Kinetically Constrained Models

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Many fascinating questions still remain unsettled for condensed matter physicists who study the *glass* and *jamming transitions*. These phenomena, which lead to the formation of amorphous solids, occur in several microscopically different systems as supercooled liquids, colloidal suspensions and vibrated granular materials [9, 27]. Basic *glassy properties* include a dramatic slowing down of dynamics when a proper external parameter is tuned (e.g. temperature is lowered for liquids) and the occurrence of a complicated relaxation: non exponential and spatially heterogeneous. When relaxation times become longer than experimental scales, equilibrium can no more be achieved: the system undergoes a dynamical arrest and freezes into an amorphous phase (the glass). The main issues in understanding these phenomena remain unsolved. In particular, it is not clear whether the dynamical arrest is due to the proximity of a phase transition and whether this is a static or purely dynamical one. The experiments on molecular liquids show that, if such an *ideal glass transition* occurs, it should have an unconventional behavior with *mixed first and second order features*. On the one hand, the divergence of relaxation times and the fact that both entropy and internal energy seem continuous is indicative of a second order transition. On the other hand a discontinuous order parameter is detected: the height of the plateau of the Fourier transform of the density–density correlation has a finite jump. This corresponds to the fact that the modulation of the microscopic density profile of the glass does not appear continuously from the flat liquid

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profile. Besides these mixed first/second order properties, another unconventional feature concerns the scaling of relaxation times which increase much more rapidly than for conventional second order transitions. Indeed most liquids display a faster than power law divergence around the glass transition, a signal of a cooperative relaxation on increasingly large scales as the temperature is decreased towards the transition. A very successful fit is the Vogel-Fulcher law: $\log \tau \simeq 1/(T - T_0)$. Finally another puzzling features is the absence of any experimental evidence of a static diverging correlation length: typical glass configurations are not very different from instantaneous configurations of the liquid and the dramatic slowing down of dynamics is apparently not due to an increasing long range order. An enormous amount of theoretical approaches has been proposed in the last fifty years to describe these phenomena. Among the theories which assume that a thermodynamic glass transition takes place at a finite temperature we recall mode coupling theories [14] and the random first order scenario [20]. Here we deal instead with Kinetically Constrained Models (KCM) which have been introduced in the 80's (see [24] for a review) and are based on the ansatz of a purely dynamical transition. The latter would be the result of the geometrical constraints on the rearrangements of molecules which become more and more important as the temperature of the liquid is lowered (the density is increased).

KCM are stochastic lattice gases with hard core exclusion, namely on each site there is one or zero particle. The configuration on a lattice Λ is thus defined by assigning to each $x \in \Lambda$ its occupation variable: $\eta_x = 1$ or $\eta_x = 0$ if the site is occupied or empty, respectively. The dynamics is given by a continuous time Markov process which consists of a sequence of jumps for models with conservative (Kawasaki) dynamics and creation/destruction of particles for models with non conservative (Glauber) dynamics. The former are also known as Kinetically Constrained Lattice Gases (KCLG) and the latter as Kinetically Constrained Spin Models (KCSM)[1] (the occupation variables can indeed be interpreted as up and down spins which can be flipped). For all the models introduced in physics literature dynamics satisfies detailed balance w.r.t. to Bernoulli product measure (see instead [6] for the extension to models which are reversible w.r.t. high temperature Gibbs measures). Thus

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[1]In physics literature the equivalent terminology "Facilitated Spin Models" is also used which, instead of emphasizing the presence of constraints, puts the accent on the complementary fact that proper events facilitate, i.e. allow, the elementary moves.

there are no static interactions beyond hard core and an equilibrium transition cannot take place. The key feature of both KCSM and KCLG is that an elementary move can occur only if the configuration verifies proper local constraints besides hard core. The latter mimic the geometric constraints on the possible rearrangements in physical systems, which could be at the root of the dynamical arrest [13, 21]. As we shall discuss, numerical simulations show that for proper choices of the constraints KCM indeed display glassy features. These include heterogeneous relaxation, faster than power law divergence of relaxation times τ and dynamical transitions. Therefore several analytical and numerical works have recently attempted to understand the mechanism which induces these glassy properties and to derive the typical time/length scales which are involved. Numerical simulations are however very delicate due to the rapid divergence of τ as the particle density p is increased as well as the non-trivial scaling of finite size effects. (Note that in order to compare with the above discussion on liquid/glass transition one should perform the change $p \rightarrow 1/(1 + \exp(-1/T))$ to have temperature rather than density as the control parameter.)

Here we review our recent mathematical results [6–8] on KCM which have contributed to settle some debated questions arising in numerical simulations. In particular in [8] we have introduced a new technique trough which we obtain upper and lower bounds on the spectral gap of the Markov process and therefore on the relaxation time τ which, as we shall see, is directly related to the inverse of the spectral gap. We focus for simplicity on KCSM and discuss only at the end the more recent results [7] for KCLG. The dynamics of a KCSM on an integer lattice $\Lambda \subset \mathbb{Z}^d$ is precisely defined as follows. Each site waits an independent mean one exponential time and then, provided that the current configuration satisfies a local constraint which does not involve η_x, it refreshes its state. This is set to occupied with probability p and empty with probability $q = 1 - p$. In other words, if $c_x(\eta)$ is the function which equals one (zero) when the constraint is (is not) verified, each site changes its current state with rate $c_x(\eta)[(1-p)\eta_x + p(1-\eta_x)]$. Since $c_x(\eta)$ does not depend on η_x detailed balance w.r.t. Bernoulli(p) measure μ_p is easily verified and μ_p is an invariant measure. As a direct consequence of the fact that the rates can degenerate to zero, there exist blocked configurations (s.t. on each site the constraint is not satisfied) as well as configurations which are not blocked but nevertheless contain a frozen backbone, i.e. a subset of sites on which for sure the constraint is not verified at any instant of time. Thus μ_p is not the

unique invariant measure, for example any measure concentrated on a blocked configuration is also invariant. By taking proper superpositions of blocked configurations it is also possible to construct stationary measures which are also translational invariant (see [22] for a more detailed discussion). In order to model the geometric constraints on highly dense liquids, c_x usually specifies the maximal number of occupied sites in a proper neighborhood of x. Thus the dynamics becomes increasingly slow as p is increased and an ergodicity breaking transition may occur at a finite critical density, $p_c < 1$. More precisely, if we denote by $\mathcal{L}$ the generator of the Markov process on $\mathbb{Z}^d$, p_c separates the density regime in which the semigroup $P_t := e^{t\mathcal{L}}$ does (does not) converge to μ_p in the large time limit, namely $\lim_{t\to\infty} P_t f = \mu_p(f)$ for all $f \in L^2(\mu_p)$ iff $p < p_c$. As it is explained in Section 2.3 of [8], the dynamical arrest at p_c corresponds to the fact that an infinite spanning cluster of mutually blocked particles occurs. One of the main issues studied in physics literature is the long-time dynamics in the ergodic regime, in particular the scaling of the typical times when p approaches p_c from below. The most studied dynamical quantities are the spin-spin time auto-correlation $C(t)$ and the persistence function $F(t)$ defined as follows

$$C(t) := \int d\mu_p(\eta(0))\eta_0(0)E_\eta\left[\eta_0(t)\right] - p^2$$

where $E_{\eta(0)}[f(\eta(t))]$ is the expectation over the Markov process at time t when the initial configuration is $\eta(0)$ and

$$F(t) := \int d\mu_p(\eta(0))\ P[\eta_0(s) = \eta_0(0)\ \forall s < t],$$

namely $F(t)$ is the probability that up to time t the occupation variable of the origin has never changed. A first key issue is whether $C(t)$ and $F(t)$ decay exponentially as for the unconstrained models (i.e. for $c_x(\eta) = 1$). Furthermore one would like to determine the scalings with p of the typical time scale τ which enter in their decay. By analyzing the spectral gap of the generator $\mathcal{L}$, namely

$$\text{gap}(\mathcal{L}) := \inf_{\substack{f \in L^2(\mu_p) \\ f \neq const.}} \frac{\mu_p(f, -\mathcal{L}f)}{\mu_p(f - \mu_p(f))^2} \tag{1}$$

and using the Poincaré inequality

$$\text{Var}(P_t(f)) \leq \exp(-2t\ \text{gap}(\mathcal{L}))\text{Var}(f) \tag{2}$$

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and the inequality established in Theorem 3.6 of [8] via a Feynman-Kac bound

$$F(t) \leq \exp(-qtc/\text{gap}) \tag{3}$$

where c is a constant independent on q, we will obtain rigorous answers to the above questions. In particular a strictly positive spectral gap together with (2) and (3) imply an exponential decay for both $C(t)$ and $F(t)$. As we will detail below, in some cases we prove and in other cases we disprove the conjectures in physics literature. Note that the above Poincaré inequality means that the inverse of the spectral gap is the worst relaxation time over all one time quantities. Thus, when referring to our results, τ will always stand for 1/gap. Analogously $\tau(L)$ will be the inverse of the spectral gap of the generator of the process restricted to a square lattice Λ_L of linear size L (with properly specified boundary conditions).

Let us recall a standard classification before introducing the specific choices of the constraints that we analyze. KCSMs can be divided into two classes: (i) non-cooperative and (ii) cooperative models. For the former it is (for the latter it is not) possible to construct an allowed path which completely empties *any* configuration which contains somewhere a proper finite cluster of vacancies. Non-cooperative models are ergodic at any density, namely $p_c = 1$, while p_c is finite for some of the cooperative models. Thus we can further classify cooperative models into: (iia) models that are ergodic in the thermodynamic limit at any $p < 1$, i.e. $p_c = 1$; (iib) models that display an ergodicity breaking transition at $p_c < 1$. Cooperative models are usually considered more interesting since their relaxation involves the collective rearrangements of increasingly large regions as the density increases, analogously to what experiments suggest near the glass transition.

Among non cooperative models we recall the Fredrickson-Andersen [13] one spin facilitated (FA1f) for which a move at x is allowed only if at least one of its nearest neighbors is empty: $c_x(\eta) = 1$ if $\sum_{y \; n.n.x}(1-\eta_x) > 0$, $c_x(\eta) = 0$ otherwise. In this case the presence of a single vacancy allows to empty the whole lattice. In [3, 4] a dynamical field theory was derived yielding an exponential decay for $C(t)$ with a typical scale for $q \to 0$ as $\tau \propto 1/q^z$ with $z=3$ for $d=1$, $z=2 + \epsilon(d)$ with $\epsilon(2) \simeq 0.3$, $\epsilon(3) \simeq 0.1$ and $\epsilon(d \geq 4) = 0$. An exact mapping into a diffusion limited aggregation model and its renormalization [18] gives instead $d=2$ as the upper critical dimension and $\epsilon(d)=0$ in $d \geq 2$. Our results are: gap $\propto q^3$ in $d = 1$, $q^2/|\log q| \leq$ gap $\leq q^2$ in $d = 2$ and

$q^2 \leq \text{gap} \leq q^{1+2/d}$ in $d = 3$. Thus we get $\epsilon(2) = 0$ and $\epsilon(3) \leq 0$, disproving the findings in [3, 4] and confirming those in [18].

Another popular model is the one-dimensional East model [10] for which the constraint requires a vacancy on the right nearest neighbor, i.e. $c_x(\eta) = (1 - \eta_{x+1})$. On a finite volume the presence of a single vacancy on the rightmost site allows to empty the whole lattice. However the East does not belong to the non-cooperative class since, due to the directed nature of constraints, the vacancy should occur in a specific position and the relaxation involves the cooperative rearrangements of large regions as $p \rightarrow 1$. The scaling $\log \tau \sim (\log(1/q))^2$ had indeed been conjectured in [10,11] and proved in [2]. In [8] we prove the exact asymptotics $\log \tau = c(\log(1/q))^2$ where $c = 1/(2 \log 2)$. Our result differs from the $c = 1/\log 2$ value incorrectly derived in [11]. As we clarify in [6], this is due to the fact that the relation between length and time scales extrapolated in [11] from coarsening dynamics does not lead to the correct equilibrium result unless relaxation on scales smaller than the typical distance of two vacancies is also taken into account.

Among cooperative models without transition (iia) we consider instead FAjf on an hyper-cubic lattice of dimension d with $2 \leq j \leq d$ [13]: the constraint requires at least j empty nearest neighbours. As can be directly checked, for all these models it is not possible to devise a finite seed of vacancies which allows emptying the whole lattice, thus the models are cooperative. Consider, e.g., the case $d = 2$, $j = 2$ (with periodic boundary conditions) and focus on a configuration which contains two adjacent rows which are completely filled. It is easy to verify that these particles can never be erased, not even if the rest of the lattice is completely empty. The upper restriction on j comes from the fact if $j > d$ there exist finite sets of forever blocked particles. Thus a fraction of the system is frozen at all densities ($p_c = 0$) and the models are not suitable to describe the slow dynamics close to glass-jamming transitions. The choices which have been most studied in physics literature are $j = 2$ both in $d = 2$ and $d = 3$ and $j = 3$ in $d = 3$. In all cases stretched exponential relaxation has been numerically detected: $C(t)$ and $F(t)$ are fitted with $\exp\left(-(t/\tau)^\beta\right)$ with β decreasing as the density p is increased [12, 15, 16]. For the scaling of τ with p, as pointed out in [24], little is known beyond the general recognition that the behavior is reminiscent to the one of supercooled liquids. Among the different forms proposed for FA2f we recall Vogel-Fulcher [15] and $\exp(c/q)$ [5]. The latter form is supported by the conjecture that relaxation occurs via the diffusion of critical droplets of size $1/q$ over distances $\exp(c/q)$ [23]. Our results are as follows. For all

$j \leq d$ and all dimensions we prove that the spectral gap is strictly positive for $p < p_c = 1$: exponential relaxation occurs both for $C(t)$ and $F(t)$ contradicting the stretched exponential conjecture of [12,15,16] and confirming the exponential decay derived in [11]. Furthermore for FA2f and FA3f we prove $\exp q^{-1} \leq \tau \leq \exp q^{-5}$ and $\exp \exp q^{-1} \leq \tau \leq \exp \exp q^{-2}$, respectively. Thus we establish a super-Arrhenius scaling compatible with [5,23] and exclude the form proposed in [15]. Also, we believe that the upper bound for FA2f can be ameliorated to $\tau \leq \exp q^{-2}$.

Among (iib) models, we consider the two dimensional North-East and the Spiral models. For the former [19] both the up $(x + \vec{e}_2)$ and right $(x + \vec{e}_1)$ neighbors should be empty in order for a move at x to be allowed ($\vec{e}_i$ are the unit basis vectors). For the Spiral model [25] the constraint is more complicated. Let the NE, NW, SW and SE neighbours of x be defined respectively as $(x + \vec{e}_2, x + \vec{e}_1 + \vec{e}_2)$, $(x - \vec{e}_1, x - \vec{e}_1 + \vec{e}_2)$, $(x - \vec{e}_2, x - \vec{e}_1 - \vec{e}_2)$ and $(x + \vec{e}_1, x + \vec{e}_1 - \vec{e}_2)$. Then the constraint at x goes as follows: (a) the two NE and/or the two SW neighbours of x should be empty and (b) the two NW and/or the two SE neighbours of x should be empty too. For both the North-East and Spiral model the cluster of frozen particles arises at $p_c = \rho_{dp}$ with ρ_{dp} the critical density of directed percolation. In the case of North-East there is a trivial one to one correspondence between directed percolation clusters and frozen clusters. As a consequence the transition is continuous, namely the density of the frozen backbone is zero at p_c. Instead for the Spiral model the mechanism is much more subtle [25]: the presence of proper directed clusters imply the occurrence of blocked clusters but the converse is not true. Indeed the proof of $p_c = \rho_{dp}$ is much more involved [25] and the transition is here due to the interaction between two independent directed percolation processes. Furthermore the transition is expected to display mixed first/second order features [25, 26]: the density of the frozen backbone is finite at p_c and the size of the frozen cluster diverges as $p \nearrow p_c$. Thus the Spiral model is a KCSM whose ergodicity breaking transition has the features of an ideal glass transition. For both North-East and Spiral models we prove that the spectral gap in infinite volume is strictly positive for any $p < p_c$. Therefore in the whole ergodic region $C(t)$ and $F(t)$ decay exponentially. At criticality, $p = \rho_{dp}$, we prove that relaxation is instead polynomial or slower than polynomial in time. Finally, for $p > \rho_{dp}$, we prove that a strong signature of the infinite volume ergodicity breaking occurs if one considers the model on a finite volume of linear size L. The relaxation time is uniform on L for $p < p_c$ and diverges as $\tau(L) \propto \exp(L\, c(p))$ for $p > p_c$.

We will sketch our technique to derive the positivity of the spectral gap and its scaling when $p \nearrow p_c$ by focusing on FA2f in $d = 2$ (see [8] for rigorous proofs). We will comment at the end on the flexibility of the tools which allow indeed to deal with all the other choices of the constraints discussed above as well as with more general models, including those with long range constraints [7] and with static interactions other than hard core [6]. Before entering in the details we wish to underline that from the mathematical point of view the main difficulties come from the existence of several invariant measures and from the fact that KCSM are not attractive, thus the usual coupling arguments cannot be applied. Both features are a direct consequence of the constraints, i.e. of the fact that the creation/destruction rates may degenerate to zero. This explains why the basic issues concerning the large time behavior of KCSM are non trivial and why they remained open for most of the interesting models, with the notable exception of the East for which in [2] the positivity of the spectral gap had been established. However the method of [2] uses the specifics of the East model and it cannot be applied neither to higher dimensions nor to the above discussed cooperative models which are relevant for physics literature.

In order to study the spectral gap of FA2f we proceed as follows. First we introduce an auxiliary KCSM model, the *General Model (GM)*, which has N-valued occupation variables and we study its relaxation time, τ_{GM}. Then we establish an upper bound on the relaxation time for FA2f, τ_{FA}, by using a renormalization procedure which forces the GM constraints on scales larger than a proper block size and leaves inside each block the original FA2f dynamics. Finally we derive a lower bound on τ_{FA} by using the knowledge of the typical regions which have to be rearranged to create/destroy a particle.

Let $n_x \in S$ be an N-valued occupation variable and choose a probability measure, ν, on S. We identify a subset G of S which we call the *good event* and we say that a site x is *good* if $n_x \in G$. GM dynamics is defined as follows. Each site x waits a mean one exponential time and then n_x is refreshed by a new value n'_x sampled from ν, provided its North, North-East and East neighbors (*i.e.* $x + \vec{e}_1, x + \vec{e}_1 + \vec{e}_2, x + \vec{e}_2$) are good. If this constraint is not satisfied n_x remains unchanged. We consider GM on a square lattice Λ_L of linear size L with good boundary conditions on the top and right boundaries to ensure ergodicity (i.e. the existence of an allowed path which connects any two configurations which in finite volume guarantees $\tau_{GM}(L) < \infty$). In order to evaluate $\tau_{GM}(L)$ we follow a *bisection-constrained method*.
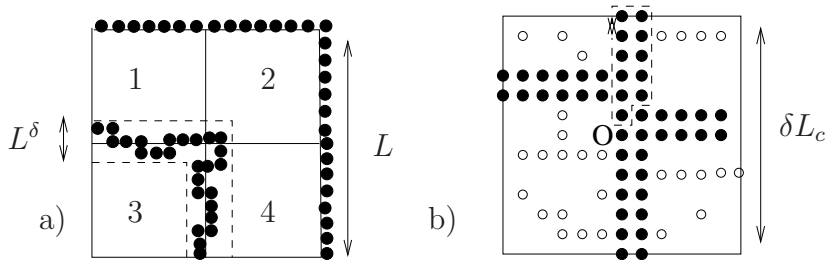
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Figure 1: a) Block dynamics for GM: percolating path of good sites (•) required to renew configuration on 3. b) Blocking event for FA2f. • (∘) stand for particles which do (do not) belong to the backbone. Sites inside dotted line form one of the sequences of $\geq \delta L_c/2$ sites to be emptied before O.

Partition Λ_L into four blocks as in fig.1a) and define the following auxiliary block accelerated dynamics. Each block waits a mean one exponential time and then its configuration is replaced by a new one chosen according to the product equilibrium probability given by ν. On the top right block (block 2 in fig.1a)) this move is always allowed. For the others, a constraint should be satisfied: on an l-shaped frame of width L^δ, $\delta < 1$, there should be a percolating cluster of good sites as in fig.1a). In other words the constraint requires the good GM boundary conditions on block 3 (see fig.1a)) and the same for blocks 1 and 4 (instead on block 2 they are automatically guaranteed by the boundary condition on Λ_L). Then

$$\tau_{GM}(L) \leq \tau_{\text{block}}^{GM}(L)\, \tau_{GM}(L/2)$$

with $\tau_{\text{block}}^{GM}(L)$ the relaxation time for the block dynamics. The above inequality (see [8] for a rigorous proof) corresponds intuitively to a two step relaxation: first on the block scale, then inside each individual block. If the probability that a site is good, $\nu(G)$, is larger than the threshold of site percolation ρ_{sp} the constraint of the block dynamics is satisfied with probability $\sim 1 - \exp(-mL^\delta)$ and $\tau_{\text{block}}^{GM}(L) \simeq \left(1 + \exp(-mL^\delta)\right)$. Then, by dividing $\Lambda_{L/2}$ into four blocks and so on up to constant size, we get

$$\tau_{GM}(L) \leq c \prod_n \tau_{\text{block}}^{GM}(2^{-n}L),$$

where c is a finite constant and the product contains $O(\log L)$ terms. Therefore we get a bound for $\tau_{GM}(L)$ which does not dependent on L provided

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the product over the τ^{GM}_{block}'s converges. From the above observation, this certainly occurs for $\nu(G) > \rho_{sp}$.

Let us now consider FA2f in $d = 2$ on Λ_L with empty boundary conditions on the top and right borders. We partition Λ_L into disjoint blocks of size $k\,L_c$, where $k \gg 1$ and $L_c = \exp(\pi^2/(18q))$. We can now define the following auxiliary dynamics: each block waits a mean one exponential time and then its configuration is replaced by a new one chosen according to μ_p provided the three neighbouring blocks in the North, East and North-East direction are *internally spanned*. By internally spanned we mean that each of these blocks can be completely emptied by a proper sequence allowed moves when we consider occupied boundary conditions on it. The probability that a block of linear size ℓ is internally spanned has been evaluated in the context of bootstrap percolation: it goes to one exponentially fast when ℓ exceeds the crossover length L_c defined above [1, 17]. Applying as before a two step relaxation argument, we get $\tau_{FA}(L) \leq \tau^{FA}_{block}(L)\,\tau_{FA}(kL_c)$ where $\tau^{FA}_{block}(L)$ is the relaxation time of the above defined block dynamics, which a priori depends on the number of blocks and therefore on L. We will now show that $\tau^{FA}_{block}(L) \simeq 1$. Take a square lattice with $(L/kL_c)^2$ sites and define on each site a $2^{(kL_c)^2}$-valued occupation variable, n_x, belonging to $S = (0,1)^{(kL_c)^2}$. It is immediate to verify that S and $S^{(L/kL_c)^2}$ are the configuration space of FA2f on a block of size kL_c and on Λ_L, respectively. Furthermore, in terms of the n_x variables, the above defined block dynamics coincides with GM with the choices: n_x is good when the corresponding block in Λ_L is internally spanned and ν equals μ_p restricted to the block. Therefore, thanks to this mapping and our result for τ_{GM}, we get $\tau^{FA}_{\text{block}}(L) = \tau^{GM}(L/kL_c) \simeq 1$ since the probability of the good event "a block of size kL_c with $k >> 1$ is internally spanned" is $\nu(G) \simeq 1$ [17]. A few remarks are in order. In our renormalization procedure we have forced on the block scale the directed GM constraint which is more restrictive than the one of FA2f. This choice, which is due to the necessity of boundary conditions which ensure ergodicity for FA2f dynamics inside each block [8], is allowed because we are deriving an upper bound on τ_{FA} and the effect of a stronger constraint is to slow down the dynamics. Furthermore, as explained above, for our choice of the block scale $\tau_{GM} \simeq 1$. This means that using GM instead of FA2f constraints on large blocks does not change the leading behavior of the upper bound. Putting above results together we conclude that $\tau_{FA}(L) \leq \tau_{FA}(kL_c)$ and, since L_c depends on p but not on L and $L_c(p) < \infty$ for $p < p_c = 1$, the relaxation time of

FA2f is uniformly bounded in L. This leads for any $p < 1$ to an exponential relaxation in the infinite volume limit for all one time functions as well as for the persistence function [8]. At the same time the density dependence of τ_{FA} is completely encoded in the size $L_c(p)$. To evaluate the latter we reduce the scale from L_c to $1/q^2$ via a strategy similar to the previous one. However, since on scales smaller than L_c the event "the block is internally spanned" becomes very unlikely, we are forced to make a different choice for the good event of the renormalized dynamics in order to keep $\tau_{\text{GM}} \simeq 1$. The new choice is suggested by the following two observations: (i) any straight empty segment of length ℓ can be displaced by one step in a given direction if there is at least one vacancy on the adjacent segment in that direction; (ii) the probability that there exists at least one vacancy on each segment of length ℓ inside a square of size L_c is very near to one as soon as $\ell \gg 1/q^2$. Thus, we choose good events which force on Λ_{kL_c} at least one straight empty segment of length $1/q^2$ and at least one vacancy on all other segments of this length. By applying again a bisection procedure together with the construction of suitable paths which allow the creation/destruction of a particle starting from straight empty segment, we get $\tau_{FA}(L_c) \leq cL_c\tau_{FA}(1/q^2)$ where the term L_c comes from the length of the path. Finally we bound $\tau_{FA}(1/q^2)$ with the highest entropy cost and get $\tau_{FA}(L) < cL_c \exp(1/q^2) = O(\exp(1/q^2))$.

In order to establish lower bounds for τ one can devise as usual a suitable choice of test functions and use the variational characterization of the spectral gap (1). In some cases it is however simpler to follow a strategy which uses the knowledge of the typical blocked configurations together with our bound for the persistence (3). Consider a set of configurations B, called the *blocking event*, and let $P_B(t)$ be the infimum over the initial configurations $\eta(0) \in B$ of the probability that the origin is occupied up to time t. The inequality (3) implies $\mu_p(B)P_B(t) \leq \exp(-tq/\tau)$. For FA2f we define the blocking event B as the set of configurations for which, after standard bootstrap percolation inside $\Lambda_{\delta L_c}$ (i.e. after removing all particles which can be removed until exhausting the set of possible movements), a backbone of particles containing the origin survives. By choosing $\delta \ll 1$ and recalling the bootstrap percolation results of [1, 17] we have $\mu_p(B) \simeq 1$. In infinite volume this backbone will eventually get unblocked thanks to the vacancies outside $\Lambda_{\delta L_c}$. However, this requires an ordered sequence of at least $\delta L_c/2$ moves (fig.1b). Thus, $P_B(t = \epsilon\delta L_c) \simeq 1$ for sufficiently small ϵ. Therefore $O(1) \leq \exp(-tq/\tau)$ for $t \simeq \epsilon\delta L_c$, i.e. $\tau \geq O(L_c)$.

In conclusion we have developed a technique which allows to obtain rig-

orous results on τ_{FA} via the knowledge of the typical region which has to be emptied around a given site in order to perform a move on it. This size can in turn be determined via a deterministic procedure which corresponds to subsequently erase all particles which are unconstrained. The latter, due to the peculiar form of FA constraints, coincides with the very much studied bootstrap algorithm. Our main new results are exponential relaxation in the whole ergodic regime as well as faster than power law divergence of τ in $p-p_c$ when $p \nearrow p_c = 1$.

In higher dimensions and for the other KCSM one can proceed analogously [8]. The only delicate point is to choose an "internally spanned event" adapted to the constraints at hand, see e.g. [8] and [6] for the natural choices in the case of the North-East and the Spiral model, respectively. In some cases, e.g. for the Spiral model [6], even the form of the blocks for the partition of Λ_L before the renormalization procedure has to be adapted to the constraints leading to a non rectangular geometry. The scaling of τ on p depends on the specific choice of the constraints but in all cases the upper bound $\tau < \tau(L_c)$ holds, where L_c is the typical size over which the proper "internally spanned event" is likely to occur. The latter can be always determined via a properly modified bootstrap-like deterministic procedure [8] and it is finite for $p < p_c$. Thus we establish that the inverse of the spectral gap is finite which implies exponential relaxation for all one times quantities (e.g. $C(t)$) as well as for $F(t)$.

Furthermore proper modifications of the bisection-constrained technique also allow to deal with models which are reversible w.r.t. a high temperature Gibbs measure instead of μ_p [6] as well as models with long range constraints [7]. In both cases we establish positivity of the spectral gap in the whole ergodic region. The result for the long range models is particularly relevant since it allows, via proper renormalization and path techniques [7], to study the models with Kawasaki dynamics, namely the KCLG. In particular, by using the positivity of the spectral gap for a proper long range KCSM, we recently established [7] polynomial decay to equilibrium in infinite volume as well as $1/L^2$ decay for the spectral gap on finite volume with boundary sources for the most popular KCLG, the so called Kob-Andersen model [21].

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