On sharp Sobolev-type inequalities for multidimensional Cauchy measures

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Abstract. We are discussing some Sobolev-type inequalities for Cauchy measures and their information-theoretic counterparts.

Contents

1. Introduction

One of the classical Sobolev inequalities on the Euclidean space $\mathbb{R}^n$ is the relation

$$\Big(\int_{\mathbb{R}^n} |f|^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{2n}} \leq S_n \Big(\int_{\mathbb{R}^n} |\nabla f|^2\Big)^{\frac{1}{2}}, \quad n \geq 3, \tag{1.1}$$

which holds true for all smooth functions f on $\mathbb{R}^n$ vanishing at infinity. The best constant

$$S_n = \frac{1}{\sqrt{\pi n(n-2)}}\,\Big(\frac{\Gamma(n)}{\Gamma(n/2)}\Big)^{\frac{1}{n}}$$

was determined in the 1970s by Aubin [1] and Talenti [13], see also [10], [3]. In the sequel, the integrals are always understood with respect to the Lebesgue measure on $\mathbb{R}^n$, if the measure is not indicated explicitly.

Information-theoretic aspects of (1.1) are recently discussed in [6]. After the change of functions $p = f^2/\int f^2$, this inequality enters the family of entropic isoperimetric inequalities

$$N_\alpha(X)\, I(X) \geq C_n(\alpha) \tag{1.2}$$

2010 *Mathematics Subject Classification.* Primary 60E, 60F.

Key words and phrases. Sobolev-type inequalities, Cauchy measures, Rényi entropy, Fisher information.

The first author was partially supported by the NSF grant DMS-1855575.

The second author was supported by the Labex MME-DII funded by ANR, reference ANR-11-LBX-0023-01 and ANR-15-CE40-0020-03 - LSD - Large Stochastic Dynamics, and the grant of the Simone and Cino Del Duca Foundation, France.

with a particular index $\alpha = \frac{n}{n-2}$. Here,

$$N_\alpha(X) = N_\alpha(p) = \Big(\int p^\alpha\Big)^{-\frac{2}{n(\alpha-1)}}$$

is the Rényi entropy power, and

$$I(X) = I(p) = \int \frac{|\nabla p|^2}{p}$$

is the Fisher information hidden in the distribution of the random vector X in $\mathbb{R}^n$ with a smooth density p. Since the function $\alpha \mapsto N_\alpha$ is non-increasing, the inequality (1.2) is getting stronger for the growing index α.

With some constants $C_n(\alpha) > 0$ independent of p, (1.2) holds true for all $\alpha \in [0,\infty]$ in the dimension $n = 1$, and for all $\alpha \in [0,\infty)$ in the dimension $n = 2$. However, $\alpha = \frac{n}{n-2}$ is the maximal possible value in (1.2) in the case $n \geq 3$.

When $\alpha = 1$, the Rényi entropy power is reduced to the Shannon entropy power

$$N_1(X) = N(X) = \exp\Big\{-\frac{2}{n}\int p\log p\Big\}.$$

In this case, being written with an optimal constant, (1.2) becomes a well-known relation due to Stam [12],

$$N(X)\, I(X) \geq 2\pi e\, n, \tag{1.3}$$

in which the standard Gaussian measure plays an extremal role (for any $n \geq 1$). Costa and Cover [9] pointed out a remarkable analogy between (1.3) and the isoperimetric inequality relating the surface of an arbitrary body in $\mathbb{R}^n$ to its volume. The terminology *isoperimetric inequality for entropies* goes back to Dembo, Costa and Thomas [11].

Rather than describing the best constant, it should be emphasized that an equality in (1.1) is always attained, and only for the functions of the form

$$f(x) = \frac{c}{(1+b\,|x-x_0|^2)^{\frac{n-2}{2}}}, \quad c \in \mathbb{R},\ b > 0,\ x_0 \in \mathbb{R}^n$$

(sometimes called the Barenblatt profiles). Up to numerical factors, they serve as densities of the generalized multidimensional Cauchy measures, also called Student's distributions. So, choosing $b = 1$ and $x_0 = 0$, we put

$$\frac{d\mathfrak{m}_s(x)}{dx} = q_s(x) = c_s\,\varphi_s(x), \quad \varphi_s(x) = \frac{1}{(1+|x|^2)^s}, \quad x \in \mathbb{R}^n.$$

The function φ_s is integrable, if and only if $s > \frac{n}{2}$, and then $c_s = c(s,n)$ is defined as a normalizing constant so that $\mathfrak{m}_s(\mathbb{R}^n) = 1$. The probability distribution $\mathfrak{m}_s$ will be called the Cauchy measure on $\mathbb{R}^n$ with parameter s.

Thus, φ_s with $s = \frac{n-2}{2}$ represents an extremizer in (1.1), which leads to the extremizer $q_{n-2} = c_{n-2}\,\varphi_{n-2}$ in the entropic isoperimetric inequality (1.2). It is indeed a probability density as long as $n \geq 5$. However, φ_{n-2} is not integrable in the dimensions $n = 3$ and $n = 4$. As a consequence, in this case there is no extremizer in (1.2) in the class of all (smooth) probability densities on $\mathbb{R}^n$. We refer an interested reader to [6] for details.

One of the aims in this note is to show the relationship of (1.1)-(1.2) with a weighted Poincare-type inequality for the Cauchy measure $\mathfrak{m}_n$ with parameter

$s = n$. It is well-defined for all $n \geq 1$ and has density

$$\frac{d\mathfrak{m}_n(x)}{dx} = \frac{c_n}{(1+|x|^2)^n}, \quad c_n = \frac{\Gamma(n)}{\pi^{\frac{n}{2}}\,\Gamma(n/2)}. \tag{1.4}$$

In particular, as a consequence of (1.1) we prove:

THEOREM 1.1. *For any C^1-smooth function $g\colon \mathbb{R}^n \to \mathbb{R}$, $n \geq 3$,*

$$\mathrm{Var}_{\mathfrak{m}_n}(g) \leq \frac{1}{4n} \int |\nabla g(x)|^2 (1+|x|^2)^2 \, d\mathfrak{m}_n(x). \tag{1.5}$$

The constant $1/(4n)$ is optimal, and an equality in (1.5) is attained for $g(x) = \frac{1}{1+|x|^2}$.

As usual,

$$\mathrm{Var}_{\mathfrak{m}_n}(g) = \int g^2 \, d\mathfrak{m}_n - \Big(\int g \, d\mathfrak{m}_n\Big)^2$$

stands for the variance of g under $\mathfrak{m}_n$.

As we will see, the inequality (1.5) expresses the fact that $p = q_{n-2}$ is a "point" of local minimum to the functional $N_\alpha(p)\, I(p)$ for $\alpha = \frac{n}{n-2}$. Equivalently, $f = \varphi_{n-2}$ is a "point" of local minimum to the functional

$$\|\nabla f\|_2 / \|f\|_{\frac{2n}{n-2}}.$$

Weighted Poincaré-type inequalities such as (1.5) have been studied quite intensively, although for a different weight function. In particular, it was shown in [4] that, for all $s \geq n$,

$$\mathrm{Var}_{\mathfrak{m}_s}(g) \leq \frac{A_s}{2(s-1)} \int |\nabla g(x)|^2 (1+|x|^2) \, d\mathfrak{m}_s(x) \tag{1.6}$$

with

$$A_s = \Big(\sqrt{1+\frac{2}{s-1}} + \sqrt{\frac{2}{s-1}}\Big)^2.$$

Up to a universal factor, (1.6) is stronger than (1.5), however, it does not contain information about extremizers. Similar weighted Poincaré-type and isoperimetric inequalities of Cheeger-type remain to hold for general convex measures, cf. [2], [7], [5].

Let us also mention that after rescaling of the space variable, (1.5) implies in the limit as $n \to \infty$ the Poincaré-type inequality with respect to the standard Gaussian measure γ_k on $\mathbb{R}^k$ (which is also true about the inequality (1.6) with $s \to \infty$ and fixed n). We provide details in the end of these notes (Section 5), while Theorem 1.1 is proved in Section 4.

In this connection it is worthwhile to note that the Stam entropic inequality (1.3) may be used to derive the Gross logarithmic Sobolev inequality in the Gauss space $(\mathbb{R}^k, \gamma_k)$, which is stronger than the Poincaré-type inequality. Hence, one may wonder whether or not a similar derivation is applicable to $\mathfrak{m}_n$ on the basis of the entropic isoperimetric inequality (1.2). We propose one variant of a log-Sobolev-type inequality in the "Cauchy space" $(\mathbb{R}^n, \mathfrak{m}_n)$ in Section 3. In Section 2 we address a closely related question: *Is it true that the Cauchy measures $\mathfrak{m}_s$ play an extremal role when minimizing the Fisher information $I(X)$ subject to certain moment-type* *constraints?* This might give another analogy with a well-known assertion that $I(X)$ is minimized for the normal distribution under a second moment assumption.

2. Minimizing the Fisher information subject to moment conditions

Recall that, given two random variables X and Z with (smooth) densities p and q respectively, the relative Fisher information is defined by

$$I(X||Z) = \int_{\{p,q>0\}} \Big|\frac{\nabla p}{p} - \frac{\nabla q}{q}\Big|^2 p.$$

To avoid technical issues, suppose that q is everywhere positive and well-behaving. Then, if $p = fq$ with f smooth and bounded,

$$I(X||Z) = \int \frac{|\nabla f|^2}{f}\, q.$$

Using an integration by part formula, we get

$$\begin{aligned} I(X) - I(Z) &= I(X||Z) + 2\int \langle \nabla f, \nabla q\rangle + \int f\,\frac{|\nabla q|^2}{q} - \int \frac{|\nabla q|^2}{q} \\ &= I(X||Z) + \int f\left[-2\Delta q + \frac{|\nabla q|^2}{q}\right] - \int \frac{|\nabla q|^2}{q}, \end{aligned} \tag{2.1}$$

where

$$\Delta g = \sum_{i=1}^n \partial^2 g/\partial_{x_i^2}$$

stands for the Laplacian operator.

As a classical example, one may consider the standard Gaussian random variable Z with density

$$q(x) = (2\pi)^{-n/2} \exp\{-|x|^2/2\}.$$

In this case, $\nabla q = -xq$ and $\Delta q = -nq + |x|^2 q$, so that the above identity amounts to

$$\begin{aligned} I(X) - I(Z) &= I(X||Z) + n - \int |x|^2 f(x) q(x) \\ &= I(X||Z) + \mathbb{E}\,|X|^2 - \mathbb{E}\,|Z|^2. \end{aligned}$$

Since the relative Fisher information is non-negative, this implies in particular that among all random vectors X in $\mathbb{R}^n$ with the second moment $\mathbb{E}\,|X|^2 = \mathbb{E}\,|Z|^2 = n$, the Fisher information $I(X)$ is minimized for the standard Gaussian distribution.

Here, we obtain a similar comparison for the Cauchy measures $\mathfrak{m}_s$ with densities

$$q_s(x) = \frac{c_s}{(1+|x|^2)^s}, \quad x \in \mathbb{R}^n, \tag{2.2}$$

where c_s is the normalizing constant so that $\int q_s = 1$. Let us denote by Z_s a random vector in $\mathbb{R}^n$ with distribution $\mathfrak{m}_s$, $s > \frac{n}{2}$. As a direct analogue of the above result for the Gaussian measure, we prove:

Theorem 2.1. *Let X be a random vector in $\mathbb{R}^n$ with a smooth density and finite Fisher information. If*

$$\mathbb{E}\,\frac{n - (s-n+2)\,|X|^2}{(1+|X|^2)^2} = \mathbb{E}\,\frac{n - (s-n+2)\,|Z_s|^2}{(1+|Z_s|^2)^2}, \tag{2.3}$$

then

$$I(X||Z) = I(X) - I(Z_s). \tag{2.4}$$

In particular, $I(X) \geq I(Z_s)$.

The expectation in (2.3) may easily be evaluated explicitly, so that one may rewrite this moment condition as

$$\mathbb{E}\,\frac{n - (s-n+2)\,|X|^2}{(1+|X|^2)^2} \; = \; \frac{\frac{n}{2}\,(s-\frac{n}{2})}{s+1}.$$

For example, for $s = n-2$ with $n \geq 5$, (2.3) is simplified to

$$\mathbb{E}\,\frac{1}{(1+|X|^2)^2} = \mathbb{E}\,\frac{1}{(1+|Z_s|^2)^2} = \frac{n-4}{4(n-1)}, \tag{2.5}$$

and then we get (2.4).

For the proof of Theorem 2.1, we need a few calculus lemmas.

Lemma 2.2. *For any* $s > n/2$ *and* $x \in \mathbb{R}^n$,

$$-2\Delta q_s(x) + \frac{|\nabla q_s(x)|^2}{q_s(x)} \; = \; 4s\,\frac{n-(s-n+2)\,|x|^2}{(1+|x|^2)^2}\; q_s(x), \tag{2.6}$$

$$\frac{|\nabla q_s(x)|^2}{q_s(x)} \; = \; 4s^2\,\frac{|x|^2}{(1+|x|^2)^2}\; q_s(x). \tag{2.7}$$

This is verified directly. Putting

$$\varphi_s(x) = (1+|x|^2)^{-s}$$

as before, we have

$$\partial_{x_i}\varphi_s(x) = -2s\,\frac{x_i}{(1+|x|^2)^{s+1}}, \quad |\nabla\varphi_s(x)| = 2s\,\frac{|x|}{(1+|x|^2)^{s+1}},$$

so that

$$\frac{|\nabla\varphi_s(x)|^2}{\varphi_s(x)} = 4s^2\,\frac{|x|^2}{(1+|x|^2)^{s+2}}.$$

This is the same as (2.7). Further differentiation gives

$$\partial^2_{x_i^2}\,\varphi_s(x) = -2s\,\left[\frac{1}{(1+|x|^2)^{s+1}} - 2(s+1)\,\frac{x_i^2}{(1+|x|^2)^{s+2}}\right],$$

$$\Delta\varphi_s(x) = -2s\,\left[\frac{n}{(1+|x|^2)^{s+1}} - 2(s+1)\,\frac{|x|^2}{(1+|x|^2)^{s+2}}\right],$$

and thus

$$\frac{|\nabla\varphi_s(x)|^2}{\varphi_s(x)} - 2\Delta\varphi_s(x) = 4s\,\frac{n-(s-n+2)\,|x|^2}{(1+|x|^2)^{s+2}}.$$

Hence, we arrive at (2.6).

For example, for $s = n-2$,

$$\frac{|\nabla\varphi_{n-2}(x)|^2}{\varphi_{n-2}(x)} - 2\Delta\varphi_{n-2}(x) = 4n(n-2)\,\frac{1}{(1+|x|^2)^n}, \tag{2.8}$$

which is a multiple of φ_n.

In order to compute the constant c_α in (2.2), we need a technical lemma. As usual,

$$B(x,y) = \int_0^1 (1-t)^{x-1}\, t^{y-1}\, dt = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}, \quad x,y>0,$$

stands for the beta function. In the sequel, ω_n denotes the volume of the unit ball of $\mathbb{R}^n$.

Lemma 2.3. *For any $s > n/2$,*

$$\frac{1}{c_s} = \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^s} = \frac{n\omega_n}{2}\, B\Big(\frac{n}{2}, s-\frac{n}{2}\Big). \tag{2.9}$$

As a consequence, we get:

Lemma 2.4. *For any $s > n/2$,*

$$\int \frac{1}{(1+|x|^2)^2}\, d\mathfrak{m}_s(x) = \frac{(s-\frac{n}{2})(s+1-\frac{n}{2})}{s(s+1)}, \tag{2.10}$$

$$\int \frac{|x|^2}{(1+|x|^2)^2}\, d\mathfrak{m}_s(x) = \frac{\frac{n}{2}\,(s-\frac{n}{2})}{s(s+1)}. \tag{2.11}$$

Proof. Using polar coordinates, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^s} &= n\omega_n \int_{\mathbb{S}^{n-1}} \int_0^\infty \frac{r^{n-1}}{(1+r^2)^s}\, dr\, d\sigma_{n-1} \\
&= n\omega_n \int_0^\infty \frac{r^{n-1}}{(1+r^2)^\alpha}\, dr.
\end{aligned}$$

Changing variable $u = \frac{1}{1+r^2}$, we get

$$\begin{aligned}
\int_0^\infty \frac{r^{n-1}}{(1+r^2)^s}\, dr &= \frac{1}{2}\int_0^1 \Big(\frac{1}{u}-1\Big)^{\frac{n-2}{2}} u^{s-2}\, du \\
&= \frac{1}{2}\int_0^1 (1-u)^{\frac{n-2}{2}}\, u^{s-2-\frac{n-2}{2}}\, du.
\end{aligned}$$

This leads to the first desired conclusion (2.9).

Applying this identity, we see that the integral in (2.10) is equal to

$$\begin{aligned}
\frac{c_s}{c_{s+2}} &= \frac{B(\frac{n}{2}, s+2-\frac{n}{2})}{B(\frac{n}{2}, s-\frac{n}{2})} \\
&= \frac{\Gamma(s+2-\frac{n}{2})}{\Gamma(s+2)}\ \frac{\Gamma(s)}{\Gamma(s-\frac{n}{2})} \\
&= \frac{(s-\frac{n}{2})(s+1-\frac{n}{2})}{s(s+1)}.
\end{aligned}$$

Applying (2.9) once more, the integral in (2.11) may be written as

$$\begin{aligned} & c_s \left[\int \frac{1}{(1+|x|^2)^{s+1}} - \int \frac{1}{(1+|x|^2)^{s+2}} \right] \\ & = c_s \left(c_{s+1}^{-1} - c_{s+2}^{-1} \right) \\ & = \frac{B(\frac{n}{2}, s+1-\frac{n}{2}) - B(\frac{n}{2}, s+2-\frac{n}{2})}{B(\frac{n}{2}, s-\frac{n}{2})} \\ & = \frac{s-\frac{n}{2}}{s} - \frac{(s-\frac{n}{2})(s+1-\frac{n}{2})}{s(s+1)} \\ & = \frac{(s-\frac{n}{2})\frac{n}{2}}{s(s+1)}. \end{aligned}$$

□

Proof of Theorem 2.1. Applying (2.6)-(2.7) in (2.1), we see that the random variable X with density $p = fq_s$ satisfies

$$\begin{aligned} I(X) - I(Z_s) \quad = \quad & I(X||Z_s) + \int 4s \, \frac{n-(s-n+2)\,|x|^2}{(1+|x|^2)^2} \, f(x)q_s(x) \\ & - \int \frac{4s^2\,|x|^2}{(1+|x|^2)^2} \, q_s(x). \end{aligned} \tag{2.12}$$

In particular, if

$$\int \frac{n-(s-n+2)\,|x|^2}{(1+|x|^2)^2} \, p(x) = s \int \frac{|x|^2}{(1+|x|^2)^2} \, q_s(x), \tag{2.13}$$

we get $I(X) - I(Z_s) = I(X||Z_s)$, that is, the desired relation (2.4). Moreover, choosing $f = 1$ in (2.12), we see that the two integrals therein must coincide, that is,

$$\int \frac{n-(s-n+2)\,|x|^2}{(1+|x|^2)^2} \, q_s(x) = s \int \frac{|x|^2}{(1+|x|^2)^2} \, q_s(x).$$

This may also be verified on the basis of Lemma 2.3. Indeed, by (2.10)-(2.11), the above first integral is equal to

$$n \, \frac{(s-\frac{n}{2})(s+1-\frac{n}{2})}{s(s+1)} - (s-n+2) \, \frac{\frac{n}{2}\,(s-\frac{n}{2})}{s(s+1)} \quad = \quad \frac{\frac{n}{2}\,(s-\frac{n}{2})}{s+1},$$

which is exactly the second integral, according to (2.11).

Thus, the moment condition (2.13) coincides with the condition (2.3). □

3. Log-Sobolev-type inequality

In analogy with the equivalence between the Stam isoperimetric inequality for entropies (that is, in the case $\alpha = 1$ as in (1.3)) and the logarithmic Sobolev inequality for the standard Gaussian measure, we derive in this section one inequality involving, as a reference measure, the Cauchy measure (2.2) with parameter $s = n-2$, that is, with density

$$q(x) = \frac{c}{(1+|x|^2)^{n-2}}, \quad x \in \mathbb{R}^n, \; n \geq 5, \tag{3.1}$$

where

$$c^{-1} = \frac{n\omega_n}{2}\, B(\frac{n}{2}, \frac{n}{2} - 2)$$

is a normalizing constant (cf. Lemma 2.3).

Recall that this function is an extremizer in the isoperimetric inequality for entropies (1.2) of order $\alpha = \frac{n}{n-2}$, so that, for all smooth densities p on $\mathbb{R}^n$

$$\Big(\int p^{\frac{n}{n-2}}\Big)^{-\frac{n-2}{n}} \int \frac{|\nabla p|^2}{p} \geq \Big(\int q^{\frac{n}{n-2}}\Big)^{-\frac{n-2}{n}} \int \frac{|\nabla q|^2}{q}. \tag{3.2}$$

Here the expression on the right-hand side represents the constant

$$C_n = 4\pi\, n(n-2) \left(\Gamma\left(\frac{n}{2}\right)/\Gamma(n)\right)^{\frac{2}{n}}. \tag{3.3}$$

Let X be random vector in $\mathbb{R}^n$ with density $p = fq$, and, as before, denote by Z a random vector with density q. Taking the logarithm in (3.2) leads to

$$-\frac{n-2}{n} \log \int f^{\frac{n}{n-2}}\, q^{\frac{n}{n-2}} + \log I(X) \geq -\frac{n-2}{n} \log \int q^{\frac{n}{n-2}} + \log I(Z).$$

Therefore, if f satisfies the moment condition (2.5), then Theorem 2.1 is applicable, and hence from (2.4) we obtain

$$\begin{aligned} \log \int f^{\frac{n}{n-2}} q^{\frac{n}{n-2}} - \log \int q^{\frac{n}{n-2}} \quad &\leq \quad \frac{n}{n-2} \big(\log I(X) - \log I(Z)\big) \\ &= \quad \frac{n}{n-2} \log\Big(1 + \frac{I(X||Z)}{I(Z)}\Big). \end{aligned}$$

Using $\log(1+x) \leq x$, this is simplified to

$$\begin{aligned} \log \int f^{\frac{n}{n-2}} q^{\frac{n}{n-2}} - \log \int q^{\frac{n}{n-2}} \quad &\leq \quad \frac{n}{n-2} \frac{1}{I(Z)} \int \frac{|\nabla f|^2}{f}\, q \\ &= \quad B_n \int \frac{|\nabla f|^2}{f}\, q. \end{aligned}$$

for some constant B_n that can be made explicit. Namely, since

$$\nabla q = -2(n-2) \frac{x}{1+|x|^2}\, q,$$

as in the relation (2.7) from Lemma 2.2 with $s = n-2$, we may apply Lemma 2.4 to get

$$I(Z) = 4(n-2)^2 \int \frac{|x|^2}{(1+|x|^2)^2}\, q = \frac{n(n-2)(n-4)}{n-1}.$$

Thus,

$$B_n = \frac{n-1}{(n-2)^2(n-4)}.$$

As a summary, we proved the following statement.

Theorem 3.1. *Let q be the density of the Cauchy distribution on $\mathbb{R}^n$ with parameter $\alpha = n-2$, $n \geq 5$, as in (3.1). For any smooth function $f : \mathbb{R}^n \to \mathbb{R}_+$ satisfying*

$$\int fq = 1 \quad \text{and} \quad \int \frac{fq}{(1+|x|^2)^2} = \frac{n-4}{4(n-1)}, \tag{3.4}$$

we have

$$\log \int f^{\frac{n}{n-2}}\, q^{\frac{n}{n-2}} - \log \int q^{\frac{n}{n-2}} \leq \frac{n-1}{(n-2)^2(n-4)} \int \frac{|\nabla f|^2}{f}\, q. \tag{3.5}$$

Turning back to the previous computations, let us note that we have actually proved a stronger inequality

$$\log \int f^{\frac{n}{n-2}}\, q^{\frac{n}{n-2}} - \log \int q^{\frac{n}{n-2}} \leq \frac{n}{n-2} \log \Big(1 + \frac{n-1}{n(n-2)(n-4)} \int \frac{|\nabla f|^2}{f}\, q\Big).$$

Similarly to the usual log-Sobolev inequality, the last integral in (3.5) describes the relative Fisher information $I(X||Z)$ of the random vector X in $\mathbb{R}^n$ with density $p = fq$ with respect to the random vector Z with density q.

Let us show that the left hand side of (3.5), which replaces the relative entropy $D(X||Z)$ in the usual log-Sobolev inequality, is always non-negative. We claim that, under the moment condition (3.4), we have

$$\int q^{\frac{n}{n-2}} \leq \int f^{\frac{n}{n-2}}\, q^{\frac{n}{n-2}}. \tag{3.6}$$

Indeed, by Holder's inequality with exponents $n/(n-2)$ and $n/2$,

$$\int \frac{fq}{(1+|x|^2)^2} \leq \Big(\int f^{\frac{n}{n-2}}\, q^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}} \Big(\int \frac{1}{(1+|x|^2)^n}\Big)^{\frac{2}{n}},$$

so that

$$\int f^{\frac{n}{n-2}}\, q^{\frac{n}{n-2}} \geq \Big(\int \frac{fq}{(1+|x|^2)^2}\Big)^{\frac{n}{n-2}} \Big(\int \frac{1}{(1+|x|^2)^n}\Big)^{-\frac{2}{n-2}}.$$

Therefore, (3.6) would follow from

$$\Big(\int \frac{1}{(1+|x|^2)^n}\Big)^{\frac{2}{n-2}} \int q^{\frac{n}{n-2}} \leq \Big(\int \frac{fq}{(1+|x|^2)^2}\Big)^{\frac{n}{n-2}}.$$

By (3.4) and (2.5), this is equivalent to

$$\Big(\int \frac{1}{(1+|x|^2)^n}\Big)^{\frac{2}{n}} \Big(\int q^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}} \leq \int \frac{1}{(1+|x|^2)^2}\, q.$$

But, since q is proportional to $(1+|x|^2)^{-(n-2)}$, the above inequality is actually an equality.

4. Proof of Theorem 1.1

In the proof of Theorem 1.1, we follow ideas from [8]. Recall that the function

$$\varphi(x) = \frac{1}{(1+|x|^2)^{n-2}}, \quad x \in \mathbb{R}^n,$$

is an extremizer in the isoperimetric inequality for entropies with order $\alpha = \frac{n}{n-2}$,

$$\Big(\int p^{\frac{n}{n-2}}\Big)^{-\frac{n-2}{n}} \int \frac{|\nabla p|^2}{p} \geq \Big(\int \varphi^{\frac{n}{n-2}}\Big)^{-\frac{n-2}{n}} \int \frac{|\nabla \varphi|^2}{p}. \tag{4.1}$$

If $n \geq 5$ (and only then), φ is integrable, and then after normalization it represents the density of the Cauchy probability measure $\mathfrak{m}_{n-2}$. But, applying (4.1) to $p = f/\int f$, one realizes that the inequality holds for any $f \geq 0$ smooth enough, not necessarily a density, like in the Sobolev inequality (1.1).

Our aim is to apply (4.1) to $p = (1 + \varepsilon g)\varphi$ and to expand in the limit $\varepsilon \to 0$. We may assume that g is smooth enough and compactly supported so that all approximations are uniform in space. We also assume that ε is small enough so that $p(x) > 0$ for all $x \in \mathbb{R}^n$. Set $\alpha = \frac{n}{n-2}$. On one hand we have

$$\begin{aligned}\int p^\alpha &= \int \varphi^\alpha \,(1+\varepsilon g)^\alpha \\ &= \int \varphi^\alpha + \varepsilon\alpha \int g\varphi^\alpha + \varepsilon^2\, \frac{\alpha(\alpha-1)}{2} \int g^2\varphi^\alpha + o(\varepsilon^2).\end{aligned}$$

Therefore,

$$\Big(\int p^\alpha\Big)^{-1/\alpha} = \Big(\int \varphi^\alpha\Big)^{-1/\alpha} \left(1 + \varepsilon\alpha\, \frac{\int g\varphi^\alpha}{\int \varphi^\alpha} + \varepsilon^2\, \frac{\alpha(\alpha-1)}{2}\, \frac{\int g^2\varphi^\alpha}{\int\varphi^\alpha} + o(\varepsilon^2)\right)^{-1/\alpha}.$$

In terms of the probability measure $\mathfrak{m}_n$ on $\mathbb{R}^n$ with density

$$\frac{\mathfrak{m}_n(dx)}{dx} = \varphi(x)^\alpha \,/ \int \varphi^\alpha,$$

the latter expression may be written as

$$\Big(\int\varphi^\alpha\Big)^{-1/\alpha}\left(1 - \varepsilon\int g\,d\mathfrak{m}_n + \varepsilon^2\Big(-\frac{\alpha-1}{2}\int g^2\,d\mathfrak{m}_n + \frac{\alpha+1}{2}\Big(\int g\,d\mathfrak{m}_n\Big)^2\Big)\right)$$

with error of order $o(\varepsilon^2)$.

On the other hand,

$$\begin{aligned}&\int \frac{|\nabla p|^2}{p} \\ &= \int \frac{(1+\varepsilon g)^2\,|\nabla\varphi|^2 + 2\varepsilon(1+\varepsilon g)\,\varphi\,\nabla\varphi\cdot\nabla g + \varepsilon^2\varphi^2\,|\nabla g|^2}{(1+\varepsilon g)\,\varphi} \\ &= \int \frac{1}{\varphi}\Big(|\nabla\varphi|^2 + 2\varepsilon\,(g\,|\nabla\varphi|^2 + \varphi\nabla\varphi\cdot\nabla g) + \varepsilon^2\,|\nabla(g\varphi)|^2\Big)\Big(1 - \varepsilon g + \varepsilon^2 g^2 + o(\varepsilon^2)\Big) \\ &= \int \frac{|\nabla\varphi|^2}{\varphi} + \varepsilon\int\Big[g\frac{|\nabla\varphi|^2}{\varphi} + 2\,\nabla\varphi\cdot\nabla g\Big] \\ &\quad + \varepsilon^2\int\Big[\frac{|\nabla(g\varphi)|^2}{\varphi} - 2g\,\nabla\varphi\cdot\nabla g - g^2\frac{|\nabla\varphi|^2}{\varphi}\Big] + o(\varepsilon^2) \\ &= \int\frac{|\nabla\varphi|^2}{\varphi} + \varepsilon\int g\Big[\frac{|\nabla\varphi|^2}{\varphi} - 2\Delta\varphi\Big] + \varepsilon^2\int\varphi|\nabla g|^2 + o(\varepsilon^2),\end{aligned}$$

where in the last line we used an integration by part to ensure that

$$\int \nabla g\cdot\nabla\varphi = -\int g\Delta\varphi.$$

Multiplying the two expressions, it follows that

$$
\begin{aligned}
&\Big(\int p^\alpha\Big)^{-1/\alpha}\int\frac{|\nabla p|^2}{p}\\
&=\Big(\int\varphi^\alpha\Big)^{-1/\alpha}\int\frac{|\nabla\varphi|^2}{\varphi}\\
&\quad+\varepsilon\Big(\int\varphi^\alpha\Big)^{-1/\alpha}\bigg[\int g\Big[\frac{|\nabla\varphi|^2}{\varphi}-2\Delta\varphi\Big]-\int g\,d\mathfrak{m}_n\int\frac{|\nabla\varphi|^2}{\varphi}\bigg]\\
&\quad+\varepsilon^2\Big(\int\varphi^\alpha\Big)^{-1/\alpha}\bigg[\int\varphi|\nabla g|^2-\int g\,d\mathfrak{m}_n\int g\Big[\frac{|\nabla\varphi|^2}{\varphi}-2\Delta\varphi\Big]\\
&\quad+\int\frac{|\nabla\varphi|^2}{\varphi}\Big(-\frac{\alpha-1}{2}\int g^2\,d\mathfrak{m}_n+\frac{\alpha+1}{2}\Big(\int g\,d\mathfrak{m}_n\Big)^2\Big)\bigg]. \qquad (4.2)
\end{aligned}
$$

By (4.1), and since ε may be both positive and negative, the coefficient in front of ε in (4.2) must be vanishing, that is,

$$\int g\Big[\frac{|\nabla\varphi|^2}{\varphi}-2\Delta\varphi\Big]=\int g\,d\mathfrak{m}_n\int\frac{|\nabla\varphi|^2}{\varphi}. \qquad (4.3)$$

Of course, this may be verified directly on the basis of Lemma 2.2 with $s=n-2$, from which we know that

$$\frac{|\nabla\varphi(x)|^2}{\varphi(x)}=4(n-2)^2\,\frac{|x|^2}{(1+|x|^2)^n}. \qquad (4.4)$$

and

$$\frac{|\nabla\varphi(x)|^2}{\varphi(x)}-2\Delta\varphi(x)=4n(n-2)\,\frac{1}{(1+|x|^2)^n}. \qquad (4.5)$$

Up to a normalizing constant, the right-hand side is the density of the probability measure $\mathfrak{m}_n$. Therefore,

$$\int g\Big[\frac{|\nabla\varphi|^2}{\varphi}-2\Delta\varphi\Big]=4n(n-2)\int g\,d\mathfrak{m}_n\int\varphi^\alpha.$$

To obtain (4.3), it remains to check that

$$4n(n-2)\int\varphi^\alpha=\int\frac{|\nabla\varphi|^2}{\varphi},$$

which follows by Lemmas 2.3-2.4 (in view of (4.4)).

Thus, the linear term in (4.2) is vanishing. As a consequence, the coefficient in front of ε^2 must be non-negative, that is,

$$
\begin{aligned}
\int\varphi\,|\nabla g|^2 \ \ \geq\ \ &\int g\,d\mathfrak{m}_n\int g\Big[\frac{|\nabla\varphi|^2}{\varphi}-2\Delta\varphi\Big]\\
&+\Big(\frac{\alpha-1}{2}\int g^2\,d\mathfrak{m}_n-\frac{\alpha+1}{2}\Big(\int g\,d\mathfrak{m}_n\Big)^2\Big)\int\frac{|\nabla\varphi|^2}{\varphi}. \quad (4.6)
\end{aligned}
$$

Recalling (4.4)-(4.5), up to the factor $4n(n-2)\int\varphi^\alpha$, the above right-hand side represents just the normalized variance

$$\Big(\int g\,d\mathfrak{m}_n\Big)^2+\frac{\alpha-1}{2}\int g^2\,d\mathfrak{m}_n-\frac{\alpha+1}{2}\Big(\int g\,d\mathfrak{m}_n\Big)^2=\frac{\alpha-1}{2}\,\mathrm{Var}_{\mathfrak{m}_n}(g).$$

As a result, (4.6) is simplified to

$$\int \varphi\, |\nabla g|^2 \geq 4n \operatorname{Var}_{\mathfrak{m}_\alpha}(g) \int \varphi^\alpha.$$

Since

$$\frac{\varphi(x)}{\varphi(x)^\alpha} = (1+|x|^2)^2,$$

we arrive at the weighted Poincaré-type inequality

$$\operatorname{Var}_{\mathfrak{m}_n}(g) \leq \frac{1}{4n} \int |\nabla g(x)|^2 (1+|x|^2)^2\, d\mathfrak{m}_n(x) \tag{4.7}$$

On this step, the assumption that g is compactly supported may be dropped.

Note also that, since the volume of the unit ball is

$$\omega_n = 2\pi^{\frac{n}{2}}/(n\Gamma(n/2)),$$

the density of $\mathfrak{m}_n$ is

$$\frac{\varphi^\alpha(x)}{\int \varphi^\alpha} = \frac{2}{n\omega_n B\left(\frac{n}{2},\frac{n}{2}\right)}\, \frac{1}{(1+|x|^2)^n} = \frac{\Gamma(n)}{\pi^{\frac{n}{2}}\,\Gamma(n/2)}\, \frac{1}{(1+|x|^2)^n}.$$

We end this section by proving that the $1/(4n)$ is optimal in (4.7). In fact,let us check that that

$$g(x) = \frac{1}{1+|x|^2}$$

is an extremizer. To that aim, we will repeatedly use the following identities:

$$B\Big(\frac{n}{2},\frac{n}{2}+1\Big) = \frac{1}{2}\, B\Big(\frac{n}{2},\frac{n}{2}\Big) \quad \text{and} \quad B\Big(\frac{n}{2},\frac{n}{2}+2\Big) = \frac{1}{4}\, \frac{n+2}{n+1}\, B\Big(\frac{n}{2},\frac{n}{2}\Big),$$

that are consequences of

$$(x+y)\, B(x,y+1) = yB(x,y).$$

We have

$$\int g\, d\mathfrak{m}_n = Z_n \int \frac{dx}{(1+|x|^2)^{n+1}} = Z_n \frac{n\omega_n}{2}\, B\Big(\frac{n}{2},\frac{n}{2}+1\Big) = \frac{1}{2}$$

and

$$\int g^2\, d\mathfrak{m}_n = Z_n \int \frac{dx}{(1+|x|^2)^{n+2}} = Z_n \frac{n\omega_n}{2}\, B\Big(\frac{n}{2},\frac{n}{2}+2\Big) = \frac{n+2}{4(n+1)}.$$

Therefore,

$$\operatorname{Var}_{\mathfrak{m}_n}(g) = \int g^2\, d\mathfrak{m}_\alpha - \Big(\int g\, d\mathfrak{m}_n\Big)^2 = \frac{1}{4}\, \Big(\frac{n+2}{n+1} - 1\Big) = \frac{1}{4(n+1)}.$$

On the other hand,

$$\begin{aligned}\int |\nabla g(x)|^2(1+|x|^2)^2\, d\mathfrak{m}_n(x) &= 4Z_n \int \frac{|x|^2}{(1+|x|^2)^{n+2}}\\ &= 4Z_n \Big(\int \frac{1}{(1+|x|^2)^{n+1}} - \int \frac{1}{(1+|x|^2)^{n+2}} \Big)\\ &= 4Z_n \frac{n\omega_n}{2} \Big(B\Big(\frac{n}{2}, \frac{n}{2}+1\Big) - B\Big(\frac{n}{2}, \frac{n}{2}+2\Big)\Big)\\ &= 4\, \frac{B(\frac{n}{2}, \frac{n}{2}+1) - B(\frac{n}{2}, \frac{n}{2}+2)}{B(\frac{n}{2}, \frac{n}{2})}\\ &= 4\Big(\frac{1}{2} - \frac{n+2}{4(n+1)}\Big)\\ &= \frac{n}{n+1}.\end{aligned}$$

Therefore, the smallest constant C such that

$$\mathrm{Var}_{\mathfrak{m}_n}(g) \le C \int |\nabla g(x)|^2 (1+|x|^2)^2\, d\mathfrak{m}_n(x)$$

holds for any g must satisfy

$$\frac{1}{4(n+1)} \le C \cdot \frac{n}{n+1}$$

from which we deduce that $C \ge 1/(4n)$ and hence that $C = 1/(4n)$ is indeed the optimal constant in the weighted Poincaré inequality (4.7)

5. Relationship with the Gaussian Poincaré-type inequality

Let us explain how (1.5) implies the Poincaré-type inequality

$$\mathrm{Var}_{\gamma_k}(g) \le \int |\nabla g|^2\, d\gamma_k \tag{5.1}$$

with respect to the standard Gaussian measure γ_k on $\mathbb{R}^k$. As is well-known, the Cauchy measure m_n may be characterized as the distribution of the random vector

$$X = \frac{Z}{\sqrt{\xi_1^2 + \cdots + \xi_n^2}},$$

where ξ_i's are independent random variables with a standard normal distribution on the real line, that are independent of a random vector $Z = (Z_1, \dots, Z_n)$ having a standard normal distribution on $\mathbb{R}^n$. Rescaling the space variable, (1.5) may be rewritten in terms of the random vector

$$Y = \sqrt{n}\, X = (Y_1, \dots, Y_n)$$

as

$$\mathrm{Var}(g(Y)) \le \frac{1}{4}\, \mathbb{E}\, |\nabla g(Y)|^2 \Big(1 + \frac{1}{n}\, |Y|^2\Big)^2.$$

If $g = g(y_1, \dots, y_k)$ depends on the first k variables ($k < n$), and

$$Z_{k,n} = (Y_1, \dots, Y_k)$$

is the k-dimensional projection of Y, we obtain that

$$\mathrm{Var}(g(Z_{k,n})) \le \frac{1}{4}\, \mathbb{E}\, |\nabla g(Z_{k,n})|^2 \Big(1 + \frac{1}{n}\, |Z_{k,n}|^2 + \frac{1}{n}\, |V_{k,n}|^2\Big)^2, \tag{5.2}$$

where

$$V_{k,n} = (Y_{k+1}, \dots, Y_n).$$

As $n \to \infty$, we have $Z_{k,n} \Rightarrow \gamma_k$ weakly in distribution, so that the variance in (5.2) is convergent to $\mathrm{Var}_{\gamma_k}(g)$ as long as the function g is bounded and continuous on $\mathbb{R}^k$. Moreover, assuming that the gradient ∇g is bounded and continuous as well, the asymptotic behavior of the right-hand side in (5.2) is easily explored. First, putting

$$\chi_n = \sqrt{\xi_1^2 + \dots + \xi_n^2},$$

we have

$$\mathbb{E}\,|Z_{k,n}|^2 = \mathbb{E}\,\sqrt{Z_1^2 + \dots + Z_k^2}\ \mathbb{E}\,\frac{\sqrt{n}}{\chi_n} = \frac{\sqrt{n}}{n-1}\,\mathbb{E}\,\chi_k\,\mathbb{E}\,\chi_n \le \frac{2kn}{n-1}$$

which is bounded in n. Since $Z_{k,n}$ and $V_{k,n}$ are asymptotically independent, we conclude that the limit of the right-hand side in (5.2) is equal to the integral in (5.1). On this step, one may use the identity

$$\mathbb{E}\,\Big(1 + \frac{1}{n}\,|Z|^2\Big)^2 = 4 + \frac{2}{n}.$$

Thus, in the limit (5.2) leads to (5.1).

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