positive. Moreover the hydrodynamic limit has been succesfully analyzed for a special class of *gradient type* in [13].

Cooperative models. In [29,30] for a large class of models it has been proven that $p_c = 1$, namely ergodicity always holds (see instead [28] for a choice of the constraints which certainly leads to $p_c < 1$). The self-diffusion coefficient is instead analyzed in [27] where positivity is proved only modulo a conjecture on the behavior of random walks on a random environment.

Finally, we recall that the Glauber version of KCLG, the so-called *Kinetically Constrained Spin Models* (KCSM), have also been very much studied in physics literature [4,10–12,24] and that some features of their long time behaviors have been rigorously analyzed in [1,8,9,16]. In particular in [9] an important correction to the "exact solution" of the East model based on non-rigorous methods was obtained.

In conclusion, apart from the ergodicity problem studied in [30], cooperative KCLG remain mathematically largely unexplored. The main contribution of this paper is to introduce for the first time suitable new ideas and techniques to partially cover this gap.

We mainly focus on the cooperative model which has been most studied in the physics literature, the so-called *Kob Andersen* (KA) model, which was introduced in [15] and subsequently studied in several works [3,17,18,21,25,26,29,30].

KA actually denotes a class of models on $\mathbb{Z}^d$ characterized by a integer parameter $j \in [2, d]$ and defined by the following nearest neighbour exchange rates: $c_{x,y} = 1$ iff at least $j - 1$ neighbours of x different from y are empty and at least $j - 1$ neighbours of y different from x are empty too. It is immediate to verify that KA is always a cooperative model. For example if $j = d = 2$, a fully occupied double stripe which spans the lattice can never be destroyed. Thus any finite cluster of vacancies cannot be mobile since it cannot overcome the double stripe. Nevertheless in [29,30] it has been proven that the infinite volume process is always ergodic at any finite density, namely $p_c = 1$. This contradicts previous claims [15,17,26] on the existence of a finite critical density.

Our two main results concern the KA model in two dimensions with $j = 2$ (which is the only possible choice when $d = 2$) but actually both extend to higher dimensions (with much more effort and more cumbersome reasoning). This choice was made in order to present the overall strategy stripped from unnecessary complications.

In Theorem 4.1 we consider the model in a box of linear size L with sources, *i.e.* Glauber moves, at the boundary sites. We establish upper and lower bounds of order $1/L^2$ (apart from logarithmic corrections) for the spectral gap at any density. Thus the scaling is the same as for the unconstrained case, in contrast with previous conjectures of an anomalous scaling at high density suggested by numerical evidences of a strong slowing down of the dynamics [15,17]. However, contrary to what happens in high temperature Ising type lattice gases, there is no uniformity in the particle density.

In Theorem 4.2 instead, we establish a diffusive $1/t$ decay (apart from logarithmic corrections) for the infinite volume time auto-correlation of local functions as for SSEP.

A rough sketch of the main new ideas which are needed to overcome the problems posed by the cooperative constraints is presented in Sect. 4. Although we have devised our techniques for the KA model, they can be easily extended to analyze other cooperative KCLG (and all non-cooperative models) via a proper modification of the choice of the constraints for the auxiliary constrained Kawasaki and Glauber dynamics discussed in Sect. 5.

2. Kinetically Constrained Lattice Gases (KCLG)

In this section we define a general setting for the class of models that will be analyzed later on and provide the main characterization of their ergodicity threshold (see Proposition 2.16).

Setting and notation.

Lattices, distances and neighbourhoods. The models considered here are defined on the integer lattice $\mathbb{Z}^d$ with sites $x = (x_1, \ldots, x_d)$ and basis vectors $\vec{e}_1 = (1, \ldots, 0)$,

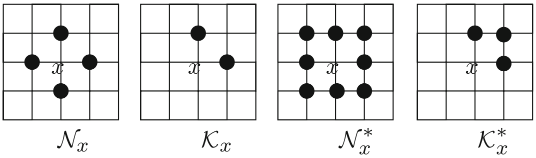

Fig. 1. The various neighborhoods of a vertex x in two dimensions

$\vec{e}_2 = (0, 1, \ldots, 0), \ldots, \vec{e}_d = (0, \ldots, 1)$. On $\mathbb{Z}^d$ we will consider the Euclidean norm $\|x\|$, the ℓ^1 (or graph theoretic) norm $\|x\|_1$ and the sup-norm $\|x\|_\infty$. The associated distances will be denoted by $d(\cdot,\cdot)$, $d_1(\cdot,\cdot)$ and $d_\infty(\cdot,\cdot)$ respectively. A bond is a couple of sites (x, y) with $d_1(x, y) = 1$ (couples are meant to be non ordered so that $(x, y) \equiv (y, x)$). For any set Λ, E_Λ will denote the set of all bonds with both sites in Λ, namely $E_\Lambda = \left\{(x, y) \in \Lambda^2 : d_1(x, y) = 1\right\}$. For any set $A \subset \mathbb{Z}^d$ and site $x \in \mathbb{Z}^d$, we denote by $A + x$ the set translated of x, namely $A + x := \{y : \exists z \in A \text{ s.t. } y = z + x\}$. For any vertex x we define its neighborhoods (see Fig. 1)

$$
\begin{aligned}
\mathcal{N}_x &= \{y \in \mathbb{Z}^d : \ d_1(x, y) = 1\},\\
\mathcal{N}^*_x &= \{y \in \mathbb{Z}^d : \ d_\infty(x, y) = 1\},\\
\mathcal{K}_x &= \{y \in \mathcal{N}_x : \ y = x + \sum_{i=1}^d \alpha_i \vec{e}_i, \ \alpha_i \geqslant 0\},\\
\mathcal{K}^*_x &= \{y \in \mathcal{N}^*_x : \ y = x + \sum_{i=1}^d \alpha_i \vec{e}_i, \ \alpha_i \geqslant 0\}.
\end{aligned}
$$

The exterior neighborhood ($\partial_+\Lambda$) and *-neighborhood ($\partial^*_+\Lambda$), the interior neighborhood ($\partial_-\Lambda$) and *-neighborhood ($\partial^*_-\Lambda$) neighborhood are defined as

$$
\begin{aligned}
\partial_+\Lambda &:= \{x \notin \Lambda : d_1(x, \Lambda) \leqslant 1\},\\
\partial^*_+\Lambda &:= \{x \notin \Lambda : d_\infty(x, \Lambda) \leqslant 1\},\\
\partial_-\Lambda &:= \{x \in \Lambda : d_1(x, \Lambda^c) \leqslant 1\},\\
\partial^*_-\Lambda &:= \{x \in \Lambda : d_\infty(x, \Lambda^c) \leqslant 1\}.
\end{aligned}
$$

Furthermore it is useful to introduce the following additional oriented sets:

$$
\begin{aligned}
&\overline{\partial}^i_+\Lambda := \{x \notin \Lambda; \ x - \vec{e}_i \in \Lambda\},\\
&\overline{\partial}_+\Lambda := \cup_{i=1}^d \overline{\partial}^i_+\Lambda,\\
&\overline{\partial}^*_+\Lambda := \left\{\cup_{x\in\Lambda}\mathcal{K}^*_x\right\} \setminus \Lambda,\\
&\overline{\partial}^i_-\Lambda := \{x \in \Lambda; \ x + \vec{e}_i \notin \Lambda\},\\
&\overline{\partial}_-\Lambda := \cup_{i=1}^d \overline{\partial}^i_-\Lambda.
\end{aligned}
$$

In order to remember the notation we may observe that: * means using the d_∞ distance, $\overline{\partial}\Lambda$ means taking only an oriented part of the boundary and +/− means exterior/interior.

Geometric sets and paths. The following notions of rectangles, cubes, cylinders, geometric paths, double-paths and crossings will be used throughout the work.

Definition 2.1 (*Rectangles, cubes and cylinders*). *A rectangle R is a set of sites of the form*

$$R := [a_1, b_1] \times \cdots \times [a_d, b_d]$$

with $a_i, b_i \in \mathbb{Z}$. Given a length $\ell \in \mathbb{Z}^+$, Q_ℓ is the cube of side ℓ,

$$Q_\ell := [0, \ell - 1] \times \ldots [0, \ell - 1].$$

Finally, for any $x \in \Lambda$, $N \in \mathbb{Z}^+$ and $i \in \{1, \ldots, d\}$, we define the cylinder of radius N around x in the i^{th} direction as

$$T^N_{x,i} = \Big\{ y \in \mathbb{Z}^d : \; y_j \in [x_j - N, x_j + N] \text{ for all } j \neq i \Big\} \tag{2.1}$$

and for any $\Lambda \subset \mathbb{Z}^d$, we let $T^N_{x,i}(\Lambda) := \Lambda \cap T^N_{x,i}$.

The following property of cylinders can be immediately verified

Claim 2.2. *Choose $x \in \mathbb{Z}^d$ and $\Lambda, \Lambda' \subset \mathbb{Z}^d$. If $z \in T^N_{x,i}(\Lambda') \cap \Lambda$, then $z \in T^N_{x,i}(\Lambda)$; if $\Lambda \subset \Lambda'$, then $T^N_{x,i}(\Lambda) \subset T^N_{x,i}(\Lambda')$.*

Definition 2.3 (*Geometric paths, double-paths and crossings*). *Given $x, y \in \mathbb{Z}^d$, a sequence $(x^{(1)}, \ldots, x^{(n)})$ is a* **geometric path** *from x to y and we denote it by γ_{xy} if: $x^{(1)} = x$, $x^{(n)} = y$, $x^{(j)} \neq x^{(j')}$ for any $j \neq j'$ and $d_1(x^{(k)}, x^{(k-1)}) = 1$ for any $k = 2, \ldots, n$. For any γ_{xy} we also define the corresponding* **geometric double-path** *as $\overline{\gamma}_{xy} := \gamma_{x,y} \cup \overline{\partial}^*_+ \gamma_{x,y}$. Given $\Lambda \subset \mathbb{Z}^d$ we say that $\gamma_{x,y}$ is inside Λ and write $\gamma_{x,y} \subset \Lambda$ if $x^{(i)} \in \Lambda$ for all $i \in \{1, \ldots, n\}$. Given $z \in \mathbb{Z}^d$ ($e \in E_{\mathbb{Z}^d}$) we say that z is inside $\gamma_{x,y}$ and write $z \in \gamma_{x,y}$ (e belongs to $\gamma_{x,y}$ or $e \in \gamma_{x,y}$) if there exists $i \in \{1, \ldots, n\}$ such that $z = x^{(i)}$ (if there exists $i \in \{1, \ldots, n-1\}$ such that $(x^{(i)}, x^{(i+1)}) = e$). Given a rectangle $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$ we say that a path $\gamma_{x,y}$ is* **crossing** *R in direction i if: $\overline{\gamma}_{x,y} \subset R$ and $x_i = a_i$ and $y_i = b_i - 1$. Finally, if $d = 2$ and $i = 1$ ($i = 2$) we say that the path is* **left-right (top-bottom) crossing**.

The following two-dimensional results can be easily verified

Claim 2.4. *Given a rectangle $R = [a_1, b_1] \times [a_2, b_2]$:*

(i) *if $\gamma_{x,y}$ is left-right crossing R and $\gamma_{\tilde{x},\tilde{y}}$ is top-bottom crossing R they have (at least) one common point $z \in \gamma_{x,y} \cap \gamma_{\tilde{x},\tilde{y}}$.*
(ii) *given $R' = [a'_1, b'_1] \times [a_2, b_2]$ ($R' = [a_1, b_1] \times [a'_2, b'_2]$) with $a'_1 \geqslant a_1$ and $b'_1 \leqslant b_1$ ($a'_2 \geqslant a_2$ and $b'_2 \leqslant b_2$), if $\gamma_{x,y}$ is left-right (top-bottom) crossing R, then it is also left-right (top-bottom) crossing R'.*

The probability space and the good event. Consider a finite probability space (W, ν) with $\nu(w) > 0$ for any $w \in W$. We will denote by $G \subset W$ a distinguished event in W which will be referred to as the "good event" and by $\rho \equiv \nu(G)$ its probability. Given (W, ν) we will consider the configuration space $\Omega = W^{\mathbb{Z}^d}$ equipped with the product measure $\mu := \prod_{x \in \mathbb{Z}^d} \nu_x$, where $\nu_x \equiv \nu$. If $W = \{0, 1\}$ then ν is completely determined by the parameter $p := \nu(1)$ and μ is a Bernoulli product measure, $\mu \equiv \mu_p$. Similarly we define Ω_Λ and μ_Λ for any subset $\Lambda \subset \mathbb{Z}^d$. Given $\omega \in \Omega$ for each $x \in \mathbb{Z}^d$ we denote by $\omega(x)$ the value of ω at site x and we say that x is *good* if $\omega(x) \in G$.

Elements of Ω (Ω_Λ) will be denoted by Greek letters ω, η ($\omega_\Lambda, \eta_\Lambda$) etc. and the variance w.r.t. μ (μ_Λ) by Var_μ ($\mathrm{Var}_{\mu_\Lambda}$). We drop the measure from the variance by adopting the simpler notation Var (Var_Λ) when confusion does not arise. We will use the shorthand notation $\mu(f)$ ($\mu_\Lambda(f)$) to denote the expected value of any $f \in L^1(\mu)$. Given a configuration $\omega \in \Omega$ and a set $\Lambda \subset \mathbb{Z}^d$, we call ω_Λ the restriction of ω to Λ. Given two configurations $\omega, \tau \in \Omega$ we call $\omega_\Lambda \cdot \tau$ the configuration that equals ω in Λ and equals τ in $\mathbb{Z}^d \backslash \Lambda$. Furthermore, for any $\Lambda_1, \Lambda_2, \Lambda \subset \mathbb{Z}^d$ with $\Lambda_1 \cup \Lambda_2 = \Lambda$ and $\Lambda_1 \cap \Lambda_2 = \emptyset$ and any two configurations $\omega, \tau \in \Omega_\Lambda$ we set $\omega_{\Lambda_1} \cdot \tau_{\Lambda_2}$ for the configuration that equals ω inside Λ_1 and τ inside Λ_2. A function $f : \Omega \to \mathbb{R}$ that depends on finitely many variables $\{\omega(x)\}_{x \in \mathbb{Z}^d}$ will be called *local*. Given a configuration $\omega \in \Omega$ for any bond $e = (x, y) \in E_{\mathbb{Z}^d}$ we denote by ω^e (or sometimes by ω^{xy}) the configuration ω with the occupation variables at x and y exchanged,

$$\omega^e(z) = \omega^{xy}(z) := \begin{cases} \omega(z) & \text{if } z \notin \{x, y\} \\ \omega(x) & \text{if } z = y \\ \omega(y) & \text{if } z = x. \end{cases}$$

We define $T_e : \Omega \to \Omega$ to be the operator acting as $T_e(\omega) = \omega^e$ and we use the symbol ∇_e to denote $\nabla_e f(\omega) := f(\omega^e) - f(\omega)$. When $W = \{0, 1\}$ we also denote by ω^x the configuration flipped at x, namely

$$\omega^x(z) := \begin{cases} \omega(z) & \text{if } z \neq x \\ 1 - \omega(z) & \text{if } z = x, \end{cases}$$

and we define as before $T_x : \Omega \to \Omega$ as $T_x(\omega) = \omega^x$ and ∇_x as $\nabla_x f(\omega) := f(\omega^x) - f(\omega)$. Finally, we introduce the notions of *G-equivalence*, *good paths* and *good crossings* (recall Definition 2.3).

Definition 2.5 (*G-equivalence*). *We say that ω and ω' are G-equivalent in Λ and write $\omega \overset{G,\Lambda}{\Longleftrightarrow} \omega'$ if for all $x \in \Lambda$ the following holds: $\omega(x) \in G$ iff $\omega'(x) \in G$.*

Definition 2.6 (*Good paths and good crossings*). *Given a configuration ω we say that a path $\gamma_{x,y}$ is* **good** *for ω if $\omega(z) \in G$ for any $z \in \overline{\gamma}_{xy} \backslash x$. Given a configuration ω, a rectangle R and a path $\gamma_{x,y}$ we say that $\gamma_{x,y}$ is a* **good crossing** *in R in direction i if: $\gamma_{x,y}$ is crossing in R in direction i, $\gamma_{x,y}$ is good and $\omega(x) \in G$. In $d = 2$ if $i = 1$ ($i = 2$) we use the notation good left-right (top-bottom) crossing.*

It is immediate to verify that if $\gamma_{x,y} \subset \Lambda$ is good for ω, then it is also good for any ω' which is G-equivalent to ω in $\Lambda \backslash x$ (where here and in the sequel we let $\Lambda \backslash x := \Lambda \backslash \{x\}$).

2.1. The Markov process and the spectral gap. The interacting particle models that we study here are Kawasaki type Markov processes in Ω which are reversible w.r.t. the product measure μ. When considered in Ω_Λ they will instead be a mixture of Glauber (on a proper inner boundary) and Kawasaki (inside Λ) dynamics.

Each model is characterized by a collection of *influence classes* $\{\mathcal{C}_e\}_{e\in E_{\mathbb{Z}^d}}$. For any bond e, $\mathcal{C}_e$ is a collection of subsets of $\mathbb{Z}^d$ which satisfies the following basic hypothesis:

(a) *independence of e*: for all $e \in E_{\mathbb{Z}^d}$ and all $A \in \mathcal{C}_e$, $e \notin A$;
(b) *translation invariance*: $\mathcal{C}_e + x = \mathcal{C}_{x+e}$ for all e;
(c) *finite range interaction:* there exists $r < \infty$ such that for any bond $e = (u, v)$, any element of $\mathcal{C}_e$ is contained in $\cup_{j=1}^r \{y : d_1(\{u, v\}, y) = j\}$.

Definition 2.7 *Given a bond $e \in E_{\mathbb{Z}^d}$ we will say that the constraint at e is satisfied by the configuration ω if $c_e(\omega)$ equals one, where*

$$c_e(\omega) = \begin{cases} 1 & \textit{if there exists a set } A \in \mathcal{C}_e \textit{ such that } \omega(y) \in G \textit{ for all } y \in A \\ 0 & \textit{otherwise}. \end{cases}$$

On the whole lattice $\mathbb{Z}^d$ the process of interest for us can be informally described as follows. Each bond $e = (x, y)$ waits an independent mean one exponential time and then, provided that the current configuration ω satisfies the constraint at e, the values $\omega(x)$ and $\omega(y)$ are exchanged. Standard methods (see *e.g.* [19]) show that the Markov semigroup P_t associated to this process is self-adjoint on $\mathbb{L}_2(\mu)$ for any choice of ν (i.e. for any product measure μ) and the corresponding infinitesimal generator $\mathcal{L}$ (i.e. the operator such that $P_t := e^{t\mathcal{L}}$) is a non-positive self-adjoint operator which acts on local functions as

$$\mathcal{L} f(\omega) = \sum_{e\in E_{\mathbb{Z}^d}} c_e(\omega) \left(f\left(\omega^e\right) - f(\omega)\right). \tag{2.2}$$

The corresponding Dirichlet form on $\mathbb{L}_2(\mu)$ is $\mathcal{D}_\mu(f) := -\mu(f \cdot \mathcal{L} f)$ which can be rewritten as

$$\mathcal{D}_\mu(f) = \sum_{e\in E_{\mathbb{Z}^d}} \mu\left(c_e (\nabla_e f)^2\right) \qquad f \in \mathbb{L}_2(\mu).$$

In the whole work, when confusion does not arise, we will omit the index μ from the Dirichlet form. It is important to notice that due to the fact that the rates are not bounded away from zero, the reversible measure μ is not in general the only invariant measure for the process. In particular there exist initial configurations that are blocked forever (all exchange rates are zero) and any measure concentrated on them is invariant too. An interesting question is therefore whether μ is ergodic or mixing for the Markov process generated by $\mathcal{L}$. To this purpose it is useful to recall the following well known result (see *e.g.* Theorem 4.13 in [19]). Denote by $\mathcal{L}_\mu$ the generator of the semigroup P_t extended by continuity to $\mathbb{L}_2(\mu)$.

Theorem 2.8. *The following are equivalent*:

(a) $\lim_{t\to\infty} P_t f = \mu(f)$ *in* $\mathbb{L}_2(\mu)$ *for all* $f \in \mathbb{L}_2(\mu)$.
(b) 0 *is a simple eigenvalue for* $\mathcal{L}_\mu$.

Clearly (a) above implies that $\lim_{t\to\infty}\mu(fP_tg)=\mu(f)\mu(g)$ for any $f,g\in\mathbb{L}_2(\mu)$, *i.e.* μ is mixing.

Up to now we have considered the infinite volume version of the models. If instead we restrict the generator (2.2) to a finite set $\Lambda\subset\mathbb{Z}^d$ with edge set E_Λ, the corresponding continuous-time Markov chain is in general not ergodic on Ω_Λ due to the presence of the constraints and to the conservative character of the dynamics. A natural possibility to restore ergodicity and to make μ_Λ an invariant measure for the chain is to freeze the external configuration to a proper reference configuration τ (the *boundary condition*) and to add a collection of sources (*i.e.* Glauber moves) on a proper set $\mathcal{S}\subset\Lambda$ (the *source set*). Let $\mathcal{M}\subset\mathbb{Z}^d\backslash\Lambda$ be s.t. $\tau(x)\in G$ ($\tau(x)\notin G$) for $x\in\mathcal{M}$ (for $x\notin\mathcal{M}$). The finite volume generator will depend on the boundary condition τ only via $\mathcal{M}$ (the *boundary set*). More precisely

Definition 2.9. *The finite volume generator with source set $\mathcal{S}$, boundary set $\mathcal{M}$, source constraints $\{c_{x,\Lambda}\}_{x\in\mathcal{S}}$ and on-site distribution ν is given by*

$$\mathcal{L}_\Lambda=\mathcal{L}^K_{\Lambda,\mathcal{M}}+\mathcal{L}^{G,\nu}_S, \tag{2.3}$$

where, for any $f:\Omega_\Lambda\to\mathbb{R}$,

$$\mathcal{L}^K_{\Lambda,\mathcal{M}}f(\omega)=\sum_{e\in E_\Lambda}c^{\mathcal{M}}_{e,\Lambda}(\omega)\left(f\left(\omega^e\right)-f(\omega)\right), \tag{2.4}$$

$$\mathcal{L}^{G,\nu}_S f=\sum_{x\in S}c_{x,\Lambda}(\omega)(\nu_x(f)-f), \tag{2.5}$$

and the finite volume exchange rates $c^{\mathcal{M}}_{e,\Lambda}$ are defined through the infinite volume constraints by

$$c^{\mathcal{M}}_{e,\Lambda}(\omega):=c_e(\omega_\Lambda\cdot\tau), \tag{2.6}$$

where τ is any configuration satisfying $\tau(z)\in G$ for all $z\in\mathcal{M}$ and $\tau(z)\notin G$ otherwise. The source rates $c_{x,\Lambda}(\omega)$ are either one or zero according to whether the particle configuration ω satisfies or not a proper constraint which does not depend on $\omega(x)$.

In the sequel we will always drop the sub/superscripts $\mathcal{M}$, $\mathcal{S}$, ν from the notation whenever confusion does not arise. Informally, the above definition means that in addition to the Kawasaki dynamics, each vertex of the source set waits an independent mean one exponential time and then, provided the corresponding source constraint is satisfied, the value $\omega(x)$ is refreshed with a new value sampled in W with ν and the whole procedure starts again.

The generator $\mathcal{L}_\Lambda$ is a non-positive self-adjoint operator on $\mathbb{L}_2(\Omega_\Lambda,\mu_\Lambda)$, where μ_Λ is now fixed by the choice of the on-site probability measure ν in the Glauber term (2.5). The corresponding Dirichlet form $\mathcal{D}_\Lambda(f)$ is given by

$$\begin{aligned}\mathcal{D}_\Lambda(f)&=\mathcal{D}^K_{\Lambda,\mathcal{M}}(f)+\mathcal{D}^G_S(f) &(2.7)\\ &=\sum_{e\in E_\Lambda}\mu_\Lambda\left(c^{\mathcal{M}}_{e,\Lambda}(\nabla_e f)^2\right)+\sum_{x\in S}\mu_\Lambda\left(c_{x,\Lambda}\operatorname{Var}_x(f)\right). &(2.8)\end{aligned}$$

Here $\operatorname{Var}_x(f)\equiv\int d\nu(\omega(x))f^2(\omega)-\left(\int d\nu(\omega(x))f(\omega)\right)^2$ denotes the local variance with respect to the variable $\omega(x)$ computed while the other variables are held fixed.

To the *generator* $\mathcal{L}_\Lambda$ we associate the Markov semigroup $P_t^\Lambda := e^{t\mathcal{L}_\Lambda}$ with reversible invariant measure μ_Λ and the *spectral gap*

$$\operatorname{gap}(\mathcal{L}_\Lambda) := \inf_{\substack{f \neq \text{const} \\ f \in \mathbb{L}_2(\mu)}} \frac{\mathcal{D}_\Lambda(f)}{\operatorname{Var}_\Lambda(f)}. \tag{2.9}$$

Remark 2.10. The chain generated by $\mathcal{L}_\Lambda$ is not ergodic for all choices of the boundary sets $\mathcal{M}$, $\mathcal{S}$. The interesting choices for us will be those for which the resulting chain is irreducible.

We conclude this paragraph with the notion of *domination*.

Definition 2.11 (*Domination*). *In the above setting let $\{\mathcal{C}'_e\}_{e\in E_{\mathbb{Z}^d}}$ be another choice of influence classes. Denote by $c'_e(\omega)$ and $\mathcal{L}'$ the corresponding rates and generator. If for all $\omega \in \Omega$ and all $e \in E_{\mathbb{Z}^d}$ it holds $c'_e(\omega) \le c_e(\omega)$, then we say that $\mathcal{L}$ is dominated by $\mathcal{L}'$ (or, equivalently, that the rates c_e are dominated by c'_e).*

The term domination here means that the KCLG associated to $\mathcal{L}'$ is *more* constrained than the one associated to $\mathcal{L}$. If $\mathcal{L}$ is dominated by $\mathcal{L}'$, for any Λ and any choice of the boundary sets $\mathcal{M}$, $\mathcal{D}'^{K}_{\Lambda,\mathcal{M}}(f) \leqslant \mathcal{D}^{K}_{\Lambda,\mathcal{M}}(f)$ holds. In particular (2.9) yields

Lemma 2.12. *Fix $\Lambda \subset \mathbb{Z}^d$. If $\mathcal{L}$ is dominated by $\mathcal{L}'$ and we define $\mathcal{L}'_\Lambda$ and $\mathcal{L}_\Lambda$ with the same choice for $\mathcal{M}$, $\mathcal{S}$ and with $c_{x,\Lambda} \equiv c'_{x,\Lambda}$, then*

$$\operatorname{gap}(\mathcal{L}'_\Lambda) \le \operatorname{gap}(\mathcal{L}_\Lambda).$$

2.2. 0-1 KCLG: ergodicity and exchangeability thresholds. In the physics literature, the on-site configuration space W is always the two-state space $W = \{0, 1\}$ which represent the empty and occupied configuration, respectively. We call such models 0-1 KCLG. The on-site distribution ν is now completely defined by specifying the parameter $p := \nu(1)$ which can be varied in $[0, 1]$. The probability μ over $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ is thus a product Bernoulli(p) measure, $\mu \equiv \mu_p$. The good set G is conventionally chosen as the empty state $\{0\}$ and we denote by $q := 1 - p$ its probability (thus q corresponds to ρ for a generic KCLG). Note that the *Simple Symmetric Exclusion Process (SSEP)* is a 0-1 KCLG with the trivial choice $c_e \equiv 1$. Recall that on a finite volume $\Lambda \subset \mathbb{Z}^d$ the generator (2.3) explicitly depends on the choice of ν which is here completely defined by the parameter q. We denote by $\mathcal{L}_\Lambda(q)$ the corresponding generator. From Definition 2.11 and Lemma 2.12 it follows immediately that the spectral gap for a 0-1 KCLG in finite volume is upper bounded by the spectral gap of SSEP in the same region. For 0-1 KCLG it is natural to define the critical value

$$q_c = \inf\{q \in [0, 1] : \ 0 \text{ is a simple eigenvalue of } \mathcal{L}_q\},$$

where, with a slight abuse of notation, we let $\mathcal{L}_q := \mathcal{L}_{\mu_{1-q}}$ be the infinite volume generator extended by continuity to $\mathbb{L}_2(\mu_{1-q})$. We now relate q_c (sometimes called *ergodicity threshold*) to another threshold of the dynamics. For this purpose we need to define the notion of *allowed paths* and *exchangeable configurations* (with respect to a given choice of the constraints), which are valid also for generic (i.e. non-0-1) KCLG.

Definition 2.13 (*Allowed paths*). *Given $\eta, \sigma \in \Omega_\Lambda$, a sequence of configurations*

$$P_{\eta,\sigma} = (\eta^{(1)}, \eta^{(2)}, \dots, \eta^{(n)})$$

starting at $\eta^{(1)} = \eta$ and ending at $\eta^{(n)} = \sigma$ is an **allowed configuration path** (*or simply allowed path*) *from η to σ inside Λ if for any $i = 1, \dots, n-1$ there exists either a bond $e_i \in E_\Lambda$ with $\eta^{(i+1)} = (\eta^{(i)})^{e_i}$ and $c^{\mathcal{M}}_{e_i,\Lambda}(\eta^{(i)}) = 1$ or a site $x_i \in \mathcal{S}$ with $\eta^{(i+1)} = (\eta^{(i)})^{x_i}$ and such that $c_{x_i}(\eta^{(i)}) = 1$. We also say that n is the length of the path and write $|P_{\eta,\sigma}| = n$. Furthermore, given $\eta \in \Omega_\Lambda$ and a geometric path $\gamma_{x,y} = (x^{(1)}, \dots x^{(n)})$, we say that γ is an* **allowed geometric path** *for η if $c^{\mathcal{M}}_{e^i,\Lambda}(\eta^{(i)}) = 1$ for any $i = 1, \dots n-1$, where we let $\eta^{(1)} = \eta$ and $\eta^{(i+1)} = (\eta^{(i)})^{e^i}$ with $e^i := x^{(i+1)}, x^{(i)}$. Note that the above definitions depend on the choice of the source and boundary set $\mathcal{S}, \mathcal{M}$. When $\Lambda = \mathbb{Z}^d$ we will mean $\mathcal{S}, \mathcal{M} = \emptyset$.*

Definition 2.14 (Λ-*Connected configurations*). *Given $\eta, \sigma \in \Omega_\Lambda$, we say that they are Λ-connected if there exists (at least) one allowed configuration path $P_{\eta,\sigma}$ inside Λ.*

Definition 2.15 ((e, Λ)-*Exchangeable configurations*). *A configuration $\eta \in \Omega_\Lambda$ is (e, Λ)-exchangeable if η and η^e are Λ-connected. We denote by $\mathcal{E}_e$ ($\mathcal{E}_e^\Lambda$) the set of $(e, \mathbb{Z}^d)$-exchangeable ((e, Λ)-exchangeable) configurations.*

With the above notation we can define the *exchangeability threshold* as

$$q_{\text{ex}} := \inf \left\{ q \in [0, 1] : \mu_{1-q}(\cap_{e \in E_{\mathbb{Z}^d}} \mathcal{E}_e) = 1 \right\}.$$

Using the simple fact that the rates $c_e(\omega)$ are increasing functions w.r.t. the partial order in Ω for which $\omega \leqslant \omega'$ iff $\omega'(x) \in G$ whenever $\omega(x) \in G$, it is easy to check that $\mu_{1-q}(\cap_{e \in E_{\mathbb{Z}^d}} \mathcal{E}_e) = 1$ if $q > q_{\text{ex}}$. We shall now prove that q_{ex} coincides with q_c by using a strategy analogous to the one of [6], Prop. 5.1.

Proposition 2.16. $q_c = q_{\text{ex}}$.

Proof. Assume that $q < q_{\text{ex}}$. Let f be the indicator function of the set of all configurations that are $(e, \mathbb{Z}^d)$-exchangeable for each e. By construction $\text{Var}(f) \neq 0$. Furthermore, since f is left invariant by the dynamics, $\mathcal{L}_q f = 0$ almost surely w.r.t. $\mu = \mu_{1-q}$. Hence 0 is not a simple eigenvalue of $\mathcal{L}_q$ and $q \leqslant q_c$.

Assume now that $q > q_{\text{ex}}$. Consider a function $f \in \mathbb{L}_2(\mu)$ with $\mathcal{L}_q f = 0$, which implies $\mathcal{D}_\mu(f) = 0$. We will now show that in turn this implies that f is constant μ-a.s. For this purpose we will show that $\mathcal{D}_\mu(f) = 0$ implies

$$\sum_{e \in \mathbb{Z}^d} \mu \left(|\nabla_e f|^2 \right) = 0. \tag{2.10}$$

Then the fact that f is constant μ-a.s. immediately follows by using the well known fact that the simple symmetric exclusion process which has the unconstrained Dirichlet form in (2.10) is ergodic at any density (see e.g. [19]). We are thus left with proving (2.10). Suppose that (2.10) does not hold, then there exists at least one bond e such that $\mu \left(|\nabla_e f|^2 \right) > 0$. We will now show that this leads to a contradiction. For any $\eta \in \mathcal{E}_e$ we can fix once and for all an allowed configuration path $P_{\eta \to \eta^e}$ and let

$A_n^e = \{\eta \in \mathcal{E}_e : |P_{\eta\to\eta^e}| = n\}$ and $A_1^e = \emptyset$ by convention. Since $q > q_{\rm ex}$, $\mu(\mathcal{E}_e) = 1$. Thus $\mu\left(\cup_{n=2}^{\infty} A_n^e\right) = 1$ and

$$\mu\left(|\nabla_e f|^2\right) = \sum_{n=2}^{\infty} \int_{A_n^e} d\mu(\eta)|f(\eta^e) - f(\eta)|^2. \tag{2.11}$$

Writing a telescopic sum, and using Cauchy-Schwartz inequality, for any $\eta \in A_n^e$ we get by the very definition of the path,

$$\begin{aligned}
|f(\eta^e) - f(\eta)|^2 &= \left(\sum_{i=1}^{n-1} f(\eta^{(i+1)}) - f(\eta^{(i)})\right)^2 \\
&\leqslant (n-1)\sum_{i=1}^{n-1}\left(f(\eta^{(i+1)}) - f(\eta^{(i)})\right)^2 \\
&= (n-1)\sum_{i=1}^{n-1} c_{e_i}(\eta^{(i)})\left(f(\eta^{(i+1)}) - f(\eta^{(i)})\right)^2,
\end{aligned}$$

where in the last step we could insert the constrained rates because the path is allowed (see in Definition 2.13). It follows that

$$\int_{A_n^e} d\mu(\eta)|f(\eta^e) - f(\eta)|^2 \leqslant C(n) \sum_{b\in E_{\mathbb{Z}^d}: d_1(e,b)\leqslant n+1} \mu\left(c_b|\nabla_b f|^2\right), \tag{2.12}$$

where the constant

$$C(n) = \max_{\omega} \max_{b\in E_{\mathbb{Z}^d}: d_1(e,b)\leqslant n+1} \sharp\{\eta : P_{\eta\to\eta^e} \ni (\omega, \omega^b)\}$$

takes into account the number of possible choices of configuration η such that the path $P_{\eta\to\eta^e}$ crosses a given couple (ω, ω^b). One can choose $C(n) \leqslant e^{cn^d}$ for some constant $c = c(q, d)$. Since by assumption $\mathcal{D}_\mu(f) = 0$, it follows that $\mu\left(c_b|\nabla_b f|^2\right) = 0$ for all $b \in E_{\mathbb{Z}^d}$. Thus (2.11) and (2.12) lead immediately to $\mu\left(|\nabla_e f|^2\right) = 0$ and the proof of (2.10) is complete. $\square$

3. Kob-Andersen (KA) Model

In this section we define the Kob-Andersen (KA) model [15] and recall some of its properties. KA is a 0-1 KCLG on $\mathbb{Z}^d$ with influence classes

$$\mathcal{C}_e = \{A \cup B : (A, B) \subset \mathcal{N}_x\backslash\{y\} \times \mathcal{N}_y\backslash\{x\} \text{ with } |A|, |B| \geq j-1\},$$

where j is a parameter satisfying $1 < j \leqslant d$. Recalling Definition 2.7 for the rates and the fact that the good event is the empty state for 0-1 KCLG, this means the following: for any two neighbouring sites x and y at least $j-1$ neighbors of x belonging to $\mathcal{N}_x\backslash\{y\}$ *and* $j-1$ neighbors of y belonging to $\mathcal{N}_y\backslash\{x\}$ should be empty in order for the exchange between x and y to be allowed. See Fig. 2 left (right) for an example of an allowed (not allowed) exchange when $d = j = 2$. Another way to formulate this rule is to say that when $\omega(x) = 1$ and $\omega(y) = 0$ (when $\omega(x) = 0$ and $\omega(y) = 1$), the jump of the particle

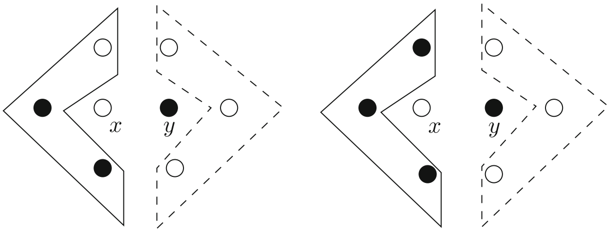

Fig. 2. The bond $e = (x, y)$ with $\mathcal{N}_x \setminus \{y\}$ (inside the continuous line) and $\mathcal{N}_y \setminus \{x\}$ (inside the dashed line). In the left (right) figure the exchange of the value at x and y is (is not) allowed for KA model with $d = 2$, $j = 2$

from x to y (from y to x) occurs iff the particle before *and* after the move has at least j empty neighbors. The choice $j = 1$ is not considered among the KA models because it corresponds to the simple symmetric exclusion process (SSEP). The choices $j > d$ are instead excluded because at any finite density zero is not a unique eigenvalue of their generator. In other words the model on infinite volume is uninteresting because it is never ergodic [29,30], namely $q_c = 1$. This can be readily verified by noticing that for any $j > d$ it is possible to construct a finite set of particles that can never be moved under the dynamics. For example for the choice $d = 2$, $j = 3$ a two by two fully occupied square can never be destroyed.

Let us now discuss boundary and source choices for the model on a finite volume, $\Lambda \subset \mathbb{Z}^d$. For simplicity we discuss only the case of a rectangular region Λ. For all $1 < j \leqslant d$ a choice which renders the generator ergodic with invariant measure μ_Λ corresponds to imposing fully occupied boundary conditions and unconstrained particle source on $\partial_-\Lambda$. This is the choice which is usually considered in the physics literature and which in our notation corresponds to the choice $\mathcal{M} = \emptyset$, $S = \partial_-\Lambda$ and $c_{x,\Lambda}(\omega) = 1$. We will make here the more constrained choice $\mathcal{M} = \emptyset$, $S = \overline{\partial}_-\Lambda$ and $c_{x,\Lambda}(\omega) = 1$ which is also ergodic. We will now recall some properties of KA obtained in [30]. We start by introducing the notion of *framed* and *frameable* configurations.

Definition 3.1 (*Framed and frameable configurations*). *Fix a set $\Lambda \subset \mathbb{Z}^d$ and a configuration $\omega \in \Omega$. We say that ω is* **Λ-framed** *if $\omega(x) = 0$ for any $x \in \partial_-\Lambda$. Let $\omega^{(\Lambda)}$ be the configuration equal to ω_Λ inside Λ and equal to 1 outside Λ. We say that ω is* **Λ-frameable** *if there exist a Λ-framed configuration $\sigma^{(\Lambda)}$ with at least one allowed configuration path $P_{\omega^{(\Lambda)} \to \sigma^{(\Lambda)}}$ inside Λ.* (*By definition any framed configuration is also frameable*).

The following results, which are valid in any dimension d and for any $1 < j \leqslant d$, have been derived in [30] and will play a key role in the proof of Theorem 4.1. For sake of completeness we present their proof in Appendix 9. Recall Definitions 2.14 and 2.15 of connectedness and exchangeability, then

Lemma 3.2. *Consider a rectangle $R \subset \mathbb{Z}^d$ and a configuration ω which is R-framed. Then, for any bond $e = (x, y) \in R$, ω is (e, R)-exchangeable, namely $\omega \in \mathcal{E}_e^R$.*

Corollary 3.3. *Consider a rectangle $R \subset \mathbb{Z}^d$ and a couple of configurations ω, σ which are both R-framed and have the same number of particles inside R, $\sum_{x \in R} \omega(x) = \sum_{x \in R} \sigma(x)$. Then σ and η are R-connected.*

Lemma 3.4. *For any ℓ, let $\mathcal{F}_\ell := \{\omega \in \Omega : \omega$ is $Q_\ell -$ frameable$\}$. Then for any $q \in (0,1]$, for any $j \in (1,d]$, any $\varepsilon > 0$, there exists $\ell_0 = \ell_0(\varepsilon, q, j)$ such that if $\ell \geq \ell_0$, then $\mu_{1-q}(\mathcal{F}_\ell) > 1 - \varepsilon$.*

As a consequence of the equivalence between the ergodicity and exchangeability thresholds (see Proposition 2.16) and of the above lemmas we get that for any $1 < j \leq d$ the ergodicity threshold q_c is zero (see [30]). We formalize the result into a theorem whose proof is also postponed to Appendix 9.

Theorem 3.5. *For any $d \geq 1$ and any $1 < j \leq d$ the ergodicity threshold q_c of the KA model on $\mathbb{Z}^d$ with parameter j verifies $q_c = 0$.*

4. Main Ideas and Results

Consider the KA model in $d = 2$ with $j = 2$. We call $\mathcal{L}^{KA}$ the generator on the infinite volume $\mathbb{Z}^2$ and $\mathcal{L}^{KA}_{Q_L}(q)$ the generator on the finite cube $Q_L \subset \mathbb{Z}^2$ with boundary-source choice $(\mathcal{M}, \mathcal{S}) = (\emptyset, \bar{\partial}_- Q_L)$ and parameter q.

Theorem 4.1. *For any $q > 0$ there exists a constant $C = C(q) > 0$ and a constant $c > 0$ independent on q such that for any L,*

$$c\left[1 - (1-q)^3\right]^{-2} L^2 \leqslant \operatorname{gap}(\mathcal{L}^{KA}_{Q_L}(q))^{-1} \leqslant C(q) L^2 (\log L)^4.$$

Theorem 4.2. *Let P_t be the semigroup associated to $\mathcal{L}^{KA}(q)$. For any $q > 0$ there exists a constant $C = C(q) > 0$ such that for any local function f,*

$$\operatorname{Var}_\mu(P_t f) \leqslant C \frac{(\log t)^5}{t} \|f\|^2_\infty \qquad \forall t > 0,$$

where $\mu = \mu_{1-q}$.

Remark 4.3. From the variational characterization of the spectral gap it follows immediately that the result of Theorem 4.1 holds also for the less constrained choice $\mathcal{S} = \partial_- Q_L$ which is usually considered in the physics literature. Actually, the proof of Theorem 4.1 will lead to the following stronger result. For any $f : \Omega_{Q_L} \to \mathbb{R}$,

$$\operatorname{Var}_{Q_L}(f) \leqslant C L^2 (\log L)^4 \mathcal{D}^K_{Q_L}(f) + C L (\log L)^2 \mathcal{D}^G_{\bar{\partial}_- Q_L}(f),$$

where $\mathcal{D}^K_{Q_L}(f)$ and $\mathcal{D}^G_{\bar{\partial}_- Q_L}(f)$ are respectively the Kawasaki and Glauber term of the Dirichlet form of $\mathcal{L}^{KA}_{Q_L}(q)$. This implies that the result of Theorem 4.1 holds also if the sources are slowed down of $(\log L)^2/L$, namely if the Glauber term of $\mathcal{L}^{KA}_\Lambda(q)$ is multiplied by a factor $(\log L)^2/L$.

Remark 4.4. With much more effort, the proof of Theorems 4.1 and 4.2 can be generalized in any dimension $d \geq 3$ for all $j, 1 < j \leqslant d$. The bounds for the spectral gap remain of order L^2 (but with higher correction in the log terms). For the decay to equilibrium we get $\operatorname{Var}_\mu(P_t f) \leqslant C \frac{(\log t)^\alpha}{t} \|f\|^2_\infty$ for some constant $\alpha(d)$. The latter result is probably not optimal: we expect the decay to be of order $C/t^{d/2}$.

Remark 4.5. Any 0-1 KCLG which is dominated by KA-2f verifies the same bounds as in Theorem 4.1. The upper (lower) bound follows from Corollary 2.12 and comparison with KA-2f (with SSEP).

Sketch of the main ideas. In order to explain our approach let us quickly review a simple route to prove Theorem 4.1 in the context of the SSEP. One first establishes a Poincaré inequality w.r.t. an auxiliary Dirichlet form with pure Glauber moves and reversible w.r.t. μ_Λ (a trivial fact since μ_Λ is a product measure) and then one transfers the Glauber moves from the bulk to the boundary using the exchange moves of the SSEP along apriori chosen geometrical paths (the so-called "path argument" see e.g. [20,23]). That gives almost immediately the diffusive scaling of the spectral gap. More or less the same technique can be applied to "non-cooperative" models.

A completely different scenario is presented when considering "cooperative models" like the Kob-Andersen model. Indeed the above Poincaré inequality is now completely useless because we are not guaranteed that from a site $x \in \Lambda$ where a Glauber move is performed we can reach the boundary by a sequence of *allowed exchange moves*. In other words nobody guarantees that in the current configuration the holes are "cooperating" in such a way that the new particle created at x can be moved to the boundary. It is precisely this loss of uniformity that requires new ideas that, to the best of our knowledge, were completely absent before our work.

The way out and a major novelty of our approach is to prove a modified Glauber-type Poincaré inequality in which the creation/annihilation move in the bulk occurs *only if* the holes in the current configuration are cooperating in a way to be able to move to the boundary the extra hole created at x. This forces us to consider an auxiliary Glauber process with very long range (essentially from the inner bulk to the boundary) constraints and now the existence of the corresponding Poincaré inequality is highly non trivial. One could naively think that the long range constraint should be of the form *"there exists a path of holes from x to the boundary along which a particle can move with legal exchanges"*. However, when the density of particles is high (and therefore the density of holes is small) the probability of such an event is exponentially small in the distance from the boundary, and most of the times the constraint will not be satisfied. That brings up the second set of new ideas, namely to consider a "renormalized Kob-Andersen model" with a much richer structure than just particle/hole and for which the "effective holes" have a high density (see model AKG below). For the new model we carry out the program just illustrated above and then finally we go back to the original Kob-Andersen model via standard comparison techniques.

5. Renormalization and Long Range Constraints

In this section we define two auxiliary models: one with purely Glauber dynamics and long range constraints (AGL) and another one with Kawasaki dynamics plus Glauber sources (AKG). Thus AGL belongs to the class of kinetically constrained spin models (KCSM) while AKG is a kinetically constrained lattice gas (KCLG). Both models are defined with an arbitrary on-site probability space (W, ν) and good event $G \subset W$, at variance with the specific choice $W = (0, 1)$, $\nu(1) = p$ and $G = 0$ of the KA model.

We will establish the positivity of the spectral gap for AGL (Theorem 5.5). This will be a key ingredient to prove both the lower bound on the spectral gap (Theorem 4.1) and the polynomial decay to equilibrium (Theorem 4.2) for KA. By combining Theorem 5.5 with proper path arguments we will deduce a $1/L^2$ lower bound for the spectral gap of AKG (Theorem 5.6). It is by using the latter result and a suitable renormalization procedure that we will cast KA into AKG and deduce the desired $1/L^2$ lower bound for the spectral gap of the KA model (Theorem 4.1). The peculiar choice of the constraints for both the auxiliary models is motivated by this final renormalization procedure and

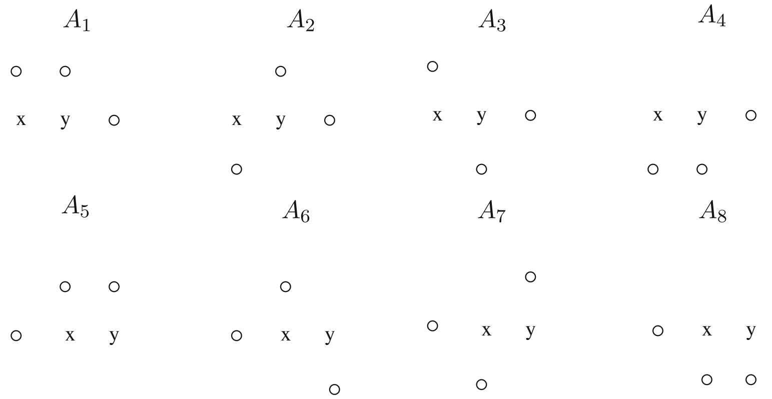

Fig. 3. The sets $A_1, \dots, A_8$ which belong to the influence class $\mathcal{C}_e$ for AKG model. We depict the case $e = x, x + e_1$

should be properly modified when one ultimately wishes to study KCLG which are different from KA.

Let us start by defining the influence classes which characterize the Kawasaki dynamics of AKG. We set

$$\mathcal{C}_e := \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}, \tag{5.1}$$

where A_i are defined as follows (see Fig. 3):

$$\begin{aligned}
A_1 &= (x + e_2, x + 2e_1, x + e_1 + e_2), & A_2 &= (x - e_2, x + 2e_1, x + e_1 + e_2),\\
A_3 &= (x + e_2, x + 2e_1, x + e_1 - e_2), & A_4 &= (x - e_2, x + 2e_1, x + e_1 - e_2),\\
A_5 &= (x - e_1, x + e_2, x + e_1 + e_2), & A_6 &= (x - e_1, x + e_2, x + e_1 - e_2),\\
A_7 &= (x - e_1, x - e_2, x + e_1 + e_2), & A_8 &= (x - e_1, x - e_2, x + e_1 - e_2),
\end{aligned}$$

if $e = x, x + e_1$ and

$$\begin{aligned}
A_1 &= (x + e_1, x + 2e_2, x + e_1 + e_2), & A_2 &= (x - e_1, x + 2e_2, x + e_1 + e_2),\\
A_3 &= (x + e_1, x + 2e_2, x + e_2 - e_1), & A_4 &= (x - e_1, x + 2e_2, x + e_2 - e_1),\\
A_5 &= (x - e_2, x + e_1, x + e_1 + e_2), & A_6 &= (x - e_2, x + e_1, x + e_2 - e_1),\\
A_7 &= (x - e_2, x - e_1, x + e_1 + e_2), & A_8 &= (x - e_2, x - e_1, x + e_2 - e_1),
\end{aligned}$$

if $e = x, x + e_2$. The following results follow immediately from the above definitions:

Remark 5.1. The influence classes of AKG dominate (see Definition 2.11) those of the KA model with $d = j = 2$. Indeed for all i there exists $A \subset \mathcal{N}_x \backslash y$ and $B \subset \mathcal{N}_y \backslash x$ s.t. $A_i = A \cup B$ and either $|A| = 1$ and $|B| = 2$ or $|A| = 2$ and $|B| = 1$. Thus A_i also belongs to the influence class of KA and the inequality among the rates of KA and AKG required to have domination immediately follows from Definition 2.7. Fix a configuration ω, then a geometric path which is good (see Definition 2.6) is also allowed for ω with the choice of AKG constraints (see Definition 2.13).

For any $\Lambda \subset \mathbb{Z}^2$ the finite volume generator of AKG, $\mathcal{L}_\Lambda^{akg}$, is then defined as in (2.3). The corresponding Kawasaki term is defined as in (2.4) with exchange rates $c_{e,\Lambda}^{\mathcal{M}}(\omega)$ defined by (2.6) with boundary set $\mathcal{M} = \overline{\partial}_+^*\Lambda$ and $c_e(\omega)$ as in Definition 2.7 with influence classes (5.1). The Glauber term is defined by (2.5) with source set $\mathcal{S} = \overline{\partial}_-\Lambda$ and constraints

$$c_{x,\Lambda}(\omega) = \begin{cases} 1 & \text{if } (\omega_\Lambda \cdot \tau)(z) \in G \text{ for any } z \in \mathcal{K}_x^* \\ 0 & \text{otherwise} \end{cases}, \tag{5.2}$$

where $\tau \in \Omega$ is any configuration such that $\tau(z) \in G$ ($\tau(z) \notin G$) if $z \in \overline{\partial}_+^*\Lambda$ ($z \notin \overline{\partial}_+^*\Lambda$).

The finite volume generator of AGL is instead of purely Glauber type and is defined by the following action on local functions:

$$\mathcal{L}_{\Lambda,N}^{agl} f(\omega) = \sum_{x\in\Lambda} c_{x,\Lambda}^N(\omega)\,(\nu_x(f) - f(\omega))\,,$$

where the constraints $c_{x,\Lambda}^N$ are defined as

$$c_{x,\Lambda}^N(\omega) = \begin{cases} \begin{cases} 1 & \text{if } |\mathcal{G}_{x,N,\Lambda}(\omega)| \geqslant 1 \\ 0 & \text{otherwise} \end{cases} & \text{if } x \notin \overline{\partial}_-\Lambda, \\ \\ \begin{cases} 1 & \text{if } (\omega_\Lambda \cdot \tau)(z) \in G \text{ for any } z \in \mathcal{K}_x^* \\ 0 & \text{otherwise} \end{cases} & \text{if } x \in \overline{\partial}_-\Lambda, \end{cases} \tag{5.3}$$

where $\tau \in \Omega$ is any configuration such that $\tau(z) \in G$ ($\tau(z) \notin G$) if $z \in \overline{\partial}_+^*\Lambda$ ($z \notin \overline{\partial}_+^*\Lambda$) and $\mathcal{G}_{x,N,\Lambda}$ is the set of all geometric paths which are allowed for ω with the AKG constraints (see Definition 2.13) and go from x to the East ($\overline{\partial}_-^1\Lambda$) or North ($\overline{\partial}_-^2\Lambda$) interior boundary of Λ never leaving a tube of width N centered at x. In formulas

$$\mathcal{G}_{x,N,\Lambda}(\omega) := \bigcup_{i=1}^{2} \bigcup_{y\in\overline{\partial}_-^i\Lambda} \{\gamma_{x,y} \subset \mathcal{G} :\ \gamma_{x,y} \text{ is allowed for AKG;} \quad \gamma_{x,y} \subset T_{x,i}^N(\Lambda)\}, \tag{5.4}$$

where $\mathcal{G}$ is the overall set of geometric paths. Recalling Definition 2.5, the following property for the set of geometric paths can be immediately verified:

Remark 5.2. Let $\omega, \omega' \in \Omega_\Lambda$ be such that $\omega \overset{G,\Lambda\setminus x}{\Longleftrightarrow} \omega'$. Then the following holds for any $x \in \Lambda$ and $N > 0$:

$$\mathcal{G}_{x,\Lambda,N}(\omega) = \mathcal{G}_{x,\Lambda,N}(\omega').$$

The Dirichlet form associated to $\mathcal{L}_{\Lambda,N}^{agl}$ is given by

$$\mathcal{D}_{\Lambda,N}(f) = \sum_{x\in\Lambda} \mu_\Lambda\left(c_{x,\Lambda}^N \operatorname{Var}_x(f)\right), \qquad f : \Omega_\Lambda \mapsto \mathbb{R}. \tag{5.5}$$

Recall from Sect. 2.1 that μ_Λ is the product measure $\mu_\Lambda := \prod_{x\in\Lambda} \nu_x$. The following results hold:

Claim 5.3. *For any $\Lambda \subset \mathbb{Z}^2$ and any choice of W, ν and G, the generator $\mathcal{L}^{agl}_{\Lambda,N}$ is reversible w.r.t. μ_Λ and it is ergodic.*

Proof. Reversibility follows immediately from the fact that $c^N_{x,\Lambda}(\omega)$ does not depend on the value of $\omega(x)$. Ergodicity follows by noticing that the constraint is verified on all sites $\tilde{\Lambda} := \{x \in \Lambda : \mathcal{K}^*_x \cap \Lambda = \emptyset\}$ whose existence can be proved by induction because Λ is finite. We can thus make the configuration of these sites good. Then we can render all sites $\{x \in (\Lambda \setminus \tilde{\Lambda}) : \mathcal{K}^*_x \cap (\Lambda \setminus \tilde{\Lambda}) = \emptyset\}$ good since they have the constraint verified. □

Claim 5.4. *For any $\Lambda \subset \mathbb{Z}^2$ and any choice of W, ν and G, the generator $\mathcal{L}^{akg}_{\Lambda}$ is reversible w.r.t. μ_Λ and ergodic on Ω_Λ.*

Proof. Reversibility follows, as for any other KCLG, from the independence of c_x on ω_x and from the independence of $c_{x,y}$ on ω_x and ω_y. In order to prove ergodicity it is sufficient to show that for any $\eta \in \Omega_\Lambda$ there exists an allowed path which connects η to σ, where σ is a completely good configuration, $\sigma(x) \in G$ for all $x \in \Lambda$. In order to construct the path we start by using the source terms to make good all sites in $\overline{\partial}^*_-\Lambda$. Then, since $\tilde{\Lambda} := \Lambda \setminus \overline{\partial}^*_-\Lambda$ is finite, there should exist at least one site $x \in \tilde{\Lambda}$ such that: $\mathcal{K}^*_x \cap \tilde{\Lambda} = \emptyset$, thus $\mathcal{K}^*_x \subset (\overline{\partial}^*_+\Lambda \cup \overline{\partial}^*_-\Lambda)$. Thus we can exchange the occupation variable in x with the (good) occupation variable of a site in $\overline{\partial}^*_-\Lambda$. The latter site can then be restored to good by using the sources. The procedure may then be iterated until making the whole configuration good. □

Recall from Sect. 2.1 that ρ is the probability of the good event $G \subset W$, $\rho := \nu(G)$. Our main results concerning the auxiliary models are the following:

Theorem 5.5. *There exist $\rho_1 \in (0,1)$ and $A > 0$ independent of W and ν such that if $\rho > \rho_1$, then for any rectangle $\Lambda = [0, L_1] \times [0, L_2]$,*

$$\mathrm{gap}(\mathcal{L}^{agl}_{\Lambda,N}) \geq \frac{1}{2},$$

provided that $N \geq A(\log(\max(L_1, L_2))^2$.

Theorem 5.6. *There exists $\rho_0 \in (0,1)$ independent of W and ν and a constant $C = C(|W|, \nu)$ such that if $\rho \geq \rho_0$, then for any cube Q_L,*

$$\mathrm{gap}(\mathcal{L}^{akg}_{Q_L})^{-1} \leqslant CL^2(\log L)^4.$$

In [9] we have devised a technique which allows to prove the positivity of the spectral gap for a large class of KCSM. However AGL does not belong to this class because its constraints are not local (while the proof in [9] relies on the hypothesis that the influence classes of the KCSM have finite range). Some additional efforts and a proper extension of the technique in [9] is thus required to establish Theorem 5.5. We postpone this rather technical proof to Sect. 7 and proceed with the proof of the result for AKG.

Proof of Theorem 5.6. We prove the theorem for $\rho_0 = \rho_1$, with ρ_1 defined by Theorem 5.5. Recalling the definition of spectral gap (2.9), the expression (5.5) for the Dirichlet form of AGL and using Theorem 5.5, for any $\rho > \rho_1$ and $f : \Omega_{Q_L} \to \mathbb{R}$, the following holds:

$$\mathrm{Var}_{Q_L}(f) \leqslant 2 \sum_{x \in Q_L} \mu_{Q_L} \left(c^N_{x,Q_L} \mathrm{Var}_x(f) \right), \tag{5.6}$$

where c^N_{x,Q_L} have been defined in (5.3), $N := A(\log L)^2$ and A and ρ_1 are those defined in Theorem 5.5. Our aim is to bound the r.h.s. with the Dirichlet form of AKG model. This will be achieved via proper path arguments. We start by rewriting $\mathrm{Var}_x(f)$ as

$$\mathrm{Var}_x(f)(\omega) = \frac{1}{2} \sum_{w, w' \in W} \mu(w)\mu(w')(f(\omega_{Q_L \setminus x} \cdot w'_x) - f(\omega_{Q_L \setminus x} \cdot w_x))^2. \tag{5.7}$$

Then for all x and ω such that $c^N_{x,Q_L}(\omega) = 1$ we choose once and for all a path in $\mathcal{G}_{x,Q_L,N}$ (5.4) which we will call $\gamma(x, \omega)$. We make the latter choice in order that $\gamma(x, \omega) = \gamma(x, \omega')$ for all ω, ω' such that $\omega \overset{G,L\setminus x}{\Longleftrightarrow} \omega'$, which is possible thanks to Remark 5.2. If we set $\gamma(x, \omega) = (x^{(1)}, \ldots, x^{(n)})$ from the above definition it follows immediately that $x^{(1)} = x, x^{(n)} \in \overline{\partial}_- Q_L \cap T^N_x(Q_L) := \overline{\partial}_- Q_L \cap (T^N_{x,1}(Q_L) \cup T^N_{x,2}(Q_L))$ and $n(x, \omega) < L(2A(\log L)^2 + 1)$. For any $\omega \in \Omega_{Q_L}$ and $w, w' \in W$, we are now ready to define a configuration path $P_{w \xrightarrow{x} w'}(\omega)$ from $\omega_{Q_L \setminus x} \cdot w_x$ to $\omega_{Q_L \setminus x} \cdot w'_x$ as follows:

$$P_{w \xrightarrow{x} w'} := (\omega^{(1)}, \ldots, \omega^{(2n)}) \tag{5.8}$$

with $\omega^{(1)} = \omega_{Q_L \setminus x} \cdot w_x$; $\omega^{(i+1)} = T_{e_i} \omega^{(i)}$, with $e_i := (\tilde{x}^{(i)}, \tilde{x}^{(i+1)}) := (x^{(i)}, x^{(i+1)})$ for all $i \in \{1, \ldots, n-1\}$; $\omega^{(n)} = \omega^{(n-1)}_{Q_L \setminus x^{(n)}} \cdot w'_{x^{(n)}}$; $\omega^{(i+1)} = T_{e_i} \omega^{(i)}$, with $e_i := (\tilde{x}^{(i)}, \tilde{x}^{(i+1)}) := (x^{(2n+1-i)}, x^{(2n+2-i)})$ for all $i \in \{n, \ldots, 2n-1\}$. It is immediate to verify that $\omega^{(2n)} = \omega_{Q_L \setminus x} \cdot w'_x$. Note that, even if we do not write it for simplicity of notation, $\omega^{(i)}$ depends on w, w', x, ω. Recall Definition 2.13. The following properties can be immediately verified by using the definition (5.3) for $c^N_{x,Q_L}(\omega)$.

Claim 5.7. *For any x, ω such that $c^N_{x,Q_L}(\omega) = 1$ and $g \in G$:*

(i) *the path $P_{w \xrightarrow{x} w'}$ (5.8) is allowed for AKG;*

(ii) *for $i \in \{1, \ldots, 2n\} \setminus n+1$, $\omega^{(i)}_{Q_L \setminus \{\tilde{x}_i, \tilde{x}_{i+1}\}} \cdot g_{\tilde{x}^{(i)}} \cdot g_{\tilde{x}^{(i+1)}} \overset{G,Q_L\setminus x}{\Longleftrightarrow} \omega$;*

(iii) $\omega^{(n+1)}_{Q_L \setminus x^{(n)}} \cdot g_{x^{(n)}} \overset{G,Q_L\setminus x}{\Longleftrightarrow} \omega$.

Thus, recalling Definition 2.13 for the meaning of $(\sigma, \sigma^e) \in P_{w \xrightarrow{x} w'}(\omega)$, via a telescopic sum and Cauchy-Schwartz inequality we get for any ω,

$$\begin{aligned} & c^N_{x,Q_L}(\omega) \left[f(\omega_{Q_L \setminus x} \cdot w'_x) - f(\omega_{Q_L \setminus x} \cdot w_x) \right]^2 \\ & \quad \leqslant 2L(2A(\log L)^2 + 1) \sum_{\substack{i=1 \\ i \neq n}}^{2n} c_{e_i,Q_L}(\omega^{(i)}) \left[\nabla_{e_i} f(\omega^{(i)}) \right]^2 \\ & \qquad + 2c_{x^{(n)},Q_L}(\omega^{(n)})(f(\omega^{(n)}_{Q_L \setminus y} \cdot w'_y) - f(\omega^{(n)}))^2, \end{aligned} \tag{5.9}$$

where c_{e,Q_L} and c_{x,Q_L} are the Kawasaki and source Glauber rates for the AKG model. Then, by plugging (5.7) and (5.9) into (5.6) we upper bound $\mathrm{Var}_{Q_L}(f)$ by

$$|W|^2 \max_{w,w' \in W} \sum_{x \in Q_L} \sum_{\omega} \mu_{Q_L}(\omega) \Bigg[2c_{x^{(n)},Q_L}(\omega^{(n)})(f(\omega^{(n)}_{Q_L \setminus y} \cdot w'_y) - f(\omega^{(n)}))^2$$

$$+ \sum_{\substack{i=1 \\ i \neq n}}^{2n} 2L(2A(\log L)^2 + 1) c_{e_i,Q_L}(\omega^{(i)}) \left[\nabla_{e_i} f(\omega^{(i)})\right]^2 \Bigg]. \tag{5.10}$$

By construction of the path $P_{w \xrightarrow{x} w'}$ any $\omega^{(i)}$ satisfies

$$\frac{\mu_{Q_L}(\omega)}{\mu_{Q_L}(\omega^{(i)})} \leqslant C := \max_{w,w' \in W} \frac{\mu(w)}{\mu(w')}.$$

Hence, inverting the summations, (5.10) is bounded above by

$$|W|^2 C L^2 (2A(\log L)^2 + 1)^2 \max_{w,w',x,\sigma,e} \sharp \left\{\omega : (\sigma, \sigma^e) \in P_{w \xrightarrow{x} w'}\right\} \mathcal{D}^K_{Q_L}(f)$$

$$+ 2|W|^2 L(2A(\log L)^2 + 1) \max_{w,w',x,\sigma,z} \sharp \left\{\omega : \omega^{(n)} = \sigma; x^{(1)} = x; x^{(n)} = z\right\} \mathcal{D}^G_{\partial_- Q_L}(f)$$

$$\leqslant 8C|W|^3 A^2 L^2 (\log L)^4 \mathcal{D}_{Q_L}(f), \tag{5.11}$$

where $\mathcal{D}_{Q_L}(f)$ is the Dirichlet form for AKG and $\mathcal{D}^K_{Q_L}(f)$, $\mathcal{D}^G_{\partial_- Q_L}(f)$ are its Kawasaki and Glauber parts. In order to derive the last inequality we have bounded the number of configurations ω such that a chosen σ, σ^e belongs to $P_{w \xrightarrow{x} w'}$. To perform this bound we used as a key ingredient the fact that from the knowledge of (σ, σ^e) we can reconstruct ω modulo the configuration in x (or completely if $(\sigma, \sigma^e) = (\omega^{(i)}, \omega^{(i+1)})$ with $i \leqslant n-1$) and from the knowledge of $\omega^{(n)}$ and $x^{(n)}$ we can reconstruct ω. This in turn is true thanks to properties (ii) and (iii) of Claim 5.7. The proof is then completed by combining the variational characterization of the spectral gap with the upper bound (5.11) for $\mathrm{Var}_{Q_L}(f)$. $\square$

6. Spectral Gap of KA Model: Proof of Theorem 4.1

Since KA dominates the Symmetric Simple Exclusion Process (SSEP), a lower bound on the inverse of the spectral gap as L^2 uniform on q follows from Claim 2.12 and from the standard results for SSEP, see e.g. [6]. In order to obtain the stronger lower bound of Theorem 4.1 which guarantees that, at variance with SSEP, the spectral gap on scale L^{-2} is not bounded away from zero at all density and instead it vanishes as $q \to 0$ we consider the test function $\sum_{x \in Q_L} \cos(\frac{\pi x}{L-1}) \cos(\frac{\pi x}{L-1}) \eta_x$. The term $(1-(1-q)^3)^2$ comes from the presence of the kinetic constraints: in order for the exchange of the occupation variables at x and y to be allowed there should be at least one empty site among the three nearest neighbours of x different from y and at least one empty site among the three neighbors of y different from x.

We will now prove the upper bound by using the $1/L^2$ bound for the spectral gap of AKG (Theorem 5.6) combined with a renormalization technique similar to the one we used in [9].

Proof of Theorem 4.1. Let ρ_0 be the threshold density defined in Theorem 5.6. Thanks to Lemma 3.4 we can choose an integer length scale ℓ such that $\mu(\mathcal{F}_\ell) > \rho_0$, where $\mathcal{F}_\ell$ is the set of configurations which are Q_ℓ-frameable. For any $z \in \mathbb{Z}^2$ we define $Q^z \subset \mathbb{Z}^2$ as $Q^z := Q_\ell + z$. Then we define the renormalized lattice $\mathbb{Z}^2(\ell) := \ell\,\mathbb{Z}^2$. Given $\tilde{L}$ s.t. $L := \tilde{L}/\ell$ is integer we also define the renormalized cube associated to $Q_{\tilde{L}} \subset \mathbb{Z}^2$ as $\tilde{Q}_L := \mathbb{Z}^2(\ell) \cap Q_{\tilde{L}}$. Note that $\tilde{Q}_L$ contains $L \times L$ sites and that $\cup_{x \in \tilde{Q}_L} Q^x = Q_{\tilde{L}}$. Consider the probability space $W = \{0,1\}^{Q_\ell}$ equipped with $\nu = \mu_{Q_\ell}$. The two probability spaces $(\{0,1\}^{\mathbb{Z}^2}, \mu)$ and $(W^{\mathbb{Z}^2(\ell)}, \prod_{x \in \mathbb{Z}^2(\ell)} \nu_x)$ coincide. Furthermore we have $\mu_{Q_{\tilde{L}}} = \nu_{\tilde{Q}_L}$, where $\nu_A = \prod_{x \in A \cap \mathbb{Z}^2(\ell)} \nu_x$. Thus if we consider AKG on $\tilde{Q}_L$ with $W = \{0,1\}^{Q_\ell}$, $\nu = \mu_{Q_\ell}$ and good event $\mathcal{F}_\ell$, by Theorem 5.6 there exists a constant $C = C(\ell, q)$ such that

$$\operatorname{Var}_{Q_{\tilde{L}}}(f) \leqslant C L^2 (\log L)^4 \sum_{e \in E_{Q_L}} \mu_{Q_{\tilde{L}}}\left(\tilde{c}_e (f \circ T_e - f)^2\right) \tag{6.1}$$

$$+ C L^2 (\log L)^4 \sum_{x \in \overline{\partial}_- Q_L} \mu_{Q_{\tilde{L}}}\left(\tilde{c}_x \operatorname{Var}_{Q^x}(f)\right), \tag{6.2}$$

where for any bond $e = (x, y) \in E_{Q_L}$, we let $T_e = T_{(x,y)} : \Omega \to \Omega$ be the operator that exchanges the configuration inside Q^x and Q^y, namely

$$T_e \omega(z) = \omega^{Q^x Q^y}(z) := \begin{cases} \omega(z) & \text{if } z \notin Q^x \cup Q^y \\ \omega(z + y - x) & \text{if } z \in Q^x \\ \omega(z + x - y) & \text{if } z \in Q^y \end{cases}$$

and $\tilde{c}_e(\omega)$ and $\tilde{c}_x(\omega)$ are defined as follows. $\tilde{c}_e$ is the indicator function of the event that there exists $A \in \mathcal{C}_e$ with $\mathcal{C}_e$ defined in (5.1) s.t. $Q^z \in \mathcal{F}_\ell$ for any $z \in A \cap Q_L$. Instead $\tilde{c}_x$ is the indicator function of the event that $Q^z \in \mathcal{F}_\ell$ for any $z \in \mathcal{K}_x^* \cap Q_L$. The proof of Theorem 4.1 is then completed by the following key Lemma 6.1 and explicit counting (left to the reader). $\square$

For any $x \in \overline{\partial}_- Q_L$ let $E(x) := E_{Q_{\tilde{L}}} \cap \left(E_{Q^x} \cup_{z \in \mathcal{K}_x^*} E_{Q^z}\right)$ and for any bond $e = (x, y)$, $E(e) := E_{Q_{\tilde{L}}} \cap \left(E_{Q^x} \cup E_{Q^y} \cup_{z \in \partial_+(x,y)} E_{Q^z}\right)$. In order to avoid confusion we call here $c_{e,\Lambda}^{KA}$ the rates of KA.

Lemma 6.1. *There exists a constant $C' = C'(\ell, q) > 0$ such that*:

(i) *for any bond $e = (x, y) \in E_{Q_L}$,*

$$\mu_{Q_{\tilde{L}}}\left(\tilde{c}_e (f \circ T_e - f)^2\right) \leqslant C' \sum_{e' \in E(e)} \mu_{Q_{\tilde{L}}}\left(c_{e', Q_{\tilde{L}}}^{KA} (\nabla_{e'} f)^2\right) + C' \sum_{z \in (Q^x \cup Q^y) \cap \overline{\partial}_- Q_{\tilde{L}}} \mu_{Q_{\tilde{L}}}\left(\operatorname{Var}_z(f)\right)$$

(*with possibly* $(Q^x \cup Q^y) \cap \overline{\partial}_- Q_{\tilde{L}} = \emptyset$);

(ii) *for any* $x \in \overline{\partial}_- Q_L$,

$$\mu_{Q_{\tilde{L}}}\left(\widetilde{c}_x \operatorname{Var}_{Q^x}(f)\right) \leqslant C' \sum_{e' \in E(x)} \mu_{Q_{\tilde{L}}}\left(c^{KA}_{e', Q_{\tilde{L}}} (\nabla_{e'} f)^2\right) + C' \sum_{z \in Q^x \cap \overline{\partial}_- Q_{\tilde{L}}} \mu_{Q_{\tilde{L}}}\left(\operatorname{Var}_z(f)\right).$$

Proof. The proof is based on path techniques. In all the proof C will denote a positive constant that depends on ℓ, q but never on L, and that might change from line to line. Finally, unless explicitly stated, an *allowed path* means allowed w.r.t. KA rates.

(i) Fix a bond $e = (x, y) \in E_{Q_L}$ and a configuration ω such that $\widetilde{c}_e(\omega) = 1$ (otherwise the result trivially holds). Since $\widetilde{c}_e(\omega) = 1$ there exists $A \subset \mathcal{C}_e$ s.t. for any $z \in A \cap Q_L$, $\omega_{Q^z} \in \mathcal{F}_\ell$ holds. We analyze separately the case $A \subset Q_L$ (a) and $A \not\subset Q_L$ (b). (a) We let $y = x + \vec{e}_1$ and $A = A_1 = x + \vec{e}_2, x + \vec{e}_1 + \vec{e}_2, x + 2\vec{e}_1$ (the other cases can be treated analogously). By subsequently applying Claim 9.2 we can construct an allowed path $\omega, \dots \omega^{(M)}$ inside $Q^x \cup Q^y \cup_{z \in A} Q^z$ such that $\omega^{(M)} = T_e(\omega)$ and $M(\omega) \leqslant C(\ell)$ uniformly in ω. Using a telescopic sum and the Cauchy-Schwartz inequality , we get

$$\begin{aligned}
\mu_{Q_{\tilde{L}}}\left(\widetilde{c}_e (f \circ T_e - f)^2\right) &= \sum_\omega \mu_{Q_{\tilde{L}}}(\omega) \widetilde{c}_e(\omega) \left(\sum_{m=1}^{M-1} f(\omega^{(m+1)}) - f(\omega^{(m)})\right)^2 \\
&\leqslant C \sum_\omega \mu_{Q_{\tilde{L}}}(\omega) \sum_{m=1}^{M-1} c_{e_m, Q_{\tilde{L}}}(\omega^{(m)}) \left(f(\omega^{(m+1)}) - f(\omega^{(m)})\right)^2 \\
&\leqslant C \sum_\omega \mu_{Q_{\tilde{L}}}(\omega) \sum_{\sigma, e'} c_{e', Q_{\tilde{L}}}(\sigma) \left(\nabla_{e'} f\right)^2 (\sigma) \mathbf{1}_{\{(\sigma, e') : (\sigma, \sigma^{e'}) \in P_{\omega, T_e(\omega)}\}} \\
&\leqslant C \sum_{e' \in E(e)} \mu_{Q_{\tilde{L}}}\left(c_{e', Q_{\tilde{L}}} (\nabla_{e'} f)^2\right),
\end{aligned}$$

where as usual, $(\sigma, \sigma^{e'}) \in P_{\omega, T_e(\omega)}$ means that there exists m such that $\sigma = \omega^{(m)}$ and $\sigma^{e'} = \omega^{(m+1)}$. In the last line we inverted the summations, used the fact that any $\sigma \in P_{\omega, T_e(\omega)}$ satisfies $\mu_{Q_{\tilde{L}}}(\sigma) = \mu_{Q_{\tilde{L}}}(\omega)$ and differs from ω on at most $9\ell^2$ sites. (b) Assume that $y = x + \vec{e}_2$, $A = A_1 = x + \vec{e}_1, x + \vec{e}_1 + \vec{e}_2, x + 2\vec{e}_2$ and that $x + \vec{e}_1, y + \vec{e}_1 \notin Q_L$ (i.e. $Q^{x+\vec{e}_1}, Q^{y+\vec{e}_1} \notin Q_{\tilde{L}}$) and $x + 2\vec{e}_2 \in Q_L$ (i.e. $Q^{x+\vec{e}_2} \in Q_{\tilde{L}}$) as in Fig. 4 (the other cases can be treated analogously). Thanks to the presence of sources on $\overline{\partial}_- Q_{\tilde{L}}$ we can create zeros on $\overline{\partial}^1_-(Q^x \cup Q^y)$. Indeed, if $z^{(1)}, \dots, z^{(m_1)}$ denote the sites inside $\overline{\partial}^1_-(Q^x \cup Q^y)$ for which $\omega(z^{(i)}) = 1$, $i = 1, \dots, m_1$, enumerated from top to bottom, the path $(\omega^{(1)}, \dots, \omega^{(m_1)})$ with $\omega^{(1)} = \omega$ and $\omega^{(m_{i+1})} = \left(\omega^{(m_i)}\right)^{z^{(i)}}$ is allowed. Furthermore $\omega^{(m_1)}(u) = 0$ for all $u \in \overline{\partial}^1_-(Q^x \cup Q^y)$ and $\omega^{(m_1)}(u) = \omega(u)$ for $u \notin \overline{\partial}^1_-(Q^x \cup Q^y)$. Since $\omega_{Q^{x+2\vec{e}_2}} \in \mathcal{F}_\ell$ (and thus $\omega^{(m_1)}_{Q^{x+2\vec{e}_2}} \in \mathcal{F}_\ell$), there exists an allowed path $(\omega^{(m_1)}, \dots, \omega^{(m_2)})$ inside $Q^{x+2\vec{e}_2}$ such that $\omega^{(m_2)}$ is $Q^{x+2\vec{e}_2}$-framed (see Fig. 4). Claim b guarantees the existence of a sequence of allowed exchanges $(\omega^{(m_2)}, \dots, \omega^{(m_3)})$ inside $Q^{x+2\vec{e}_2}$ which bring the empty upper line of this frame adjacent to the bottom empty line. Then it is easy to verify that we can rigidly shift the double empty line downwards thanks to the presence of the empty sites on $\overline{\partial}^1_-(Q^x \cup Q^y)$. Therefore, for any chosen bond $e' = x, x + \vec{e}_1 \in Q^x \cup Q^y$

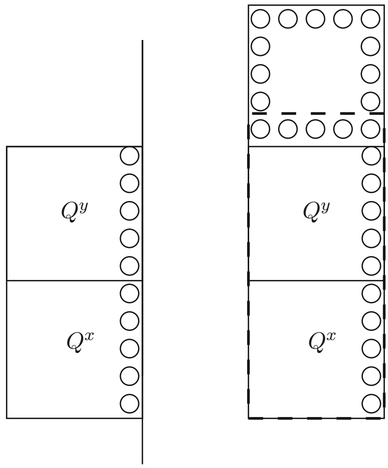

Fig. 4. Proof of Lemma 6.1(i), case (b). The configurations $\omega^{(m_1)}$ on the left and $\omega^{(m_2)}$ on the right. The dashed line corresponds to the rectangle R

we can shift the double empty line till the position $x\cdot\vec{e}_2+1$ and then perform the exchange. The case $e' = x, x+\vec{e}_2$ can be dealt by first creating a second empty column adjacent to the one on $\overline{\partial}_-^1(Q^x \cup Q^y)$ (via exchanges plus source terms) and then shifting horizontally this double empty column till the position $x \cdot \vec{e}_1 + 1$. In conclusion, since we can perform any internal exchange in $Q^x \cup Q^y$, we can construct a path $(\omega^{(m_3)}, \ldots, \omega^{(m_4)})$ with $\omega^{(m_4)} = T_e(\omega^{(m_3)})$. As before, we can now reconstruct the initial configuration ω outside $Q^x \cup Q^y$ and then also on $\overline{\partial}_-^1(Q^x \cup Q^y)$ by using the sources again. Thus we have constructed an allowed path $P_{\omega,T_e\omega} = (\omega^{(1)}, \ldots, \omega^{(M)})$ inside $Q^x \cup Q^y \cup Q^{x+2\vec{e}_2}$ with $M(\omega) \leqslant C(\ell)$ uniformly in ω. Using that $\mu_{Q_L}(\omega) \leqslant C\mu_{Q_L}(\omega^{(m)})$ for any m, the same kind of computation as before (telescopic sum, Cauchy-Schwartz inequality, inverting the summations, explicit counting...) leads to

$$\mu_{Q_{\tilde{L}}}\left(\widetilde{c}_e(f\circ T_e - f)^2\right) = \sum_{\omega} \mu_{Q_{\tilde{L}}}(\omega)\widetilde{c}_e(\omega)\left(\sum_{m=1}^{M-1} f(\omega^{(m+1)}) - f(\omega^{(m)})\right)^2$$
$$\leqslant C \sum_{e'\in E(e)} \mu_{Q_{\tilde{L}}}\left(c_{e',Q_{\tilde{L}}}(\nabla_{e'} f)^2\right) + C \sum_{z\in\overline{\partial}_-^1(Q^x\cup Q^y)\cap\overline{\partial}_- Q_L} \mu_{Q_{\tilde{L}}}\left(\mathrm{Var}_z(f)\right).$$

(ii) Let $x \in \overline{\partial}_- Q_L$ and ω be such that $\widetilde{c}_x(\omega) = 1$. Assume that $x+\vec{e}_1 \notin Q_L$ and $z = x+\vec{e}_2 \in Q_L$ as in Fig. 5 (the other cases are similar). Using the Poincaré inequality for the unconstrained Glauber dynamics inside Q^x (or the tensorisation property, see e.g. [2]) leads to

$$\mathrm{Var}_{Q^x}(f) \leqslant pq \sum_{y\in Q^x} \mu_{Q^x}\left((\nabla_y f)^2\right).$$

Thus we have to estimate terms of the form $f(\omega^y) - f(\omega)$. This will be done by constructing a proper allowed path which is depicted in Fig. 5. Assume that $\omega(y) = 1$ (the case $\omega(y) = 0$ can be treated analogously). Since $\widetilde{c}_x(\omega) = 1$, Q^z is frameable.

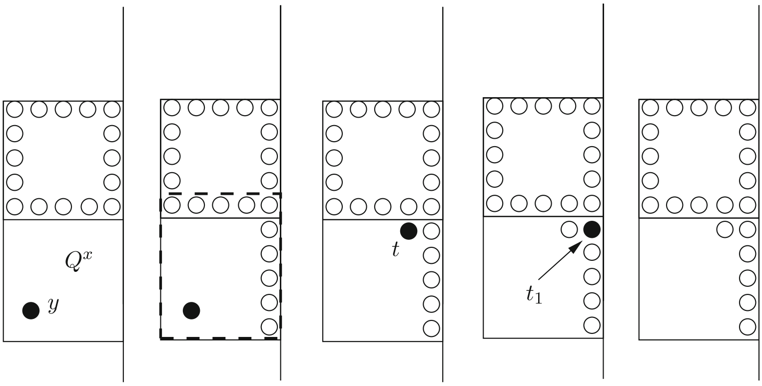

Fig. 5. From left to right, the configurations $\omega^{(m_1)}$, $\omega^{(m_2)}$, $\omega^{(m_3)}$, $\omega^{(m_3+1)}$ and $\omega^{(m_3+2)}$. The dotted line corresponds to the rectangle R

Hence, there exists an allowed path $(\omega^{(1)}, \ldots, \omega^{(m_1)})$ inside Q^z s.t. $\omega^{(1)} = \omega$ and $\omega^{(m_1)}$ is Q^z-framed. Thanks to the sources on $\overline{\partial}_-^1 Q^x$, we can create zeros on $\overline{\partial}_-^1 Q^x$. Indeed, if $z^{(m_1+1)}, \ldots, z^{(m_2)}$ denote the sites inside $\overline{\partial}_-^1 Q^x$ for which $\omega(z^{(m)}) = 1$, enumerated from top to bottom, the path $(\omega^{(m_1)}, \ldots, \omega^{(m_2)})$ with $\omega^{(m_1+i)} = \left(\omega^{(m_1)}\right)^{z^{(i)}}$ is allowed. Furthermore $\omega^{(m_2)}(z) = 0$ for all $z \in \overline{\partial}_-^1 Q^x$ and $\omega^{(m_2)}(z) = \omega(z)$ for $z \notin \overline{\partial}_-^1 (Q^x \cup Q^y)$. Set $R := Q^x \cup \overline{\partial}_+^2 Q^x$ and let t be the unique site such that $\mathcal{K}_t^* \subset \overline{\partial}_- R$ and $t_1 = t + e_1$. Suppose $\omega(t) = 0$ (the case $\omega(t) = 1$ can be treated analogously). Then we can shift the top empty line of the frame of Q^z near the bottom one and, by using this double empty line plus the empty column on $\overline{\partial}_-^1 Q^x$ we can perform any exchange inside Q^x analogously to what we did in point (i). In particular we can construct a path $\omega^{(m_2)}, \ldots \omega^{(m_3)}$ which moves the particle from y to t and then perform the exchange on t, t_1 and use the source to force on t_1 an empty site. In other words the path $(\omega^{(m_3)}, \omega^{(m_3+1)}, \omega^{(m_3+2)})$ with $\omega^{(m_3+1)} = \left(\omega^{(m_3)}\right)^{t,t_1}$ and $\omega^{(m_3+2)} = \omega^{(m_3+1)}_{Q_{\tilde{L}} \setminus t_1} \cdot 0_{t_1}$ is allowed (see Fig. 5). Thus we have reached a configuration which corresponds to ω^y on all site in $Q^x \setminus \overline{\partial}_-^1 Q^x$. Then we can reconstruct the configuration ω^y following the inverse of the path $(\omega, \omega^{(m_2)})$. Thus we have shown the existence of an allowed path $P_{\omega,\omega^y} = (\omega^{(1)}, \ldots, \omega^{(M)})$ with $M(\omega) \leqslant C(\ell)$ uniformly in ω. Using the same routine arguments (Cauchy-Schwartz inequality, telescopic sum,…) as in point (i) leads to the expected result. $\square$

7. Spectral Gap of AGL: Proof of Theorem 5.5

In this section we prove Theorem 5.5 which establishes the positivity of the spectral gap for the auxiliary model AGL. This result has in turn been used as a key ingredient for the proof of the $1/L^2$ lower bound for the spectral gap of KA and will be also used in Sect. 8 to prove the polynomial decay to equilibrium. The main tool here is an extension of the *bisection-constrained method* we introduced in [9] which we have here properly modified to account for the long range constraints of AGL. As a result the proof is quite technical and lengthy due to easy but cumbersome geometric results. These are necessary to establish the existence of the paths which guarantee that the long range constraints of AGL are satisfied.

We start by a monotonicity remark

Remark 7.1. Fix $\Lambda, \tilde{\Lambda} \subset \mathbb{Z}^2$ and $N, N' > 0$ with $N \leqslant N'$ and $\Lambda \subset \tilde{\Lambda}$. For any $x \in \Lambda$ and for all ω:

(i) $c^N_{x,\Lambda}(\omega) \leqslant c^{N'}_{x,\Lambda}(\omega)$;
(ii) if $\overline{\partial}_- \Lambda \subset \overline{\partial}_- \tilde{\Lambda}$, then $c^N_{x,\Lambda}(\omega) \leqslant c^N_{x,\tilde{\Lambda}}(\omega)$.

Proof of Theorem 5.5. In what follows we will drop the superscript *agl* from the generator. Thanks to the monotonicity of the spectral gap established by Lemma 2.12 and to the property of the rates in Remark 7.1 (i), it is enough to prove the result when $N = A(\log(\max(L_1, L_2))^2$. We start by recalling a simple geometric result of [5] which we will use. Let $l_k := (3/2)^{k/2}$, and let $\mathbb{F}_k$ be the set of all rectangles $\Lambda \subset \mathbb{Z}^2$ which, modulo translations and permutations of the coordinates, are contained in $[0, l_{k+1}] \times [0, l_{k+2}]$. The main property of $\mathbb{F}_k$ is that each rectangle in $\mathbb{F}_k \backslash \mathbb{F}_{k-1}$ can be obtained as a “slightly overlapping union” of two rectangles in $\mathbb{F}_{k-1}$. More precisely we have:

Lemma 7.2 ([5], Prop. 3.2). *For all $k \in \mathbb{Z}_+$, for all $\Lambda \in \mathbb{F}_k \backslash \mathbb{F}_{k-1}$ there exists a finite sequence $\{\Lambda^{(i)}_1, \Lambda^{(i)}_2\}^{s_k}_{i=1}$ in $\mathbb{F}_{k-1}$, where $s_k := \lfloor l_k^{1/3} \rfloor$, such that, letting $\delta_k := \frac{1}{8}\sqrt{l_k} - 2$,*

(i) $\Lambda = \Lambda^{(i)}_1 \cup \Lambda^{(i)}_2$,
(ii) $d(\Lambda \backslash \Lambda^{(i)}_1, \Lambda \backslash \Lambda^{(i)}_2) \geq \delta_k$,
(iii) $\left(\Lambda^{(i)}_1 \cap \Lambda^{(i)}_2\right) \cap \left(\Lambda^{(j)}_1 \cap \Lambda^{(j)}_2\right) = \emptyset$, *if $i \neq j$.*

Let $\bar{k}$ be such that $\Lambda \in \mathbb{F}_{\bar{k}} \backslash \mathbb{F}_{\bar{k}-1}$. Then $\max(L_1, L_2) \geqslant l_{\bar{k}-1+1} = l_{\bar{k}}$, thus $N = A(\log(\max(L_1, L_2)))^2 \geqslant N_{\bar{k}}$, where $N_k := A(\log l_k)^2$. By using again the monotonicity properties of Lemma 2.12 and Remark 7.1 (i) we immediately get

$$\mathrm{gap}(\mathcal{L}_{\Lambda,N})^{-1} \leqslant \mathrm{gap}(\mathcal{L}_{\Lambda,N_{\bar{k}}})^{-1} \leqslant \gamma_{\bar{k}} \leqslant \sup_k \gamma_k,$$

where we define

$$\gamma_k := \sup_{k'=1,\ldots,k} \left(\sup_{\Lambda \in \mathbb{F}_{k'} \backslash \mathbb{F}_{k'-1}} \mathrm{gap}(\mathcal{L}_{\Lambda,N_{k'}})^{-1} \right). \tag{7.1}$$

Therefore, to prove the theorem, it is enough to show that there exist $\rho_1 \in (0, 1)$ and $A > 0$ independent of W and ν such that for any $\rho > \rho_1$,

$$\sup_k \gamma_k \leqslant 2. \tag{7.2}$$

The strategy to prove (7.2) will be to establish a proper iterative inequality between γ_k and γ_{k-1}. Let us fix k, $\Lambda \in \mathbb{F}_k \backslash \mathbb{F}_{k-1}$, and let $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_1, \Lambda_2 \in \mathbb{F}_{k-1}$ satisfying the properties described in Lemma 7.2 above. Without loss of generality we can assume that the faces of Λ_1 and of Λ_2 parallel to $\vec{e}_1$ lay on the faces of Λ and that, along that direction, Λ_1 comes before Λ_2 (see Fig. 6). Set $\widetilde{I} \equiv \Lambda_1 \cap \Lambda_2$ and write, for concreteness, $\widetilde{I} = [\tilde{a}_1, \tilde{b}_1] \times [\tilde{a}_2, \tilde{b}_2]$. Lemma 7.2 implies that the width of $\widetilde{I}$ in the first direction, $\tilde{b}_1 - \tilde{a}_1$, is at least δ_k. Set $B_1 = \Lambda \backslash \Lambda_2$, $B_2 = \Lambda_2$ and $I = [a_1, b_1] \times [a_2, b_2]$ with $a_1 = \tilde{a}_1$, $b_1 = \tilde{a}_1 + \frac{A}{2} \log(3/2) \log(l_k)$, $a_2 = \tilde{a}_2$ and $b_2 = \tilde{b}_2$. Notice that our choice implies $b_1 - a_1 \leq N_k - N_{k-1}$. We also assume that k is sufficiently large so that $\delta_k \geqslant b_1 - a_1$. Then the following geometric properties can be immediately verified.

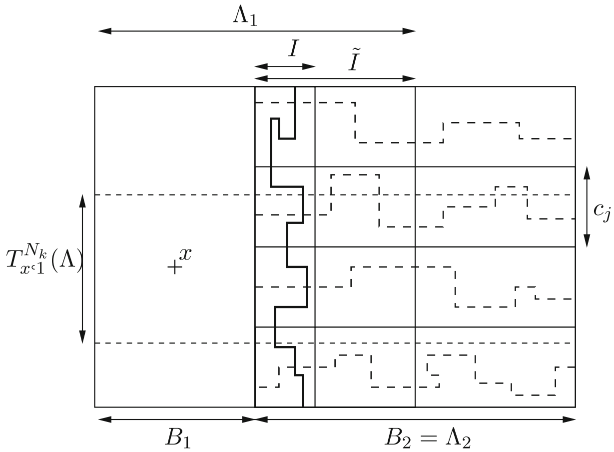

Fig. 6. We depict the regions Λ_1, $\Lambda_2 = B_2$, $\Lambda = \Lambda_1 \cup \Lambda_2$, $\tilde{I} = \Lambda_1 \cap \Lambda_2$, $I \subset \tilde{I}$, and the cylinder $T^{N_k}_{x,1}(\Lambda)$ (inside the dotted-dashed line). The set Λ_2 is divided into rectangles of width c_j, $\Lambda_2 = \cup_{i=1}^m R_j$ (here $m = 4$). The dashed lines inside the R_j's stand for the paths which are good left-right crossings and thus guarantee that $\omega \in \cap_{i=1}^m \mathcal{R}_i$. The bold dashed line inside I is a good top-bottom crossing which guarantees $\omega \in \mathcal{I}$

Claim 7.3. (i) $I \subset \widetilde{I}$;
(ii) *If* $x \in I$, *then* $I \subset T^{N_k}_{x,2}$;
(iii) *If* x *is such that* $I \cap T^{N_{k-1}}_{x,2} \neq \emptyset$, *then* $I \cup T^{N_{k-1}}_{x,2} \subset T^{N_k}_{x,2}$.

We will now define a constrained block dynamics on Λ with blocks B_1 and B_2 and prove that it has a positive spectral gap. Then from its Dirichlet form we will reconstruct the Dirichlet form of AGL (5.5) and establish the desired recursive inequality between γ_k and γ_{k-1}. To this purpose, in analogy with the strategy adopted in [9], we have to define a proper good event on the block B_2 which should occur in order to allow refreshing of the configuration on B_1. Recall Definition 2.6 and define the event

$$\mathcal{I} := \{\omega : \omega \text{ has a good top-bottom crossing in } I\}.$$

In analogy with what is done in [14, see Proof of Lemma (11.73)], we define the following natural partial order on the set of top-bottom crossing paths in I. We say that γ is to the right of γ' if it lies inside the connected (with respect to d_1 distance) region of I which stays to the right of γ'. For any $\omega \in \mathcal{I}$ we can then define the geometric set Π_ω which is its right-most good top-bottom crossing and let $\overline{\Pi}_\omega$ be the corresponding double-path (see Fig. 8). Finally we set

$$\mathcal{C}_\Pi := \{\Gamma \subset B_2 : \exists\, \omega \text{ s.}t.\ \overline{\Pi}_\omega = \Gamma\}.$$

By geometrical considerations we have the following.

Claim 7.4. *For all* $\omega \in \mathcal{I}$,

(i) *there exists a unique right-most good top-bottom crossing* Π_ω *of* ω *in* I;
(ii) *for any* $\Gamma \in \mathcal{C}_\Pi$ *the event* $\{\omega : \overline{\Pi}_\omega = \Gamma\}$ *does not depend on the values of* ω *to the left of* Γ.

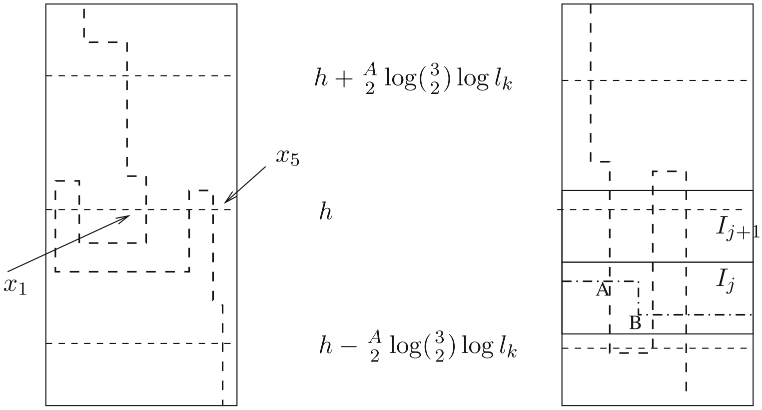

Fig. 7. Left: An example of top bottom crossing with k-bounded oscillation inside the stripe I. Right: The bold dashed line is a good top-bottom crossing γ which does not have k-bounded oscillations: at h there is an oscillation which goes beyond $h - A/2\log(3/2)\log l_k$. The dotted-dashed line is a good left-right bottom crossing of the rectangle I_j. Note that γ cannot be the rightmost good top-bottom crossing, $\gamma \neq \Pi_\omega$. Indeed a good top-bottom crossing to the right of γ can be constructed by using γ and the part of the good left-right crossing from site A to B

Next we need to introduce the notion of paths which "do not oscillate too much" in the vertical direction, namely

Definition 7.5 (*Path with k-bounded oscillation*). *Consider a geometric path $\gamma_{xy} = (x^{(1)} = x, \ldots, x^{(n)} = y)$. Let*

$$\underline{x}_1 = min\{x_1^{(1)}, \ldots, x_1^{(n)}\}, \quad \overline{x}_1 = max\{x_1^{(1)}, \ldots, x_1^{(n)}\},$$
$$\underline{x}_2 = min\{x_2^{(1)}, \ldots, x_2^{(n)}\}, \quad \overline{x}_2 = max\{x_2^{(1)}, \ldots, x_2^{(n)}\}.$$

We say that γ_{xy} has k-bounded oscillations if for all $h \in [\underline{x}_2, \overline{x}_2]$ the following h-condition holds. Let $1 \le i_1 < i_2 < \cdots < i_m \le n$ be the indexes of the points of $\gamma_{x,y}$ with height h, namely $\{x^{(i_1)}, x^{(i_2)}, \ldots, x^{(i_m)}\} := \gamma_{xy} \cap ([\underline{x}_1, \overline{x}_1] \times \{h\})$. Then the h-condition (depicted in Fig. 7) requires that for all $i = i_1, i_1 + 1, i_1 + 2, \ldots, i_m$,

$$x^{(i)} \in [\underline{x}_1, \overline{x}_1] \times [h - A/2\log(3/2)\log l_k, h + A/2\log(3/2)\log l_k].$$

With this notation we can define the event

$$\mathcal{J} := \{\omega \in \mathcal{I} : \Pi_\omega \text{ has } k\text{-bounded oscillations}\}$$

and the geometric set $\widetilde{\mathcal{C}}_\Pi \subset \mathcal{C}_\Pi$ as

$$\widetilde{\mathcal{C}}_\Pi := \{\Gamma \subset B_2 : \exists\, \omega \text{ s.}t.\ \overline{\Pi}_\omega = \Gamma \text{ and } \Pi_\omega \text{ has } k\text{-bounded oscillations}\}.$$

Lemma 7.2 guarantees that $B_2 = \Lambda_2 \in \mathbb{F}_{k-1}$. Thus there exist integers a, b which are bounded from above by l_{k+1} such that Λ_2 is a translated copy of $[0, a] \times [0, b]$. If $b \geqslant N_{k-1}/2$ we decompose Λ_2 into m disjoint rectangles $\Lambda_2 = \cup_{j=1}^m R_j$ (see Fig. 6) with each R_j being a translation of $[0, a] \times [0, c_j]$, where c_j satisfies $N_{k-1} \leqslant 2c_j \leqslant 2N_{k-1} - 1$ for any $j = 1, \ldots, m$ and m verifies $m \leqslant (2l_{k+1}/N_{k-1})$ (the bounds $N_{k-1}/2 < b \leqslant l_{k+1}$ guarantee that we can perform such a procedure). Thanks to the bounds on c_j the following property can then be immediately verified:

Claim 7.6. *For any $x \in \Lambda$ there exists $j \in 1, \dots, m$ such that $R_j \subset T_{x,1}^{N_{k-1}}(\Lambda_2)$.*

We are now ready to define the good event on the block B_2 which will enter in the definition of the block dynamics.

Definition 7.7 (B_2*-good configurations*). *Call b the size of B_2 in direction* 2.
If $b < N_{k-1}/2$ we say that ω is B_2-good iff $\omega \in \mathcal{I}$.
If $b \geqslant N_{k-1}/2$ we say that ω is B_2-good iff $\omega \in \cap_{j=1}^m \mathcal{R}_j \cap \mathcal{J}$, where
$\mathcal{R}_j := \{\omega : \omega$ has a good left-right crossing in $R_j\}$ (see Fig. 6).

The block dynamics, which is again defined on $\Omega = W^{\mathbb{Z}^2}$ and reversible w.r.t. $\mu = \prod_{x\in\mathbb{Z}^2} \nu_x$, is then defined as follows. The block B_2 waits a mean one exponential random time and then its current configuration is refreshed with a new one sampled from μ_{B_2}. The block B_1 does the same but now the configuration is refreshed *only* if the current configuration ω is B_2-good. Thus the generator of this auxiliary chain acts on local functions as

$$\mathcal{L}_{block} f(\omega) = c_1(\omega) \left(\mu_{B_1}(f) - f(\omega)\right) + \mu_{B_2}(f) - f(\omega), \tag{7.3}$$

where c_1 is the characteristic function of the event that ω is B_2-good, namely

$$c_1(\omega) := \begin{cases} 1\!\!1_{\mathcal{J}}(\omega) \prod_{j=1}^m 1\!\!1_{\mathcal{R}_j}(\omega) & \text{if } b \geqslant N_{k-1}/2 \\ 1\!\!1_{\mathcal{I}} & \text{if } b < N_{k-1}/2 \end{cases}, \tag{7.4}$$

where we recall that b is the vertical size of Λ_2 (and of Λ). The Dirichlet form associated to (7.3) is

$$\mathcal{D}_{block}(f) = \mu_\Lambda \left(c_1 \operatorname{Var}_{B_1}(f) + \operatorname{Var}_{B_2}(f)\right). \tag{7.5}$$

Denote by $\gamma_{\text{block}}(\Lambda)$ the inverse spectral gap of $\mathcal{L}_{block}$. The following bound, whose proof relies on the fact that $c_1(\omega)$ depends only on ω_{B_2}, can be proven as in [9, Prop. 4.4].

Proposition 7.8. *Let $\varepsilon_k \equiv \max_I \mathbb{P}(\omega$ is not B_2-good$)$, where the $\max_I$ is taken over the s_k possible choices of the pair (Λ_1, Λ_2). Then*

$$\gamma_{\text{block}}(\Lambda) \le \frac{1}{1-\sqrt{\varepsilon_k}}.$$

Thus, by using the standard Poincaré inequality for the block auxiliary chain and Proposition 7.8 as well as (7.5), we get that for any $f : \Omega_\Lambda \mapsto \mathbb{R}$,

$$\operatorname{Var}_\Lambda(f) \le \left(\frac{1}{1-\sqrt{\varepsilon_k}}\right) \mu_\Lambda \left(c_1 \operatorname{Var}_{B_1}(f) + \operatorname{Var}_{B_2}(f)\right). \tag{7.6}$$

We will now reconstruct the Dirichlet form of AGL (5.5) from the two terms on the right hand side of (7.6). Let us start with the second term. By construction, there exists $k' \in \{1, \dots, k-1\}$ such that $B_2 \in \mathbb{F}_{k'} \setminus \mathbb{F}_{k'-1}$. Thus, using the definition (7.1) for γ_{k-1}, we have

$$\mu_\Lambda \left(\operatorname{Var}_{B_2}(f)\right) \le \gamma_{k-1} \sum_{x\in B_2} \mu_\Lambda \left(c_{x,B_2}^{N_{k'}} \operatorname{Var}_x(f)\right).$$

By using both monotonicity properties stated in Remark 7.1, it is immediate to verify the following property

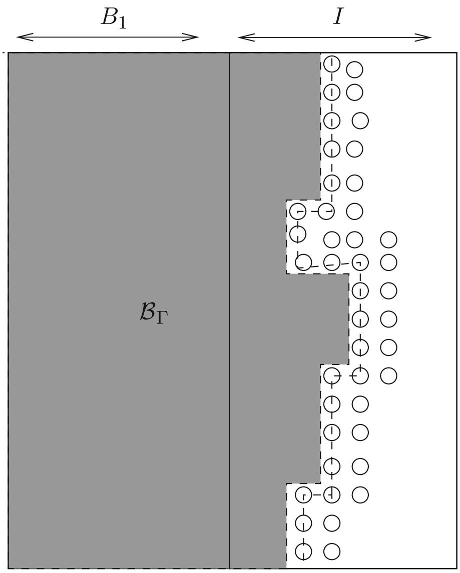

Fig. 8. The volume B_1, the stripe I and the rightmost good top bottom crossing Π_ω (empty sites within the dashed line). The whole set of empty sites is instead $\overline{\Pi}_\omega$. The set $\mathcal{B}_\omega$ is the region which lies inside the shaded region

Claim 7.9. *For all $x \in B_2$, all ω and all $k' \leqslant k-1$, $c^{N_{k'}}_{x,B_2}(\omega) \leqslant c^{N_k}_{x,\Lambda}(\omega)$.*

Therefore we have

$$\mu_\Lambda\left(\mathrm{Var}_{B_2}(f)\right) \le \gamma_{k-1} \sum_{x\in B_2} \mu_\Lambda\left(c^{N_k}_{x,\Lambda}\,\mathrm{Var}_x(f)\right). \tag{7.7}$$

The r.h.s. of the latter is nothing but the contribution carried by the set B_2 to the full Dirichlet form (5.5) when $N = N_k$.

Let us now examine the more complicated term $\mu_\Lambda\left(c_1\,\mathrm{Var}_{B_1}(f)\right)$.

For any B_2-good configuration ω recall that Π_ω is the right-most good top-bottom crossing of ω in I. Then divide $B_1 \cup I$ into two connected components (with respect to the distance d_1): the sites on the right of $\overline{\Pi}_\omega$, and those on the left. We shall call $\mathcal{B}_\omega$ the sites of $B_1 \cup I \setminus \overline{\Pi}_\omega$ on the left of $\overline{\Pi}_\omega$ (see Fig. 8). Notice that if ω and ω' are B_2-good and $\Pi_\omega = \Pi_{\omega'}$, then $\mathcal{B}_\omega = \mathcal{B}'_\omega$. In other words $\mathcal{B}_\omega$ is unequivocally defined by Π_ω. Thus, with a slight abuse of notation, for any $\Gamma \in \mathcal{C}_\Pi$ we let $\mathcal{B}_\Gamma$ be the $\mathcal{B}_\omega$ which corresponds to all ω with $\overline{\Pi}_\omega = \Gamma$.

If we observe that $\mathrm{Var}_{B_1}(f)$ and $c_1(\omega)$ depend only on ω_{B_2}, we use the independence of $\mathbb{1}_{\{\overline{\Pi}_\omega=\Gamma\}}$ from ω_{I_Γ} (Claim 7.4 (ii)) and let $I_\Gamma := \mathcal{B}_\Gamma \cap I$ we can write

$$\begin{aligned}\mu\left(c_1\,\mathrm{Var}_{B_1}(f)\right) &= \sum_{\Gamma\in\widetilde{\mathcal{C}}_\Pi} \mu\left(\mathbb{1}_{\{\overline{\Pi}_\omega=\Gamma\}}\mathbb{1}_{\cap_{j=1}^m \mathcal{R}_j}\,\mathrm{Var}_{B_1}(f)\right)\\ &= \sum_{\Gamma\in\widetilde{\mathcal{C}}_\Pi}\sum_{\omega_{B_2\setminus I}} \mu(\omega_{B_2\setminus I}) \sum_{\omega_I} \mu(\omega_I)\,\mathbb{1}_{\{\overline{\Pi}_\omega=\Gamma\}}\mathbb{1}_{\cap_{j=1}^m \mathcal{R}_j}\,\mathrm{Var}_{B_1}(f) \qquad (7.8)\\ &= \sum_{\Gamma\in\widetilde{\mathcal{C}}_\Pi}\sum_{\omega_{B_2\setminus I}} \mu(\omega_{B_2\setminus I}) \sum_{\omega_{I\setminus I_\Gamma}} \mu(\omega_{I\setminus I_\Gamma})\,\mathbb{1}_{\{\overline{\Pi}_\omega=\Gamma\}}\mathbb{1}_{\cap_{j=1}^m \mathcal{R}_j} \sum_{\omega_{I_\Gamma}} \mu(\omega_{I_\Gamma})\,\mathrm{Var}_{B_1}(f)\end{aligned}$$

when $b \geqslant N_{k-1}/2$ and

$$\begin{aligned}\mu\left(c_1 \operatorname{Var}_{B_1}(f)\right) &= \sum_{\Gamma\in\mathcal{C}_\Pi} \mu\left(\mathbb{1}_{\{\overline{\Pi}_\omega=\Gamma\}} \operatorname{Var}_{B_1}(f)\right) \\ &= \sum_{\Gamma\in\mathcal{C}_\Pi} \sum_{\omega_{B_2\setminus I}} \mu(\omega_{B_2\setminus I}) \sum_{\omega_I} \mu(\omega_I) \mathbb{1}_{\{\overline{\Pi}_\omega=\Gamma\}} \operatorname{Var}_{B_1}(f) \\ &= \sum_{\Gamma\in\mathcal{C}_\Pi} \sum_{\omega_{B_2\setminus I}} \mu(\omega_{B_2\setminus I}) \sum_{\omega_{I\setminus I_\Gamma}} \mu(\omega_{I\setminus I_\Gamma}) \mathbb{1}_{\{\overline{\Pi}_\omega=\Gamma\}} \sum_{\omega_{I_\Gamma}} \mu(\omega_{I_\Gamma}) \operatorname{Var}_{B_1}(f)\end{aligned} \tag{7.9}$$

when $b < N_{k-1}/2$. Then we can in both cases upper bound the last term by using the convexity of the variance which implies

$$\sum_{\omega_{I_\Gamma}} \nu(\omega_{I_\Gamma}) \operatorname{Var}_{B_1}(f) \le \operatorname{Var}_{\mathcal{B}_\Gamma}(f). \tag{7.10}$$

Now for any $x \in \mathcal{B}_\Gamma$ and any $\omega_{\mathcal{B}_\Gamma} \in \Omega_{\mathcal{B}_\Gamma}$, let $\overline{c}_x(\omega_{\mathcal{B}_\Gamma})$ be the indicator function of the event that there exists a direction $i \in \{1, 2\}$ and a geometric path $\gamma_{x,y}$ inside $T_{x,i}^{N_{k-1}}(\mathcal{B}_\Gamma)$ from x to some $y \in \overline{\partial}_-\mathcal{B}_\Gamma \cup (\partial_+\Gamma \cap \mathcal{B}_\Gamma)$ which is allowed for the AKG model if $\tau(t) \in G$ for any $t \in \overline{\partial}_+^*\mathcal{B}_\Gamma$ (see Fig. 9). It is then possible to define the generator $\overline{\mathcal{L}}_{\mathcal{B}_\Gamma}$ obtained from $\mathcal{L}_{\Lambda,N_k}$ by substituting the rates $c_{x,\Lambda}^{N_k}$ with $\overline{c}_x$. The generator $\overline{\mathcal{L}}_{\mathcal{B}_\Gamma}$ with Dirichlet form $\overline{\mathcal{D}}_{\mathcal{B}_\Gamma}$ is ergodic and reversible with respect to $\mu_{\mathcal{B}_\Gamma}$. Denote by $\operatorname{gap}(\overline{\mathcal{L}}_{\mathcal{B}_\Gamma})$ the associated spectral gap. Applying the Poincaré inequality leads to

$$\operatorname{Var}_{\mathcal{B}_\Gamma}(f) \le \operatorname{gap}(\overline{\mathcal{L}}_{\mathcal{B}_\Gamma})^{-1} \sum_{x\in\mathcal{B}_\Gamma} \mu_{\mathcal{B}_\Gamma}\left(\overline{c}_x \operatorname{Var}_x(f)\right). \tag{7.11}$$

Claim 7.10. *For any ω and any $x \in \mathcal{B}_\Gamma$,*

$$c_{x,B_1\cup I}^{N_{k-1}}(\omega) \leqslant \overline{c}_x(\omega).$$

Proof. The result follows immediately from the definition of $\bar{c}$ (see Fig. 9). □

Claim 7.11.

$$\operatorname{gap}(\overline{\mathcal{L}}_{\mathcal{B}_\Gamma})^{-1} \le \operatorname{gap}(\mathcal{L}_{B_1\cup I,N_{k-1}})^{-1} \le \gamma_{k-1}.$$

Proof. For any $f \in \mathbb{L}_2(\Omega_{\mathcal{B}_\Gamma}, \mu_{\mathcal{B}_\Gamma})$ we have $\mathcal{D}_{B_1\cup I}^N(f) \le \mathcal{D}_{\mathcal{B}_\Gamma}^N(f)$ and $\operatorname{Var}_{\mathcal{B}_\Gamma}(f) = \operatorname{Var}_{B_1\cup I}(f)$. The first property follows by using Claim 7.10. The second property follows from the product structure of the measure $\mu_{B_1\cup I}$. The first inequality of the claim then follows at once from the variational characterization of the spectral gap. Furthermore, since $B_1 \cup I \in \mathbb{F}_{k-1}$ there should exist $k' \in \{1, \ldots, k-1\}$ such that $B_1 \cup I \in \mathbb{F}_{k'}\setminus\mathbb{F}_{k'-1}$. Thus since $N_{k'} \leqslant N_{k-1}$ from the monotonicity Remark 7.1(i) we get $\operatorname{gap}(\mathcal{L}_{B_1\cup I,N_{k-1}})^{-1} \leqslant \operatorname{gap}(\mathcal{L}_{B_1\cup I,N_{k'}})^{-1}$. We can now use the definition (7.1) of γ_{k-1} to get the second inequality. □

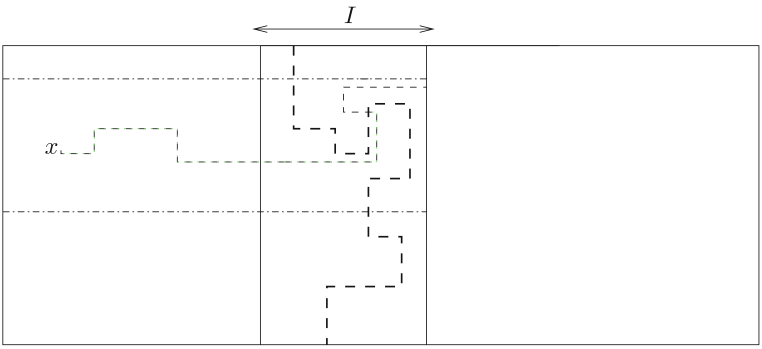

Fig. 9. Inside the dotted-dashed line we depict the cylinder $T^{N_{k-1}}_{x,B_1\cup I}$. The bold dashed line represents Γ. The light dashed line is instead a path belonging to $\mathcal{G}_{x,N_{k-1},B_1\cup I}$. It is immediate to verify that this implies the existence of an AKG allowed path inside $\mathcal{B}_\Gamma$ from x to a site $y \in \partial_+\Gamma \cap \mathcal{B}_\Gamma$, thus $\bar{c}_x = 1$. From the drawings it is also clear that it *does not* necessarily imply a path which ends at $y \in \partial^*_-\mathcal{B}_\Gamma$, hence the necessity to introduce the additional rates $\bar{c}_x$ in $\mathcal{B}_\Gamma$ instead of using $c^{N_{k-1}}_{x,\mathcal{B}_\Gamma}$

By Putting together (7.11) with Claim 7.11 yields

$$\mathrm{Var}_{\mathcal{B}_\Gamma}(f) \le \gamma_{k-1} \sum_{x\in\mathcal{B}_\Gamma} \mu_{\mathcal{B}_\Gamma}\left(\bar{c}_x \,\mathrm{Var}_x(f)\right)$$

which, together with (7.8), (7.9) and (7.10) gives

$$\mu_\Lambda\left(c_1 \,\mathrm{Var}_{B_1}(f)\right) \leqslant \gamma_{k-1} \sum_{\Gamma\in\widetilde{\mathcal{C}}_\Pi} \mu_\Lambda\left(\sum_{x\in\mathcal{B}_\Gamma}\left(\mathbb{1}_{\cap_{j=1}^m \mathcal{R}_j}\mathbb{1}_{\{\overline{\Pi}_\omega=\Gamma\}}\bar{c}_x \,\mathrm{Var}_x(f)\right)\right) \tag{7.12}$$

when $b \geqslant N_{k-1}/2$ and

$$\mu_\Lambda\left(c_1 \,\mathrm{Var}_{B_1}(f)\right) \leqslant \gamma_{k-1} \sum_{\Gamma\in\mathcal{C}_\Pi} \mu_\Lambda\left(\sum_{x\in\mathcal{B}_\Gamma}\left(\mathbb{1}_{\{\overline{\Pi}_\omega=\Gamma\}}\bar{c}_x \,\mathrm{Var}_x(f)\right)\right) \tag{7.13}$$

when $b < N_{k-1}/2$. We now wish to upper-bound the terms which appear in front of Var_x in the right-hand sides of (7.12) and (7.13) with the long range rates $c^{N_k}_{x,\Lambda}$, in order to upper-bound the right-hand side of (7.6) with the full Dirichlet form (5.5) by using (7.7) and (7.12) or (7.7) and (7.13) according to the value of the height b of the rectangle Λ. Once this is achieved we will divide the left and right side by $\mathrm{Var}_\Lambda(f)$ and take the sup on f in order to gain an inequality between γ_k and γ_{k-1} which, properly iterated, will lead to the desired bound (7.2).

Claim 7.12. *For any ω, $\Gamma \in \widetilde{\mathcal{C}}_\Pi$ and $x \in \mathcal{B}_\Gamma$,*

$$\mathbb{1}_{\cap_{j=1}^m \mathcal{R}_j}\mathbb{1}_{\{\overline{\Pi}_\omega=\Gamma\}}\bar{c}_x(\omega) \leqslant \mathbb{1}_{\{\overline{\Pi}_\omega=\Gamma\}} c^{N_k}_{x,\Lambda}(\omega). \tag{7.14}$$

For any ω, $\Gamma \in \mathcal{C}_\Pi$ and $x \in \mathcal{B}_\Gamma$,

$$\mathbb{1}_{\{\overline{\Pi}_\omega=\Gamma\}}\bar{c}_x(\omega) \leqslant \mathbb{1}_{\{\overline{\Pi}_\omega=\Gamma\}} c^{N_k}_{x,\Lambda}(\omega). \tag{7.15}$$

Proof. It is sufficient to show that when $\bar{c}_x(\omega) = 1$ the right hand side also equals one. We recall that $\bar{c}_x(\omega) = 1$ guarantees that there exists (at least) one direction i and a geometric path $\gamma_{x,y}$ inside $T_{x,i}^{N_{k-1}}(\mathcal{B}_\Gamma)$ from x to some $y \in \overline{\partial}_-\mathcal{B}_\Gamma \cup (\partial_+\Gamma \cap \mathcal{B}_\Gamma)$ which is AKG allowed. Let $\gamma_{x,y}$ be one of such paths and distinguish three cases: (a) $y \in \overline{\partial}_-\Lambda$; (b) $y \notin \overline{\partial}_-\Lambda$ and $i = 1$; (c) $y \notin \overline{\partial}_-\Lambda$ and $i = 2$.

(a) From the definition of the rates in formula (5.3) it follows immediately that $c_{x,\Lambda}^{N_k} = 1$ since $T_{x,i}^{N_k}(\Lambda) \supset T_{x,i}^{N_{k-1}}(\mathcal{B}_\Gamma)$, thus $\gamma_{x,y} \in \mathcal{G}_{x,N_k,\Lambda}$.

(b) In this case $y \in \partial_+\Gamma \cap \mathcal{B}_\Gamma$, thus there exists y' with $d_1(y, y') = 1$ and $y' \in \Gamma = \overline{\Pi}_\omega$. This implies that $\omega(y')$ is good and either $y' \in \Pi_\omega$ or there exists $y'' \in \Pi_\omega$ with $d_1(y', y'') \in (1, 2)$. By using this together with the existence of the AKG allowed path $\gamma_{x,y}$ allows to conclude that there always exists a path $\gamma_{y,y''}$ AKG allowed. We should now distinguish the case (i) $b < N_{k-1}/2$ and (ii) $b \geqslant N_{k-1}/2$. Case (i) can be handled very simply by noticing that in this case $\forall x \in \mathcal{B}_\Gamma$, $I \subset T_{x,1}^{N_k}(\Lambda)$ holds. Therefore $c_{x,\Lambda}^{N_k} = 1$ thanks to the path $\gamma_{x,z} := \gamma_{x,y} \cdot \gamma_{y,y''} \cdot \gamma_{y'',z} \subset T_{x,1}^{N_k}(\Lambda)$, where $z \in \partial_-^2\Lambda$ and $\gamma_{y'',z}$ is a subset of the good top-bottom crossing $\Pi(\omega)$. Case (ii) requires a bit more work and the use of the left-rightmost crossings in the R_j rectangles. Claim 7.6 and the fact that $\mathbb{1}_{\cap_{i=1}^m \mathcal{R}_j}(\omega) = 1$ guarantee that there exists a good path $\gamma_{w,z}$ inside $T_{x,1}^{N_{k-1}}(\Lambda)$ with $w \in \Pi_\omega \cap T_{x,1}^{N_{k-1}}(\Lambda)$ and $z \in \partial_-^*\Lambda \cap T_{x,1}^{N_{k-1}}(\Lambda)$. Since $\Gamma \in \widetilde{\mathcal{C}}$, $\Pi(\omega)$ has k-bounded oscillations. Thus by recalling Definition 7.5 and noticing that $N_{k-1} + A/2 \log(3/2) \log(l_k) \leq N_k$, there exists a good path $\gamma_{y'',w} \subset \Pi_\omega$ inside $T_{x,1}^{N_k}$.

(c) Define as in the previous case y'' and $\gamma_{y,y''}$. By construction (since $y'' \in \Pi_\omega$) there exists a good path $\gamma_{y'',z} \subset I$ (which is a subset of the top bottom crossing of I) for some $z \in \partial_-^2\Lambda$. Furthermore Claim 7.3(iii) guarantees that in this case $I \subset T_{x,2}^{N_k}(\Lambda)$. Thus $c_{x,\Lambda}^{N_k}(\omega) = 1$ is guaranteed if we consider the overall path $\gamma_{x,z} := \gamma_{x,y} \cdot \gamma_{y,y''} \cdot \gamma_{y'',z}$. $\square$

If we finally plug (7.14) in the r.h.s. of (7.12) or (7.15) in the r.h.s. of (7.13) we obtain in both cases

$$\begin{aligned}\mu_\Lambda\left(c_1 \operatorname{Var}_{B_1}(f)\right) &\leq \gamma_{k-1}\mu_\Lambda\left(\sum_{x\in\mathcal{B}_\Gamma} c_{x,\Lambda}^{N_k} \operatorname{Var}_x(f)\right) \\ &\leq \gamma_{k-1}\mu_\Lambda\left(\sum_{x\in B_1\cup I} c_{x,\Lambda}^{N_k} \operatorname{Var}_x(f)\right).\end{aligned} \tag{7.16}$$

Thus, by using (7.6), (7.7), (7.16) and (5.5) we get

$$\operatorname{Var}_\Lambda(f) \leq \left(\frac{1}{1-\sqrt{\varepsilon_k}}\right)\gamma_{k-1}\left(\mathcal{D}_{\Lambda,N_k}(f) + \sum_{x\in I}\mu_\Lambda\left(c_{x,\Lambda}^{N_k} \operatorname{Var}_x(f)\right)\right). \tag{7.17}$$

Recalling Lemma 7.2 we can now averaging over the $s_k = \lfloor l_k^{1/3} \rfloor$ possible choices of the sets $\Lambda_1^{(k)}, \Lambda_2^{(k)}$ which verify $I \subset \Lambda_1^k \cap \Lambda_2^k$ and $\left(\Lambda_1^{(k)} \cap \Lambda_2^{(k)}\right) \cap \left(\Lambda_1^{(k')} \cap \Lambda_2^{(k')}\right) = \emptyset$

for $k \neq k'$. Thus we get

$$\mathrm{Var}_\Lambda(f) \le \left(\frac{1}{1-\sqrt{\varepsilon_k}}\right) \gamma_{k-1} \left(1+\frac{1}{s_k}\right) \mathcal{D}_{\Lambda,N_k}(f) \tag{7.18}$$

which in turn, dividing by Var_Λ and taking the sup over f, implies

$$\gamma_k \le \left(\frac{1}{1-\sqrt{\varepsilon_k}}\right)\left(1+\frac{1}{s_k}\right)\gamma_{k-1} \le \gamma_{k_0} \prod_{j=k_0}^{k} \left(\frac{1}{1-\sqrt{\varepsilon_j}}\right)\left(1+\frac{1}{s_j}\right), \tag{7.19}$$

where k_0 is the smallest integer such that $\delta_{k_0} > 1$ and ε_k has been defined in Proposition 7.8. By plugging the results of Claim 7.13 and 7.14 below into (7.19) the proof of Theorem 5.5 is completed with the choice $\rho_1 = \max(\tilde{\rho}_1, \bar{\rho}_1)$. $\square$

Claim 7.13. *There exists $\tilde{\rho}_1 \in (0,1)$ and $A_0 > 0$ such that for $\rho > \tilde{\rho}_1$ and $A > A_0$,*

$$\varepsilon_k \leqslant \frac{2}{l_k}. \tag{7.20}$$

Claim 7.14. *For all $\epsilon \in (0,1)$ there exists $\bar{\rho}_1 \in (0,1)$ such that for $\rho > \bar{\rho}_1$,*

$$\gamma_{k_0} \leqslant 2 - \epsilon. \tag{7.21}$$

Proof of Claim 7.13. Recalling the definition of ε_k given in Proposition 7.8 and Definition 7.7 for B_2-good configurations we should distinguish two cases: (i) $b \geqslant N_{k-1}/2$ and (ii) $b < N_{k-1}/2$, where b is the length of Λ_2 in direction 2.

(i) By using the FKG inequality we get

$$\nu(\{\omega \text{ is } B_2\text{-good}\}) = \mu\left(\cap_{j=1}^{m} \mathcal{R}_j \cap \mathcal{J}\right) \geqslant \mu(\mathcal{J}) \prod_{j=1}^{m} \mu(\mathcal{R}_j). \tag{7.22}$$

Next we can decompose I into m' smaller disjoint rectangles, $I = \cup_{j=1}^{m'} I_j$, with each I_j being a translation of $[0,a] \times [0,h_j]$, where $a = A/2\log(3/2)\log(l_k)$, $m' < 2l_{k+1}/(A/2\log(3/2)\log l_k)$ and h_j verifies $A\log(3/2)/4\log l_k < 2h_j < A/2\log(3/2)\log l_k$ (this procedure is possible thanks to the bounds on b, $N_{k-1}/2 \leqslant b \leqslant l_{k+1}$). Define the events

$$\mathcal{I}_j := \{\omega : \omega \text{ has a left-right good crossing in } I_j\}.$$

It is then easy to prove by recalling Definition 7.5 and by an inspection of the right Fig. 7 that

$$\mathcal{J} \subset \cap_{j=1}^{m'} \mathcal{I}_j \cap \mathcal{I}, \tag{7.23}$$

namely the occurrence of a good top bottom crossing in I plus good left right crossing in each I_j guarantee that the rightmost top bottom crossing of I has k-bounded oscillations.

By using standard percolation results, there exists $\rho_1 < 1$ and $\alpha > 0$ such that if $\rho > \rho_1$ and k is sufficiently large, then

$$\mu(\mathcal{R}_j) \geqslant 1 - l_{k+1} \exp(-\alpha c_j) \qquad \forall j = 1, \dots, m, \tag{7.24}$$
$$\mu(\mathcal{I}_j) \geqslant 1 - (b_1 - a_1) \exp(-\alpha h_j) \qquad \forall j = 1, \dots, m', \tag{7.25}$$
$$\mu(\mathcal{I}) \geqslant 1 - l_{k+1} \exp(-\alpha[b_1 - a_1]), \tag{7.26}$$

where we recall that c_j is the height of the rectangle R_j which verifies $c_j \geqslant N_k/2$, $b_1 - a_1 = A/2 \log(3/2) \log l_k$ is the width of the strip I, m verifies $m \leqslant (2l_{k+1}/N_{k-1})$, and $N_k = A(\log(l_k))^2$. Provided that A is large enough the desired inequality (7.20) immediately follows from (7.22), (7.23), (7.24), (7.25) and (7.26).

(ii) In this case

$$\nu(\{\omega \text{ is } B_2\text{-good}\}) = \mu(\mathcal{I}) \tag{7.27}$$

and again, provided that A is large enough, the desired inequality (7.20) immediately follows from (7.26). □

Proof of Claim 7.14. Choose $k \leqslant k_0$ and a rectangle $R \in \mathbb{F}_k \backslash \mathbb{F}_{k-1}$. Let ℓ_1 (ℓ_2) be the length of R in the $\vec{e}_1$ ($\vec{e}_2$) direction. We label the $\ell_1\ell_2$ sites of R from the bottom left one from left to right and bottom to top as $x_1, \dots, x_{\ell_1\ell_2}$. We also let $B_0 = R$, $B_1 = x_1$ and $B_2 = R\backslash x_1$ and we consider the following block dynamics. The block B_2 waits a mean one exponential random time and then the current configuration inside it is refreshed with a new one sampled from μ_{B_2}. The block B_1 does the same but now the configuration is refreshed *only* if the current configuration ω in B_2 is such that the path $\gamma_{x_1,x_{\ell_1}} = x_1, \dots, x_{\ell_1}$ which goes straight towards the right of x_1 up to the border of R is good. By using Poincaré inequality together with the same strategy as in [9, Prop. 4.4] to evaluate the spectral gap of this auxiliary dynamics we get

$$\begin{aligned}
\mathrm{Var}_R(f) &\leqslant \frac{1}{1 - \sqrt{1 - q^{2\ell_1}}} \mu_R \left(\mathbb{1}_{\{\gamma_{x_1,x_{\ell_1}} \text{ is good}\}} \mathrm{Var}_{B_1}(f) + \mathrm{Var}_{B_2}(f) \right) \\
&\leqslant \frac{1}{1 - \sqrt{1 - q^{2\ell_1}}} \mathcal{D}_{R,N_k}(f) \left(\mathrm{gap}(\mathcal{L}_{B_2,N_k})^{-1} \right),
\end{aligned} \tag{7.28}$$

where to get the last inequality we use the fact that $\bar{\partial}_- B_2 \subset \bar{\partial}_- R$ and $\mathbb{1}_{\{\gamma_{x_1,x_{\ell_1}} \text{ is good}\}} \leqslant c_{x,R}^{N_k}$. The variational characterization of the spectral gap together with (7.28) leads to

$$\left(\mathrm{gap}(\mathcal{L}_{R,N_k})\right)^{-1} \leqslant \frac{1}{1 - \sqrt{1 - q^{2\ell_1}}} \left(\mathrm{gap}(\mathcal{L}_{R\backslash x_1,N_k})\right)^{-1}. \tag{7.29}$$

We can then let $\tilde{B}_0 := R\backslash x_1$ and divide it into $\tilde{B}_1 := x_2$ and $\tilde{B}_2 := R\backslash(x_1 \cup x_2)$ and proceed analogously to get inequality (7.29) with $R\backslash x_1$ in the left-hand side and $R\backslash(x_1 \cup x_2)$ on the right-hand. By proceeding iteratively we finally get

$$\gamma_{k_0} \leqslant \frac{1}{(1 - \sqrt{1 - q^{2k_0}})^{k_0}}, \tag{7.30}$$

which concludes the proof of the claim. □

8. Polynomial Decay to Equilibrium: Proof of Theorem 4.2

In order to establish polynomial decay to equilibrium in infinite volume we start by reducing as usual the dynamics to a finite volume thanks to the finite speed of propagation. Then we follow a soft spectral theoretic argument introduced in [7] which requires a bound of the variance with the Dirichlet form of the process. Establishing this bound is the difficult step here due to the presence of the kinetic constraints. In order to obtain this result we use the positivity of the spectral gap of AGL (Theorem 5.5) combined with path and renormalization arguments.

Proof of Theorem 4.2. Let f be a local function of zero mean value and fix a large time t. Consider $\tilde{L} = at$ with the constant a defined by Lemma 8.1 below. By translation invariance of the system we can assume that f has support Δ_f at the center of the cube $Q_{\tilde{L}}$. Then, we have

$$\mathrm{Var}_{\mu}(P_t f) \leqslant 2\|P_t f - P_t^{Q_{\tilde{L}}} f\|_{\infty}^2 + 2\,\mathrm{Var}_{\mu_{Q_{\tilde{L}}}}(P_t^{Q_{\tilde{L}}} f), \tag{8.1}$$

where $P_t^{Q_{\tilde{L}}} = e^{t\mathcal{L}_{Q_{\tilde{L}}}}$ and $\mathcal{L}_{Q_{\tilde{L}}}$ is defined with boundary-source choice $(\mathcal{M}, S) = (\emptyset, \overline{\partial}_- Q_{\tilde{L}})$. A standard property known as finite speed of propagation (see [22]) asserts that for a proper C, if a is chosen large enough for all t,

$$\|P_t f - P_t^{Q_{\tilde{L}}} f\|_{\infty}^2 \leqslant C e^{-t/C}\|f\|_{\infty}^2 \tag{8.2}$$

holds. Putting together (8.1) and (8.2) with Lemma 8.1 concludes the proof. □

Lemma 8.1. *There exists $a > 0$ such that for all $t > 0$ if we let $\tilde{L} := at$,*

$$\mathrm{Var}_{\mu_{Q_{\tilde{L}}}}(P_t^{Q_{\tilde{L}}} f) \leqslant C\frac{(\log t)^5}{t}\|f\|_{\infty}^2$$

holds.

Proof of Lemma 8.1. Let $g = P_{2t}^{Q_{\tilde{L}}} f$. By reversibility $\mathrm{Var}_{\mu_{Q_{\tilde{L}}}}(P_t^{Q_{\tilde{L}}} f) = \mu_{Q_{\tilde{L}}}(f, g)$. Thus if

$$\left[\mu_{Q_{\tilde{L}}}(f, g)\right]^2 \leqslant \|f\|_{\infty}^2 \frac{(\log t)^5}{t}\mathrm{Var}_{\mu_{Q_{\tilde{L}}}}(P_t^{Q_{\tilde{L}}} f), \tag{8.3}$$

the desired result follows immediately. We are therefore left with proving (8.3). Note that by definition $\frac{\partial}{\partial t}\mathrm{Var}_{\mu_{Q_{\tilde{L}}}}(P_t^{Q_{\tilde{L}}} f) = -2\mathcal{D}_{Q_{\tilde{L}}}(P_t^{Q_{\tilde{L}}} f) \leqslant 0$. Hence, $\mathrm{Var}_{\mu_{Q_{\tilde{L}}}}(P_t^{Q_{\tilde{L}}} f)$ is a decreasing function in t and

$$\mathrm{Var}_{\nu_{\tilde{Q}_L}}(g) = \mathrm{Var}_{\mu_{Q_{\tilde{L}}}}(P_{2t}^{Q_{\tilde{L}}} f) \leqslant \mathrm{Var}_{\mu_{Q_{\tilde{L}}}}(P_t^{Q_{\tilde{L}}} f). \tag{8.4}$$

Fix t and a and choose $\ell > 0$ s.t.: $L := \tilde{L}/\ell = at/\ell$ is integer, f has support $\Delta_f \subset Q^x := Q_\ell + x$ for a proper x and $\mu(\mathcal{F}_\ell) > \max(\rho_0, \tilde{\rho}_0)$, where ρ_0 and $\tilde{\rho}_0$ are the thresholds defined in Theorem 5.6 and Claim 8.2 respectively (this is possible thanks to Lemma 3.4). Then define the renormalized lattice and the renormalized cube as in Sect. 6, namely $\mathbb{Z}^2(\ell) := \ell\mathbb{Z}^2$ and $\tilde{Q}_L := \mathbb{Z}^2(\ell) \cap Q_{\tilde{L}}$. As already noticed in Sect. 6, if we consider the probability space $W = \{0,1\}^{Q_\ell}$ equipped with $\nu = \mu_{Q_\ell}$ the two probability spaces $(\{0,1\}^{\mathbb{Z}^2}, \mu)$ and $(W^{\mathbb{Z}^2(\ell)}, \prod_{x\in\mathbb{Z}^2(\ell)} \nu_x)$ coincide. Furthermore $\mu_{Q_{\tilde{L}}} = \nu_{\tilde{Q}_L}$, where $\nu_A = \prod_{x\in A\cap\mathbb{Z}^2(\ell)} \nu_x$. Choose $B > 0$ such that $M := B\log L$ divides L and L/M is odd and consider the following rectangles on $\mathbb{Z}^2(\ell)$:

$$H_k := [0, L-1] \times [(k-1)M, kM-1], \qquad k = 1, \dots, \frac{L}{M},$$
$$V_k := [(k-1)M, kM-1] \times [0, L-1], \qquad k = 1, \dots, \frac{L}{M}.$$

It is immediate to verify that the renormalized cube $\tilde{Q}_L$ can be written as the disjoint union of both sets of rectangles, $\tilde{Q}_L = \cup_k V_k = \cup_k H_k$ (see Fig. 10). Define on $W = \{0,1\}^{Q_\ell}$ the good event $G = \mathcal{F}_\ell$. With this choice and recalling Definition 2.6 we let

$$\mathcal{V}_k := \{\omega : \omega \text{ has a good top-bottom crossing in } V_k\},$$
$$\mathcal{H}_k := \{\omega : \omega \text{ has a good left-right crossing in } H_k\},$$
$$\Theta_{L,B} := \{\omega : \omega \in \cap_{k=1}^{L/M} (\mathcal{V}_k \cap \mathcal{H}_k)\}.$$

Then, by using the Cauchy-Schwartz inequality and the fact that $\nu_{Q_{\tilde{L}}}(f) = 0$ we get

$$\left[\mu_{Q_{\tilde{L}}}(f,g)\right]^2 = \left[\nu_{\tilde{Q}_L}(f,g)\right]^2 \leqslant 2\nu_{\tilde{Q}_L}(\mathbf{1}_\Theta f g)^2 + 2\nu_{\tilde{Q}_L}\left((1-\mathbf{1}_\Theta) f (g - \nu_{\tilde{Q}_L}(g))\right)^2, \tag{8.5}$$

where from now on we drop the indexes L and B from Θ. Let us deal first with the second term. By using again the Cauchy-Schwartz inequality we get

$$\begin{aligned}\nu_{\tilde{Q}_L}\left((1-\mathbf{1}_\Theta) f (g - \nu_{\tilde{Q}_L}(g))\right)^2 &\leqslant \nu_{\tilde{Q}_L}(1-\mathbf{1}_\Theta)\,\mathrm{Var}_{\nu_{\tilde{Q}_L}}(g)\|f\|_\infty^2, \\ &\leqslant \frac{C}{L}\mathrm{Var}_{\nu_{\tilde{Q}_L}}(g)\|f\|_\infty^2 \leqslant \frac{C}{at}\mathrm{Var}_{\mu_{Q_{\tilde{L}}}}(P_t^{Q_{\tilde{L}}} f)\|f\|_\infty^2,\end{aligned} \tag{8.6}$$

where the second inequality relies on Claim 8.2 below and we used (8.4) in order to derive the third inequality.

Let us now consider the first term of (8.5). Without loss of generality we can assume that f has support $\Delta_f \subset (\cup_{x\in H_{k_0}} Q_x) \cap (\cup_{x\in V_{k_0}} Q_x)$ with $k_0 = L/2M$ (see Fig. 10). As explained in Sect. 7 (see before Claim 7.4), there is a natural partial order on the set of top-bottom crossing paths in V_k that allows to the define the right-most one. Thus, for any $\omega \in \mathcal{V}_k$, we define $\Pi_\omega^{V,k}$ to be its right-most good top-bottom crossing. Analogously for each $\omega \in \mathcal{H}_k$ we define $\Pi_\omega^{H,k}$ to be its up-most good left-right crossing. As usual, we let $\overline{\Pi}_\omega^{V,k}$ and $\overline{\Pi}_\omega^{H,k}$ be the corresponding double paths. For any $\omega \in \Theta$ we can then let $\Pi_\omega := \{\Pi_\omega^{H,k}, \Pi_\omega^{V,k}\}_{k=1}^{L/M}$ and define the geometric set

$$\mathcal{C}_\Pi := \{\Gamma \ : \ \exists\, \omega \in \Theta \text{ s.}t.\ \Pi_\omega = \Gamma\}.$$

By geometrical considerations one can verify that points (i) and (ii) of Claim 7.4 are valid for all $\omega \in \mathcal{V}_k$, and analogous statements are valid for all $\omega \in \mathcal{H}_k$. Let $R_{k_0} := (H_{k_0} \cup H_{k_0+1}) \cap (V_{k_0} \cup V_{k_0+1})$. For any chosen $\omega \in \Theta$ we divide the region R_{k_0} into two connected (with respect to the distance d_1) components : the sites on the right of

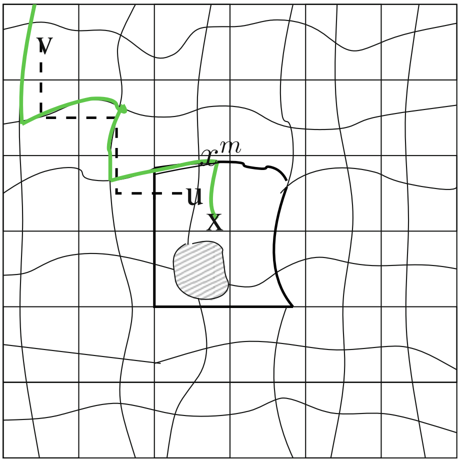

Fig. 10. The box $\tilde{Q}_L \subset \mathbb{Z}^2(\ell)$ divided into the rectangles H_k and V_k with $k = 1, \dots L/M$ (here $L/M = 6$). The shaded region represents the support of f, $\Delta_f \subset H_{k_0} \cap V_{k_0}$. We depict a configuration $\omega \in \Theta$. The continuous non straight lines which form a grid represent the rightmost good top-bottom crossing of each V_k and the up-most good left-right crossing of each H_k. The region delimited by the bold black line is $\mathcal{B}_\omega$. We choose a site $x \in \mathcal{B}_\omega$, assume that $\bar{c}_x(\omega) = 1$ and we associate to x the corresponding site $u \in \hat{Q}_{L/M}$. Then we fix $v \in \partial_- \hat{Q}_{L/M}$ and we choose a geodesic path $\gamma_{u,v}$ (dashed line). The green path is the geometric path γ_{x,x^*} which is allowed for the AKG model and is composed of two parts. The first path is from x to $\partial_- \mathcal{B}_\Gamma \cup (\partial_+ \Gamma \cap \mathcal{B}_\Gamma)$ and is guaranteed by $\bar{c}_x = 1$. The second path is guaranteed by $\omega \in \Theta$ and uses the grid Γ of good crossing following the sites which belong to the geodesic path $\gamma_{u,v}$

$\overline{\Pi}_\omega^{V,k_0+1}$ and those on the left. We call $\mathcal{B}_\omega^V$ the latter set. Analogously we consider the two connected components which correspond to the sites above $\overline{\Pi}_\omega^{H,k_0+1}$ and those below. We call $\mathcal{B}_\omega^H$ the latter set. Then we define $\mathcal{B}_\omega := \mathcal{B}_\omega^V \cap \mathcal{B}_\omega^H$ and, with a slight abuse of notation, for any $\Gamma \in \mathcal{C}_\Pi$ we let $\mathcal{B}_\Gamma$ be the $\mathcal{B}_\omega$ which corresponds to all ω with $\Pi_\omega = \Gamma$. With this notation we have

$$\nu_{\tilde{Q}_L}(\mathbf{1}_\Theta f g)^2 \le \|f\|_\infty^2 \sum_{\Gamma \in \mathcal{C}_\Pi} \nu_{\tilde{Q}_L}\left(\mathrm{Var}_{\mathcal{B}_\Gamma}(g)\chi_\Gamma\right),$$

where we used the fact that $\chi_\Gamma(\sigma)$ does not depend on the value of σ inside $\mathcal{B}_\Gamma$ and the hypothesis $\nu_{\mathcal{B}_\Gamma}(f) = 0$. Then, by using Lemma 8.3 below and Lemma 6.1 and some explicit counting we get

$$\left[\nu_{\tilde{Q}_L}(\mathbf{1}_\Theta f g)\right]^2 \le C(\log L)^5 \|f\|_\infty^2 \left(\mathcal{D}^K_{Q_{\tilde{L}}}(g) + \mathcal{D}^G_{\partial_- Q_{\tilde{L}}}(g)\right). \tag{8.7}$$

By the spectral decomposition of $-\mathcal{L}_{Q_{\tilde{L}}}$ in $\mathbb{L}_2(\mu_{Q_{\tilde{L}}})$ and the bound $2t\lambda e^{-2t\lambda} \leqslant 1/e$, we have

$$\begin{aligned}\mathcal{D}^K_{Q_{\tilde{L}}}(g) + \mathcal{D}^G_{\partial_- Q_{\tilde{L}}}(g) &= \mu_{Q_{\tilde{L}}}\left(g(-\mathcal{L}_{Q_{\tilde{L}}})g\right) = \int_0^\infty \lambda e^{-2t\lambda} dE_\lambda(g) \\ &\leqslant \frac{1}{2et}\int_0^\infty dE_\lambda(g) = \frac{1}{2et}\mathrm{Var}_{\mu_{Q_{\tilde{L}}}}(g) \leqslant \frac{1}{2et}\mathrm{Var}_{\mu_{Q_{\tilde{L}}}}(P_t^{Q_{\tilde{L}}} f),\end{aligned} \tag{8.8}$$

where the last inequality comes from (8.4). Therefore the desired inequality (8.3) follows from (8.5), (8.6), (8.7), (8.8) and $\tilde{L} = at$ and the proof is concluded. □

Claim 8.2. *There exists $\tilde{\rho}_0 < 1$ s.t. for $\mu(\mathcal{F}_\ell) \geqslant \tilde{\rho}_0$ we have*

$$\nu_{\tilde{Q}_L}(1 - \mathbf{1}_\Theta) \le \frac{C}{L}.$$

Proof of the claim. By translation invariance, the probability that there is a good left-right crossing in H_k or that there is a good top bottom crossing in V_k do not depend on k. Hence, let $\alpha(L) = \nu_{\tilde{Q}_L}(\mathcal{V}_k) = \nu_{\tilde{Q}_L}(\mathcal{H}_k)$. Using the FKG inequality, we get that

$$\nu_{\tilde{Q}_L}(1 - \mathbf{1}_\Theta) \le 1 - \alpha^{2L/M}.$$

Standard percolation results [14] guarantee the existence of a constant $c > 0$ such that, provided the probability for a site to be good is above a certain percolation threshold $\tilde{\rho}_0$, then

$$\alpha \ge 1 - Le^{-cM} = 1 - Le^{-cB\log L}.$$

Hence, provided B is large enough,

$$\nu_{\tilde{Q}_L}(1 - \mathbf{1}_\Theta) \le 1 - \exp\left\{\frac{2L}{M}\log(1 - Le^{-cB\log L})\right\} \le \frac{C}{L}.$$

This achieves the proof of the claim. □

Let T_e, $\widetilde{c}_e$ and $\widetilde{c}_x$ be defined as in Sect. 6, then

Lemma 8.3. *For any f there exists a positive constant C such that*

$$\sum_{\Gamma\in\mathcal{C}_\Pi} \nu_{\tilde{Q}_L}\left(\mathrm{Var}_{\mathcal{B}_\Gamma}(f)\chi_\Gamma\right) \leqslant C(\log L)^5 \sum_{e\in E_{\tilde{Q}_L}} \nu_{\tilde{Q}_L}\left(\widetilde{c}_e(T_e f - f)^2\right)$$
$$+ C(\log L)^5 \sum_{x\in\overline{\partial}_-\tilde{Q}_L} \nu_{\tilde{Q}_L}\left(\widetilde{c}_x \,\mathrm{Var}_x(f)\right).$$

Proof of the lemma. The proof of this result makes use of the positivity of the spectral gap of the AGL model (Theorem 5.5) and involves a renormalization technique in the same spirit as the one used to prove the lower bound for the spectral gap of KA (Theorem 4.1). Fix $\Gamma \in \mathcal{C}_\Pi$. Let $N := A(\log(2M))^2$ with A defined as in Theorem 5.5. For any $x \in \mathcal{B}_\Gamma$ and any $\omega \in \Omega_{\mathcal{B}_\Gamma}$, we let $\overline{c}_x(\omega)$ be the indicator function of the event that there exists a geometric path $\gamma_{x,y}$ inside $T^N_{x,1}(\mathcal{B}_\Gamma)$ or inside $T^N_{x,2}(\mathcal{B}_\Gamma)$ with $y \in \overline{\partial}_-\mathcal{B}_\Gamma \cup (\partial_+\Gamma \cap \mathcal{B}_\Gamma)$ and such that this path is allowed with the choice of the AKG constraints for the configuration $(\omega_{\mathcal{B}_\Gamma}\cdot\tau)(z)$ with $\tau(t) \in G$ for any $t \in \overline{\partial}^*_+\mathcal{B}_\Gamma$. It is then possible to define the correspondent Glauber generator in the volume $\mathcal{B}_\Gamma$ as

$$\overline{\mathcal{L}}_{\mathcal{B}_\Gamma} f = \sum_{x\in\mathcal{B}_\Gamma} \overline{c}_x(\nu_x(f) - f).$$

Applying the Poincaré inequality leads to

$$\mathrm{Var}_{\mathcal{B}_\Gamma}(g) \le \mathrm{gap}(\overline{\mathcal{L}}_{\mathcal{B}_\Gamma})^{-1} \sum_{x\in\mathcal{B}_\Gamma} \nu_{\mathcal{B}_\Gamma}\left(\overline{c}_x \,\mathrm{Var}_x(g)\right). \tag{8.9}$$

Analogously to Claim 7.10 and 7.11, one can prove the following bounds with respect to the rates and the spectral gap of the AGL model

Claim 8.4. *For any ω and $x \in \mathcal{B}_\Gamma$,*

$$c^N_{x,R_{k_0}}(\omega) \leqslant \bar{c}_x(\omega)$$

and

$$\text{gap}(\overline{\mathcal{L}}_{\mathcal{B}_\Gamma})^{-1} \leqslant \text{gap}(\mathcal{L}^{agl}_{R_{k_0},N})^{-1}.$$

If we choose B (the constant which enters in the definition of M, the short size of the renormalized rectangles) in order that there exists $k \in \mathbb{N}$ s.t. $2M = 2B \log L = l_k$ with $l_k = (3/2)^{k/2}$, then $N = A(\log l_k)^2 = N_k$, where N_k is the one used in Sect. 7. Furthermore since R_{k_0} is a cube of linear size $2M = l_k$ it belongs to $\mathbb{F}_{k-1} \backslash \mathbb{F}_{k-2}$, where $\mathbb{F}_k$ are the set of rectangles defined in Sect. 7. Therefore by recalling the definition of γ_k (Eq. 7.1) and the result of Theorem 5.5 we get

$$\text{gap}(\mathcal{L}^{agl}_{R_{k_0},N})^{-1} \leqslant 2. \tag{8.10}$$

This, together with (8.9) yields

$$\nu_{\tilde{Q}_L}\left(\text{Var}_{\mathcal{B}_\Gamma}(g)\chi_\Gamma\right) \leqslant 2 \sum_{x \in \mathcal{B}_\Gamma} \nu_{Q_L}\left(\bar{c}_x \chi_\Gamma \, \text{Var}_x(g)\right). \tag{8.11}$$

As usual we can rewrite the variance as

$$\text{Var}_x(g)(\omega) = \frac{1}{2} \sum_{w,w' \in W} \nu(w)\nu(w')\left(g(\omega_{\tilde{Q}_L \backslash x} \cdot w') - g(\omega_{\tilde{Q}_L \backslash x} \cdot w)\right)^2. \tag{8.12}$$

Our aim is now to reconstruct the move from $\omega_{\tilde{Q}_L \backslash x} \cdot w'$ to $\omega_{\tilde{Q}_L \backslash x} \cdot w$ via proper paths by using the properties which are guaranteed if $\bar{c}_x \chi_\Gamma = 1$. We start by noticing that the renormalized cube $\tilde{Q}_L$ can also be seen as the union of squares whose side has length M, i.e. as a subset $\hat{Q}_{L/M}$ of $\mathbb{Z}^2(M)$. Then, for any $(u, v) \in \hat{Q}_{L/M} \times \overline{\partial}_- \hat{Q}_{L/M}$ we choose once and for all a path $\gamma_{u,v}$ with $\gamma_{uv} = (u = t^{(1)}, t^{(2)}, \ldots, t^{(m-1)}, v = t^{(m)})$ among the geodesic paths inside $\hat{Q}_{L/M}$ from u to v such that, for any $t \in \gamma_{uv}$, the Euclidean distance between t and the straight line segment $[u, v]$ is at most $\sqrt{2}/2$ (see Fig. 10). Fix $x \in \mathcal{B}_\Gamma$ and suppose $\bar{c}_x = 1$. Then there exists at least one geometric path $(x_1 = x, \ldots, x^m)$ which is allowed for ω with AKG constraints and with $x^m \in \overline{\partial}_- \mathcal{B}_\Gamma \cup (\partial_+ \Gamma \cap \mathcal{B}_\Gamma)$. Then let $u \in \hat{Q}_{L/M}$ be the square which contains x^m, i.e. $x^m \in Q^u$ (see Fig. 10). If $\chi_\Gamma = 1$, by using the grid Γ of vertical and horizontal good crossings, we can now construct in a unique way a geometric path $x^m, \ldots, x^n = x^*$ inside $\tilde{Q}_L$ which is allowed for AKG constraints and such that $x^m, \ldots, x^{m_1} \in Q^{t^{(1)}}$, $x^{(m_1+1)}, \ldots, x^{(m_2)} \in Q^{t^{(2)}}$ and so on (see Fig. 10). In conclusion we have constructed an AKG allowed path $\gamma_{xx^*} = (x^1, \ldots, x^n = x^*)$ with $x^* \in \overline{\partial}_- \tilde{Q}_L \cap Q_v$ (overall green path in the figure). We construct such path for any couple ω, x with $\bar{c}_x(\omega) = 1$ and $\omega \in \Theta$ and we perform this choice in order that the path is the same for any two configurations which are G-equivalent inside $\tilde{Q}_L \backslash x$. For $i = 1, \ldots, n-1$, let $e_i = (x^{(i)}, x^{(i+1)}) \in E_{\tilde{Q}_L}$. Then, for any $w, w' \in W$, we can define the path

$$P_{w \xrightarrow{x} w'} = P_{w \xrightarrow{x} w'}(\omega, v, \Gamma) = (\omega^{(1)}, \ldots, \omega^{(2n)})$$

from $\omega^{(1)} = \omega_{\tilde{Q}_L \setminus x} \cdot w$ to $\omega^{(2n)} = \omega_{\tilde{Q}_L \setminus x} \cdot w'$ by $\omega^{(i+1)} = T_{e_i}\omega^{(i)}$ for $i = 1, \dots, n-1$, $\omega^{(n)} = \omega^{(n-1)}_{\tilde{Q}_L \setminus x^*} \cdot w'$ and $\omega^{(i+1)} = T_{e_{2n-i}}\omega^{(i)}$ for $i = n+1, \dots, 2n-1$. It is then easy to verify that $P_{w \xrightarrow{x} w'}$ is an allowed path for the AKG model on $\tilde{Q}_L$, more precisely for any $i \neq n$, $\widetilde{c}_{e_i}(\omega^{(i)}) = 1$, and $\widetilde{c}_{x^*}(\omega^{(n)}) = 1$.

For any $e = (z, z') \in E_{\tilde{Q}_L}$, define $t(e) := (t : z \in Q^t)$ and the weight function ψ by $\psi(e) := j + 1$, where $j := d_1(t(e), u)$. We will denote by

$$|P_{w \xrightarrow{x} w'}|_\psi := 2 \sum_{i=1}^{n-1} \frac{1}{\psi(e_i)}$$

the weighted length of the path $P_{w \xrightarrow{x} w'}$. Using a telescopic sum, and the Cauchy-Schwartz inequality with weight ψ, we get that for any $w, w' \in W$,

$$\begin{aligned}
&\bar{c}_x(\omega)\chi_\Gamma(\omega)(f(\omega_{Q_L \setminus x} \cdot w') - f(\omega_{Q_L \setminus x} \cdot w))^2 \\
&= \bar{c}_x(\omega)\chi_\Gamma(\omega)\left(\sum_{i=1}^{2n-1} f(\omega^{(i+1)}) - f(\omega^{(i)})\right)^2 \\
&\leqslant \chi_\Gamma(\omega) 2 |P_{w \xrightarrow{x} w'}|_\psi \sum_{\sigma, e} \psi(e)\widetilde{c}_e(\sigma)(\nabla_e f)^2(\sigma)\mathbf{1}_{\{(\sigma,\sigma^e) \in P_{w \xrightarrow{x} w'}\}} \\
&\quad + \chi_\Gamma(\omega) 2 \widetilde{c}_{x^*}(\omega^{(n)})(f(\omega^{(n)}_{\tilde{Q}_L \setminus x^*} \cdot w') - f(\omega^{(n)}_{\tilde{Q}_L \setminus x^*} \cdot w))^2.
\end{aligned} \tag{8.13}$$

By construction, uniformly in x, ω and $w, w' \in W$, we have

$$|P_{w \xrightarrow{x} w'}|_\psi \leqslant CM^2 \sum_{j=1}^{2L/M} \frac{1}{j} \leqslant CM^2 \log(L/M) \leqslant C(\log L)^3. \tag{8.14}$$

We get from (8.11), (8.12), (8.13) and (8.14) that

$$\begin{aligned}
\nu_{\tilde{Q}_L}\left(\mathrm{Var}_{\mathcal{B}_\Gamma}(f)\chi_\Gamma\right) \leq C(\log L)^3 &\sum_{x \in \mathcal{B}_\Gamma} \sum_{\omega} \nu_{\tilde{Q}_L}(\omega)\widetilde{c}_x(\omega)\chi_\Gamma(\omega) \\
&\times \sum_{w, w' \in W} \nu(w)\nu(w') \sum_{\sigma, e} \psi(e)\widetilde{c}_e(\sigma)(\nabla_e f)^2(\sigma)\mathbf{1}_{\{(\sigma,\sigma^e) \in P_{w \xrightarrow{x} w'}\}} \\
&+ 2 \sum_{x \in \mathcal{B}_\Gamma} \sum_{\omega} \nu_{\tilde{Q}_L}(\omega)\chi_\Gamma(\omega)\widetilde{c}_{x^*}(\omega^{(n)}) \,\mathrm{Var}_{x^*}(f)(\omega^{(n)}).
\end{aligned}$$

Note that by construction, any $\sigma \in P_{w \xrightarrow{x} w'}$ satisfies $\frac{\nu_{\tilde{Q}_L}(\omega)}{\nu_{\tilde{Q}_L}(\sigma)} \leqslant C$. Hence, using the trivial bound $\overline{c}_x \leq 1$, taking the average with respect to the $2L/M$ possible $v \in \overline{\partial}_- \hat{Q}_{L/M}$, and inverting the summations, we get

$$
\begin{aligned}
&\sum_{\Gamma\in\mathcal{C}_\Pi} \nu_{\tilde{Q}_L}\left(\mathrm{Var}_{\mathcal{B}_\Gamma}(f)\chi_\Gamma\right)\\
&\quad\le \frac{CM}{L}(\log L)^3\sum_{\sigma,e}\nu_{\tilde{Q}_L}(\sigma)\widetilde{c}_e(\sigma)(\nabla_e f)^2(\sigma)\\
&\qquad\times\max_{w,w'\in W}\left\{\psi(e)\sum_{v\in\overline{\partial}_-\hat{Q}_{L/M}}\ \sum_{\Gamma\in\mathcal{C}_\Pi}\ \sum_{x\in\mathcal{B}_\Gamma}\ \sum_{\omega}\chi_\Gamma(\omega)\mathbf{1}_{\{(\sigma,\sigma^e)\in P_{w\xrightarrow{x}w'}\}}\right\}\\
&\qquad+\frac{CM}{L}(\log L)^3\sum_{\sigma}\ \sum_{x^*\in\overline{\partial}^*_-Q_L}\mu_{Q_L}(\sigma)\widetilde{c}_{x^*}(\sigma)\,\mathrm{Var}_{x^*}(f)(\sigma)\\
&\qquad\times\left\{\sum_{v\in\overline{\partial}_-\hat{Q}_{L/M}}\ \sum_{\Gamma\in\mathcal{C}_\Pi}\ \sum_{x\in\mathcal{B}_\Gamma}\ \sum_{\omega}\chi_\Gamma(\omega)\mathbf{1}_{\{O(\omega,x)=(\sigma,x^*)\}}\right\}. \qquad (8.15)
\end{aligned}
$$

Note that ω can be reconstructed from x, σ and e, at the exception of one site where the value of the configuration might be unknown. This, analogously to what occurred in the proof of 5.6, is true thanks to the fact that we have chosen the geometric path from x to the border in such a way that paths are equal for any two configurations which are G-equivalent inside $\tilde{Q}_L\backslash x$. Hence for any σ, e,

$$
\begin{aligned}
&\max_{w,w'\in W}\sum_{v\in\overline{\partial}_-\hat{Q}_{L/M}}\ \sum_{\Gamma\in\mathcal{C}_\Pi}\ \sum_{x\in\mathcal{B}_\Gamma}\ \sum_{\omega}\chi_\Gamma(\omega)\mathbf{1}_{\{(\sigma,\sigma^e)\in P_{w\xrightarrow{x}w'}\}}\\
&\qquad\leqslant |W|M^2\max_{w,w'\in W}\max_{\Gamma,\omega,x}\sharp\left\{v: t(e)\in\gamma_{u,v}\right\}, \qquad (8.16)
\end{aligned}
$$

where v is running over $\overline{\partial}_-\hat{Q}_{L/M}$, $x\in Q_{k_0}$, $\Gamma\in\mathcal{C}_\Pi$ and $t(e)\in\hat{Q}_{L/M}$ is such that if $e=(z,z')$ then $z\in Q^{t(e)}$. Recalling that γ_{uv} is a geodesic path, by the Theorem of Thales (see Fig. 11) given Γ, u and e, one has

$$
\sharp\{v: t(e)\in\gamma_{uv}\}\le C\frac{L}{M\psi(e)}. \qquad (8.17)
$$

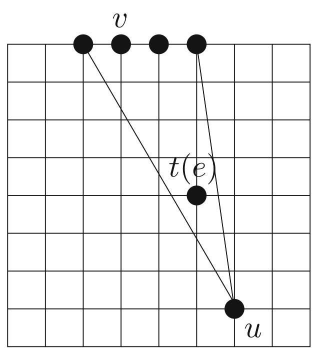

Fig. 11. The set of admissible v such that $t(e)\in\gamma_{uv}$

It follows that

$$\max_{\sigma,e}\ \max_{w,w'\in W}\left\{\psi(e)\sum_{v\in\bar\partial_-\hat Q_{L/M}}\ \sum_{\Gamma\in\Omega_\Pi}\ \sum_{x\in\mathcal{B}_\Gamma}\ \sum_{\omega}\chi_\Gamma(\omega)\mathbf{1}_{\{(\sigma,\sigma^e)\in P_{w\xrightarrow{x}w'}\}}\right\}\leqslant CLM. \quad (8.18)$$

Then, a similar reasoning gives

$$\max_{\sigma,x^*}\sum_{v\in\bar\partial_-\hat Q_{L/M}}\ \sum_{\Gamma\in\mathcal{C}_\Pi}\ \sum_{x\in\mathcal{B}_\Gamma}\ \sum_{\omega}\chi_\Gamma(\omega)\mathbf{1}_{\{O(\omega,x)=(\sigma,x^*)\}}\leqslant CM^2. \quad (8.19)$$

Plugging (8.16), (8.17), (8.18) and (8.19) into (8.15) the proof of the lemma is concluded. □

9. Appendix: Properties of KA Model

In this section we prove Lemma 3.2 and 3.4 in the case $d = j = 2$, since they have been used in the proof of Theorem 4.1. We also prove that $p_c = 1$ for the two-dimensional KA model, Theorem 3.5. The proofs follow the arguments sketched in [30].

We start with the trivial observation that if a region is framed then its empty borders can be rigidly shifted in the interior of the region by proper allowed paths. More precisely we have

Claim 9.1. *Consider a rectangle $R = [0, n] \times [0, m]$ and fix a configuration ω which is R-framed. Then*

(a) *Let σ be such that $\sigma(z) = \omega(z - \vec e_2)$ if $z = i\vec e_1 + m\vec e_2$ with $i \in \{1, \dots, n-1\}$, $\sigma(z) = \omega(z)$ otherwise. There exists an allowed path $P_{\omega,\sigma}$ inside R.*
(b) *Let σ be such that $\sigma(z) = \omega(z - \vec e_1)$ if $z = n\vec e_1 + i\vec e_2$ with $i \in \{0 \dots, m-1\}$, $\sigma(z) = \omega(z)$ otherwise. There exists an allowed path $P_{\omega,\sigma}$ inside R.*

Proof. (a) Consider the geometric path $x_1, \dots, x_{n-2}$ with $x_1 = (n-1)\vec e_1 + (m-1)\vec e_2$, $x_{i+1} = x_i - \vec e_1$. It is immediate to verify that $\omega^{(1)}, \dots, \omega^{(n+1)}$ with $\omega^{(1)} = \omega$, $\omega^{(i+1)} = (\omega^{(i)})^{x_i, x_i - \vec e_2}$ is an allowed path from ω to σ. (b) The proof follows along the same lines as (a). □

Proof of Lemma 3.2. Let $R = [0, n] \times [0, m]$. In order to prove the result it is clearly sufficient to show that for any framed ω and any $e = (x, y) \in E_R$ such that $\omega(x) \neq \omega(y)$ there exists an allowed path P_{ω,ω^e}. Suppose that $y = x + \vec e_1$ (the other cases can be treated analogously) and let $x = x_1\vec e_1 + x_2\vec e_2$. By repeatedly using the path constructed in the proof of Claim b (a) we can construct an allowed path from ω to $\tilde\omega$ with $\tilde\omega(z) = 0$ if $z \cdot \vec e_2 = x_2 + 1$, $\tilde\omega(z) = \omega(z)$ if $z \cdot \vec e_2 \leqslant x_2$ and $\tilde\omega(z) = \omega(z - \vec e_2)$ if $z \cdot \vec e_2 > x_2 + 1$. Then $c_e(\tilde\omega) = 1$ and we can perform the exchange on x, y. Finally, using the reverse of the path to go from ω to $\tilde\omega$, we reconstruct the initial configuration on all sites $z \neq x, y$. □

Claim 9.2. *Choose $e = (x, y) \in E_{\mathbb{Z}^2}$ and $\ell \in \mathbb{N}$ and let $Q^x := Q_\ell + x$, $Q^y := Q_\ell + y$. If there exists $A \in \mathcal{C}_e$ with $\mathcal{C}_e$ defined in (5.1) s.t. $\omega_{Q^z} \in \mathcal{F}_\ell$ for all $z \in A$, then there exists an allowed path inside $Q^x \cup Q^y \cup_{z\in A} Q^z$ from ω to $\omega^{e'}$ for all $e' \in E_{Q_x \cup Q_y}$.*

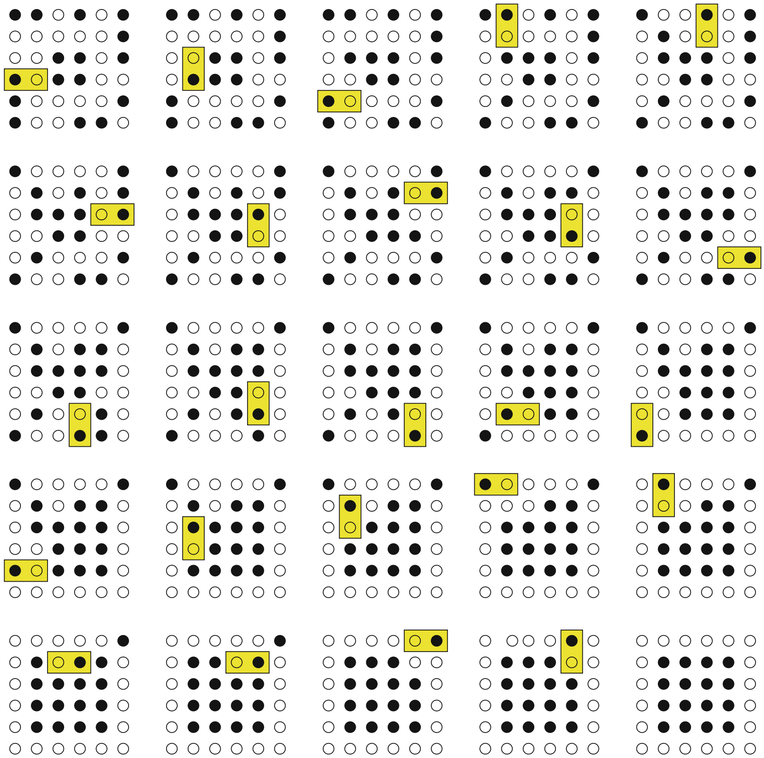

Fig. 12. The path to frame a 6×6 configuration which has at least two empty sites on each of the four independent sides of the central framed 4×4 square

Proof. The result can be proved analogously to Lemma 3.2. We give a rough sketch of the procedure since some additional efforts are required in the construction of the allowed path. Let $e = x, x + \vec{e}_1, e' \in Q_x$ and $A = x - \vec{e}_1, x + \vec{e}_2, x + \vec{e}_1 - \vec{e}_2$. We first construct the frames inside the Q^z with $z \in A$. Then we construct a double empty line adjacent to $Q^x \cup Q^y$ by properly shifting the part of the frames in Q^z which are far away (thanks to Claim b). Finally we can rigidly shift the double empty line of $Q^{x+\vec{e}_1}$ inside $Q^x \cup Q^y$ and bring it as before near the desired bound to perform the exchange. The reason why we construct and shift the double empty lines is a technical trick which is necessary since, at variance with the situation of Lemma 3.2, we do not have here complete frames but frames which cover only two adjacent sides of Q_x. $\square$

Consider a rectangle $R = [a_1, b_1] \times [a_2, b_2]$ and let $\widetilde{\partial}^i_+ R := \{x \notin \Lambda : x + \vec{e}_i \in \Lambda\}$. Divide $\partial^*_+ R$ into four non-intersecting sets $R_1 := \overline{\partial}^1_+ R \cup ((b_1 + 1)\vec{e}_1 + (a_2 - 1)\vec{e}_2)$, $R_2 := \overline{\partial}^2_+ R \cup ((b_1 + 1)\vec{e}_1 + (b_2 + 1)\vec{e}_2)$, $R_3 := \widetilde{\partial}^1_+ R \cup ((a_1 - 1)\vec{e}_1 + (b_2 + 1)\vec{e}_2)$, and $R_4 := \widetilde{\partial}^2_+ R \cup ((a_1 - 1)\vec{e}_1 + (a_2 - 1)\vec{e}_2)$. We call the R_i's the *independent sides* of R. The following property can be easily verified:

Claim 9.3. *If η is R-framed and there exists at least two empty sites inside each of the four independent sides of R, then η is $R \cup \overline{\partial}^*_+ R$ frameable* (*see for example Fig.* 12).

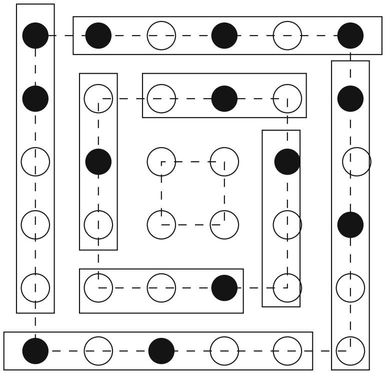

Fig. 13. The bottom left site is $x - 2(\vec{e}_1 + \vec{e}_2)$ and the whole region is $Q(3, x - 2(\vec{e}_1 + \vec{e}_2))$. The sites touched by the dashed lines correspond (from inside to outside) to $Q(1, x)$, $\partial_+^* Q(1, x)$ and $\partial_+^* Q(2, x - (\vec{e}_1 + \vec{e}_2))$. The continuous lines delimit the four independent sides of $\partial_+^* Q(1, x)$ and $\partial_+^* Q(2, x - (\vec{e}_1 + \vec{e}_2))$. The depicted configuration belongs to $\mathcal{F}^0(x, 2)$

Proof of Lemma 3.4. For any integer n, any $x \in \mathbb{Z}^2$, let $Q(n, x) = Q_{2n} + x$. Note that $Q(n+1, x - \vec{e}_1 - \vec{e}_2) = Q(n, x) \cup \partial_+^* Q(n, x)$. Let $\mathcal{F}^0(x, n)$ be the set of configurations such that $Q(1, x)$ is empty and for each $i \in (1, n-1)$ there are at least two empty sites on each independent side of $Q(i, x - (i-1)(\vec{e}_1 + \vec{e}_2))$ (see Fig. 13). A direct calculation gives

$$\mu(\mathcal{F}^0(n, x)) = q^4 \prod_{k=2}^{n} \left(1 - (1-q)^{2k-1} - (2k-1)q(1-q)^{2k-2}\right)^4 .$$

Thus for any $q > 0$, $\mu(\mathcal{F}^0(n, x))$ converges to a non-zero limit (independent from x) when n tends to infinity. Moreover, the limit $n \to \infty$ and $q \to 0$ is computed in [30]:

$$\lim_{q \to 0} \lim_{n \to \infty} q \log \mu(\mathcal{F}^0(n, x)) = 4\alpha := 2 \int_0^1 \frac{\log(1 - y + y \log y)}{y} dy.$$

Fix $q \in (0, 1)$. From the limit above, there exists $n_0 = n_0(q)$ such that for any $n \geq n_0$,

$$\mu(\mathcal{F}^0(n, x)) \geqslant e^{-\alpha/q}. \tag{9.1}$$

Now for any n, any $x \in \mathbb{Z}^2$, we define $\mathcal{F}^1(n, x)$ as the set of all configurations $\omega \in \Omega$ such that every horizontal and vertical row (of length n) inside $Q(n, x)$ has at least two empty sites. All the sets $\mathcal{F}^1(n, x)$ with $x \in \mathbb{Z}^2$ have the same probability. By FKG inequality one gets

$$\mu(\mathcal{F}^1(n, x)) \geq \left(1 - (1-q)^n - nq(1-q)^{n-1}\right)^{2n} . \tag{9.2}$$

Finally, we divide the box Q_ℓ into a collection of smaller boxes of size $\sqrt{\ell}$. Assuming that $\sqrt{\ell} \in \mathbb{N}$ we let $\mathbb{Z}^2(\sqrt{\ell}) := \sqrt{\ell}\mathbb{Z}^2$ and $\tilde{Q}_{\sqrt{\ell}} := \mathbb{Z}(\sqrt{\ell}) \cap Q_\ell$. Then $Q_\ell = \cup_{x \in \tilde{Q}_{\sqrt{\ell}}} Q(\sqrt{\ell}, x)$ (see Fig. 14). Then we set $\overline{Q}_{\sqrt{\ell}} := (x \in \tilde{Q}_{\sqrt{\ell}} : x = 2i\vec{e}_1 + 2j\vec{e}_2$ with $i, j \in \mathbb{N})$ and define

$$\mathcal{A}_\ell := \{\omega : \exists x \in \overline{Q}_{\sqrt{\ell}} \text{ s.t. } \omega \in \mathcal{F}^0(\sqrt{\ell}, x\sqrt{\ell})\}$$

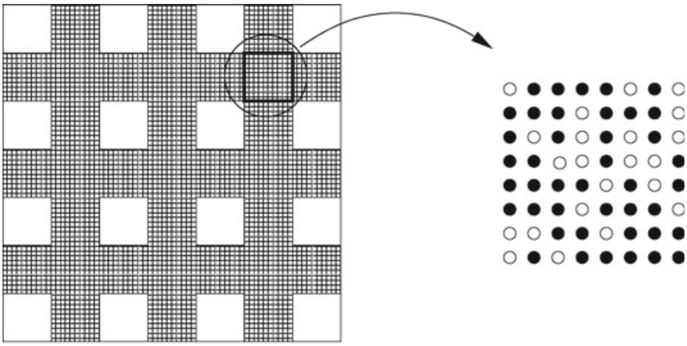

Fig. 14. The large box is Q_ℓ divided into small boxes of size $\sqrt{\ell}$. The bottom left site of each small box is a point in $\tilde{Q}_{\sqrt{\ell}}$. The bottom left sites of the white boxes are the points in $\overline{Q}_{\sqrt{\ell}}$. On the right is an example of configuration $\omega \in \mathcal{F}^1(\sqrt{\ell}, y\sqrt{\ell})$ for some $y \in Q_{\sqrt{\ell}} \setminus \tilde{Q}_{\sqrt{\ell}}$

and

$$\mathcal{B}_\ell := \{\omega : \omega \in \mathcal{F}^1(\sqrt{\ell}, x) \ \forall x \text{ s.t. } x = z\sqrt{\ell} \text{ with } z \in \tilde{Q}_{\sqrt{\ell}} \setminus \overline{Q}_{\sqrt{\ell}}\}.$$

By using Claim 9.3, it follows by construction that $\mathcal{A}_\ell \cap \mathcal{B}_\ell \subset \mathcal{F}_\ell$.

Furthermore, thanks to (9.1) and (9.2) and using the fact that the events which define $\mathcal{A}_\ell$ and $\mathcal{B}_\ell$ are independent, we get

$$\mu(\mathcal{F}_\ell) \geq \mu(\mathcal{A}_\ell)\mu(\mathcal{B}_\ell) \geq \left(1 - \left[1 - \mu(\mathcal{F}^0(\sqrt{\ell}, 0))\right]^{\ell/2}\right) \left(\mu(\mathcal{F}^1(\sqrt{\ell}, 0))\right)^{\ell/2}$$
$$\geq \left(1 - e^{-c\ell e^{-\alpha/q}}\right) e^{-c\ell^2(1-q)^{\sqrt{\ell}}}$$

for a proper constant $c = c(q)$. Thus for any ϵ there exists $\ell_0(\epsilon, q)$ such that for $\ell > \ell_0$ we get $\mu(\mathcal{F}_\ell) > 1 - \epsilon$. $\square$

Proof of Theorem 3.5. Let Λ_ℓ be a cube of size $a\ell$. Then $\lim_{\ell\to\infty} \mu(\cap_{e\in E_{\Lambda_\ell}} \mathcal{E}_e) = \mu(\cap_{e\in E_{\mathbb{Z}^2}} \mathcal{E}_e)$. For a given bond $e = (x, y)$ we let $\mathcal{F}_\ell^e$ be the set of configuration inside $\mathcal{F}_\ell$ which remain Q_ℓ-frameable even if we fill both x and y, namely $\mathcal{F}_\ell^e := \{\omega \in \Omega : \omega_{Q_\ell \setminus (x,y)} \cdot 1_x \cdot 1_y$ is $Q_\ell -$ frameable $\}$. By proceeding along the same lines as the proof of the above Lemma 3.4, it is easy to verify that there exists $c_1, c_2 > 0$ s.t. $\mu(\mathcal{F}_\ell^e) > 1 - c_1 \exp(-c_2\ell)$. Furthermore $\mathcal{F}_\ell^e \subset \mathcal{E}_e^{Q_\ell} \subset \mathcal{E}_e$. Indeed if $\eta \in \mathcal{F}_\ell^e$ then both η and η^e are frameable, thus there exists an allowed path from η to η^e which goes through the corresponding framed configurations which in turn are connected thanks to Corollary 3.3. Therefore $\mu(\cap_{e\in E_\Lambda} \mathcal{E}_e) \geqslant 1 - |E_\Lambda|(1-\mu(\mathcal{F}_\ell^e)) \geqslant 1 - 2\ell^2 a^2 c_1 \exp(-\ell c_2)$, which goes to one when $\ell \to \infty$ and the proof is concluded. $\square$

References

1. Aldous, D., Diaconis, P.: The asymmetric one-dimensional constrained Ising model: rigorous results. J. Stat. Phys. **107**(5-6), 945–975 (2002)
2. Ané, C., Blachère, S., Chafaï, D., Fougères, P., Gentil, I., Malrieu, F., Roberto, C., Scheffer, G.: *Sur les inégalités de Sobolev logarithmiques*, Paris: Société Mathématique de France, 2000, with a preface by Dominique Bakry and Michel Ledoux

3. Barrat, A., Kurchan, J., Loreto, V., Sellitto, M.: Edwards measures for powders and glasses. Phys. Rev. Lett. **85**, 5034–5038 (2000)
4. Berthier, L., Garrahan, J.P., Whitelam, S.: Dynamic criticality in glass forming liquids. Phys. Rev. Lett. **92**, 185705–185709 (2004)
5. Bertini, L., Cancrini, N., Cesi, F.: The spectral gap for a Glauber-type dynamics in a continuous gas. Ann. Inst. H. Poincaré Prob. Stat. **38**(1), 91–108 (2002)
6. Bertini, L., Toninelli, C.: Exclusion processes with degenerate rates: Convergence to equilibrium and tagged particle. J. Stat. Phys. **117**, 549–580 (2004)
7. Cancrini, N., Martinelli, F.: On the spectral gap of Kawasaki dynamics under a mixing condition revisited. J. Math. Phys. **41**(3), 1391–1423 (2000)
8. Cancrini, N., Martinelli, F., Roberto, C., Toninelli, C.: *Facilitated spin models: recent and new results*. Proceedings of Prague summer school 2006 on Mathematical Statistical Mechanics, available at http://arxiv.org/abs/0712.1934v1[math.PR], 2007
9. Cancrini, N., Martinelli, F., Roberto, C., Toninelli, C.: Kinetically constrained spin models. Probab. Th. and Rel. Fields **140**, 459–504 (2008)
10. Eisinger, S., Jackle, J.: A hierarchically constrained kinetic ising model. Z. Phys. B **84**, 115–124 (1991)
11. Evans, M.R., Sollich, P.: Glassy time-scale divergence and anomalous coarsening in a kinetically constrained spin chain. Phys. Rev. Lett **83**, 3238–3241 (1999)
12. Fredrickson, G.H., Andersen, H.C.: Kinetic ising model of the glass transition. Phys. Rev. Lett. **53**, 1244–1247 (1984)
13. Goncalves, P., Landim, C., Toninelli, C.: Hydrodinamic limit for a particle system with degenerate rates. Annals of IHP Prob. Stat. **45**, 887–909 (2009)
14. Grimmett, G.: *Percolation*. Second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. **321**, Berlin: Springer-Verlag, 1999
15. Kob, W., Andersen, H.C.: Kinetic lattice-gas model of cage effects in high-density liquids and a test of mode-coupling theory of the ideal-glass transition. Phys. Rev. E **48**, 4359–4363 (1993)
16. Kordzakhia, G., Lalley, S.: Ergodicity and mixing properties of the northeast models. J. Appl. Probab. **43**(3), 782–792 (2006)
17. Kurchan, J., Peliti, L., Sellitto, M.: Aging in lattice-gas models with constrained dynamics. Europhys. Lett **39**(4), 365–370 (1997)
18. Lawlor, A., De Gregorio, P., Bradley, P., Sellitto, M., Dawson, K.A.: Geometry of dynamically available empty space is the key to near-arrest dynamics. Phys. Rev. E **72**, 021401 (2005)
19. Liggett, T.M.: *Interacting particle systems*. New York: Springer-Verlag, 1985
20. Lu, S.L., Yau, H.-T.: Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics. Commun. Math. Phys. **156**(2), 399–433 (1993)
21. Marinari, E., Pitard, E.: Spatial correlations in the relaxation of the kob-andersen model. Europhys. Lett. **69**(2), 35–241 (2005)
22. Martinelli, F.: *Lectures on Glauber dynamics for discrete spin models*, In: Lectures on probability theory and statistics (Saint-Flour, 1997), Berlin: Springer, 1999, pp. 93–191
23. Quastel, J.: Diffusion of color in the simple exclusion process. Comm. Pure Appl. Math. **45**(6), 623–679 (1992)
24. Ritort, F., Sollich, P.: Glassy dynamics of kinetically constrained models. Adv. in Phys. **52**(4), 219–342 (2003)
25. Sellitto, M., Arenzon, J.J.: Free-volume kinetic models of granular matter. Phys. Rev. E **62**, 7793 (2000)
26. Franz, S., Mulet, R., Parisi, G.: Kob-andersen model: A nonstandard mechanism for the glassy transition. Phys. Rev. E **65**(2), 021506 (1987)
27. Toninelli, C., Biroli, G.: Dynamical arrest, tracer diffusion and kinetically constrained lattice gases. J. Stat. Phys. **117**, 7–54 (2004)
28. Toninelli, C., Biroli, G.: A new class of cellular automata with a discontinuous glass transition. J. Stat. Phys. **130**, 83–112 (2008)
29. Toninelli, C., Biroli, G., Fisher, D.S.: Spatial structures and dynamics of kinetically constrained models for glasses. Phys. Rev. Lett. **92**(1-2), 185504 (2004)
30. Toninelli, C., Biroli, G., Fisher, D.S.: Kinetically constrained lattice gases for glassy systems. J. Stat. Phys. **120**(1-2), 167–238 (2005)

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