Modified logarithmic Sobolev inequalities on $\mathbb{R}$

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February 28, 2008

Abstract

We provide a sufficient condition for a measure on the real line to satisfy a modified logarithmic Sobolev inequality, thus extending the criterion of Bobkov and Götze. Under mild assumptions the condition is also necessary. Concentration inequalities are derived. This completes the picture given in recent contributions by Gentil, Guillin and Miclo.

1 Introduction

In this paper we are interested in Sobolev type inequalities satisfied by probability measures. It is well known that they allow to describe their concentration properties as well as the regularizing effects of associated semigroups. Several books are available on these topics and we refer to them for more details (see e.g. [1, 16]). Establishing such inequalities is a difficult task in general, especially in high dimensions. However, it is very natural to investigate such inequalities for measures on the real line. Indeed many high dimensional results are obtained by induction on dimension, and having a good knowledge of one dimensional measures becomes crucial. Thanks to Hardy-type inequalities, it is possible to describe very precisely the measures on the real line which satisfy certain Sobolev inequalities. Our goal here is to extend this approach to the so-called modified logarithmic Sobolev inequalities. They are introduced below.

Let γ denote the standard Gaussian probability measure on $\mathbb{R}$. The Gaussian logarithmic Sobolev asserts that for every smooth $f : \mathbb{R} \to \mathbb{R}$

$$\mathbf{Ent}_\gamma(f^2) \leq 2 \int (f')^2 d\gamma,$$

where the entropy functional with respect to a probability measure μ is defined by

$$\mathbf{Ent}_\mu(f) = \int f \log f \, d\mu - \left(\int f \, d\mu \right) \log \left(\int f \, d\mu \right).$$

This famous inequality implies the Gaussian concentration inequality, as well as hypercontractivity and entropy decay along the Ornstein-Uhlenbeck semigroup. Since the logarithmic-Sobolev inequality implies a sub-Gaussian behavior of tails, it is not verified for many measures and one has to consider weaker Sobolev inequalities. In the case of the symmetric exponential measure $d\nu(t) = e^{-|t|}dt/2$, an even more classical fact is available, namely a Poincaré or spectral gap inequality: for every smooth function f:

$$\mathbf{Var}_\nu(f) \leq 4 \int (f')^2 d\nu. \tag{1}$$

This property implies an exponential concentration inequality, as noted by Gromov and Milman [14], as well as a fast decay of the variance along the corresponding semigroup. If one compares to the log-Sobolev inequality, the spectral gap inequality differs by its left side only. In order to describe more precisely the concentration phenomenon for product of exponential measures, and to recover a celebrated result by Talagrand [22], Bobkov and Ledoux [6] introduced a so-called modified logarithmic Sobolev inequality for the exponential measure. Here the entropy term remains but the term involving the derivatives is changed. Their result asserts that every smooth $f : \mathbb{R} \to \mathbb{R}$ with $|f'/f| \le c < 1$ verifies

$$\mathbf{Ent}_\nu(f^2) \le \frac{2}{1-c} \int (f')^2 d\nu. \tag{2}$$

The latter may be rewritten as

$$\mathbf{Ent}_\nu(f^2) \le \int H\left(\frac{f'}{f}\right) f^2 d\nu, \tag{3}$$

where $H(t) = 2t^2/(1-c)$ if $|t| \le c$ and $H(t) = +\infty$ otherwise. Such general modified log-Sobolev inequalities have been established by Bobkov and Ledoux [7] for the probability measures $d\nu_p(t) = e^{-|t|^p} dt/Z_p$, $t \in \mathbb{R}$ in the case $p > 2$ (a more general result is valid for measures $e^{-V(x)}dx$ on $\mathbb{R}^n$ where V is strictly uniformly convex). These measures satisfy a modified log-Sobolev inequality with function $H(t) = c_p|t|^q$ where $q = p/(p-1) \in [1, 2]$ is the dual exponent of $p \ge 2$. The inequality can be reformulated as $\mathbf{Ent}_{\nu_p}(|g|^q) \le \tilde{c}_p \int |g'|^q d\nu_p$. These q-log-Sobolev inequalities are studied in details by Bobkov and Zegarlinski in [8].

The case $p \in (1, 2)$ is more delicate: the inequality cannot hold with $H(t) = c_p|t|^q$ since this function is too small close to zero. Indeed for $f = 1 + \varepsilon g$ when g is bounded and ε very small, the left hand side of (3) is equivalent to $\varepsilon^2 \mathbf{Var}_\nu(g)$ whereas the right hand side is comparable to $\int H(\varepsilon g')\, d\nu$. Hence $H(t)$ cannot be much smaller than t^2 when t goes to zero. If it compares to t^2 then in the limit one recovers a spectral gap inequality. Gentil, Guillin and Miclo [10] established a modified log-Sobolev inequality for ν_p when $p \in (1, 2)$, with a function $H_p(t)$ comparable to $k_p \max(t^2, |t|^q)$. In the subsequent paper [11] they extend their method to even log-concave measures on the line, with tail behavior between exponential and Gaussian. Their method is rather involved. It relies on classical Hardy types inequalities, adapted to inequalities involving terms as $\int (f')^2 d\mu$, where μ is carefully chosen.

Our alternative approach is to develop Hardy type methods directly for inequalities involving terms as $\int H(f'/f) f^2 d\mu$. This is done abstractly in Section 2, but more work is needed to present the results in an explicit and workable form. Section 3 provides a simple sufficient condition for a measure to satisfy a modified log-Sobolev inequality with function $H(t) = k_p \max(t^2, |t|^q)$ for $p \in (1, 2)$, and recovers in a soft way the result of [10]. Under mild assumptions, the condition is also necessary and we have a reasonable estimate of the best constant in the inequality. Next in Section 4 we consider the same problem for general convex functions H. The approach remains rather simple, but technicalities are more involved. However Theorem 20 provides a neat sufficient condition, which recovers the result of [11] for log-concave measures but also applies without this restriction. In Section 5 we describe concentration consequences of modified logarithmic Sobolev inequalities, obtained by the Herbst method.

Logarithmic Sobolev inequalities are known to imply inequalities between transportation cost and entropy [19, 4]. Our criterion can be compared with the one recently derived by Gozlan [13]. It confirms that modified logarithmic Sobolev inequalities are strictly stronger than the corresponding transportation cost inequalities, as discovered by Cattiaux and Guillin [9] for the classical logarithmic Sobolev inequality and Talagrand's transportation cost inequality. For log-concave measures on $\mathbb{R}$ the results of Gozlan yield precise modified logarithmic Sobolev inequalities. By different methods, based on isoperimetric inequalities, Kolesnikov [15] recently established more general modified F-Sobolev inequalities for log-concave probability measures on $\mathbb{R}^n$.

We end this introduction by setting the notation. It will be convenient to work with locally Lipschitz functions $f : \mathbb{R}^d \to \mathbb{R}$, for which the norm of the gradient (absolute value of the derivative when $d = 1$) can be defined as a whole by

$$|\nabla f|(x) = \lim_{r\to 0^+} \sup_{y;\,|x-y|\le r} \frac{|f(x) - f(y)|}{|x-y|},$$

where the denominator is the Euclidean norm of $x-y$. By Rademacher's theorem, f is Lebesgue almost everywhere differentiable, and at these points the above notion coincides with the Euclidean norm of the gradient of f.

We recall that a Young function is an even convex function $\Phi : \mathbb{R} \to [0, +\infty)$ with $\Phi(0) = 0$ and $\lim_{x\to+\infty} \Phi(x) = +\infty$. Following [20] we say that Φ is a nice Young function if it also verifies $\Phi'(0) = 0$, $\lim_{x\to+\infty} \frac{\Phi(x)}{x} = +\infty$ and vanishes only at 0. We refer to the Appendix for more details about these functions and their Legendre transforms.

Given a nice Young function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ we define its *modification*

$$\begin{array}{rcl} H_\Phi : \mathbb{R} & \to & \mathbb{R}^+ \\ x & \mapsto & x^2 \mathbb{1}_{[0,1]} + \frac{\Phi(|x|)}{\Phi(1)} \mathbb{1}_{]1,\infty)}. \end{array} \tag{4}$$

A probability measure μ on $\mathbb{R}$ satisfies a modified logarithmic Sobolev inequality with function H_Φ, if there exists some constant $\kappa \in (0, \infty)$ such that every locally Lipschitz $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\mathbf{Ent}_\mu(f^2) \le \kappa \int H_\Phi \left(\frac{f'}{f}\right) f^2 d\mu.$$

We consider functions Φ such that $\Phi(x) \ge cx^2$ for $x \ge 1$, hence the inequalities we study are always weaker than the classical logarithmic Sobolev inequality. On the other hand, as recalled in the introduction, they imply the Poincaré Inequality.

2 Hardy inequalities on the line

Hardy type techniques allowed recent progress in the understanding of several functional inequalities on the real line. We refer to the book [1] for the history of the topic. In this section we show that the modified log-Sobolev inequality can also be addressed by such methods. We will use the following extension of Hardy's inequalities to measures on $\mathbb{R}^+$ equipped with its Borel σ-field. This statement appears in the work of Muckenhoupt [18]. The case of measures with densities with respect to Lebesgue's measure was established earlier by Artola (unpublished), Talenti [24] and Tomaselli [25].

Theorem 1. *Let μ, ν be Borel measures on $\mathbb{R}^+$ and $p > 1$. Then the best constant A such that every locally Lipschitz function f verifies*

$$\int_{[0,+\infty)} |f - f(0)|^p d\mu \le A \int_{[0,+\infty)} |f'|^p d\nu$$

is finite if and only if

$$B := \sup_{x>0} \mu\big([x, +\infty)\big) \left(\int_0^x \frac{dt}{n(t)^{\frac{1}{p-1}}}\right)^{p-1}$$

is finite. Here n is the density of the absolutely continuous part of ν. Moreover, when it is finite $B \le A \le \frac{p^p}{(p-1)^{p-1}} B$.

As an easy consequence, one gets a characterization of measures satisfying a spectral gap inequality together with a good estimate of the optimal constant (see e.g [1]). The next statement also gives an improved lower bound on the best constant C_P recently obtained by Miclo [17].

Theorem 2. *Let μ be a probability measure on $\mathbb{R}$ with median m and let $d\nu(t) = n(t)\,dt$ be a measure on $\mathbb{R}$. The best constant C_P such that every locally Lipschitz $f : \mathbb{R} \to \mathbb{R}$ verifies*

$$\mathbf{Var}_\mu(f) \le C_P \int (f')^2 d\nu \tag{5}$$

verifies $\max(B_+, B_-) \le C_P \le 4\max(B_+, B_-)$*, where*

$$B_+ = \sup_{x>m} \mu\big([x,+\infty)\big) \int_m^x \frac{1}{n}, \quad B_- = \sup_{x<m} \mu\big((-\infty,x]\big) \int_x^m \frac{1}{n}.$$

Bobkov and Götze [5] used Hardy inequalities to obtain a similar result for the best constant in logarithmic Sobolev inequalities: they showed that up to numerical constants, the best C_{LS} such that for all locally Lipschitz f

$$\mathbf{Ent}_\mu(f) \le C_{LS} \int (f')^2 d\nu,$$

is the maximum of

$$\sup_{x>m} \mu\big([x,+\infty)\big) \log\left(\frac{1}{\mu\big([x,+\infty)\big)}\right) \int_m^x \frac{1}{n}$$

and of the corresponding term involving the left side of the median. In [3], we improved their method and extended it to inequalities interpolating between Poincaré and log-Sobolev inequalities (but involving $\int (f')^2 d\nu$).

Using classical arguments (see e.g. the Appendix of [17]) it is easy to see that the Poincaré, the logarithmic Sobolev and the modified logarithmic Sobolev constants are left unchanged if one restrict oneself to the absolutely continuous part of the measure ν in the right hand side. So, without loss of generality, in the sequel we will always assume that ν is absolutely continuous with respect to the Lebesgue measure.

The next two statements show that similar results hold for modified log-Sobolev inequalities provided one replaces the term $\int_m^x 1/n$ by suitable quantities. Obtaining workable expressions for them is not so easy, and will be addressed in the next sections.

Proposition 3. *Let μ be a probability measure with median m and ν a non-negative measure, on $\mathbb{R}$. Assume that ν is absolutely continuous with respect to Lebesgue measure and that the following Poincaré inequality is satisfied: for all locally Lipschitz f*

$$\mathbf{Var}_\mu(f) \le C_P \int (f')^2 d\nu.$$

Let Φ be a nice Young function such that $\Phi(t)/t^2$ is non-decreasing for $t > 0$. Define for $x > m$ the number α_x^+ and for $x < m$ the number α_x^- as follows

$$\alpha_x^+ := \inf\left\{\int_m^x \Phi\left(\frac{f'}{f}\right) f^2 d\nu, f\, \textit{non-decreasing}, f(m) = 1, f(x) = 2\right\},$$

$$\alpha_x^- := \inf\left\{\int_x^m \Phi\left(\frac{f'}{f}\right) f^2 d\nu, f\, \textit{non-increasing}, f(x) = 2, f(m) = 1\right\}.$$

Denote

$$
\begin{aligned}
B^{+}(\Phi): &= \sup_{x>m} \mu([x,\infty)) \log\left(\frac{1}{\mu([x,\infty))}\right)\frac{1}{\alpha_x^+},\\
B^{-}(\Phi): &= \sup_{x<m} \mu((-\infty,x]) \log\left(\frac{1}{\mu((-\infty,x])}\right)\frac{1}{\alpha_x^-}.
\end{aligned}
$$

Then for any for any locally Lipschitz $f:\mathbb{R}\to\mathbb{R}$

$$
\mathbf{Ent}_\mu(f^2) \le \Big(235 C_P + 8\Phi(1)\max\big(B_+(\Phi), B_-(\Phi)\big)\Big)\int H_\Phi\left(\frac{f'}{f}\right) f^2 d\nu.
$$

Proof. In the above statement, there is nothing canonical about 2 in the definition of α_x^+ and α_x^-. We could replace it by a parameter $\sqrt{\rho}>1$. Optimising over ρ would yield non-essential improvements in the results of this paper. However, for this proof we keep the parameter, as we find it clearer like this. We set $\rho=4$ and any value stricty bigger than 1 would do.

Without loss of generality we start with a non-negative function f on $\mathbb{R}$. We consider the associated function

$$
\begin{aligned}
g(x) &= f(m) + \int_m^x f'(u)\mathbb{1}_{f'(u)>0}\,du \quad \text{if} \quad x\ge m\\
g(x) &= f(m) + \int_m^x f'(u)\mathbb{1}_{f'(u)<0}\,du \quad \text{if} \quad x< m.
\end{aligned}
$$

We follow the method of Miclo-Roberto [21, Chapter 3] (see also Section 5.5 of [2] where it is extended). We will omit a few details, which are available in these references. We introduce for $x,t>0$, $\Psi_t(x) = x\log(x/t) - (x-t)$. By convexity of the function $x\log x$ it is easy to check that

$$
\mathbf{Ent}_\mu(f^2) = \int \Psi_{\mu(f^2)}(f^2)d\mu = \inf_t \int \Psi_t(f^2)d\mu \le \int \Psi_{\mu(g^2)}(f^2)d\mu.
$$

Defining $\Omega := \{x;\ f^2(x)\ge 2\rho\,\mu(g^2)\}$, we get

$$
\mathbf{Ent}_\mu(f^2) \le \int_{\Omega^c}\Psi_{\mu(g^2)}(f^2)d\mu + \int_{\Omega\cap[m,+\infty)}\Psi_{\mu(g^2)}(f^2)d\mu + \int_{\Omega\cap(-\infty,m]}\Psi_{\mu(g^2)}(f^2)d\mu. \tag{6}
$$

The first term is bounded as follows. One can check that for any $x\in[0,\sqrt{2\rho}t]$, it holds $\Psi_{t^2}(x^2)\le(1+\sqrt{2\rho})^2(x-t)^2$. Thus

$$
\begin{aligned}
&\int_{\Omega^c}\Psi_{\mu(g^2)}(f^2)d\mu \le (1+\sqrt{2\rho})^2\int_{\Omega^c}\left(f-\sqrt{\mu(g^2)}\right)^2 d\mu\\
\le\ & 2(1+\sqrt{2\rho})^2\int\big(f-g\big)^2 d\mu + 2(1+\sqrt{2\rho})^2\int\big(g-\sqrt{\mu(g^2)}\big)^2 d\mu
\end{aligned}
$$

The last term of the above expression is bounded from above by applying the Poincaré inequality to g. Using the definition of g and applying Hardy's inequality on $(-\infty,m]$ and $[m,+\infty)$ allows to upper bound the term $\int(f-g)^2d\mu$. By Theorems 1 and 2 the best constants in Hardy inequality compare to the Poincaré constant. Finally one gets

$$
\int_{\Omega^c}\Psi_{\mu(g^2)}(f^2)d\mu \le 16(1+\sqrt{2\rho})^2 C_P\int (f')^2 d\nu.
$$

The second term in (6) is

$$
\int_{[m,+\infty)\cap\{f^2\ge 2\rho\,\mu(g^2)\}}\left(f^2\log\Big(\frac{f^2}{\mu(g^2)}\Big) - (f^2-\mu(g^2))\right)d\mu
$$

$$\leq \int_{[m,+\infty)\cap\{g^2\geq 2\rho\,\mu(g^2)\}} g^2 \log\Big(\frac{g^2}{\mu(g^2)}\Big) d\mu = \int_{\Omega_1} g^2 \log\Big(\frac{g^2}{\mu(g^2)}\Big) d\mu,$$

where we have set for $k \in \mathbb{N}$, $\Omega_k := \big\{x \geq m;\ g^2(x) \geq 2\rho^k \mu(g^2)\big\}$. Since g is non-decreasing on the right of m, we have $\Omega_{k+1} \subset \Omega_k = [a_k, \infty)$ for some $a_k \geq m$. Also by Markov's inequality $\mu(\Omega_k) \leq 1/(2\rho^k)$. Furthermore, on $\Omega_k \setminus \Omega_{k+1}$, $2\rho^k \mu(g^2) \leq g^2 < 2\rho^{k+1}\mu(g^2)$. Thus we have

$$\begin{aligned}
\int_{\Omega_1} g^2 \log \frac{g^2}{\mu(g^2)} d\mu &= \sum_{k\geq 1} \int_{\Omega_k\setminus\Omega_{k+1}} g^2 \log \frac{g^2}{\mu(g^2)} d\mu \\
&\leq \sum_{k\geq 1} \mu(\Omega_k) 2\rho^{k+1}\mu(g^2) \log(2\rho^{k+1}) \\
&\leq 2\sum_{k\geq 1} \mu(\Omega_k) 2\rho^{k+1}\mu(g^2) \log(2\rho^{k}) \\
&\leq 2\sum_{k\geq 1} \mu(\Omega_k) \log \frac{1}{\mu(\Omega_k)} 2\rho^{k+1}\mu(g^2) \\
&\leq 2B^+(\Phi) \sum_{k\geq 1} 2\rho^{k+1}\mu(g^2)\alpha^+_{a_k}(\rho)
\end{aligned}$$

where we used $\log(2\rho^{k+1}) \leq 2\log(2\rho^k)$ for $k \geq 1$ and the definition of $B^+(\Phi)$. Now consider the function $g_k = \mathbb{1}_{[m,a_{k-1}[} + \mathbb{1}_{[a_{k-1},a_k[} \frac{g}{\sqrt{2\rho^{k-1}\mu(g^2)}} + \sqrt{\rho}\mathbb{1}_{[a_k,\infty)}$. Since g_k is non-decreasing, $g_k(m) = 1$ and $g_k(a_k) = \sqrt{\rho}$, we have

$$\alpha^+_{a_k}(\rho) \leq \int_m^{a_k} \Phi\left(\frac{g_k'}{g_k}\right) g_k^2 d\nu \leq \frac{1}{2\rho^{k-1}\mu(g^2)} \int_{a_{k-1}}^{a_k} \Phi\left(\frac{g'}{g}\right) g^2 d\nu.$$

Thus,

$$\begin{aligned}
\int_{\Omega_1} g^2 \log \frac{g^2}{\mu(g^2)} d\mu &\leq 2\rho^2 B^+(\Phi) \sum_{k\geq 1} \int_{a_{k-1}}^{a_k} \Phi\left(\frac{g'}{g}\right) g^2 d\nu \\
&\leq 2\rho^2 B^+(\Phi) \int_{\Omega_0} \Phi\left(\frac{f'}{g}\right) g^2 d\nu \\
&\leq 2\rho^2 B^+(\Phi) \int_{[m,+\infty)} \Phi\left(\frac{f'}{f}\right) f^2 d\nu,
\end{aligned}$$

where we have used that $f \leq g$ and the monotonicity of $\Phi(t)/t^2$.

The third term in (6) is estimated in a similar way. Finally one gets

$$\begin{aligned}
\mathbf{Ent}_\mu(f^2) \;\leq\; & 16(1+\sqrt{2\rho})^2 C_P \int \left(\frac{f'}{f}\right)^2 f^2 d\mu + \\
& 2\rho^2 \max(B_+(\Phi), B_-(\Phi)) \int \Phi\left(\frac{f'}{f}\right) f^2 d\mu.
\end{aligned}$$

Our hypotheses ensure that $H_\Phi(x) \geq \max(x^2, \Phi(x)/\Phi(1))$, hence

$$\mathbf{Ent}_\mu(f^2) \leq \Big(16(1+\sqrt{2\rho})^2 C_P + 2\rho^2\Phi(1)\max(B_+(\Phi), B_-(\Phi))\Big) \int H_\Phi\left(\frac{f'}{f}\right) f^2 d\mu.$$

□

Proposition 4. *Let μ be a probability measure with median m and ν a non-negative measure, on $\mathbb{R}$. Assume that ν is absolutely continuous with respect to Lebesgue measure. Let Φ be a nice Young function and H_Φ its modification (see (4)).*

Define the quantities α_x^+ for $x > m$ and α_x^- for $x < m$ as follows

$$\widetilde{\alpha}_x^+ := \inf\left\{ \int_m^x H_\Phi\left(\frac{f'}{f}\right) f^2 d\nu, f \, \textit{non-decreasing}, f(m) = 0, f(x) = 1 \right\},$$

$$\widetilde{\alpha}_x^- := \inf\left\{ \int_x^m H_\Phi\left(\frac{f'}{f}\right) f^2 d\nu, f \, \textit{non-increasing}, f(x) = 1, f(m) = 0 \right\}.$$

Let

$$\begin{aligned} \widetilde{B}^+ &:= \sup_{x>m} \mu([x,\infty)) \log\left(1 + \frac{1}{2\mu([x,\infty))}\right) \frac{1}{\widetilde{\alpha}_x^+}, \\ \widetilde{B}^- &:= \sup_{x<m} \mu((-\infty,x]) \log\left(1 + \frac{1}{2\mu((-\infty,x])}\right) \frac{1}{\widetilde{\alpha}_x^-}. \end{aligned}$$

If C is a constant such that for any locally Lipschitz $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbf{Ent}_\mu(f^2) \leq C \int H_\Phi\left(\frac{f'}{f}\right) f^2 d\nu, \tag{7}$$

then

$$C \geq \max(\widetilde{B}^+, \widetilde{B}^-).$$

Proof. Fix $x_0 > m$ and consider a non-decreasing function f with $f(m) = 0$ and $f(x_0) = 1$. Consider the function $\widetilde{f} = f \mathbb{1}_{[m,x_0[} + \mathbb{1}_{[x_0,\infty)}$. Following [3] and starting with the variational expression of entropy (see e.g. [1, chapter 1]),

$$\begin{aligned} \mathbf{Ent}_\mu(\widetilde{f}^2) &= \sup\left\{ \int \widetilde{f}^2 g \, d\mu, \int e^g d\mu \leq 1 \right\} \\ &\geq \sup\left\{ \int_{[m,+\infty)} \widetilde{f}^2 g \, d\mu, g \geq 0 \text{ and } \int_{[m,+\infty)} e^g d\mu \leq 1 \right\} \\ &\geq \sup\left\{ \int_{[x_0,+\infty)} g \, d\mu, g \geq 0 \text{ and } \int_{[m,+\infty)} e^g d\mu \leq 1 \right\} \\ &= \mu([x_0,\infty)) \log\left(1 + \frac{1}{2\mu([x_0,\infty))}\right) \end{aligned}$$

where the first inequality relies on the fact that $\widetilde{f} = 0$ on $(-\infty, m]$ (hence the best is to take $g = -\infty$ on $(-\infty, m]$). The latter equality follows from [3, Lemma 6] which we recall below. Applying the modified logarithmic Sobolev inequality to $\widetilde{f}$, we get

$$\mu([x_0,\infty)) \log\left(1 + \frac{1}{2\mu([x_0,\infty))}\right) \leq C \int_m^{x_0} H_\Phi\left(\frac{f'}{f}\right) f^2 d\nu.$$

Optimizing over all non-decreasing functions f with $f(m) = 0$ and $f(x_0) = 1$, we get

$$\mu([x_0,\infty)) \log\left(1 + \frac{1}{2\mu([x_0,\infty))}\right) \leq C \widetilde{\alpha}_{x_0}^+.$$

Hence $C \geq \widetilde{B}^+$. A similar argument on the left of the median yields $C \geq \widetilde{B}^-$. □

Lemma 5 ([3]). *Let Q be a finite measure on a space X. Let $K > Q(X)$ and let $A \subset X$ be measurable with $Q(A) > 0$. Then*

$$\sup\left\{\int_X \mathbb{1}_A h\, dQ;\ \int_X e^h dQ \le K \,\text{and}\, h \ge 0\right\} = Q(A)\log\left(1 + \frac{K - Q(X)}{Q(A)}\right).$$

Remark 6. For $x \in (0, \frac{1}{2})$, $\frac{3}{4}\log\frac{1}{x} \le \log(1 + \frac{1}{2x}) \le \log\frac{1}{x}$. Hence B^+ is comparable to

$$\sup_{x>m} \mu([x,\infty))\log\left(\frac{1}{\mu([x,\infty))}\right)\frac{1}{\widetilde{\alpha}_x^+}$$

and similarly for $\widetilde{B}^-$.

In order to turn the previous abstract results into efficient criteria, we need more explicit estimates of the quantities α_x and $\widetilde{\alpha}_x$.

3 The example of power functions: $\Phi(x) = |x|^q$, $q \ge 2$.

In this section we set $\Phi(x) = \Phi_q(x) = |x|^q$, with $q \ge 2$. Its modification is $H(x) = H_q(x) = \max(x^2, |x|^q)$. The constants $\alpha_x^\pm$ and $\widetilde{\alpha}_x^\pm$ are defined accordingly as in Proposition 3 and Proposition 4.

The quantity α_x^+ is estimated thanks to the following easy lemma. A similar bound is available for α_x^- when $x < m$.

Lemma 7. *Assume that ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with density n. Then for $x > m$*

$$\frac{1}{\alpha_x^+} \le 2^{q-2}\left(\int_m^x n^{\frac{-1}{q-1}}\right)^{q-1}.$$

Proof. Fix $x > m$. Let q^* be such that $\frac{1}{q} + \frac{1}{q^*} = 1$. Consider a non-decreasing function f with $f(m) = 1$ and $f(x) = 2$. We assume without loss of generality that $\int_m^x |f'|^q f^{2-q} d\nu$ and $\int_m^x n^{-q^*/q}$ are finite. By Hölder's inequality (valid also when n vanishes), we have

$$1 = \int_m^x f' \le \left(\int_m^x |f'|^q n\right)^{\frac{1}{q}}\left(\int_m^x n^{-\frac{q^*}{q}}\right)^{\frac{1}{q^*}} \le \left(2^{q-2}\int\left|\frac{f'}{f}\right|^q f^2 d\nu\right)^{\frac{1}{q}}\left(\int_m^x n^{-\frac{q^*}{q}}\right)^{\frac{1}{q^*}},$$

where we used the bounds $f \le 2$ and $q \ge 2$. The result follows at once. □

Next we study the quantities $\widetilde{\alpha}_x^+$. They are estimated by testing the inequality on specific functions, as in the proofs of Hardy's inequality. However the presence of the modification H_q creates complications, and we are lead to make additional assumptions. We also omit the corresponding bound on $\widetilde{\alpha}_x^-$.

Lemma 8. *Let ν be a non-negative measure absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with density n. Assume that there exists $\varepsilon > 0$ such that for every $x > m$, it holds*

$$(q-1)n(x)^{\frac{-1}{q-1}} \ge \varepsilon \int_m^x n(u)^{\frac{-1}{q-1}} du. \tag{8}$$

Then for $x > m$, the quantity

$$\widetilde{\alpha}_x^+ = \inf\left\{\int_m^x H_q\left(\frac{f'}{f}\right) f^2 d\nu, f \text{ non-decreasing}, f(m) = 0, f(x) = 1\right\}.$$

verifies

$$\frac{1}{\widetilde{\alpha}_x^+} \geq \frac{\min\left(\varepsilon^{q-2},1\right)}{(q-1)^{q-1}} \left(\int_m^x n(u)^{\frac{-1}{q-1}} du\right)^{q-1}.$$

Proof. Fix $x > m$. Then define

$$f_x(t) = \left(\frac{\displaystyle\int_m^t n^{\frac{-1}{q-1}}}{\displaystyle\int_m^x n^{\frac{-1}{q-1}}} \right)^{q-1} \mathbb{1}_{[m,x]} + \mathbb{1}_{(x,\infty)}.$$

Note that f_x is non-decreasing and satisfies $f_x(m) = 0$ and $f_x(x) = 1$. Thus,

$$\widetilde{\alpha}_x^+ \leq \int_m^x H_q\left(\frac{f_x'}{f_x}\right) f_x^2 d\nu.$$

Furthermore (8) yields for $t \in (m, x)$,

$$\frac{f_x'(t)}{f_x(t)} = \frac{(q-1)n(t)^{\frac{-1}{q-1}}}{\displaystyle\int_m^t n^{\frac{-1}{q-1}}} \geq \varepsilon.$$

Since $H_q(t) \leq \max\left(\frac{1}{\varepsilon^{q-2}}, 1\right) t^q$ for $t \in [\varepsilon, \infty)$, it follows, after some computations, that

$$\begin{aligned}
\int_m^x H_q\left(\frac{f_x'}{f_x}\right) f_x^2 d\nu &\leq \max\left(\frac{1}{\varepsilon^{q-2}}, 1\right) \int_m^x \left(\frac{f_x'}{f_x}\right)^q f_x^2 d\nu \\
&= \max\left(\frac{1}{\varepsilon^{q-2}}, 1\right) \frac{(q-1)^{q-1}}{\left(\displaystyle\int_m^x n^{\frac{-1}{q-1}}\right)^{q-1}}.
\end{aligned}$$

This is the expected result. □

The next result provides a simple condition ensuring Hypothesis (8) to hold

Lemma 9. *For a function $n(x) = e^{-V(x)}$ defined for $x \geq m$. Assume that for $x \in [m, m+K]$ one has $|V(x)| \leq C$ and that V restricted to $[m+K, +\infty)$ is C^1 and verifies $V'(x) \geq \delta > 0$, $x \geq m+K$. Then for $x \geq m$, one has*

$$(q-1)n(x)^{\frac{-1}{q-1}} \geq \varepsilon \int_m^x n^{\frac{-1}{q-1}},$$

where $\varepsilon = \dfrac{1}{\frac{1}{\delta} + \frac{K}{q-1} e^{2C/(q-1)}} > 0.$

Proof. Note that $V(x) \geq -C$ is actually valid for all $x \geq m$. If $x \leq m+K$, simply write

$$\int_m^x n^{-\frac{1}{q-1}} = \int_m^x e^{\frac{V}{q-1}} \leq K e^{\frac{C}{q-1}} \leq K e^{\frac{2C}{q-1}} e^{\frac{V(x)}{q-1}} = K e^{\frac{2C}{q-1}} n(x)^{\frac{-1}{q-1}}.$$

If $x > m+K$, then

$$\begin{aligned}
\int_m^x e^{\frac{V}{q-1}} &\leq K e^{\frac{C}{q-1}} + \int_{m+K}^x e^{\frac{V}{q-1}} \\
&\leq K e^{\frac{2C}{q-1}} e^{\frac{V(x)}{q-1}} + \frac{1}{\delta} \int_{m+K}^x V' e^{\frac{V}{q-1}} \\
&= K e^{\frac{2C}{q-1}} e^{\frac{V(x)}{q-1}} + \frac{q-1}{\delta}\left(e^{\frac{V(x)}{q-1}} - e^{\frac{V(m+K)}{q-1}}\right) \\
&\leq \left(K e^{\frac{2C}{q-1}} + \frac{q-1}{\delta}\right) e^{\frac{V(x)}{q-1}}.
\end{aligned}$$

□

Theorem 10. *Let μ be a probability measure on $\mathbb{R}$ with median m. Let ν be a positive measure absolutely continuous with respect to the Lebesgue measure with density n. Let $C_P \in (0,+\infty]$ be the optimal constant so that the Poincaré inequality (5) holds. Fix $q \geq 2$ and define*

$$\begin{aligned} B_q^+ &:= \sup_{x>m} \mu([x,\infty)) \log \frac{1}{\mu([x,\infty))} \left(\int_m^x n^{\frac{-1}{q-1}} \right)^{q-1}, \\ B_q^- &:= \sup_{x<m} \mu((-\infty,x]) \log \frac{1}{\mu((-\infty,x])} \left(\int_x^m n^{\frac{-1}{q-1}} \right)^{q-1}. \end{aligned}$$

Let $\kappa_q \in (0,+\infty]$ be the best constant such that every locally Lipschitz $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\mathbf{Ent}_\mu(f^2) \leq \kappa_q \int H_q \left(\frac{f'}{f} \right) f^2 d\nu. \tag{9}$$

Then

$$\kappa_q \leq 235 C_P + 2^{q+1} \max(B_q^+, B_q^-).$$

If there exists $\varepsilon > 0$ such that for all $x \neq m$,

$$(q-1) n(x)^{\frac{-1}{q-1}} \geq \varepsilon \int_{\min(x,m)}^{\max(x,m)} n^{\frac{-1}{q-1}},$$

then it is also true that

$$\kappa_q \geq \max \left(2C_P, \frac{3 \min\left(\varepsilon^{q-2}, 1\right)}{4(q-1)^{q-1}} \max\left(B_q^+, B_q^-\right) \right).$$

Proof. The upper bound is immediate from Proposition 3 and Lemma 7 (and its obvious counterpart on the left of the median). The lower bound $\kappa_q \geq 2C_P$ is well known, see [10]. It follows from applying the modified log-Sobolev inequality to $f = 1 + tg$ where g is a bounded function and t goes to zero. Indeed $t^{-2}\mathbf{Ent}_\mu((1+tg)^2)$ tends to $2\mathbf{Var}_\mu(g)$ in this case. The lower bound in terms of $B_q^\pm$ is a direct consequence of Proposition 4, Remark 6 and Lemma 8. □

The following classical lemma (see e.g. [1, Chapter 6]) allows to estimate the integrals appearing in $B_q^\pm$.

Lemma 11. *Let $\Psi : [a,+\infty) \to \mathbb{R}^+$ be a locally bounded function. Assume that it is $\mathcal{C}^2$ in a neighborhood of $+\infty$ and satisfies $\liminf_\infty \Psi' > 0$.*

1. *If $\lim_\infty \Psi''(x)/\Psi'(x)^2 = 0$ then for x growing to infinity*

$$\int_a^x e^{\Psi(t)} dt \sim \frac{e^{\Psi(x)}}{\Psi'(x)}, \qquad \text{and} \qquad \int_x^{+\infty} e^{-\Psi(t)} dt \sim \frac{e^{-\Psi(x)}}{\Psi'(x)}.$$

2. *If for $x \geq x_0$ and $\varepsilon, A > 0$, it holds $-1+\varepsilon \leq \frac{\Psi''(x)}{\Psi'(x)^2} \leq A$, then for $x \geq x_0$*

$$\frac{1}{1+A} \frac{e^{-\Psi(x)}}{\Psi'(x)} \leq \int_x^{+\infty} e^{-\Psi(t)} dt \leq \frac{1}{\varepsilon} \frac{e^{-\Psi(x)}}{\Psi'(x)}.$$

As an application we obtain a workable criterion for satisfying a modified log-Sobolev inequality with function H_q.

Theorem 12. *Let $q \geq 2$. Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}$. Assume that $V : \mathbb{R} \to \mathbb{R}$ is locally bounded, and C^2 in neighborhoods of $+\infty$ and $-\infty$ with*

(i) $\liminf_{|x|\to\infty} \operatorname{sign}(x)V'(x) > 0$

(ii) $\lim_{|x|\to\infty} \frac{V''(x)}{V'(x)^2} = 0.$

Then, there exists $\kappa < +\infty$ such that for every locally Lipschitz f,

$$\mathbf{Ent}_\mu(f^2) \leq \kappa \int H_q\left(\frac{f'}{f}\right) f^2 d\mu$$

if and only if

$$\limsup_{|x|\to\infty} \frac{V(x)}{|V'(x)|^q} < \infty.$$

Remark 13. The condition on $V''/(V')^2$ can be relaxed to $-1 < \liminf \frac{V''}{(V')^2} \leq \limsup \frac{V''}{(V')^2} < \frac{1}{q}$. See Section 4 where this is done in the general case.

Proof. Combining Theorem 2 (for $\nu = \mu$) with Lemma 11 shows that μ satisfies a Poincaré inequality. The hypotheses of Lemma 9 are satisfied, therefore we may apply the two results in Theorem 10. It follows that μ satisfies the modified log-Sobolev inequality if and only if the quantities B_q^+ and B_q^- are finite. The potential V being locally bounded we only have to care about large values of the variables. Applying Lemma 11 again, we see that for x large

$$\mu([x,+\infty)) \log\left(\frac{1}{\mu([x,+\infty))}\right)\left(\int_m^x e^{\frac{V}{q-1}}\right)^{q-1} \sim \frac{V(x) + \log V'(x)}{V'(x)^q}.$$

Hence B_q^+ is finite if and only if $\frac{V+\log V'}{(V')^q}$ has a finite upper limit at $+\infty$. By (i), the term V' is bounded away from 0 in the large. Thus $\log(V')/(V')^q$ is bounded and only $V/(V')^q$ matters. A similar argument allows to deal with B_q^-. ☐

As a direct consequence we recover Theorem 3.1 of Gentil, Guillin and Miclo [10].

Corollary 14. *Fix $q \geq 2$ and define its dual exponent q^* by $\frac{1}{q} + \frac{1}{q^*} = 1$. Let $p \geq 1$ and $d\mu_p(x) = Z_p^{-1}e^{-|x|^p}dx$. Then there exists a constant $C_{p,q} < +\infty$ such that every locally Lipschitz $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$\mathbf{Ent}_{\mu_p}(f^2) \leq C_{p,q} \int H_q\left(\frac{f'}{f}\right) f^2 d\mu_p$$

if and only if $p \geq q^*$.

Remark 15. Bobkov and Ledoux [6] proved that a measure satisfies a Poincaré inequality if and only if it satisfies a modified logarithmic Sobolev inequality with function $H(t) = t^2\mathbb{1}_{|t|\leq t_0}$. This equivalence yields an improvement of the concentration inequalities that one can deduce from a Poincaré inequality. It is natural to conjecture equivalences between general modified log-Sobolev inequalities and inequalities involving $\int (f')^2 d\mu$. Under the hypotheses of the above theorem, Proposition 15 in [3] shows that the condition $\limsup_{|x|\to\infty} \frac{V(x)}{|V'(x)|^q} < \infty$ is also equivalent to μ satifying the following Latała-Oleszkiewicz inequality: there exists $\lambda < +\infty$ such that for all locally Lipschitz f,

$$\sup_{\theta\in[1,2)} \frac{\int f^2 d\mu - \left(\int |f|^\theta d\mu\right)^{2/\theta}}{(2-\theta)^{2/q}} \leq \lambda \int (f')^2 d\mu.$$

Hence, under the hypotheses of Theorem 12, a measure satisfies the latter inequality if and only if it satisfies a modified log-Sobolev inequality with function H_q.

Remark 16. It is known that general modified log-Sobolev inequalities imply so-called transportation cost inequalities, see [4]. Criteria for measures on the line to satisfy such inequalities have been obtained recently by Gozlan [13], after a breakthrough of Cattiaux and Guillin [9]. It is interesting to compare his result with Theorem 12.

4 More general cases

The results of the previous section extend to more general functions Φ. Now, we show how to reach them. In order to obtain workable versions of Proposition 3 we need explicit lower bounds on α_x^+ and α_x^-. Actually our methods also allow bounds in the other direction, but we omit them as they have no other use than showing that the bounds are rather good. By symmetry we shall discuss only α_x^+.

In all this section, Φ stands for a nice Young function, Φ^* for its conjugate and ν for a non-negative measure on $\mathbb{R}$.

Given $x > m$, we have set

$$\alpha_x^+ = \inf\left\{\int_m^x \Phi\left(\frac{f'}{f}\right) f^2 d\nu, f \text{ non-decreasing}, f(m)=1, f(x)=2\right\}.$$

The following simple lower bound is available

$$\begin{aligned}\alpha_x^+ &\geq \inf\left\{\int_m^x \Phi\left(\frac{f'}{2}\right) d\nu, f \text{ non-decreasing}, f(m)=1, f(x)=2\right\}\\ &\geq \inf\left\{\int_m^x \Phi\left(\frac{g}{2}\right) d\nu, g\geq 0, \int_m^x g(u)\,du = 1\right\} = \beta_x\left(\frac{1}{2}\right),\end{aligned}$$

where we have set for $a > 0$,

$$\beta_x(a) := \inf\left\{\int_m^x \Phi(g)\,d\nu\,;\ g \geq 0 \text{ and } \int_m^x g(t)\,dt = a\right\}.$$

The infimum is evaluated in the next lemma. A similar result has been recently established by Arnaud Gloter [12]. The statement involves the following new notation. The left inverse of a non-decreasing function f is defined by $f^{-1}(x) := \inf\{y;\ f(y) \geq u\}$. Also for a non-decreasing function Ψ on $\mathbb{R}^+$ with limits 0 at 0 and $+\infty$ at $+\infty$ but not necessarily convex, we define for a measurable function on $\mathbb{R}$

$$\|g\|_\Psi := \inf\left\{\delta > 0;\ \int_{\mathbb{R}} \Psi\left(\frac{|g|}{\delta}\right) \leq 1\right\},$$

which needs not be a norm.

Lemma 17. *Assume that ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with density n. Then,*

$$\beta_x(a) \geq \int_m^x \Phi\left({\Phi_r'}^{-1}\left(\frac{\gamma_{x,a}}{n}\right)\right) d\nu$$

where

$$\gamma_{x,a} := \sup\left\{\lambda \geq 0;\ \int_m^x {\Phi_r'}^{-1}\left(\frac{\lambda}{n(u)}\right) du \leq a\right\} = \left(\left\|\frac{\mathbb{1}_{[m,x]}}{n}\right\|_{\frac{1}{a}{\Phi_r'}^{-1}}\right)^{-1},$$

and ${\Phi_r'}^{-1}$ is the left inverse of the right derivative of Φ.

Moreover, if Φ_r' is strictly increasing and satisfies the following doubling condition: there exists $K > 1$ such that for all $x \geq 0$, $\Phi_r'(Kx) \geq 2\Phi_r'(x)$, then when $\gamma_{x,a} \neq 0$,

$$\int_m^x {\Phi_r'}^{-1}\left(\frac{\gamma_{x,a}}{n(u)}\right) du = a \quad \textit{and} \quad \beta_x(a) = \int_m^x \Phi\left({\Phi_r'}^{-1}\left(\frac{\gamma_{x,a}}{n}\right)\right) d\nu.$$

Proof. If the set of points in $[m,x]$ where n vanishes has positive Lebesgue measure, it is plain that $\beta_x(a) = \gamma_{x,a} = 0$ and the claimed result is obvious. Hence we may assume that almost every $t \in [m,x]$ verifies $n(t) > 0$. We also assume that $\gamma_{x,a} > 0$ otherwise there is nothing to prove. Let us start with a nonnegative function g on $[m,x]$ with $\int_m^x g = a$ and $\int_m^x \Phi(g)\, d\nu < \infty$. For $\lambda > 0$, and almost every $t \in [m,x]$, $n(t) \neq 0$ and Young's inequality yields

$$g(t) \leq \frac{n(t)}{\lambda}\left(\Phi(g(t)) + \Phi^*\Big(\frac{\lambda}{n(t)}\Big)\right),$$

where $\Phi^*(u) := \sup_{y\geq 0}\{uy - \phi(y)\}$. The analysis of equality cases in Young's inequality leads us to introduce

$$g_\lambda(t) := \inf\left\{x \geq 0;\ \Phi_r'(x) \geq \frac{\lambda}{n(t)}\right\} = {\Phi_r'}^{-1}\Big(\frac{\lambda}{n(t)}\Big).$$

Since Φ_r' is right continuous and vanishes at 0, one has $\Phi_r'(g_\lambda(t)) \geq \frac{\lambda}{n(t)} \geq \Phi_\ell'(g_\lambda(t))$ (at least when $n(t) \neq 0$). By convexity this yields

$$\Phi^*\Big(\frac{\lambda}{n(t)}\Big) = \sup_{y\geq 0}\left\{\frac{\lambda}{n(t)}y - \Phi(y)\right\} = \frac{\lambda}{n(t)}g_\lambda(t) - \Phi\big(g_\lambda(t)\big).$$

Combining this with the latter inequality gives

$$n(t)\Phi(g(t)) \geq n(t)\Phi(g_\lambda(t)) + \lambda(g(t) - g_\lambda(t)).$$

If λ is chosen so that $\int_m^x g_\lambda \leq a$, integrating the previous relation on $[m,x]$ implies that $\int_m^x \Phi(g)\, d\nu \geq \int_m^x \Phi(g_\lambda)\, d\nu$. Optimizing on g and λ satisfying the above conditions, we obtain

$$\beta_x(a) \geq \sup\left\{\int_m^x \Phi\left({\Phi_r'}^{-1}\Big(\frac{\lambda}{n}\Big)\right) d\nu\right\},$$

where the supremum is taken above all λ with $\int_m^x {\Phi_r'}^{-1}(\frac{\lambda}{n}) \leq a$. By definition $\gamma_{x,a}$ is the supremum of such λ's. Using that a left inverse is left continuous, we conclude that

$$\beta_x(a) \geq \int_m^x \Phi\left({\Phi_r'}^{-1}\Big(\frac{\gamma_{x,a}}{n}\Big)\right) d\nu.$$

If we also know that Φ_r' is strictly increasing, then its left inverse is continuous. Moreover the doubling condition: $2\Phi_r'(x) \leq \Phi_r'(Kx)$ translates to the left inverse as a so-called Δ_2 condition: for all $x \geq 0$, ${\Phi_r'}^{-1}(2x) \leq K{\Phi_r'}^{-1}(x)$. Hence for every positive real numbers $\lambda_1 < \lambda_2$ and every $x \geq 0$,

$${\Phi_r'}^{-1}(\lambda_1 x) \leq {\Phi_r'}^{-1}(\lambda_2 x) \leq {\Phi_r'}^{-1}\left(2^{\left\lceil\frac{\log(\lambda_2/\lambda_1)}{\log 2}\right\rceil}\lambda_1 x\right) \leq K^{\left\lceil\frac{\log(\lambda_2/\lambda_1)}{\log 2}\right\rceil}{\Phi_r'}^{-1}(\lambda_1 x).$$

Consequently the family of integrals $\left(\int_m^x {\Phi_r'}^{-1}(\frac{\lambda}{n})\right)_{\lambda>0}$ are either simultaneously infinite or simultaneously finite. In the former situation one gets $\gamma_{x,a} = 0$ whereas in the latter, the function $\lambda \mapsto \int_m^x {\Phi_r'}^{-1}(\frac{\lambda}{n})$ is continuous by dominated convergence and varies from 0 to $+\infty$

(recall that we reduced to $n > 0$ almost everywhere on $[m, x]$). Hence it achives the value $a > 0$ for at least one λ and the smallest of them is $\gamma_{x,a}$. The function $g := \frac{\gamma_{x,a}}{n}$ satisfies $\int_m^x g = a$ and

$$\int_m^x \Phi(g)\, d\nu = \int_m^x \Phi\left({\Phi'_r}^{-1}\left(\frac{\gamma_{x,a}}{n}\right)\right) d\nu.$$

Hence the latter quantity coincides with $\beta_x(a)$. $\square$

Under natural assumptions on the rate of growth of Φ we obtain a simpler bound on $\beta_x(a)$.

Proposition 18. *Assume that ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with density n. Assume that Φ is a strictly convex nice Young function such that on $\mathbb{R}^+$ the function $\Phi(x)/x^2$ is non-decreasing and the function $\Phi(x)/x^\theta$ is non-increasing, where $\theta > 2$. Then for all $a > 0$,*

$$\beta_x(a) \geq \frac{a\, \gamma_{x,a}}{\theta}.$$

Proof. Assume as we may that $\gamma_{x,a} > 0$. We check that the hypothesis of the stronger part of the previous lemma are satisfied. The strict convexity of Φ ensures that Φ'_r is strictly increasing. It remains to check the doubling condition for this function. By differentiation, the monotonicity of $\Phi(x)/x^2$ and $\Phi(x)/x^\theta$ yields for $x \geq 0$,

$$2\Phi(x) \leq x\Phi'_r(x) \leq \theta\Phi(x).$$

Combining these inequalities with the monotonicity of $\Phi(x)/x^2$ yields

$$\Phi'_r(\theta y) \geq 2\frac{\Phi(\theta y)}{\theta y} \geq 2\theta\frac{\Phi(y)}{y} \geq 2\Phi'_r(y),$$

as needed. Applying the previous lemma, we obtain that $a = \int_m^x {\Phi'_r}^{-1}(\frac{\gamma_{x,a}}{n})$, and

$$\begin{aligned}
\beta_x(a) &= \int_m^x \Phi\left({\Phi'_r}^{-1}\left(\frac{\gamma_{x,a}}{n}\right)\right) d\nu \\
&\geq \frac{1}{\theta}\int_m^x {\Phi'_r}^{-1}\left(\frac{\gamma_{x,a}}{n}\right)\Phi'_r\left({\Phi'_r}^{-1}\left(\frac{\gamma_{x,a}}{n}\right)\right) n \\
&\geq \frac{\gamma_{x,a}}{\theta}\int_m^x {\Phi'}^{-1}\left(\frac{\gamma_{x,a}}{n}\right) = \frac{a\,\gamma_{x,a}}{\theta},
\end{aligned}$$

where we have used $F\big(F^{-1}(u)\big) \geq u$, valid for any right-continuous function F. $\square$

Remark 19. When $\Phi(x) = |x|^q$, $\gamma_{x,a}$ and $\beta_x(a)$ are multiples of $(\int_m^x n^{\frac{-1}{q-1}})^{q-1}$. This is consistent with Lemma 7.

Combining the Proposition 3 with the observation that $\alpha_x^+ \geq \beta_x(1/2)$ and Proposition 18, we obtain the following criterion:

Theorem 20. *Let $\theta \geq 2$. Let Φ be a strictly convex nice Young function such that $\frac{\Phi(x)}{x^2}$ is non-decreasing and $\frac{\Phi(x)}{x^\theta}$ is non-increasing. Let μ be a probability measure on $\mathbb{R}$ with median m, and let $d\nu(x) = n(x)\,dx$ be a measure on $\mathbb{R}$. Assume that they satisfy a Poincaré inequality (5) with constant C_P. Then for every locally Lipschitz function f on $\mathbb{R}$, the following modified log-Sobolev inequality holds:*

$$\mathbf{Ent}_\mu(f^2) \leq \Big(235C_P + 16\,\theta\,\Phi(1)\max\big(C_-(\Phi), C_+(\Phi)\big)\Big)\int_{\mathbb{R}} H_\Phi\left(\frac{f'}{f}\right) f^2 d\nu,$$

with

$$C_+(\Phi) := \sup_{x>m} \mu\big([x,+\infty)\big) \log\Big(\frac{1}{\mu\big([x,+\infty)\big)}\Big) \left\| \frac{\mathbb{1}_{[m,x]}}{n} \right\|_{2\Phi_r'^{-1}},$$

$$C_-(\Phi) := \sup_{x<m} \mu\big((-\infty,x]\big) \log\Big(\frac{1}{\mu\big((-\infty,x]\big)}\Big) \left\| \frac{\mathbb{1}_{[x,m]}}{n} \right\|_{2\Phi_r'^{-1}}.$$

Lemma 21. *Let Φ be a differentiable, strictly convex nice Young function. Assume that there exists $\theta > 1$ such that $\Phi(x)/x^\theta$ is non-increasing on $\mathbb{R}^+$. Let $V : [m,+\infty) \to \mathbb{R}$ such that for all $x \in [m, m+K]$, it holds $|V(x)| \le C$. Also assume that V is C^2 on $[m+K,+\infty)$ and verifies for $x \ge m+K$,*

$$V'(x) > 0 \quad \text{and} \quad \frac{V''(x)}{V'(x)^2} \le \frac{1}{\theta}.$$

Then for $x \in]m, m+K]$, it holds $\left\| \frac{\mathbb{1}_{[m,x]}}{e^{-V}} \right\|_{2\Phi'^{-1}} \le \frac{e^C}{\Phi'\big(\frac{1}{4K}\big)}$ and for all $x > m+K$,

$$\left\| \frac{\mathbb{1}_{[m,x]}}{e^{-V}} \right\|_{2\Phi'^{-1}} \le \max\left(\frac{e^C}{\Phi'\big(\frac{1}{4K}\big)}, \frac{e^{V(x)}}{\Phi'\Big(\frac{V'(x)}{4\theta(\theta-1)}\Big)} \right).$$

Proof. Our hypotheses ensure that Φ' is a bijection of $[0;+\infty)$; its inverse is $\Phi^{*\prime}$. In order to show that $\|f\|_\Psi \le \lambda$ it is enough to prove that $\int \Psi(|f|/\lambda) \le 1$. Hence our task is to find $\varepsilon > 0$ with $\int_m^x 2\Phi'^{-1}(\varepsilon e^V) \le 1$. We deal with the case $x \ge m+K$ (the remaining case is simpler and actually contained in the beginning of the following argument):

$$\begin{aligned}
\int_m^x \Phi'^{-1}(\varepsilon e^V) &= \int_m^{m+K} \Phi'^{-1}\big(\varepsilon e^{V(t)}\big)dt + \int_{m+K}^x \Phi'^{-1}\big(\varepsilon e^{V(t)}\big)dt \\
&\le K\Phi'^{-1}(\varepsilon e^C) + \int_{m+K}^x \Phi^{*'}\big(\varepsilon e^{V(t)}\big)dt.
\end{aligned}$$

The first term in the above sum is less than $1/4$ as soon as $\varepsilon \le e^{-C}\Phi'\big(\frac{1}{4K}\big)$. The last term is estimated by integration by parts:

$$\begin{aligned}
&\int_{m+K}^x \Phi^{*\prime}\big(\varepsilon e^{V(t)}\big)dt = \int_{m+K}^x \varepsilon V'(t)e^{V(t)}\Phi^{*\prime}\big(\varepsilon e^{V(t)}\big)\frac{1}{\varepsilon V'(t)e^{V(t)}}\,dt \\
=\ & \frac{\Phi^*\big(\varepsilon e^{V(x)}\big)}{\varepsilon e^{V(x)}V'(x)} - \frac{\Phi^*\big(\varepsilon e^{V(m+K)}\big)}{\varepsilon e^{V(m+K)}V'(m+K)} + \int_{m+K}^x \frac{\Phi^*\big(\varepsilon e^{V(t)}\big)}{\varepsilon e^{V(t)}}\left(1 + \frac{V''(t)}{V'(t)^2}\right) dt \\
\le\ & \frac{\Phi^*\big(\varepsilon e^{V(x)}\big)}{\varepsilon e^{V(x)}V'(x)} + \Big(1+\frac{1}{\theta}\Big)\int_{m+K}^x \frac{\Phi^*\big(\varepsilon e^{V(t)}\big)}{\varepsilon e^{V(t)}}dt \\
\le\ & \frac{\Phi^*\big(\varepsilon e^{V(x)}\big)}{\varepsilon e^{V(x)}V'(x)} + \Big(1-\frac{1}{\theta^2}\Big)\int_{m+K}^x \Phi^{*\prime}\big(\varepsilon e^{V(t)}\big)\,dt,
\end{aligned}$$

where we have used in the last line the inequality $\Phi^*(x) \le \big(1-\frac{1}{\theta}\big)\, x\Phi^{*\prime}(x)$, which follows from our hypotheses by Lemma 30. The term $\int \Phi^{*\prime}\big(\varepsilon e^V\big)$ appears on both sides of the inequality. So after rearrangement we get

$$\int_{m+K}^x \Phi^{*\prime}\big(\varepsilon e^{V(t)}\big)dt \le \theta^2 \frac{\Phi^*\big(\varepsilon e^{V(x)}\big)}{\varepsilon e^{V(x)}V'(x)} \le \theta(\theta-1)\frac{\Phi^{*\prime}\big(\varepsilon e^{V(x)}\big)}{V'(x)} = \theta(\theta-1)\frac{\Phi'^{-1}\big(\varepsilon e^{V(x)}\big)}{V'(x)}.$$

Hence $\int_{m+K}^x \Phi^{*\prime}\big(\varepsilon e^{V(t)}\big)dt \leq 1/4$ holds when

$$\varepsilon \leq e^{-V(x)}\Phi'\left(\frac{V'(x)}{4\theta(\theta-1)}\right).$$

Finally for

$$\varepsilon_0 := \min\left(e^{-C}\Phi'\Big(\frac{1}{4K}\Big), e^{-V(x)}\Phi'\left(\frac{V'(x)}{4\theta(\theta-1)}\right)\right),$$

we have shown that $\int_m^x 2\Phi'^{-1}\big(\varepsilon_0 e^V\big) \leq 1$. This concludes the proof. □

Lemma 21 allows to get more explicit versions of Theorem 20. Here is an example

Theorem 22. *Let Φ be a strictly convex differentiable nice Young function on $\mathbb{R}^+$. Assume that $\Phi(x)/x^2$ is non-decreasing and that there exists $\theta > 2$ such that $\Phi(x)/x^\theta$ is non-increasing.*

Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}$. Assume that V is locally bounded, of class $\mathcal{C}^2$ in neighborhoods of $+\infty$ and $-\infty$ such that:

1. $\liminf_{|x|\to+\infty} \operatorname{sign}(x)V'(x) > 0$,
2. $-1 < \liminf_{|x|\to+\infty} \frac{V''(x)}{V'(x)^2} \leq \limsup_{|x|\to+\infty} \frac{V''(x)}{V'(x)^2} < \frac{1}{\theta}$,
3. $\limsup_{|x|\to+\infty} \frac{V(x)}{\Phi\big(|V'(x)|\big)} < +\infty$.

Then there exists a constant $\kappa < +\infty$ such that for all locally Lipschitz f on $\mathbb{R}$

$$\mathbf{Ent}_\mu(f^2) \leq \kappa \int_{\mathbb{R}} H_\Phi\left(\frac{f'}{f}\right) f^2 d\mu.$$

Proof. Combining hypothesis (i) with Theorem 2 for $\nu = \mu$ and Lemma 11 shows that μ satisfies a Poincaré inequality. Our task is therefore to show that the numbers $C_+(\Phi), C_-(\Phi)$ in the statement of Theorem 20 are finite. By symmetry we only deal with $C_+(\Phi)$. Since V is locally bounded and $t\log(1/t)$ is upper bounded on $(0,1]$, Lemma 21 allows us to reduce the problem to the finiteness of the upper limit when $x \to +\infty$ of

$$\mu\big([x,+\infty)\big)\log\left(\frac{1}{\mu\big([x,+\infty)\big)}\right)\frac{e^{V(x)}}{\Phi'\left(\frac{V'(x)}{4\theta(\theta-1)}\right)}.$$

For shortness we set $T := 4\theta(\theta-1) > 1$. Our assumptions imply that there exists $\varepsilon >$ such that for x large enough $1 \geq V''(x)/V'(x)^2 \geq -1+\varepsilon$. Thus, the second part of Lemma 11 shows that the above quantity is at most

$$\frac{V(x)+\log\big(2V'(x)\big)}{\varepsilon V'(x)\Phi'\left(\frac{V'(x)}{T}\right)} \leq \frac{V(x)+\log\big(2V'(x)\big)}{\varepsilon T\Phi\left(\frac{V'(x)}{T}\right)} \leq T^{\theta-1}\frac{V(x)+\log\big(2V'(x)\big)}{\varepsilon\Phi\big(V'(x)\big)},$$

where we have used that $\Phi(x)/x^\theta$ is non-increasing. Finally since $V'(x)$ is bounded below by a positive number for large x, the ratio of $\log V'$ to $\Phi(V')$ is upper bounded in the large. Condition (iii) allows to conclude. □

As a direct consequence we recover the result by Gentil-Guillin and Miclo [11] with slightly different conditions.

Corollary 23. *Let Ψ be an even convex function on $\mathbb{R}$ such that $d\mu_\Psi(x) = e^{-\Psi(x)}dx$ is a probability measure. Let $\alpha \in (1,2]$. Assume that for $x \geq x_0$, Ψ is of class $\mathcal{C}^2$ with $\Psi(x)/x^2$ non-increasing and $\Psi(x)/x^\alpha$ non-decreasing, and that* $\limsup_\infty \frac{\Psi''}{\Psi'^2} < 1 - \frac{1}{\alpha}$.

Then there exists $C, D \in (0, +\infty)$ such that, setting $\mathcal{H}(x) = C\big(x^2 \mathbb{1}_{|x|<D} + \Psi^*(|x|)\mathbb{1}_{|x|\geq D}\big)$, every locally Lipschitz $f : \mathbb{R} \to \mathbb{R}$ verifies

$$\mathbf{Ent}_{\mu_\Psi}(f^2) \leq \int_{\mathbb{R}} \mathcal{H}\left(\frac{f'}{f}\right) f^2 d\mu_\Psi.$$

Remark 24. If for some $\varepsilon \in (0,1)$, Ψ^ε is concave in the large, then $\lim_\infty \frac{\Psi''}{\Psi'^2} = 0$.

Proof. We apply Theorem 22 with a suitable function Φ. We choose $x_1 > x_0$ such that $\Psi(x_1) > 1$ and $\Psi'(x_1) > 1$. Our monotonicity assumptions ensure that for $x \geq x_0$, $\alpha\Psi(x) \leq x\Psi'(x) \leq 2\Psi(x)$. Let $\beta = \frac{x_1\Psi'(x_1)}{\Psi(x_1)} \in [\alpha, 2]$, and set for $x \geq 0$

$$f(x) = \Psi(x_1)\left(\frac{x}{x_1}\right)^\beta \mathbb{1}_{x<x_1} + \Psi(x)\mathbb{1}_{x\geq x_1}.$$

One easily checks that f is convex of class C^1, and that on $\mathbb{R}^+$, $f(x)/x^\alpha$ is non-decreasing whereas $f(x)/x^2$ is non-increasing. By Lemma 30 the conjugate function is such that $f^*(x)/x^2$ is non-decreasing and $f^*(x)/x^{\alpha^*}$ is non-increasing for $x > 0$ and $\alpha^* = \alpha/(\alpha-1) \geq 2$. One easily checks that for a suitable constant b and for $x \geq 0$

$$f^*(x) = bx^{\beta^*}\mathbb{1}_{x<\Psi'(1)} + \Psi^*(x)\mathbb{1}_{x\geq\Psi'(1)}.$$

Finally we set $\Phi(x) = f^*(x) + x^2$ in order to have a strictly convex function with the same monotonicity properties, to which Theorem 22 may be applied for $V = \Psi$. Note that obviously $\lim_{+\infty}\Psi' = +\infty$. Our assumptions imply that $0 \leq \liminf \frac{\Psi''}{\Psi'^2} \leq \limsup \frac{\Psi''}{\Psi'^2} < 1 - \frac{1}{\alpha} = \frac{1}{\alpha^*}$. Our task is to show the boundedness of the upper limit at $+\infty$ of $\frac{\Psi}{\Phi(\Psi')}$. For x large enough,

$$\frac{\Psi(x)}{\Phi(\Psi'(x))} \leq \frac{\Psi(x)}{\Psi^*(\Psi'(x))} \leq \frac{\alpha^*\Psi(x)}{\Psi'(x)\Psi^{*\prime}(\Psi'(x))} = \frac{\alpha^*\Psi(x)}{\Psi'(x)x} \leq \frac{\alpha^*}{\alpha},$$

where we have used, in differential form, the fact that in the large $\Psi^*(x)/x^{\alpha^*} = f^*(x)/x^{\alpha^*}$ is non-increasing and $\Psi(x)/x^\alpha$ is non-decreasing. Since Ψ is even, Theorem 22 ensures that the measure μ_Ψ satisfies a modified log-Sobolev inequality with function H_Φ. One easily checks that for suitable choice of C, D, this function H_Φ is upper-bounded by the function $\mathcal{H}$ of the claim.

□

Remark 25. Combining Proposition 18 and Lemma 21 provides the existence of constants C, C' such that for x large enough it holds $\alpha_x^+ \geq Ce^{-V(x)}\Phi'(C'V'(x))$. Under additional assumptions and at the price of heavy technicalities, it is actually possible to show that there exists a constant K such that for x large enough $\widetilde{\alpha}_x^+ \leq Ke^{-V(x)}\Phi'\big(V'(x)\big)$. Hence our sufficient condition is also necessary in this case.

5 Concentration of measure phenomenon

By Herbst argument, logarithmic Sobolev inequalities imply Gaussian concentration, see e.g. [1, 16]. Bobkov and Ledoux showed that their modified inequality implies an improved form of exponential concentration for products measures [6], thus extending a well-known result by

Talagrand for the exponential measure [22]. In this section we show that the argument may be adapted to more general modified inequalities.

For a convex function $H : [0, +\infty) \to \mathbb{R}^+$ we define

$$\omega_H(x) = \sup_{t>0} \frac{H(tx)}{H(t)}, \quad x \geq 0.$$

Clearly $\omega_H(0) = 0$ and on $(0, +\infty)$ it is either identically infinite or everywhere finite (exactly when H satisfies the Δ_2 condition). One easily checks that $\omega_H \geq H/H(1)$ is convex and satisfies $\omega_H(ab) \leq \omega_H(a)\omega_H(b)$ for all $a, b \geq 0$. Moreover if $H(x)/x^2$ is non decreasing for $x > 0$ then so is the function $\omega_H(x)/x^2$.

Proposition 26. *Let μ be a probability measure on $\mathbb{R}$ and μ^n the n-fold product measure on $\mathbb{R}^n$. Let $H : \mathbb{R} \to [0, +\infty]$ be an even convex function. Assume that $x \mapsto H(x)/x^2$ is non-decreasing on $(0, +\infty)$. If there exists $\kappa < +\infty$ such that every locally Lipschitz $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$\mathbf{Ent}_\mu(f^2) \leq \kappa \int H\left(\frac{f'}{f}\right) f^2 d\mu, \tag{10}$$

then every locally Lipschitz $F : \mathbb{R}^n \to \mathbb{R}$ with $\sum_{i=1}^n H(\partial_i F) \leq a$ μ^n-a.e. verifies

$$\mu^n\left(\{F - \mu^n(F) \geq r\}\right) \leq e^{-K\omega_H^*\left(\frac{2r}{K}\right)} \qquad \forall r \geq 0$$

where ω_H^* is the conjugate of ω_H and $K = a\kappa$.

Proof. We may assume that ω_H is everywhere finite otherwise there is nothing to prove. Fix $F : \mathbb{R}^n \to \mathbb{R}$ with $\sum_{i=1}^n H(\partial_i F) \leq a$. Assume first that F is integrable. By tensorisation of the modified logarithmic Sobolev Inequality (10) (see [10]), any locally Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$ verifies

$$\mathbf{Ent}_{\mu^n}(f^2) \leq \kappa \int \sum_{i=1}^n H\left(\frac{\partial_i f}{f}\right) f^2 d\mu^n.$$

Plugging $f := e^{\frac{\lambda}{2}F}$, $\lambda \in \mathbb{R}^+$, leads to

$$\begin{aligned}\mathbf{Ent}_{\mu^n}(e^{\lambda F}) &\leq \kappa \int \sum_{i=1}^n H\left(\frac{\lambda}{2}\partial_i F\right) e^{\lambda F} d\mu^n \\ &\leq \kappa\, a\, \omega_H\left(\frac{\lambda}{2}\right) \int e^{\lambda F} d\mu^n.\end{aligned}$$

Define $\Psi(\lambda) := \int e^{\lambda F} d\mu^n$. Then $\mathbf{Ent}_{\mu^n}(e^{\lambda F}) = \lambda\Psi'(\lambda) - \Psi(\lambda)\log\Psi(\lambda)$. Hence, by definition of K,

$$\lambda\Psi'(\lambda) - \Psi(\lambda)\log\Psi(\lambda) \leq K\omega_H\left(\frac{\lambda}{2}\right)\Psi(\lambda) \qquad \forall \lambda \geq 0.$$

In particular, dividing by $\lambda^2\Psi(\lambda)$,

$$\frac{d}{d\lambda}\left(\frac{\log\Psi(\lambda)}{\lambda}\right) \leq K\frac{\omega_H\left(\frac{\lambda}{2}\right)}{\lambda^2} \qquad \forall \lambda > 0.$$

Note that $\lim_0 \frac{\log\Psi(\lambda)}{\lambda} = \mu^n(F)$. Hence integrating leads to

$$\int e^{\lambda(F - \mu^n(F))} d\mu^n \leq \exp\left\{K\lambda \int_0^\lambda \frac{\omega_H\left(\frac{u}{2}\right)}{u^2} du\right\}.$$

Chebichev Inequality finally gives for any $r \geq 0$, any $\lambda > 0$,

$$\mu^n(\{F - \mu^n(F) \geq r\}) \quad \leq \quad e^{-\lambda r} \int e^{\lambda(F-\mu^n(F))} d\mu^n$$

which leads to

$$\mu^n(\{F - \mu^n(F) \geq r\}) \leq \exp\left\{-K \sup_{\lambda>0}\left[\frac{2r}{K}\frac{\lambda}{2} - \lambda\int_0^\lambda \frac{\omega_H\left(\frac{u}{2}\right)}{u^2}du\right]\right\}.$$

The conclusion follows from the inequality

$$\lambda\int_0^\lambda \frac{\omega_H\left(\frac{u}{2}\right)}{u^2}du \leq \omega_H\left(\frac{\lambda}{2}\right),$$

which is proved as follows: let $\theta(\lambda) := \int_0^\lambda \frac{\omega_H(u/2)}{u^2}du$. The result is equivalent to $\theta(\lambda) \leq \lambda\theta'(\lambda)$. Now since $\theta'(\lambda) = \frac{\omega_H(\lambda/2)}{\lambda^2} = \frac{1}{4}\frac{\omega_H(\lambda/2)}{(\lambda/2)^2}$ is non decreasing, θ is convex. In turn, since $\theta(0) = 0$, $\theta(\lambda) \leq \lambda\theta'(\lambda)$ as expected.

The proof is complete for F integrable. A standard truncation argument, see e.g. [1, Lemma 7.3.3], shows that F is automatically integrable. □

Theorem 27. *Let μ be a probability measure on $\mathbb{R}$, which we assume to be absolutely continuous with respect to Lebesgue's measure. Let $H : \mathbb{R} \to \mathbb{R}^+$ be an even convex function, with $H(0) = 0$. Assume that $x \mapsto H(x)/x^2$ is non-decreasing for $x > 0$ and that H^* is strictly convex. If there exists $\kappa < +\infty$ such that every locally Lipschitz $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$\mathbf{Ent}_\mu(f^2) \leq \kappa \int H\left(\frac{f'}{f}\right) f^2 d\mu,$$

then every Borel set $A \subset \mathbb{R}^n$ with $\mu^n(A) \geq \frac{1}{2}$ satisfies

$$1 - \mu^n\left(A + \left\{x : \sum_{i=1}^n H^*(x_i) < r\right\}\right) \leq e^{-Kr} \qquad \forall r \geq 0$$

where $K = \omega_H(2)\,\kappa\,\omega_H^*\left(\frac{1}{\omega_H(2)\,\kappa}\right)$.

Remark 28. The hypothesis of strict convexity of H^* is here for technical reasons. In practice H^* often fails to be strictly convex on a set $[a, b] \subset (0, +\infty)$. In this case it is easy to build an even strictly convex function $I \geq H^*$ which actually coincides with H^* outside of a slightly larger interval and satisfies $I'_r \leq 2H^{*\prime}_r$ on $\mathbb{R}^+$. Following the proof of the theorem with I instead of H^* then yields the concentration inequality claimed in the above theorem, only with a worse constant.

Proof. We start with establishing a useful inequality verified by H. Since $H(x)/x^2$ is non-decreasing on $(0, +\infty)$ it follows that $H^*(x)/x^2$ is non-increasing on this interval, and taking right derivatives that $2H^*(x) \geq x(H^*)'_r(x)$ for $x > 0$ (actually Lemma 30 is valid without differentiability). Next we use the easy inequality $H^*(x) \geq H\big(H^*(x)/x\big)$ for $x > 0$ (it is usually written in the following nicer but more restrictive form $H^{-1}(x)H^{*-1}(x) \geq x$). It follows that

$$H^*(x) \geq H\left(\frac{(H^*)'_r(x)}{2}\right) \geq \frac{1}{\omega_H(2)} H\big((H^*)'_r(x)\big). \tag{11}$$

Let $A \subset \mathbb{R}^n$ with $\mu^n(A) \geq \frac{1}{2}$ and $F_A(x) = \inf_{z\in A} \sum_{i=1}^n H^*(x_i - z_i)$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. For $r > 0$ set further $F = \min(F_A, r)$. We claim that Lebesgue a.e and thus μ^n-a.s., it holds

$$\sum_{i=1}^n H(\partial_i F) \leq \omega_H(2)\, r. \tag{12}$$

First let us develop the consequence of this claim. Note that $F_A = 0$ on A. Thus, $\int F d\mu^n \leq r(1-\mu^n(A)) \leq \frac{r}{2}$. Hence, since $\{F \geq r\} \subset \left\{F - \mu^n(F) \geq \frac{r}{2}\right\}$, Proposition 26 ensures that

$$\mu^n\left(\{F \geq r\}\right) \leq \mu^n\left(\left\{F - \mu^n(F) \geq \frac{r}{2}\right\}\right) \leq \exp\left\{-\omega_H(2)\, r\, \kappa\, \omega_H^*\left(\frac{1}{\omega_H(2)\kappa}\right)\right\}.$$

This leads to the expected result since one can easily see that

$$\{F < r\} = \{F_A < r\} \subset A + \left\{x : \sum_{i=1}^n H^*(x_i) < r\right\}.$$

Finally we establish the claim (12). Since H^* is convex and always finite, it is locally Lipschitz and one easily checks that this property passes to F. Hence F is almost everywhere differentiable and the set $\{x;\ \nabla F(x) \neq 0 \text{ and } F = r\}$ is negligeable. Hence we may restrict to points where $F < r$ and thus $F = F_A < r$ and F_A is differentiable. Denote $\mathcal{H}(x) = \sum_{i=1}^n H^*(x_i)$.

We shall first prove that when F_A is differentiable at x, there exits a unique $a \in \overline{A}$ such that $F_A(x) = \mathcal{H}(x-a)$. Assume that F_A is differentiable at x and that there exist $a \neq b$ in $\overline{A}$ such that $F_A(x) = \min_{c\in\overline{A}} \mathcal{H}(x-c) = \mathcal{H}(x-a) = \mathcal{H}(x-b)$. Consider the function $L : [0,1] \to \mathbb{R}$ defined by $L(u) = \mathcal{H}\big(x - (ua + (1-u)b)\big)$. Since it is strictly convex and $L(0) = F_A(x) = L(1)$ it follows that $L'_r(0) < 0 < L'_\ell(1)$. Since $b \in \overline{A}$ it holds for $t \in [0,1]$

$$F_A(x + t(b-a)) \leq \mathcal{H}(x + t(b-a) - b) = L(t),$$

with equality at $t = 0$. It follows that $DF_A(x).(b-a) \leq L'_r(0) < 0$. On the other hand, since $a \in \overline{A}$, it holds for $t \in [-1, 0]$,

$$F_A(x + t(b-a)) \leq \mathcal{H}(x + t(b-a) - a) = L(1+t),$$

with equality at $t = 0$. It follows that $DF_A(x).(b-a) \geq L'_\ell(1) > 0$ which contradicts our previous bound.

To complete the proof of the claim, we consider a point x where F_A is differentiable and $F_A(x) < r$ and we consider $a \in \overline{A}$ the unique minimizer for $\mathcal{H}(x - \cdot)$ on $\overline{A}$. An easy consequence of the uniqueness is that for every sequence y^k converging to x and $a^k \in \overline{A}$ such that $F_A(y^k) = \mathcal{H}(y^k - a^k)$, the sequence a^k converges to a. Let t_k be a sequence of positive numbers converging to zero. Then, denoting by e^i the i-th vector in the canonical basis of $\mathbb{R}^n$,

$$\begin{aligned}
F_A(x + t_k e^i) - F_A(x) &= \inf_{c\in\overline{A}} \mathcal{H}(x + t_k e^i - c) - \mathcal{H}(x-a) \\
&\leq \mathcal{H}(x + t_k e^i - a) - \mathcal{H}(x-a) = H^*(x_i + t_k - a_i) - H^*(x_i - a_i).
\end{aligned}$$

Dividing by $t_k > 0$ and taking limits yields $\partial_i F_A(x) \leq H_r^{*\prime}(x_i - a_i) \leq H_r^{*\prime}(|x_i - a_i|)$. Similarly, if we denote by a^k a minimizer of $c \in \overline{A} \mapsto \mathcal{H}(x + t_k e^i - c)$

$$\begin{aligned}
F_A(x + t_k e^i) - F_A(x) &= \mathcal{H}(x + t_k e^i - a^k) - \inf_{c\in\overline{A}} \mathcal{H}(x-c) \\
&\geq \mathcal{H}(x + t_k e^i - a^k) - \mathcal{H}(x - a^k) = H^*(x_i + t_k - a_i^k) - H^*(x_i - a_i^k) \\
&\geq t_k H_r^{*\prime}(x_i - a_i^k),
\end{aligned}$$

by convexity. Recall that a^k converges to a. Hence letting k to infinity we get $\partial_i F_A(x) \geq H_\ell^{*\prime}(x_i - a_i) \geq -H_r^{*\prime}(|x_i - a_i|)$. Eventually when $F_A(x) = F(x) < r$

$$\begin{aligned} \sum_{i=1}^n H\big(\partial_i F(x)\big) &\leq \sum_{i=1}^n H\big(H_r^{*\prime}(|x_i - a_i|)\big) \leq \omega_H(2) \sum_{i=1}^n H^*(x_i - a_i) \\ &= \omega_H(2)\mathcal{H}(x-a) = \omega_H(2) F_A(x) < \omega_H(2) r, \end{aligned}$$

using (11) and the definition of a as a minimizer. □

If $H = H_\Phi$ is the modification of an even convex $\Phi : \mathbb{R} \to \mathbb{R}^+$ with $\Phi(x)/x^2$ non-decreasing on $\mathbb{R}^+$ one easily checks that there exists x_0 such that $H_\Phi^*(x)$ is comparable to x^2 up to multiplicative constants if $|x| \leq x_0$, and $H_\Phi^*(x) = \Phi^*(x)$ otherwise. Then, separating coordinates x_i of absolute value less or more than x_0, one gets that there exists a constant c (depending on Φ) such that for any r,

$$\left\{ x : \sum_{i=1}^n H_\Phi^*(x_i) < r \right\} \subset \sqrt{cr} B_2 + \left\{ x : \sum_{i=1}^n \Phi^*(x_i) < cr \right\}.$$

Let $\omega_{\Phi^*}(t) := \sup_{x>0} \frac{\Phi^*(tx)}{\Phi^*(x)}$ for $t > 0$ and $B_{\Phi^*} := \{x : \sum_{i=1}^n \Phi^*(x_i) < 1\}$. For any x such that $\sum_{i=1}^n \Phi^*(x_i) < s$, we have

$$\sum_{i=1}^n \Phi^* \left(\omega_{\Phi^*}^{-1}\left(\frac{1}{s}\right) x_i \right) \leq \omega_{\Phi^*}\left(\omega_{\Phi^*}^{-1}\left(\frac{1}{s}\right)\right) \sum_{i=1}^n \Phi^*(x_i) < 1.$$

Thus $\{x : \sum_{i=1}^n \Phi^*(x_i) < s\} \subset \frac{1}{\omega_{\Phi^*}^{-1}(\frac{1}{s})} B_{\Phi^*}$. Hence, under the hypotheses of Theorem 27 we have for any Borel set $A \subset \mathbb{R}^n$ with $\mu^n(A) \geq \frac{1}{2}$,

$$\mu^n \left(A + \sqrt{r} B_2 + \frac{1}{\omega_{\Phi^*}^{-1}\left(\frac{1}{r}\right)} B_{\Phi^*} \right) \geq \mu^n \left(A + \left\{ x : \sum_{i=1}^n H_\Phi^*(x_i) < r \right\} \right) \geq 1 - e^{-Cr} \qquad \forall r \geq 0 \tag{13}$$

for some constant C independent on r. Such concentration inequalities were established by Talagrand [22, 23] for the exponential measure and later for even log-concave measures, via inf-convolution inequalities (which are strongly related to transportation cost inequalities). More recently Gozlan derived such inequalities from his criterion for transportation inequalities on the line [13]. We conclude this section with concrete examples.

Example 29. Let $\Phi_q(x) = |x|^q$, $q \geq 2$ and $H_q(x) = H_{\Phi_q}(x) = \max(x^2, |x|^q)$. Straightforward calculations give

$$H_q^*(x) = \begin{cases} x^2/4 & \text{if } x \leq 2 \\ x - 1 & \text{if } 2 \leq x \leq q \\ (q-1)\,(x/q)^{\frac{q}{q-1}} & \text{if } x \geq q \end{cases}.$$

Here $\omega_{\Phi_q^*} = \Phi_q^* = C_q |x|^{q^*}$ with $\frac{1}{q} + \frac{1}{q^*} = 1$. Let $B_{q^*} := \left\{ x : \sum_{i=1}^n |x_i|^{q^*} < 1 \right\}$ be the ℓ^{q^*}-unit ball in $\mathbb{R}^n$. If μ satisfies the modified logarithmic Sobolev Inequality (9), there exists a constant C_q' (depending only on q) such that

$$1 - \mu^n \left(A + \sqrt{r} B_2 + r^{\frac{1}{q^*}} B_{q^*} \right) \leq e^{-C_q' r} \qquad \forall r \geq 0$$

for any A with $\mu^n(A) \geq \frac{1}{2}$. In particular, thanks to Corollary 14, the measures $d\mu_\beta(x) = Z_\beta^{-1} e^{-|x|^\beta} dx$ satisfy the latter concentration result for any $\beta \geq q^* > 1$.

Note that the limit case $q^* = 1$ or $q = +\infty$ is not treated in our argument. It corresponds to the case when $H(x) = x^2 \mathbb{1}_{|x|<c} + \infty \mathbb{1}_{|x|\geq c}$ treated by Bobkov and Ledoux [6]. Our "extension" does not cover this case since for technical reasons we considered only functions H taking finite values. On the other hand combining Corollary 23 with the above theorem and remark, yields similar concentration properties for a wide class of even log-concave measures with an intermediate behaviour between exponential and Gaussian.

6 Appendix on Young functions

In this section we collect some useful facts about Orlicz spaces. We refer to [20] for details.

Definition 1 (Young function)**.** A function $\Phi : \mathbb{R} \to [0, \infty]$ is a *Young function* if it is convex, even, such that $\Phi(0) = 0$, and $\lim_{x\to+\infty} \Phi(x) = +\infty$.

The Legendre transform Φ^* of Φ is defined by $\Phi^*(y) = \sup_{x\geq 0}\{x|y| - \Phi(x)\}$. It is a lower semi-continuous Young function called the *complementary function* or *conjugate* of Φ. Among the Young functions, we call *nice Young function* those which take only finite values and such that $\Phi(x)/x \to \infty$ as $x \to \infty$, $\Phi(x) = 0 \Leftrightarrow x = 0$ and $\Phi'(0) = 0$.

For any nice Young function Φ, the conjugate of Φ^* is Φ and for any $x > 0$,

$$x \leq \Phi^{-1}(x)(\Phi^*)^{-1}(x) \leq 2x.$$

The simplest example of nice Young function is $\Phi(x) = \frac{|x|^p}{p}$, $p > 1$, for which, $\Phi^*(x) = \frac{|x|^q}{q}$, with $1/p + 1/q = 1$.

Now let $(\mathcal{X}, \mu)$ be a measurable space, and Φ a Young function. The space

$$\mathbb{L}_\Phi(\mu) = \left\{ f : \mathcal{X} \to \mathbb{R} \text{ measurable}; \exists \alpha > 0, \int_{\mathcal{X}} \Phi(\alpha f) < +\infty \right\}$$

is called the *Orlicz space* associated to Φ. When $\Phi(x) = |x|^p$, then $\mathbb{L}_\Phi(\mu) = \mathbb{L}^p(\mu)$, the standard Lebesgue space. There are two natural equivalent norms which give to $\mathbb{L}_\Phi(\mu)$ a structure of Banach space. Namely

$$\|f\|_\Phi = \inf \left\{ \lambda > 0; \int_{\mathcal{X}} \Phi\left(\frac{f}{\lambda}\right) d\mu \leq 1 \right\}$$

and

$$N_\Phi(f) = \sup \left\{ \int_{\mathcal{X}} |fg| d\mu; \int_{\mathcal{X}} \Phi^*(g) d\mu \leq 1 \right\}.$$

Note that we invert the notation with respect to [20]. For $\alpha \in [1, \infty]$ we denote the dual coefficient $\alpha^* \in [1, \infty]$. It is defined by the equality $\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1$.

The two next lemmas are very simple, but convenient (proofs are left to the reader).

Lemma 30. *Let $\alpha \in (1, +\infty)$. Let Φ be a differentiable, strictly convex nice Young function. Then the following assertions are equivalent:*

1. *The function $\Phi(x)/x^\alpha$ is non-decreasing for $x > 0$.*
2. *For $x \geq 0$, $x\Phi'(x) \geq \alpha\Phi(x)$.*
3. *For $x \geq 0$, $x\Phi^{*\prime}(x) \leq \alpha^*\Phi^*(x)$.*
4. *The function $\Phi^*(x)/x^{\alpha^*}$ is non-increasing for $x > 0$.*

Note that Φ and Φ^* play symmetric roles so that similar equivalent formulations exist for the property: $\Phi(x)/x^\alpha$ is non-increasing for $x \geq 0$.

Lemma 31. *Let $0 < \alpha < \theta$. Let Φ be a differentiable function on $[0, +\infty)$ such that the function $\Phi(x)/x^\alpha$ is non-decreasing and $\Phi(x)/x^\theta$ is non-increasing. Then for $x > 0$, and $t \geq 1$ it holds*

$$\Phi(tx) \leq t^\theta \Phi(x), \quad \Phi'(tx) \leq \theta t^{\theta-1} \frac{\Phi(x)}{x} \leq \frac{\theta}{\alpha} t^{\theta-1} \Phi'(x).$$

For for $x > 0$, and $t \in (0, 1]$ it holds

$$\Phi(tx) \leq t^\alpha \Phi(x), \quad \Phi'(tx) \leq \theta t^{\alpha-1} \frac{\Phi(x)}{x} \leq \frac{\theta}{\alpha} t^{\alpha-1} \Phi'(x).$$

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Mathematics Subject Classification: 26D10, 60E15.
Keywords: Sobolev inequalities, concentration.

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