Spectral gap and logarithmic Sobolev constant of Kawasaki dynamics under a mixing condition revisited

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Abstract. We consider a conservative stochastic spin exchange dynamics reversible with respect to the canonical Gibbs measure of a lattice gas model. We assume that the corresponding grand canonical measure satisfies a suitable strong mixing condition. We discuss the main ideas we used to reprove the well known results of Lu and Yau, and of Yau stating that the inverse of the spectral gap and the logarithmic Sobolev constant in a box of side L grow like L^2.

Key Words: Kawasaki dynamics, spectral gap, logarithmic Sobolev constant.

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1. Introduction

In the last years the problem of computing the relaxation time of stochastic Monte Carlo algorithm for classical spin models on $\mathbb{Z}^d$ has been intensively studied. We will focus our attention on lattice gases with Kawasaki dynamics. The lattice gases can be described as follows. Let Q_L be a cube of width L in $\mathbb{Z}^d$. At each lattice site of Q_L we associate an occupation number of particle $\sigma(x) \in \{0, 1\}$. The equilibrium states are described by the Gibbs measures on Q_L, characterized by a Hamiltonian and a boundary condition. There are two ensembles of interest the grand canonical one with temperature and chemical potential as external fixed variables and the canonical one where the chemical potential is substituted by the number of particles. In Kawasaki dynamics each particle performs a random walk with the following properties. Jumps to occupied sites are suppressed so that there is at most one particle per site; no creation or annihilation of particles is allowed so that the total number of particles is conserved; jump rates are determined by nearby particles according to some fixed local rules such that the canonical Gibbs measure is reversible. The models just described are interacting random walk and have a natural interpretation as discretizations of interacting Brownian motions. The limiting case of zero interaction is known as symmetric simple exclusion process, where the dynamics is given by the symmetric random walk and the invariant measures are simply a product of Bernoulli measures.

It is well known that the fundamental ingredients to study the relaxation time are the *spectral gap* (SG) of the generator and the *logarithmic Sobolev constant* (LSC). By the SG one obtains the time of convergence to equilibrium in L^2 norm (with respect to the canonical Gibbs measure), while the LSC allows to convert the L^2 convergence into a stronger statement.

The fundamental results of [LY] and [Y] on SG and LSC state that, under a suitable mixing condition on the grand canonical Gibbs measure, the inverse of the SG and the LSC in a box of side L scale like L^2. The mixing condition for the two dimensional Ising model holds for any temperature above the critical one. While in the phase coexistence region, at least for the two dimensional Ising model with periodic or free boundary condition, the SG becomes exponentially small in the side of the box [CCM]. The diffusive scaling L^2 for the relaxation time of Kawasaki dynamics, proved in [LY] and [Y], is a key stone in the study of the hydrodynamical limit of the Ising model [VY] and its proof required the development of a rather sophisticated technology which posed new, non trivial, problems on the theory of canonical Gibbs measures and their accurate comparison with the grand canonical ones (see also [BZ1], [BZ2], [CM1], [BCO], and [CZ]).

Unfortunately the proofs given in [LY] and particularly in [Y] are quite difficult to study and the application of their techniques to other related problems, for example lattice gases with random interaction in the so–called Griffiths phase, seems to require a considerable effort. With this motivation in [CM2] and [CMR] the results of [LY] and [Y] are reproved by different means in a way that looks, at least to us, intuitevely more appealing and natural to apply in other contexts. In particular in [CM3], the techniques developed in [CM2] as been applied to the bond dilute Ising model below the percolation threshold. Anyway we must note that our proofs would never found their way without some very nice ideas we have found in [LY] and [Y].

In this note we illustrate in simple terms the strategy behind the proofs in [CM2] and [CMR].

2. Notation and results

In this section we first define the setting in which we will work (spin model, Gibbs measure, dynamics), then we define the basic mixing condition on the Gibbs measure and subsequently state the main theorem on this work.

2.1 The lattice and the configuration space

The lattice. We consider the d dimensional lattice $\mathbb{Z}^d$ with *sites* $x = (x_1, \dots, x_d)$ and norms

$$|x|_p = (\sum_{i=1}^{d} |x_i|^p)^{1/p} \quad p \geq 1 \quad \text{and} \qquad |x| = |x|_\infty = \max_{i \in \{1,\dots,d\}} |x_i| \,.$$

The associated distance functions are denoted by $d_p(\cdot,\cdot)$ and $d(\cdot,\cdot)$. By Q_L we denote the cube of all $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ such that $x_i \in \{0, \dots, L-1\}$. If $x \in \mathbb{Z}^d$, $Q_L(x)$ stands for $Q_L + x$. We also let B_L be the ball (w.r.t $d(\cdot,\cdot)$) of radius L centered at the origin, *i.e.* $B_L = Q_{2L+1}((-L, \dots, -L))$. If Λ is a finite subset of $\mathbb{Z}^d$ we write $\Lambda \subset\subset \mathbb{Z}^d$. The cardinality of Λ is denoted by $|\Lambda|$. $\mathbb{F}$ is the set of all nonempty finite subsets of $\mathbb{Z}^d$. $[x, y]$ is the *closed segment* with endpoints x and y. The *edges* of $\mathbb{Z}^d$ are those $e = [x, y]$ with x, y nearest neighbors in $\mathbb{Z}^d$. We denote by $\mathcal{E}_\Lambda$ the set of all edges such that both endpoints are in Λ.

Given $\Lambda \subset \mathbb{Z}^d$ we define its interior and exterior boundaries as respectively, $\partial^- \Lambda = \{x \in \Lambda : d(x, \Lambda^c) \leq 1\}$ and $\partial^+ \Lambda = \{x \in \Lambda^c : d(x, \Lambda) \leq 1\}$, and more generally we define the boundaries of width n as $\partial_n \Lambda = \{x \in \Lambda : d(x, \Lambda^c) \leq n\}$, $\partial_n^+ \Lambda = \{x \in \Lambda^c : d(x, \Lambda) \leq n\}$.

Regular sets. A finite subset Λ of $\mathbb{Z}^d$ is said to be *l–regular*, $l \in \mathbb{Z}_+$, if Λ is the union of a finite number of cubes $Q_l(x^i)$ where $x^i \in l\mathbb{Z}^d$. We denote the class of all such sets by $\mathbb{F}_l$. Notice that any set is 1–regular *i.e.* $\mathbb{F}_{l=1} = \mathbb{F}$.

The configuration space. Our *configuration space* is $\Omega = S^{\mathbb{Z}^d}$, where $S = \{0, 1\}$, or $\Omega_V = S^V$ for some $V \subset \mathbb{Z}^d$. The single spin space S is endowed with the discrete topology and Ω with the corresponding product topology. Given $\sigma \in \Omega$ and $\Lambda \subset \mathbb{Z}^d$ we denote by σ_Λ the natural projection over Ω_Λ. If U, V are disjoint, $\sigma_U \tau_V$ is the configuration on $U \cup V$ which is equal to σ on U and τ on V. Given $V \in \mathbb{F}$ we define the *number of particles* $N_V : \Omega \mapsto \mathbb{N}$ as

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$$N_V(\sigma) = \sum_{x \in V} \sigma(x) \tag{2.1}$$

while the *density* is given by $\rho_V = N_V / |V|$.

If f is a function on Ω, Δ_f denotes the smallest subset of $\mathbb{Z}^d$ such that $f(\sigma)$ depends only on σ_{Δ_f}. f is called *local* if Δ_f is finite. The *l–support* of a function $\Delta_f^{(l)}$, $l \in \mathbb{Z}_+$, is the smallest l–regular set V such that $\Delta_f \subset V$. $\mathcal{F}_\Lambda$ stands for the $\sigma-$algebra generated by the set of projections $\{\pi_x\}$, $x \in \Lambda$, from Ω to $\{0, 1\}$, where $\pi_x : \sigma \mapsto \sigma(x)$. When $\Lambda = \mathbb{Z}^d$ we set $\mathcal{F} = \mathcal{F}_{\mathbb{Z}^d}$ and $\mathcal{F}$ coincides with the Borel $\sigma-$algebra on Ω with respect to the topology introduced above. By $\|f\|_\infty$ we mean the supremum norm of f.

2.2 The interaction and the Gibbs Measures.

potential **Definition 2.1.** *A finite range, translation–invariant potential* $\{\Phi_\Lambda\}_{\Lambda\in\mathbb{F}}$ *is a collection of real, local functions on* Ω *with the following properties*

(1) $\Phi_\Lambda = \Phi_{\Lambda+x}$ *for all* $\Lambda\in\mathbb{F}$ *and all* $x\in\mathbb{Z}^d$

(2) For each Λ *the support of* Φ_Λ *coincides with* Λ

(3) There exists $r>0$ *such that* $\Phi_\Lambda = 0$ *if* $\operatorname{diam}\Lambda > r$. r *is called the range of the interaction.*

(4) $\|\Phi\| := \sum_{\Lambda\ni 0}\|\Phi_\Lambda\|_\infty < \infty$

Given a collection of real numbers $\underline{\lambda} = \{\lambda_x\}_{x\in\mathbb{Z}^d}$ and a *potential* Φ, we define $\Phi^{\underline{\lambda}}$ as

$$\Phi^{\underline{\lambda}}_\Lambda(\sigma) = \begin{cases} (h+\lambda_x)\sigma(x) & \text{if } \Lambda = \{x\} \\ \Phi_\Lambda(\sigma) & \text{otherwise} \end{cases}$$

where h is the chemical potential (one body part of Φ).
Given a *potential* Φ ($\Phi^{\underline{\lambda}}$) and $V\in\mathbb{F}$, we define the Hamiltonian $H^\Phi_V:\Omega\mapsto\mathbb{R}$ by

$$H^\Phi_V(\sigma) = -\sum_{\Lambda:\,\Lambda\cap V\neq\emptyset}\Phi_\Lambda(\sigma)$$

For $\sigma,\tau\in\Omega$ we also let $H^{\Phi,\tau}_V(\sigma) = H^\Phi_V(\sigma_V\tau_{V^c})$ and τ is called the *boundary condition.* For each $V\in\mathbb{F}$, $\tau\in\Omega$ the (finite volume) conditional Gibbs measure on $(\Omega,\mathcal{F})$, are given by

finvolmea
$$d\mu^{\Phi,\tau}_V(\sigma) = \begin{cases} \left(Z^{\Phi,\tau}_V\right)^{-1}\exp[-H^{\Phi,\tau}_V(\sigma)] & \text{if } \sigma(x)=\tau(x) \text{ for all } x\in V^c \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

where $Z^{\Phi,\tau}_V$ is the proper normalization factor called partition function. Notice that in (2.2) we have absorbed in the interaction Φ the usual inverse temperature factor β in front of the Hamiltonian. In most notation we will drop the superscript Φ if that does not generate confusion. Moreover, whenever we consider $\Phi^{\underline{\lambda}}$ instead of Φ, we will write $H^{\tau,\underline{\lambda}}_V$ for the finite volume Hamiltonian and $\mu^{\tau,\underline{\lambda}}_V$ for the corresponding finite volume Gibbs measure.

Given a measurable bounded function f on Ω, $\mu_V(f)$ denotes the *function* $\sigma\mapsto\mu^\sigma_V(f)$ where $\mu^\sigma_V(f)$ is just the average of f w.r.t. μ^σ_V. Analogously, for any event X, $\mu^\tau_V(X) := \mu^\tau_V(\mathbb{1}_X)$, where $\mathbb{1}_X$ is the characteristic function of X. $\mu^\tau_V(f,g)$ stands for the covariance or *truncated correlation* (with respect to μ^τ_V) of f and g. The set of measures (2.2) satisfies the DLR compatibility conditions

DLR
$$\mu^\tau_\Lambda(\mu_V(X)) = \mu^\tau_\Lambda(X) \qquad \forall X\in\mathcal{F} \qquad \forall V\subset\Lambda\subset\subset\mathbb{Z}^d \tag{2.3}$$

Gibbs **Definition 2.2.** *A probability measure μ on $(\Omega, \mathcal{F})$ is called a Gibbs measure for Φ if*

DLRi
$$\mu(\mu_V(X)) = \mu(X) \qquad \forall\, X \in \mathcal{F} \qquad \forall\, V \in \mathbb{F} \tag{2.4}$$

see e.g. [G].

We introduce the *canonical Gibbs measures* on $(\Omega, \mathcal{F})$ defined as

cano
$$\nu^{\tau}_{\Lambda,N} = \mu^{\tau}_{\Lambda}(\cdot \mid N_\Lambda = N) \qquad N \in \{0, 1, \ldots, |\Lambda|\} \tag{2.5}$$

2.3 The dynamics

We consider the so–called Kawasaki dynamics in which particles (spins with $\sigma(x) = +1$) can jump to nearest neighbor empty ($\sigma(x) = 0$) locations, keeping the total number of particles constant. For $\sigma \in \Omega$, let σ^{xy} be the configuration obtained from σ by exchanging the spins $\sigma(x)$ and $\sigma(y)$. Let $t_{xy}\sigma = \sigma^{xy}$ and define $(T_{xy}f)(\sigma) = f(t_{xy}\sigma)$. The stochastic dynamics we want to study is determined by the Markov generators L_V, $V \subset\subset \mathbb{Z}^d$, defined by

gnrt
$$(L_V f)(\sigma) = \sum_{[x,y] \in \mathcal{E}_V} c_{xy}(\sigma)\, (\nabla_{xy} f)(\sigma) \qquad \sigma \in \Omega\,, \quad f : \Omega \mapsto \mathbb{R} \tag{2.6}$$

where $\nabla_{xy} = T_{xy} - \mathbb{1}$. The nonnegative real quantities $c_{xy}(\sigma)$ are the *transition rates* for the process.

The general assumptions on the transition rates are

(1) *Finite range.* $c_{xy}(\sigma)$ depends only on the spins $\sigma(z)$ with $d(\{x, y\}, z) \le r$

(2) *Detailed balance.* For all $\sigma \in \Omega$ and $[x, y] \in \mathcal{E}_{\mathbb{Z}^d}$

dbal
$$\exp\big[- H_{\{x,y\}}(\sigma)\big]\, c_{xy}(\sigma) = \exp\big[- H_{\{x,y\}}(\sigma^{xy})\big]\, c_{xy}(\sigma^{xy}) \tag{2.7}$$

(3) *Positivity and boundedness.* There exist positive real numbers $c_m(\beta)$ $c_M(\beta)$ such that

bounded
$$c_m \le c_{xy}(\sigma) \le c_M \qquad \forall x, y \in \mathbb{Z}^d, \sigma \in \Omega\,. \tag{2.8}$$

We denote by $L^{\tau}_{V,N}$ the operator L_V acting on $L^2(\Omega, \nu^{\tau}_{V,N})$ (this amounts to choosing τ as the boundary condition and N as the number of particles). Assumptions (1), (2) and (3) guarantee that there exists a unique Markov process whose generator is $L^{\tau}_{V,N}$, and whose semigroup we denote by $(T^{V,N,\tau}_t)_{t \ge 0}$. $L^{\tau}_{V,N}$ is a bounded operator on $L^2(\Omega, \nu^{\tau}_{V,N})$ and $\nu^{\tau}_{V,N}$ is its unique invariant measure. Moreover $\nu^{\tau}_{V,N}$ is *reversible* with respect to the process, *i.e.* $L^{\tau}_{V,N}$ is self–adjoint on $L^2(\Omega, \nu^{\tau}_{V,N})$.

A first fundamental quantity associated with the dynamics of a reversible system is the spectral gap of the generator, *i.e.*

$$\mathrm{gap}(L^{\tau}_{V,N}) = \inf \mathrm{spec}\, (-L^{\tau}_{V,N} \restriction \mathbb{1}^{\perp})$$

where $\mathbb{1}^{\perp}$ is the subspace of $L^2(\Omega, \nu^{\tau}_{V,N})$ orthogonal to the constant functions. We let $\mathcal{E}$ to be the Dirichlet form associated with the generator $L^{\tau}_{V,N}$,

dir
$$\mathcal{E}^{\tau}_{V,N}(f,f) = \langle f,\, -L^{\tau}_{V,N} f\rangle_{L^2(\Omega,\nu^{\tau}_{V,N})} = \frac{1}{2} \sum_{[x,y]\in\mathcal{E}_V} \nu^{\tau}_{V,N}\left[\, c_{xy}\, (\nabla_{xy} f)^2\,\right] \tag{2.9}$$

and $\mathrm{Var}^{\tau}_{V,N}$ the variance relative to the probability measure $\nu^{\tau}_{V,N}$. Then the gap can also be characterized as

gap
$$g^{\tau}_{V,N} := \mathrm{gap}(L^{\tau}_{V,N}) = \inf_{\substack{f\in L^2(\Omega,\nu^{\tau}_{V,N}),\\ \mathrm{Var}^{\tau}_{V,N}(f)\neq 0}} \frac{\mathcal{E}^{\tau}_{V,N}(f,f)}{\mathrm{Var}^{\tau}_{V,N}(f)}\,. \tag{2.10}$$

A second relevant quantity is the logarithmic Sobolev constant $c^{\tau}_{V,N}$ defined as the smallest constant c such that

csob
$$\mathrm{Ent}^{\tau}_{V,N}(f^2) \le \frac{c}{2}\, \mathcal{E}^{\tau}_{V,N}(f,f) \tag{2.11}$$

for all non negative functions f with $\nu^{\tau}_{V,N}(f^2) = 1$, where $\mathrm{Ent}^{\tau}_{V,N}(f^2) = \nu^{\tau}_{V,N}(f^2 \ln f^2)$. For the connection between spectral gap, logarithmic Sobolev constant and speed of relaxation to equilibrium we refer the reader to [DiSa].

2.4 Definition of the mixing condition and main results.

In order to formulate our basic mixing condition on the two (or more) body part of the interaction Φ we fix positive numbers C, m, l with $l \in \mathbf{N}$. We then say that a collection of real numbers $\underline{\lambda} := \{\lambda_x\}_{x\in\mathbb{Z}^d}$ is l–regular if, for all $i \in \mathbb{Z}^d$ and all $x \in Q_l(x^i)$, $x^i \in l\mathbb{Z}^d$, $\lambda_x = \lambda_{x^i}$.

Definition of property $USMT(C,m,l)$**.** *For any l–regular set Λ, any l–regular $\underline{\lambda}$, any boundary condition τ and any pair of bounded local functions f and g*

$$|\mu^{\tau,\underline{\lambda}}_{\Lambda}(f,g)| \le C \sup_{\tau} \mu^{\tau,\underline{\lambda}}_{\Lambda}(|f|) \sup_{\tau} \mu^{\tau,\underline{\lambda}}_{\Lambda\setminus\Delta^{(l)}_f}(|g|) \sum_{x\in\partial^-_r \Delta^{(l)}_f} \sum_{y\in\partial^-_r \Delta^{(l)}_g} e^{-m|x-y|}$$

provided that $d(\Delta^{(l)}_f, \Delta^{(l)}_g) \ge l$.

Remark. The expert reader may have noticed that our condition is different, and in principle stronger, than the one used in [LY] and [Y] because we require the exponential decay of covariances uniformly in the chemical potential even when the latter *varies* over the atoms of a partition of Λ while in the above mentioned papers the chemical potential is assumed to be constant over Λ. In two dimension, followig the ideas of [MOS], one can prove [BCO] that the two conditions are equivalent. In higher dimension one can construct examples in which a kind of phase transition occurs along the interface between two subsets with different chemical potential, even if for all l–regular sets Λ the covariances decay exponentially fast uniformly w.r.t. constant chemical potentials.

We are finally in a position to formulate the results on the SG and on the LSC of the generator of Kawasaki dynamics in a finite volume main **Theorem 2.3.** *Assume that there exist positive numbers C, m, l, with $l \in \mathbf{N}$, such that property $USMT(C, m, l)$ holds. Then there exist positive constants c_1, c_2 such that*

main.lb
$$c_1 L^{-2} \leq \operatorname{gap}(L^{\tau}_{Q_L,N}) \leq c_2 L^{-2} \tag{2.12}$$

and

main.lbs
$$c_1 L^2 \leq c^{\tau}_{Q_L,N} \leq c_2 L^2 \tag{2.13}$$

for all boundary condition τ and particles number N.

A nice consequence of the estimate (2.12) is an inverse polynomial bound on the time decay to equilibrium in $L^2\left(d\nu^{\tau}_{\Lambda,N}\right)$ of local observables (see [CM2] for the proof).

local **Corollary 2.4.** *Assume that there exist positive numbers C, m, l, with $l \in \mathbf{N}$, such that property $USMT(C, m, l)$ holds. Then for any $\epsilon \in (0,1)$ and any local function f with $0 \in \Delta_f$ there exists a positive constant $C_{f,\epsilon}$ such that for any integer L multiple of l and any integer $N \in \{1, \ldots, (2L)^d\}$*

$$\operatorname{Var}^{\tau}_{\Lambda,N}\left(e^{tL^{\tau}_{\Lambda,N}} f\right) \leq C_{f,\epsilon}\,\frac{1}{t^{\alpha-\epsilon}}$$

where $\Lambda := B_L$ and $\alpha = \frac{1}{2}$ in $d = 1$, $\alpha = 1$ for $d > 1$.

Remark. The expected decay is $t^{-\frac{d}{2}}$, exactly as for the simple exclusion, i.e. $\beta = 0$ case, has been proved in [BZ2], at least for functions f that have non zero grand canonical covariance with the number of particles. We refer to [JLQY] where a very sharp result of this kind for the zero–range process is obtained. Notice that the power α that appears in our bound coincides with $\frac{d}{2}$ in one and two dimensions but not in higher dimensions.

2.5 The main ideas of the proof.

We confine ourselves with the proof of the upper bound of the inverse of the spectral gap (ISG) and of the logarithmic Sobolev constant (LSC) since the lower bounds are easily proved by plugging a suitable test function (a slowly varying function of the local density) inside the definition of the spectral gap (2.10) or inside the logarithmic Sobolev inequality (2.11).
To illustrate better our strategy we discuss the two proofs in parallel even if, to obtain the upper bound of the logarithmic Sobolev inequality, we use the Poincaré bound

poinc
$$\operatorname{Var}(f) \leq \; k\, L^2\, \mathcal{E}(f,f) \tag{2.14}$$

For simplicity we carry out the discussion in two dimensions but the extension to higher dimensions is straightforward (see [CM2] or [CMR]). The proof is based on a recursive analysis, introduced in [M] for Glauber dynamics, on the behavior of the ISG and of the LSC when the linear size of the volume under consideration is doubled. The method works as follows. Let $g(L)^{-1}$ and $c(L)$ the largest (over the boundary conditions and number of particles) among the ISG and LSC respectively, in a square of side L with given boundary conditions and fixed number of particles. We look for a recursive inequality of the form

rico

$$\begin{aligned} g(2L)^{-1} &\le \frac{3}{2}\,g(L)^{-1} + kL^2 \\ c(2L) &\le \frac{3}{2}\,c(L) + kL^2 \end{aligned} \tag{2.15}$$

where

$$g(L) = \min_{N,\tau} g^{\tau}_{Q_L,N} \qquad \text{and} \qquad c_s(L) = \max_{N,\tau} c^{\tau}_{Q_L,N}$$

Inequality (2.15), upon iteration, prove the bounds $g(L)^{-1} \le k'L^2$ and $c(L) \le k'L^2$.
For this purpose let Λ be a square of side $2L$ and divide it into two (almost) halves Λ_1 and L_2 in such a way that the overlap between Λ_1 and Λ_2 is a thin layer of width δL, $\delta \ll 1$. Although the truncated correlations in the grand canonical Gibbs measure decay exponentially fast, due to the conservation of the number of particles which introduces a global constraint in the system, the dynamics does not separate into two weakly dependent components as happens in the case of the non conservative Glauber dynamics (see e.g. [M] or more recently [Ce]). Note that even at infinite temperature ($\beta = 0$) the dynamics does not factorize. More precisely, the relaxation time in a volume with linear size $2L$ is related to the relaxation time of the modified dynamics in which the two rectangles do not exchange particles but feel each other only through the transition rates *and* the relaxation time of the process of exchange of particles between the two halves of Q_{2L}. Such a simple observation suggests to try to separate the two effects which are, a priori, strongly interlaced and to analyze them separately.
Denote by ν the canonical Gibbs measure $\nu^{\tau}_{\Lambda,N}$ and define the two σ-algebras $\mathcal{F}_1 := \mathcal{F}_{\Lambda_1^c}$ and $\mathcal{F}_2 := \mathcal{F}_{\Lambda_2^c}$ namely the σ-algebras generated by the lattice gas variables outside Λ_1 and Λ_2 respectively. Let n_0 and n_1 be the random varaiables counting the number of particles in $\Lambda_1 \cap \Lambda_2$ and $\Lambda \setminus \Lambda_2$ respectively and let $\mathrm{Var}_\nu(f \,|\, n_0, n_1)$ and $\mathrm{Ent}_\nu(f^2 \,|\, n_0, n_1)$ be the variance of f and the entropy of f^2 w.r.t. the canonical measure conditioned on n_0, n_1. Then, using the formula of the "conditional variance" and of the "conditional entropy", we can write

$$\mathrm{Var}_\nu(f) = \nu(\mathrm{Var}_\nu(f \,|\, n_0, n_1)) + \mathrm{Var}_\nu(\nu(f \,|\, n_0, n_1)) \tag{2.16}$$

$$\mathrm{Ent}_\nu(f^2) = \nu(\mathrm{Ent}_\nu(f^2 \,|\, n_0, n_1) + \mathrm{Ent}_\nu(\nu(f^2 \,|\, n_0, n_1)) \tag{2.17}$$

The second term in (2.16) and (2.17) can in turn be expanded as

varb

$$\mathrm{Var}_\nu(\nu(f \,|\, n_0, n_1)) = \nu(\mathrm{Var}_\nu(\nu(f \,|\, n_0, n_1) \,|\, n_0)) + \mathrm{Var}_\nu(\nu(f \,|\, n_0)) \tag{2.18}$$

entb

$$\mathrm{Ent}_\nu(\nu(f \,|\, n_0, n_1)) = \nu(\mathrm{Ent}_\nu(\nu(f^2 \,|\, n_0, n_1) \,|\, n_0)) + \mathrm{Ent}_\nu(\nu(f^2 \,|\, n_0)) \tag{2.19}$$

Bound of the first terms in the r.h.s. of (2.16) and (2.17). If the two σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ were *weakly dependent* in the sense that for some $\epsilon = \epsilon(L) \ll 1$

weak1

$$||\nu_2(g) - \nu(g)||_\infty \le \epsilon\, ||g||_\infty \qquad \forall g \in L^\infty(\Omega, \mathcal{F}_{\Lambda_1^c}, \nu) \tag{2.20}$$

where $\nu_i := \nu(\cdot \,|\, \mathcal{F}_i),\ i = 1, 2$, then it follows that (*almost factorization of the variance*)

factvar
$$\mathrm{Var}_\nu(f) \le\ (1+\epsilon)\,\nu(\mathrm{Var}_{\nu_1}(f) + \mathrm{Var}_{\nu_2}(f)) \quad f \in L^2(\nu) \tag{2.21}$$

while if

weak2
$$\|\nu(g \,|\, \mathcal{F}_2) - \nu(g)\|_\infty \le\ \epsilon\,\nu(g) \tag{2.22}$$

for all non–negative functions g measurable w.r.t. $\mathcal{F}_1$ then it follows that (*almost factorization of the entropy*)

factent
$$\mathrm{Ent}_\nu(f^2) \le (1+\epsilon)\,\nu\big(\mathrm{Ent}_{\nu_1}(f^2) + \mathrm{Ent}_{\nu_2}(f^2)\big) \tag{2.23}$$

Inequalities (2.20) and (2.22) mean that there is a weak dependence on the boundary conditions, for more details on (2.21) see [M], [CM2] and more recently [BCC] while for more details on (2.23) see [Ce]. Notice that in the first term in the r.h.s of (2.16) and of (2.17) we need to bound the variance and the entropy with respect to a *multi canonical* measure in which the number of particles in each atom of the partition $\{R_1 := \Lambda \setminus \Lambda_1, R_2 := \Lambda_1 \cap \Lambda_2, R_3 := \Lambda \setminus \Lambda_2\}$ is frozen. As shown in [CM1] such a new measure has better chances to satisfy the "weak dependence" conditions (2.20) and (2.22) than the original measure ν precisely because of the extra conservation laws. The first step is thus to prove, using property USMT, that the multi canonical measure $\nu(\cdot \,|\, n_0, n_1)$ satisfies conditions (2.20) and (2.22) so that (2.21) and (2.23) for this measure hold see [CM2] and [CMR]. Then we may bound the first term in the r.h.s of (2.16) and of (2.17) by the largest among the ISG and of LSC respectively of each of the three sets times the Dirichlet form of the Kawasaki dynamics. Notice that for each of the three sets the linear dimension in one direction has been (at least) almost halved. Thus the first of the r.h.s. of (2.16) can be bounded by

var1
$$(1+\epsilon)\,\max_{\tau,N,i}(g^\tau_{R_i,N})^{-1}\,\mathcal{E}_\nu(f,f) \tag{2.24}$$

while the first term on the r.h.s. of (2.17) by

ent1
$$(1+\epsilon)\,\max_{\tau,N,i} c^\tau_{R_i,N}\,\mathcal{E}_\nu(f,f) \tag{2.25}$$

It is thus clear that the these terms are responsible for the first terms in the r.h.s. of (2.15).
We note here, for the expert reader, that the importance in the case of the entropy of such an inequality resides in the fact that one is spared from the cumbersome computation of quantities like $[\nabla_{xy}\nu(f^2)^{\frac{1}{2}}]^2$.

Bound of the second term on the r.h.s. of (2.16) and (2.17). Let us examine the terms in (2.18); the necessary steps are almost identical for all of them and therefore, for shortness, we treat only the second one. We have to bound the variance with respect to the distribution of a one dimensional discrete random variable, the number of particles n_0. Although such a distribution is difficult to compute exactly, one has a sufficiently good control to be able to establish, a sharp Poincaré inequality with respect to the Dirichlet form of a reversible Metropolis birth and death process

var2
$$\mathrm{Var}_\nu(g(n_0)) \le\ k(N)\,\nu\big((\frac{d}{dn_0}g)^2\big) \tag{2.26}$$

for any g depending only on n_0. Following the same reasoning for the second term in (2.19) we have to bound the entropy w.r.t. to the distribution of n_0, one can establish a sharp logarithmic Sobolev inequality with respect to the Dirichlet form of a reversible Metropolis birth and death process

$$\mathrm{Ent}_\nu(g^2(n_0)) \leq \, k(N)\, \nu\big((\frac{d}{dn_0} g)^2\big) \qquad (2.27)$$

for any g depending only on the number of particles n_0 and $\frac{d}{dn_0}$ is the discrete derivative. Physically the birth and death process corresponds to the creation of an extra particle e.g. in R_1 and the contemporary annihilation of a particle in e.g. R_2 that is to the *exchange* of particles among the three sets. Since each particle moves , essentially, by a sort of perturbed random walk, and on average it has to travel a distance $O(L)$, it is not surprising that these terms are responsible for the L^2 terms in (2.15). To estimate $k(N) = O(N)$ in the case of ISG Cheeger's constant is used (see [LS]), while in the case of LSC Hardy inequality (see [Mi] and [An]); on the other hand the bound of the discrete gradient of $g = \nu(f \,|\, n_0)$ in the case of ISG or $\nu(f^2 \,|\, n_0)$ in the case of LSC is technical: USMT, equivalence of ensembles, some ideas of [LY] and [Y] and concentration inequalities (see [CMR]) in the particular case of LSC are used. In particular (2.26) can be bounded by

$$C_\epsilon \, L^2 \, \mathcal{E}_\nu(f,f) + \epsilon \; \mathrm{Var}_\nu(f)$$

and (2.27) by

$$C_\epsilon \; \mathrm{Var}_\nu(f^2) + C_\epsilon \, L^2 \, \mathcal{E}_\nu(f,f) + \epsilon \; \mathrm{Ent}_\nu(f^2)$$

Putting together the bounds above we have for the variance

$$\mathrm{Var}(f) \leq \; [(1+\epsilon) \; \max_{\tau,N,i} (g^\tau_{R_i,N})^{-1} \; + \; C_\epsilon L^2] \, \mathcal{E}_\nu(f,f)$$

and, by the Poincaré bound (2.14), for the entropy

$$\mathrm{Ent}(f^2) \leq \; [(1+\epsilon) \max_{\tau,N,i} c^\tau_{R_i,N} + C_\epsilon L^2] \, \mathcal{E}(f,f)$$

Once such a step has been carried out it is not too difficult to complete the scale reduction from $2L$ to L by one more iteration and obtain the recursive bounds (2.15).

3. Open problems

A natural problem is to estimate the relaxation time to equilibrium in the low temperature *end* low density part of the one phase region. In this region USMT holds but not uniformly in the chemical potential if it varies over the atoms of a partition. As such uniformity is one of the main ingredients of the above proofs, the techniques above discussed cannot be easily extended to this case.

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