Claude Sabbah

DE RHAM AND DOLBEAULT COHOMOLOGY OF $\mathscr{D}$-MODULES

LECTURE NOTES (OBERWOLFACH, MAY-JUNE 2007)

PRELIMINARY VERSION

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LECTURE 1

INTRODUCTION TO $\mathscr{D}$-MODULES

In this first lecture, we introduce the sheaf of differential operators and its (left or right) modules. Our main concern is to develop the relationship between two *a priori* different notions:

(1) the classical notion of a $\mathscr{O}_X$-module with a *flat* connection,
(2) the notion of a left $\mathscr{D}_X$-module.

Both notions are easily seen to be equivalent. However, the extension of the equivalence to complexes (or to the derived category) is less clear. The notion of differential complex is useful to express this equivalence, but will not be considered here.

The relationship between left and right $\mathscr{D}_X$-modules, although simple, is also somewhat subtle, and we insist on the basic isomorphisms.

The results in this lecture are mainly algebraic, and do not involve any analytic property. They are given in the algebraic situation, but can be adapted to the complex analytic setting. One can find many of these notions in the classical books [1–3, 7, 15, 18, 19, 22]. Some of them are also directly inspired from the work of M. Saito [25] about Hodge $\mathscr{D}$-modules.

In this chapter, X denotes a smooth scheme of finite type over a field $\boldsymbol{k}$ of characteristic 0 (which can be the field $\mathbb{C}$ of complex numbers) or a complex analytic manifold. We will denote by $\mathscr{O}_X$ the corresponding sheaves of functions. By a function, we will mean a local section of $\mathscr{O}_X$.

1.1. The sheaf of differential operators

We will denote by Θ_X the sheaf of vector fields on X. This is the $\mathscr{O}_X$-locally free sheaf generated in local étale coordinates by $\partial_{x_1}, \dots, \partial_{x_n}$. It is a sheaf of $\mathscr{O}_X$-Lie algebras which is locally free as a $\mathscr{O}_X$-module.

In a dual way, we denote by Ω^1_X the sheaf of 1-forms on X. We will set $\Omega^k_X = \wedge^k \Omega^1_X$. We denote by $d : \Omega^k_X \to \Omega^{k+1}_X$ the (Kähler) differential.

By definition, the vector fields act (on the left) on functions by derivation, in a compatible way with the Lie algebra structure.

Let ω_X denote the sheaf $\Omega_X^{\dim X}$ of forms of maximal degree. Then there is a natural *right* action (in a compatible way with the Lie algebra structure) of Θ_X on ω_X: the action is given by $\omega \cdot \xi = -\mathcal{L}_\xi \omega$, where $\mathcal{L}_\xi$ denotes the Lie derivative, equal to the composition of the interior product ι_ξ by ξ with the differential d, as it acts on forms of maximal degree (cf. Exercise E.1.1).

***Definition 1.1.1* (The sheaf of differential operators).** For any open set U of X, the ring $\mathscr{D}_X(U)$ of *differential operators on* U is the subring of $\mathrm{Hom}_{\boldsymbol{k}}(\mathscr{O}_U, \mathscr{O}_U)$ generated by

– multiplication by functions on U,
– derivation by vector fields on U.

The sheaf $\mathscr{D}_X$ is defined by $\Gamma(U, \mathscr{D}_X) = \mathscr{D}_X(U)$ for any open set U of X.

(cf. Exercise E.1.2 for more details on Hom.) By construction, the sheaf $\mathscr{D}_X$ acts on the left on $\mathscr{O}_X$, i.e., $\mathscr{O}_X$ is a left $\mathscr{D}_X$-module.

***Example 1.1.2* (The one-variable Weyl algebra).** Assume that X is the affine line $\mathbb{A}^1_{\boldsymbol{k}}$ with coordinate t. Then $\mathscr{D}(X)$ is called the one-variable Weyl algebra. This is the quotient algebra of the free algebra generated by the polynomial algebras $\boldsymbol{k}[t]$ and $\boldsymbol{k}[\partial_t]$ by the relation $[\partial_t, t] = 1$. We will denote it by $\boldsymbol{k}[t]\langle\partial_t\rangle$. It is a non-commutative algebra. Its elements are the *differential operators* with polynomial coefficients. Such an operator can be written in a unique way as

$$P(t, \partial_t) = a_d(t)\partial_t^d + \cdots + a_1(t)\partial_t + a_0(t)$$

where the a_i are polynomials in t, and $a_d \neq 0$. We call d the *degree* of the operator. The product of two such operators can be reduced to this form by using the commutation relation

$$\partial_t a(t) = a(t)\partial_t + a'(t). \tag{1.1.3}$$

Its degree is equal to the sum of the degrees of its factors. The *symbol* of an operator is the class of the operator modulo the operators of strictly lower degree. The set of symbols of differential operators, equipped with the induced operations, is identified with the algebra of polynomials of two variables with coefficients in $\boldsymbol{k}$; it is thus a Noetherian ring. This enables one to prove that the Weyl algebra is itself left and right Noetherian.

The Weyl algebra contains as sub-algebras the algebras $\boldsymbol{k}[t]$ (operators of degree 0) and $\boldsymbol{k}[\partial_t]$ (operators with constant coefficients).

***Definition 1.1.4* (The filtration of $\mathscr{D}_X$ by the order).** The increasing family of subsheaves $F_k\mathscr{D}_X \subset \mathscr{D}_X$ is defined inductively:

– $F_k\mathscr{D}_X = 0$ if $k \leqslant -1$,
– $F_0\mathscr{D}_X = \mathscr{O}_X$ (*via* the canonical injection $\mathscr{O}_X \hookrightarrow \mathrm{Hom}_{\boldsymbol{k}}(\mathscr{O}_X, \mathscr{O}_X)$),

– the local sections P of $F_{k+1}\mathscr{D}_X$ are characterized by the fact that $[P,\varphi]$ is a local section of $F_k\mathscr{D}_X$ for any function φ.

(cf. Exercises E.1.4 and E.1.5.)

1.2. Left and right

Let us recall the basic lemmas for generating left or right $\mathscr{D}$-modules. We refer for instance to [6, § 1.1] for more details.

***Lemma 1.2.1* (Generating left or right $\mathscr{D}_X$-modules).** *Let $\mathscr{M}^l$ (resp. $\mathscr{M}^r$) be a $\mathscr{O}_X$-module and let $\varphi^l:\Theta_X\otimes_{\boldsymbol{k}_X}\mathscr{M}^l\to\mathscr{M}^l$ (resp. $\varphi^r:\mathscr{M}^r\otimes_{\boldsymbol{k}_X}\Theta_X\to\mathscr{M}^r$) be a $\boldsymbol{k}$-linear morphism such that, for any local sections f of $\mathscr{O}_X$, ξ,η of Θ_X and m of $\mathscr{M}^l$ (resp. of $\mathscr{M}^r$), one has*

(1) $\varphi^l(f\xi\otimes m)=f\varphi^l(\xi\otimes m)$,
(2) $\varphi^l(\xi\otimes fm)=f\varphi^l(\xi\otimes m)+\xi(f)m$,
(3) $\varphi^l([\xi,\eta]\otimes m)=\varphi^l(\xi\otimes\varphi^l(\eta\otimes m))-\varphi^l(\eta\otimes\varphi^l(\xi\otimes m))$,

resp.

(1) $\varphi^r(mf\otimes\xi)=\varphi^r(m\otimes f\xi)$,
(2) $\varphi^r(m\otimes f\xi)=\varphi^r(m\otimes\xi)f-m\xi(f)$,
(3) $\varphi^r(m\otimes[\xi,\eta])=\varphi^r(\varphi^r(m\otimes\xi)\otimes\eta)-\varphi^r(\varphi^r(m\otimes\eta)\otimes\xi)$.

Then there exists a unique structure of left (resp. right) $\mathscr{D}_X$-module on $\mathscr{M}^l$ (resp. $\mathscr{M}^r$) such that $\xi m=\varphi^l(\xi\otimes m)$ (resp. $m\xi=\varphi^r(m\otimes\xi)$) for any ξ,m.

(cf. Exercises E.1.8–E.1.13.)

***Definition 1.2.2* (Right-left transformation).** Any left $\mathscr{D}_X$-module $\mathscr{M}^l$ gives rise to a right one $\mathscr{M}^r$ by setting (cf. [6] for instance) $\mathscr{M}^r=\omega_X\otimes_{\mathscr{O}_X}\mathscr{M}^l$ and, for any vector field ξ and any function f,

$$(\omega\otimes m)\cdot f=f\omega\otimes m=\omega\otimes fm,\quad(\omega\otimes m)\cdot\xi=\omega\xi\otimes m-\omega\otimes\xi m.$$

Conversely, set $\mathscr{M}^l=\mathscr{H}om_{\mathscr{O}_X}(\omega_X,\mathscr{M}^r)$, which also has in a natural way the structure of a left $\mathscr{D}_X$-module (cf. Exercise E.1.11(2)).

Properties of this involution are given in Exercises E.1.14 and E.1.15.

Example 1.2.3. The *transposition* is the involution $P\mapsto{}^tP$ of $\boldsymbol{k}[t]\langle\partial_t\rangle$ defined by the following properties:

(1) ${}^t(P\cdot Q)={}^tQ\cdot{}^tP$ for all $P,Q\in\boldsymbol{k}[t]\langle\partial_t\rangle$,
(2) ${}^tP=P$ for any $P\in\boldsymbol{k}[t]$,
(3) ${}^t\partial_t=-\partial_t$.

If M^r is a right module over $\boldsymbol{k}[t]\langle\partial_t\rangle$, we can equip it canonically with a structure of left module M^l by setting, for any $m \in M^r$,

$$P \cdot m \overset{\mathrm{def}}{=} m\,{}^tP.$$

Conversely, to any left module is canonically associated a right module.

1.3. Examples of $\mathscr{D}$-modules

We list here some classical examples of $\mathscr{D}$-modules. One may get many other examples by applying various operations on $\mathscr{D}$-modules.

1.3.a. Let $\mathscr{I}$ be a sheaf of left ideals of $\mathscr{D}_X$. Then the quotient $\mathscr{M} = \mathscr{D}_X/\mathscr{I}$ is a left $\mathscr{D}_X$-module. If $\mathscr{I}$ is locally generated by a finite set $P_1, \dots, P_k$ of differential operators then, locally, $\mathscr{M}$ is the $\mathscr{D}_X$-module associated with $P_1, \dots, P_k$.

Notice that different choices of generators of $\mathscr{I}$ give rise to the same $\mathscr{D}_X$-module $\mathscr{M}$ (cf. Exercise E.1.16). It may be sometime difficult to guess that two sets of operators generate the same ideal. Therefore, it is useful to develop a systematic procedure to construct from a system of generators a *division basis* of the ideal in order to have a decision algorithm.

1.3.b. If I is an ideal of $\mathscr{O}_X$, it generates a left ideal $\mathscr{I} = \mathscr{D}_X \cdot I$ in $\mathscr{D}_X$. For instance, if $X = \mathbb{A}^n$, let I be the ideal generated by a coordinate x_n. Then $\mathscr{D}_{\mathbb{A}^n}/\mathscr{D}_{\mathbb{A}^n} \cdot x_n = \mathscr{D}_{\mathbb{A}^{n-1}}[\partial_{x_n}]$.

More generally, let $\mathscr{L}$ be a $\mathscr{O}_X$-module. There is a very simple way to get a left (resp. right) $\mathscr{D}_X$-module from $\mathscr{L}$: consider $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{L}$ (resp. $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$) equipped with the natural left (resp. right) action of $\mathscr{D}_X$. This is called an *induced* $\mathscr{D}_X$-module. Although this construction is very simple, it is also very useful to get cohomological properties of $\mathscr{D}_X$-modules.

1.3.c. One of the main geometrical examples of $\mathscr{D}_X$-modules are the vector bundles on X equipped with an *integrable* connection. Recall that left $\mathscr{D}_X$-modules are $\mathscr{O}_X$-modules with an integrable connection. Among them, the coherent $\mathscr{O}_X$-modules are particularly interesting. We will see (cf. §2.2.c), that such modules are $\mathscr{O}_X$-locally free, i.e., correspond to vector bundles of finite rank on X.

More generally, we have:

Proposition 1.3.1. *The category of $\mathscr{O}_X$-modules equipped with an integrable connection (the morphisms being compatible with the connections) is equivalent to the category of* left *$\mathscr{D}_X$-modules.*

(cf. Exercises E.1.6 and E.1.7.)

It may happen that, for some X, such a category of $\mathscr{O}_X$ locally free sheaves with an integrable connection does not give any interesting geometric object (cf. Exercise E.1.17).

However, on Zariski open sets of X, there may exist interesting vector bundles with connection.

In the complex analytic setting, this leads to the notion of meromorphic vector bundle with connection. Let D be a divisor in X and denote by $\mathscr{O}_X(*D)$ the sheaf of meromorphic functions on X with poles along D at most. This is a sheaf of left $\mathscr{D}_X$-modules, being a $\mathscr{O}_X$-module equipped with the natural connection $d : \mathscr{O}_X(*D) \to \Omega^1_X(*D)$. By definition, a *meromorphic bundle* is a locally free $\mathscr{O}_X(*D)$ module of finite rank. When it is equipped with an integrable connection, it becomes a left $\mathscr{D}_X$-module.

In the algebraic setting, this notion is equivalent to the notion of vector bundle on $U = X \smallsetminus D$.

1.3.d. One may *twist* the previous examples. Assume that there exists a *closed* form ω on X. Define $\nabla : \mathscr{O}_X \to \Omega^1_X$ by the formula $\nabla = d + \omega$. As ω is closed, ∇ is an integrable connection on the trivial bundle $\mathscr{O}_X$.

Usually, there only exist meromorphic closed form on X, with poles on some divisor D. Then ∇ is an integrable connection on $\mathscr{O}_X(*D)$.

If ω is exact, $\omega = df$ for some meromorphic function f on X, then ∇ may be written as $e^{-f} \circ d \circ e^f$.

More generally, if $\mathscr{M}$ is any meromorphic bundle with an integrable connection ∇, then, for any such ω, $\nabla + \omega$ defines a new $\mathscr{D}_X$-module structure on $\mathscr{M}$.

1.3.e. One may construct new examples from old ones by using various operations.

– Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. Then $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)$ has a natural structure of right $\mathscr{D}_X$-module. Using a resolution $\mathscr{N}^\bullet$ of $\mathscr{M}$ by left $\mathscr{D}_X$-modules which are acyclic for $\mathscr{H}om_{\mathscr{D}_X}(\bullet, \mathscr{D}_X)$, one gets a right $\mathscr{D}_X$-module structure on the $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)$.

– Given two left (resp. a left and a right) $\mathscr{D}_X$-modules $\mathscr{M}$ and $\mathscr{N}$, the same argument allows one to put on the various $\mathscr{T}or_{i,\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ a left (resp. a right) $\mathscr{D}_X$-module structure.

– We will see in Lecture 4 the geometric operation "direct image" (push-forward) of a $\mathscr{D}_X$-module by a holomorphic map. The "inverse image" (pull-back) also exists, but it will not be explained in these notes.

1.3.f. Fourier transform. The relation $[\partial_t, t] = 1$ defining the Weyl algebra can also be written as $[(-t), \partial_t] = 1$, so that we have an isomorphism of algebras

$$\begin{aligned} \boldsymbol{k}[t]\langle\partial_t\rangle &\longrightarrow \boldsymbol{k}[\tau]\langle\partial_\tau\rangle \\ t &\longmapsto -\partial_\tau, \\ \partial_t &\longmapsto \tau. \end{aligned}$$

Any module M over $\boldsymbol{k}[t]\langle\partial_t\rangle$ becomes *ipso facto* a module over $\boldsymbol{k}[\tau]\langle\partial_\tau\rangle$.

This transformation is not at all trivial. Moreover, it does not behave well with respect to the correspondence of Proposition 1.3.1 (cf. Exercise E.1.19).

1.3.g. Duality. Another important operation on $\mathscr{D}_X$-modules is duality. However, as for sheaves of $\mathscr{O}_X$-modules, the dual of a $\mathscr{D}_X$-module consists in general of a complex.

Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. Then $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}_X)$ does not have a natural structure of left $\mathscr{D}_X$-module. Nevertheless, as $\mathscr{D}_X$ is also a right $\mathscr{D}_X$-module over itself, the previous objects gets a natural right action of $\mathscr{D}_X$, hence is a right $\mathscr{D}_X$-module. More generally, using standard techniques of homological algebra, the sheaves $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}_X)$ are naturally right $\mathscr{D}_X$-modules. Here, the right-left transformation of Definition 1.2.2 proves useful to recover a left $\mathscr{D}_X$-module.

It is instructive to compare the two notions of duality for a vector bundle with flat connection (in the sense of vector bundles with connections and in the sense of $\mathscr{D}_X$-modules).

Let $\mathscr{M}$ be a locally free $\mathscr{O}_X$-module of finite rank equipped with a flat connection $\nabla:\mathscr{M}\to\Omega^1_X\otimes_{\mathscr{O}_X}\mathscr{M}$. Let $\mathscr{M}^\vee$ denote the dual bundle $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X,\mathscr{M})$, and let $\nabla^\vee$ the natural connection on $\mathscr{M}^\vee$, defined by the property that, for any local section φ of $\mathscr{M}^\vee$ and any local section m of $\mathscr{M}$, we have $d(\varphi(m))=\nabla^\vee\varphi(m)+\varphi(\nabla m)$. Then $\nabla^\vee$ is flat, hence $(\mathscr{M}^\vee,\nabla^\vee)$ comes from a unique left $\mathscr{D}_X$-module that we call $\mathscr{M}^*$. How can we recover $\mathscr{M}^*$ by using one of the $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}_X)$?

$\mathscr{M}^*$ is the left $\mathscr{D}_X$-module assocated with the right $\mathscr{D}_X$-module $\mathscr{E}xt^{\dim X}_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}_X)$.

We will give the proof in the case X is a Zariski open set in $\mathbb{A}^1$ for simplicity. Hence, $\mathscr{O}(X)=\boldsymbol{k}[t,1/p]$ for some non-zero polynomial $p\in\boldsymbol{k}[t]$ and we consder a free $\mathscr{O}(X)$-module M with a connection, that is, a left action of $\mathscr{O}(X)\langle\partial_t\rangle$.

We will now construct a free resolution of M as a $\mathscr{O}(X)\langle\partial_t\rangle$-module. In order to do that, let us forget for a while the connection on M and consider it only as a $\mathscr{O}(X)$-module. The induced $\mathscr{O}(X)\langle\partial_t\rangle$-module $\mathscr{O}(X)\langle\partial_t\rangle\otimes_{\mathscr{O}(X)}M$ (cf. §1.3.b) is free and can be identified with $\boldsymbol{k}[\partial_t]\otimes_{\boldsymbol{k}}M$. Any element can be written in a unique way as $\sum_{k\geqslant0}\partial_t^k\otimes m_k$, and the (left) action of $\mathscr{O}(X)\langle\partial_t\rangle$ is given by the formulas

$$\partial_t\Big(\sum_{k\geqslant0}\partial_t^k\otimes m_k\Big)=\sum_{k\geqslant0}\partial_t^{k+1}\otimes m_k \tag{1.3.2}$$

$$f(t)(\partial_t^k\otimes m_k)=\partial_t^k\otimes f(t)m_k+\sum_{\ell=0}^{k-1}\partial_t^\ell\otimes g_\ell(t)m_k, \tag{1.3.3}$$

where have set $[f(t),\partial_t^k]=\sum_{\ell=0}^{k-1}\partial_t^\ell g_\ell(t)$ in the algebra $\mathscr{O}(X)\langle\partial_t\rangle$.

Let us consider the homomorphism

$$\begin{aligned}\mathscr{O}(X)\langle\partial_t\rangle\underset{\mathscr{O}(X)}{\otimes}M&\xrightarrow{\ \partial_t\otimes1-1\otimes\partial_t\ }\mathscr{O}(X)\langle\partial_t\rangle\underset{\mathscr{O}(X)}{\otimes}M\\ \sum_{k\geqslant0}\partial_t^k\otimes m_k&\longmapsto\sum_{k\geqslant0}\partial_t^k\otimes(m_{k-1}-\partial_tm_k),\end{aligned}\tag{1.3.4}$$

where we now use the action of ∂_t on M. Let us show that the morphism (1.3.4) is $\mathscr{O}(X)\langle\partial_t\rangle$-linear: we will for instance check that

$$f\cdot(\partial_t\otimes 1-1\otimes\partial_t)(1\otimes m)=(\partial_t\otimes 1-1\otimes\partial_t)(1\otimes fm),$$

letting the other properties as an exercise; we have

$$\begin{aligned}f\cdot(\partial_t\otimes 1-1\otimes\partial_t)(1\otimes m)&=\partial_t\otimes fm-1\otimes f'm-1\otimes f\partial_t m\\&=\partial_t\otimes fm-1\otimes\partial_t(fm)\\&=(\partial_t\otimes 1-1\otimes\partial_t)(1\otimes fm).\end{aligned}$$

Let us show the injectivity of (1.3.4): let $\sum_{k\geqslant 0}\partial_t^k\otimes m_k$ be such that $m_{k-1}-\partial_t m_k=0$ for any $k\geqslant 0$ (setting $m_{-1}=0$); as $m_k=0$ for $k\gg 0$, we deduce that $m_k=0$ for any k.

We then identify the cokernel of (1.3.4) to M (as a $\mathscr{O}(X)\langle\partial_t\rangle$-module) by the mapping

$$\sum_{k\geqslant 0}\partial_t^k\otimes m_k\longmapsto\sum_{k\geqslant 0}\partial_t^k m_k.$$

Let us now come back to the computation of Ext^1 with this resolution. A left $\mathscr{O}(X)\langle\partial_t\rangle$-linear morphism $\boldsymbol{k}[\partial_t]\otimes_{\boldsymbol{k}}M\to\mathscr{O}(X)\langle\partial_t\rangle$ is determined by its restriction to $1\otimes M$, which must be $\mathscr{O}(X)$-linear. We deduce an identification

$$\mathrm{Hom}_{\mathscr{O}(X)\langle\partial_t\rangle}\Big(\boldsymbol{k}[\partial_t]\underset{\boldsymbol{k}}{\otimes}M,\mathscr{O}(X)\langle\partial_t\rangle\Big)=M^*\underset{\boldsymbol{k}}{\otimes}\boldsymbol{k}[\partial_t],$$

where the structure of right $\mathscr{O}(X)\langle\partial_t\rangle$-module on the right-hand term is given formulas analogous to (1.3.2) and (1.3.3). Therefore, applying the functor

$$\mathrm{Hom}_{\mathscr{O}(X)\langle\partial_t\rangle}(\bullet,\mathscr{O}(X)\langle\partial_t\rangle)^l$$

to the complex (1.3.4), we get a complex of the same kind, where we have replaced M with M^*. Taking up the argument used for (1.3.4), we deduce an identification

$$\mathrm{Ext}^1_{\mathscr{O}(X)\langle\partial_t\rangle}(M,\mathscr{O}(X)\langle\partial_t\rangle)^l\simeq M^*.$$

Exercises for Lecture 1

***Exercise E.1.1* (The Lie derivative).** Let ω be a local section of ω_X and set $d=\dim X$. For any local vector field ξ, the interior product $\iota_\xi\omega$ is the $(d-1)$-form defined by the formula:

$$\forall\,\xi_2,\dots,\xi_d\in\Theta_X,\quad \iota_\xi\omega(\xi_2,\dots,\xi_d)=\omega(\xi,\xi_2,\dots,\xi_d)$$

and the Lie derivative $\mathscr{L}_\xi\omega$ by $\mathscr{L}_\xi\omega=(d\iota_\xi+\iota_\xi d)\omega=d\iota_\xi\omega$ (as $d\omega=0$).

Prove that the formula $\omega\cdot\xi\overset{\mathrm{def}}{=}-\mathscr{L}_\xi\omega$ is a right action of the Lie algebra Θ_X on ω_X.

***Exercise E.1.2* (The sheaf $\mathscr{H}om$).** Let X be a topological space and let $\mathscr{F}$ and $\mathscr{G}$ be two sheaves of $\mathscr{A}$-modules on X, $\mathscr{A}$ being a sheaf of rings on X. We denote by $\mathrm{Hom}_{\mathscr{A}}(\mathscr{F},\mathscr{G})$ the $\Gamma(X,\mathscr{A})$-module of morphisms of sheaves of $\mathscr{A}$-modules from $\mathscr{F}$ to $\mathscr{G}$. An element ϕ of $\mathrm{Hom}_{\mathscr{A}}(\mathscr{F},\mathscr{G})$ is a collection of morphisms $\phi(U) \in \mathrm{Hom}_{\mathscr{A}(U)}(\mathscr{F}(U),\mathscr{G}(U))$, on open subsets U of X, compatible with the restrictions.

Show that the presheaf $\mathscr{H}om_{\mathscr{A}}(\mathscr{F},\mathscr{G})$ defined by

$$\Gamma(U,\mathscr{H}om_{\mathscr{A}}(\mathscr{F},\mathscr{G})) = \mathrm{Hom}_{\mathscr{A}_{|U}}(\mathscr{F}_{|U},\mathscr{G}_{|U})$$

is a *sheaf* (notice that $U \mapsto \mathrm{Hom}_{\mathscr{A}(U)}(\mathscr{F}(U),\mathscr{G}(U))$ is not a presheaf, because there are no canonical morphisms of restriction).

Exercise E.1.3. Show that a differential operator P *of order* $\leqslant 1$ satisfying $P(1) = 0$ is a derivation of $\mathscr{O}_X$, i.e., a section of Θ_X.

Exercise E.1.4 **(Local computations).** Let U be an open set of $\mathbb{A}^n$ with coordinates $x_1,\dots,x_n$. Denote by $\partial_{x_1},\dots,\partial_{x_n}$ the corresponding vector fields.

(1) Show that the following relations are satisfied in $\mathscr{D}(U)$:

$$\begin{aligned}
[\partial_{x_i},\varphi] &= \frac{\partial\varphi}{\partial x_i}, \quad \forall\,\varphi\in\mathscr{O}(U),\ \forall\, i\in\{1,\dots,n\},\\
[\partial_{x_i},\partial_{x_j}] &= 0 \quad \forall\, i,j\in\{1,\dots,n\},\\
\partial_x^\alpha\cdot\varphi &= \sum_{0\leqslant\beta\leqslant\alpha}\frac{\alpha!}{(\alpha-\beta)!\beta!}\partial_x^{\alpha-\beta}(\varphi)\partial_x^\beta,\\
\varphi\cdot\partial_x^\alpha &= \sum_{0\leqslant\beta\leqslant\alpha}\frac{\alpha!}{(\alpha-\beta)!\beta!}(-1)^{|\alpha-\beta|}\partial_x^\beta\partial_x^{\alpha-\beta}(\varphi),
\end{aligned}$$

with standard notation concerning multi-indices α,β.

(2) Show that any element $P\in\mathscr{D}(U)$ can be written in a unique way as $\sum_\alpha a_\alpha\partial_x^\alpha$ or $\sum_\alpha \partial_x^\alpha b_\alpha$with $a_\alpha,b_\alpha\in\mathscr{O}(U)$. Conclude that $\mathscr{D}_X$ is a locally free left and right module over $\mathscr{O}_X$.

(3) Show that $\max\{|\alpha|\ ;\ a_\alpha\neq 0\} = \max\{|\alpha|\ ;\ b_\alpha\neq 0\}$. It is denoted by $\mathrm{ord}_x P$.

(4) Show that $\mathrm{ord}_x P$ does not depend on the coordinate system chosen on U.

(5) Identify $F_k\mathscr{D}_X$ with the subsheaf of local sections of $\mathscr{D}_X$ having order $\leqslant k$ (in some or any local coordinate system). Show that it is a locally free $\mathscr{O}_X$-module of finite rank.

(6) Show that the filtration $F_\bullet\mathscr{D}_X$ is exhaustive (i.e., $\mathscr{D}_X = \cup_k F_k\mathscr{D}_X$) and that it satisfies

$$F_k\mathscr{D}_X\cdot F_\ell\mathscr{D}_X = F_{k+\ell}\mathscr{D}_X.$$

Exercise E.1.5 **(The graded sheaf $\mathrm{gr}^F\mathscr{D}_X$).** The goal of this exercise is to show that the sheaf of graded rings $\mathrm{gr}^F\mathscr{D}_X$ may be canonically identified with the sheaf of graded rings $\mathrm{Sym}\,\Theta_X$. If one identifies Θ_X with the sheaf of functions on the cotangent space T^*X which are linear in the fibres, then $\mathrm{Sym}\,\Theta_X$ is the sheaf of functions on T^*X which are polynomial in the fibres. In particular, $\mathrm{gr}^F\mathscr{D}_X$ is a sheaf of commutative rings.

(1) Identify the $\mathscr{O}_X$-module $\operatorname{Sym}^k \Theta_X$ with the sheaf of symmetric $\boldsymbol{k}$-linear forms $\boldsymbol{\xi} : \mathscr{O}_X \otimes_{\boldsymbol{k}} \cdots \otimes_{\boldsymbol{k}} \mathscr{O}_X$ on the k-fold tensor product, which behave like a derivation with respect to each factor.

(2) Show that $\operatorname{Sym} \Theta_X \overset{\text{def}}{=} \oplus_k \operatorname{Sym}^k \Theta_X$ is a sheaf of graded $\mathscr{O}_X$-algebras on X and identify it with the sheaf of functions on T^*X which are polynomial in the fibres.

(3) Show that the map $F_k\mathscr{D}_X \to \mathscr{H}\!om_{\boldsymbol{k}}\big(\overset{k}{\otimes}_{\boldsymbol{k}}\mathscr{O}_X, \mathscr{O}_X\big)$ which sends any section P of $F_k\mathscr{D}_X$ to

$$\varphi_1 \otimes \cdots \otimes \varphi_k \longmapsto [\cdots[[P,\varphi_1]\varphi_2]\cdots\varphi_k]$$

induces an isomorphism of $\mathscr{O}_X$-modules $\operatorname{gr}^F_k \mathscr{D}_X \to \operatorname{Sym}^k \Theta_X$.

(4) Show that the induced morphism

$$\operatorname{gr}^F \mathscr{D}_X \overset{\text{def}}{=} \oplus_k \operatorname{gr}^F_k \mathscr{D}_X \longrightarrow \operatorname{Sym} \Theta_X$$

is an isomorphism of sheaves of $\mathscr{O}_X$-algebras.

***Exercise E.1.6* (The universal connection)**

(1) Show that the natural left multiplication of Θ_X on $\mathscr{D}_X$ can be written as a *connection*

$$\nabla : \mathscr{D}_X \longrightarrow \Omega^1_X \underset{\mathscr{O}_X}{\otimes} \mathscr{D}_X,$$

i.e., as a $\boldsymbol{k}$-linear morphism satisfying the *Leibniz rule* $\nabla(fP) = df \otimes P + f\nabla P$ (where f is any local section of $\mathscr{O}_X$ and P any local section of $\mathscr{D}_X$).

(2) Extend this connection for any $k \geqslant 1$ as a $\boldsymbol{k}$-linear morphism

$${}^{(k)}\nabla : \Omega^k_X \underset{\mathscr{O}_X}{\otimes} \mathscr{D}_X \longrightarrow \Omega^{k+1}_X \underset{\mathscr{O}_X}{\otimes} \mathscr{D}_X$$

satisfying the Leibniz rule written as

$${}^{(k)}\nabla(\omega \otimes P) = d\omega \otimes P + (-1)^k \omega \wedge \nabla P.$$

(3) Show that ${}^{(k+1)}\nabla \circ {}^{(k)}\nabla = 0$ for any $k \geqslant 0$ (i.e., ∇ is *flat*).

(4) Show that the morphisms ${}^{(k)}\nabla$ are *right* $\mathscr{D}_X$-linear (but not left $\mathscr{O}_X$-linear).

Exercise E.1.7. More generally, show that a left $\mathscr{D}_X$-module $\mathscr{M}$ is nothing but a $\mathscr{O}_X$-module with a *flat* connection $\nabla : \mathscr{M} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{M}$. Define similarly the iterated connections ${}^{(k)}\nabla : \Omega^k_X \underset{\mathscr{O}_X}{\otimes} \mathscr{M} \to \Omega^{k+1}_X \otimes_{\mathscr{O}_X} \mathscr{M}$. Show that

$${}^{(k+1)}\nabla \circ {}^{(k)}\nabla = 0.$$

***Exercise E.1.8* ($\mathscr{O}_X$ is a left $\mathscr{D}_X$-module).** Use the left action of Θ_X on $\mathscr{O}_X$ to define on $\mathscr{O}_X$ the structure of a left $\mathscr{D}_X$-module.

***Exercise E.1.9* (ω_X is a right $\mathscr{D}_X$-module).** Use the right action of Θ_X on ω_X to define on ω_X the structure of a right $\mathscr{D}_X$-module.

***Exercise E.1.10* (Tensor products over $\mathscr{O}_X$)**

(1) Let $\mathscr{M}$ and $\mathscr{N}$ be two left $\mathscr{D}_X$-modules.

(a) Show that the $\mathscr{O}_X$-module $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ has the structure of a left $\mathscr{D}_X$-module by setting, by analogy with the Leibniz rule,

$$\xi \cdot (m \otimes n) = \xi m \otimes n + m \otimes \xi n.$$

(b) Notice that, in general, $m \otimes n \mapsto (\xi m) \otimes n$ (or $m \otimes n \mapsto m \otimes (\xi n)$) does not define a left $\mathscr{D}_X$-action on the $\mathscr{O}_X$-module $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$.

(c) Let $\varphi : \mathscr{M} \to \mathscr{M}'$ and $\psi : \mathscr{N} \to \mathscr{N}'$ be $\mathscr{D}_X$-linear morphisms. Show that $\varphi \otimes \psi$ is $\mathscr{D}_X$-linear.

(2) Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module and $\mathscr{N}$ be a right $\mathscr{D}_X$-module. Show that $\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M}$ has the structure of a right $\mathscr{D}_X$-module by setting

$$(n \otimes m) \cdot \xi = n\xi \otimes m - n \otimes \xi m.$$

Exercise E.1.11 **($\mathscr{H}om$ over $\mathscr{O}_X$)**

(1) Let $\mathscr{M}, \mathscr{N}$ be left $\mathscr{D}_X$-modules. Show that $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ has a natural structure of left $\mathscr{D}_X$-module defined by

$$(\xi \cdot \varphi)(m) = \xi \cdot (\varphi(m)) + \varphi(\xi \cdot m),$$

for any local sections ξ of Θ_X, m of $\mathscr{M}$ and φ of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$.

(2) Similarly, if $\mathscr{M}, \mathscr{N}$ are right $\mathscr{D}_X$-modules, then $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ has a natural structure of left $\mathscr{D}_X$-module defined by

$$(\xi \cdot \varphi)(m) = \varphi(m \cdot \xi) - \varphi(m) \cdot \xi.$$

Exercise E.1.12 **(Tensor product of a left $\mathscr{D}_X$-module with $\mathscr{D}_X$)**

Let $\mathscr{M}^l$ be a left $\mathscr{D}_X$-module. Notice that $\mathscr{M}^l \otimes_{\mathscr{O}_X} \mathscr{D}_X$ has two commuting structures of $\mathscr{O}_X$-module. Similarly $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^l$ has two such structures. The goal of this exercise is to extend them as $\mathscr{D}_X$-structures and examine their relations.

(1) Show that $\mathscr{M}^l \otimes_{\mathscr{O}_X} \mathscr{D}_X$ has the structure of a left and of a right $\mathscr{D}_X$-module *which commute*, given by the formulas:

$$\text{(left)} \qquad \begin{cases} f \cdot (m \otimes P) = (fm) \otimes P = m \otimes (fP), \\ \xi \cdot (m \otimes P) = (\xi m) \otimes P + m \otimes \xi P, \end{cases}$$

$$\text{(right)} \qquad \begin{cases} (m \otimes P) \cdot f = m \otimes (Pf), \\ (m \otimes P) \cdot \xi = m \otimes (P\xi), \end{cases}$$

for any local vector field ξ and any local function f.

(2) Similarly, $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^l$ also has such structures *which commute*, given by formulas:

$$\text{(left)} \qquad \begin{cases} f \cdot (P \otimes m) = (fP) \otimes m, \\ \xi \cdot (P \otimes m) = (\xi P) \otimes m, \end{cases}$$

$$\text{(right)} \qquad \begin{cases} (P \otimes m) \cdot f = P \otimes (fm) = (Pf) \otimes m, \\ (P \otimes m) \cdot \xi = P\xi \otimes m - P \otimes \xi m. \end{cases}$$

(3) Show that both morphisms

$$\begin{array}{cc} \mathscr{M}^l \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^l & \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^l \longrightarrow \mathscr{M}^l \otimes_{\mathscr{O}_X} \mathscr{D}_X \\ m \otimes P \longmapsto (1 \otimes m) \cdot P & P \otimes m \longmapsto P \cdot (m \otimes 1) \end{array}$$

are left and right $\mathscr{D}_X$-linear, induce the identity $\mathscr{M}^l \otimes 1 = 1 \otimes \mathscr{M}^l$, and their composition is the identity of $\mathscr{M}^l \otimes_{\mathscr{O}_X} \mathscr{D}_X$ or $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^l$, hence both are reciprocal isomorphisms.

(4) Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module and let $\mathscr{L}$ be a $\mathscr{O}_X$-module. Justify the following isomorphisms of left $\mathscr{D}_X$-modules and right $\mathscr{O}_X$-modules:

$$\begin{aligned} \mathscr{M} \otimes_{\mathscr{O}_X} (\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{L}) &\simeq (\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \otimes_{\mathscr{O}_X} \mathscr{L} \\ &\simeq (\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}) \otimes_{\mathscr{O}_X} \mathscr{L} \simeq \mathscr{D}_X \otimes_{\mathscr{O}_X} (\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{L}). \end{aligned}$$

Assume moreover that $\mathscr{M}$ and $\mathscr{L}$ are $\mathscr{O}_X$-locally free. Show that $\mathscr{M} \otimes_{\mathscr{O}_X} (\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{L})$ is $\mathscr{D}_X$-locally free.

***Exercise E.1.13* (Tensor product of a right $\mathscr{D}_X$-module with $\mathscr{D}_X$)**

Same exercise with a right $\mathscr{D}_X$-module $\mathscr{M}^r$. Use the following formulas for $\mathscr{M}^r \otimes_{\mathscr{O}_X} \mathscr{D}_X$:

(left) $$\begin{cases} f \cdot (m \otimes P) = (mf) \otimes P = m \otimes (fP), \\ \xi \cdot (m \otimes P) = (m\xi) \otimes P - m \otimes \xi P, \end{cases}$$

(right) $$\begin{cases} (m \otimes P) \cdot f = m \otimes (Pf), \\ (m \otimes P) \cdot \xi = m \otimes (P\xi), \end{cases}$$

and for $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^r$:

(left) $$\begin{cases} f \cdot (P \otimes m) = (fP) \otimes m, \\ \xi \cdot (P \otimes m) = (\xi P) \otimes m, \end{cases}$$

(right) $$\begin{cases} (P \otimes m) \cdot f = P \otimes (mf) = (Pf) \otimes m, \\ (P \otimes m) \cdot \xi = P \otimes m\xi - (\xi P) \otimes m. \end{cases}$$

***Exercise E.1.14* (Compatibility of right-left transformations)**

Show that the natural morphisms

$$\mathscr{M}^l \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\omega_X, \omega_X \otimes_{\mathscr{O}_X} \mathscr{M}^l), \qquad \omega_X \otimes_{\mathscr{O}_X} \mathscr{H}om_{\mathscr{O}_X}(\omega_X, \mathscr{M}^r) \longrightarrow \mathscr{M}^r$$

are isomorphisms of $\mathscr{D}_X$-modules.

***Exercise E.1.15* (Local expression of the left-right transformation)**

Let U be an open set of $\mathbb{A}^n$.

(1) Show that there exists a unique $\boldsymbol{k}$-linear involution $P \mapsto {}^tP$ from $\mathscr{D}(U)$ to itself such that

- $\forall \varphi \in \mathscr{O}(U)$, ${}^t\varphi = \varphi$,
- $\forall i \in \{1, \dots, n\}$, ${}^t\partial_{x_i} = -\partial_{x_i}$,
- $\forall P, Q \in \mathscr{D}(U)$, ${}^t(PQ) = {}^tQ \cdot {}^tP$.

(2) Let $\mathscr{M}$ be a left (resp. right) $\mathscr{D}_U$-module and let ${}^t\mathscr{M}$ be $\mathscr{M}$ equipped with the right (resp. left) $\mathscr{D}_U$-module structure

$$P \cdot m \overset{\mathrm{def}}{=} {}^tPm.$$

Show that ${}^t\mathscr{M} \xrightarrow{\sim} \mathscr{M}^r$ (resp. ${}^t\mathscr{M} \xrightarrow{\sim} \mathscr{M}^l$).

Exercise E.1.16. Show that the two sets of differential operators $\{\partial_{x_1}, \dots, \partial_{x_n}\}$ and $\{\partial_{x_1}, x_1\partial_{x_2} + \cdots + x_{n-1}\partial_{x_n}\}$ generate the same ideal of $\mathscr{D}_{\mathbb{A}^n}$.

Exercise E.1.17. Classify all vector bundles with connection on the projective line $\mathbb{P}^1(\boldsymbol{k})$. (Hint: Start with rank-one bundles; use then the existence of a decomposition as the direct sum of rank-one bundles).

Exercise E.1.18 **(Products of operators and exact sequences of modules)**

(1) Prove that, in the Weyl algebra $\boldsymbol{k}[t]\langle\partial_t\rangle$, we have

$$PQ = 0 \implies P = 0 \text{ or } Q = 0$$

(consider the highest degree terms in ∂_t).

(2) Show that, if $Q \neq 0$, right multiplication by Q induces an isomorphism

$$\boldsymbol{k}[t]\langle\partial_t\rangle/(P) \underset{\sim}{\xrightarrow{\cdot Q}} (Q)/(PQ),$$

if (R) denotes the left ideal $\boldsymbol{k}[t]\langle\partial_t\rangle \cdot R$.

(3) Prove that the sequence of left $\boldsymbol{k}[t]\langle\partial_t\rangle$-modules is exact:

$$0 \longrightarrow \boldsymbol{k}[t]\langle\partial_t\rangle/(P) \xrightarrow{\cdot Q} \boldsymbol{k}[t]\langle\partial_t\rangle/(PQ) \longrightarrow \boldsymbol{k}[t]\langle\partial_t\rangle/(Q) \longrightarrow 0.$$

Exercise E.1.19 **(Examples of Fourier transforms)**

(1) Compute the Fourier transform of $(\mathbb{C}[t], d)$.

(2) (This part uses the lectures on regular singularity.) Show that, if $P \in \boldsymbol{k}[t]\langle\partial_t\rangle$ is a differential operator with a regular singularity at infinity, then the Fourier transform of P has only two singularities, one at 0 which is regular, and one at infinity, whic may be irregular.

LECTURE 2

CHARACTERISTIC VARIETY

One of the most basic geometric object attached to a $\mathscr{D}$-module is its characteristic variety. This variety gives some control on the solutions of a $\mathscr{D}$-module. We will not insist on this aspect in these notes (cf. however Exercise E.2.4). Our main concern will be Bernstein's inequality (the dimension of the characteristic variety is bounded from below by the dimension of the underlying variety), and more precisely the involutivity theorem, although we will not give a proof of the latter.

2.1. Good filtrations and coherence

2.1.a. Good filtrations. Let $\mathscr{M}$ be a $\mathscr{D}_X$-module and let $F_\bullet\mathscr{M}$ be an increasing filtration of $\mathscr{M}$ by $\mathscr{O}_X$-submodules indexed by $\mathbb{Z}$. We say that this filtration is a F-filtration if it is exhaustive (i.e., $\bigcup_\ell F_\ell\mathscr{M} = \mathscr{M}$) and

$$\forall\, k, \ell \in \mathbb{Z}, \quad F_k\mathscr{D}_X \cdot F_\ell\mathscr{M} \subset F_{k+\ell}\mathscr{M}.$$

In other words, $(\mathscr{M}, F_\bullet)$ is a filtered module over the filtered ring $(\mathscr{D}_X, F_\bullet)$.

Definition 2.1.1. A F-filtration $F_\bullet\mathscr{M}$ is *good* if the following two properties are fulfilled:

(1) for any $\ell \in \mathbb{Z}$, $F_\ell\mathscr{M}$ is a $\mathscr{O}_X$-coherent module,

(2) locally on X there exists $\ell_0 \in \mathbb{N}$ such that, for any $k \in \mathbb{N}$,

  (a) $F_k\mathscr{D}_X \cdot F_{\ell_0}\mathscr{M} = F_{\ell_0+k}\mathscr{M}$,

  (b) $F_{-k}\mathscr{D}_X \cdot F_{-\ell_0}\mathscr{M} = F_{-(\ell_0+k)}\mathscr{M}$, that is, $F_\ell\mathscr{M} = 0$ for any $\ell \ll 0$.

Proposition 2.1.2. *Any two good F-filtrations $F_\bullet\mathscr{M}$ and $F'_\bullet\mathscr{M}$ are locally comparable, i.e., for any $x \in X$ there exists $\ell_1 \in \mathbb{N}$ such that, in some open neighbourhood of x the following inclusions hold for any $\ell \in \mathbb{Z}$:*

$$F_{\ell-\ell_1}\mathscr{M} \subset F'_\ell\mathscr{M} \subset F_{\ell+\ell_1}\mathscr{M}.$$

Examples 2.1.3

(1) The *stupid filtration* of $\mathscr{O}_X$ defined by $F_k\mathscr{O}_X = 0$ for $k \leqslant -1$ and $F_k\mathscr{O}_X = \mathscr{O}_X$ if $k \geqslant 0$ is good.

(2) Let V be a vector bundle on X with a flat connection $\nabla : V \to \Omega^1_X \otimes V$. Let $F^\bullet V$ be a finite decreasing exhaustive filtration by holomorphic subbundles. Define the increasing filtration $F_\bullet V$ by $F_p V = F^{-p} V$. Then $F_\bullet V$ is a F-filtration of (V, ∇) if and only if $F^\bullet V$ satisfies *Griffiths' transversality property*: $\nabla F^p V \subset \Omega^1_X \otimes F^{p-1} V$ for all $p \in \mathbb{Z}$. Then, the filtration is good.

(3) Let D be a reduced divisor in X. Then the filtration $F_k \mathscr{O}_X(*D) \overset{\mathrm{def}}{=} \mathscr{O}_X(kD)$ is a F-filtration. When D is smooth, this is a good filtration and in particular $\mathscr{O}_X(*D)$ is $\mathscr{D}_X$-coherent (see the computation in Example 2.2.5(2). When D is not smooth, this is not necessarily true, but one can show that $\mathscr{O}_X(*D)$ has a good filtration (i.e., is $\mathscr{D}_X$-coherent). This is not trivial at all and relies on Bernstein's functional equation. This equation implies that, if U is affine and $f \in \mathscr{O}(U)$, there exists $k_0 \geqslant 0$ such that, for any $k \geqslant 0$, $f^{-k_0-k} \in \mathscr{D}(U) \cdot f^{-k_0}$.

2.1.b. Coherence. Let us begin by recalling the definition of coherence. Let $\mathscr{A}$ be a sheaf of rings on a space X. A sheaf of $\mathscr{A}$-modules $\mathscr{F}$ is said to be *$\mathscr{A}$-coherent* if it is locally of finite type and if, for any open set U of X and any $\mathscr{A}$-linear morphism $\varphi : \mathscr{A}^r_{|U} \to \mathscr{F}_{|U}$, the kernel of φ is locally of finite type. The sheaf $\mathscr{A}$ is a coherent sheaf of ring if it is coherent as a (left and right) module over itself. If $\mathscr{A}$ is coherent, the a sheaf $\mathscr{F}$ is $\mathscr{A}$-coherent if and only if it has locally a finite presentation.

Recall (cf. Exercise E.1.5) that $\mathrm{gr}^F \mathscr{D}_X$ is identified with the sheaf of commutative rings $\mathrm{Sym}\,\Theta_X$, which is a coherent sheaf on X.

Proposition 2.1.4. *A F-filtration is good if and only if $F_\ell \mathscr{M} = 0$ for $\ell \ll 0$ and $\mathrm{gr}^F \mathscr{M}$ is coherent over $\mathrm{gr}^F \mathscr{D}_X$.*

We say that $\mathscr{M}$ is coherent over $\mathscr{D}_X$ if it has a good F-filtration in the neighbourhood of any point of X (in the complex analytic setting, a good filtration may exist locally but possibly not globally).

One should be careful that this definition is tautological for $\mathscr{D}_X$. But fortunately this does not lead to a contradiction as one can show that $\mathscr{D}_X$ is coherent as a sheaf of rings.

Proposition 2.1.5. *Let $\mathscr{M}$ be a $\mathscr{D}_X$-module with a good filtration $F_\bullet \mathscr{M}$ and let $\mathscr{N}$ be a coherent $\mathscr{D}_X$-submodule of $\mathscr{M}$. Then the filtration $F_\bullet \mathscr{N} \overset{\mathrm{def}}{=} \mathscr{N} \cap F_\bullet \mathscr{M}$ is good.*

If $\varphi : \mathscr{M} \to \mathscr{N}$ is a morphism of coherent $\mathscr{D}_X$-modules, then a good filtration on $\mathscr{M}$ induces a good filtration on $\mathrm{Ker}\,\varphi$ *and* $\mathrm{Coker}\,\varphi$.

2.2. Support and characteristic variety

2.2.a. Support. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Being a sheaf on X, $\mathscr{M}$ has a support $\mathrm{Supp}\,\mathscr{M}$, which is the closed subset complement to the set of $x \in X$ in the neighbourhood of which $\mathscr{M}$ is zero. Recall that the support of a coherent $\mathscr{O}_X$-module is a closed algebraic (or analytic) subset of X. Such a property extends to coherent $\mathscr{D}_X$-modules:

Proposition 2.2.1. *The support* $\operatorname{Supp}\mathscr{M}$ *of a coherent* $\mathscr{D}_X$*-module* $\mathscr{M}$ *is a closed algebraic (or analytic) subset of* X*.*

Proof. The property of being an algebraic (or analytic) subset being local, we may assume that $\mathscr{M}$ is generated over $\mathscr{D}_X$ by a coherent $\mathscr{O}_X$-submodule $\mathscr{F}$. Then the support of $\mathscr{M}$ is equal to the support of $\mathscr{F}$. □

2.2.b. Characteristic variety. However, the support is usually not the right geometric object attached to a $\mathscr{D}_X$-module $\mathscr{M}$, as it does not provide enough information on $\mathscr{M}$. A finer object is the *characteristic variety* that we introduce below. The following lemma will justify its definition.

Lemma 2.2.2. *Let* $\mathscr{M}$ *be a coherent* $\mathscr{D}_X$*-module. Then there exists a coherent sheaf* $\mathscr{I}(\mathscr{M})$ *of ideals of* $\operatorname{gr}^F\mathscr{D}_X$ *such that, for any open set* U *of* X *and any good filtration* $F_\bullet\mathscr{M}_{|U}$*, we have* $\mathscr{I}(\mathscr{M})_{|U} = \operatorname{Rad}(\operatorname{ann}_{\operatorname{gr}^F\mathscr{D}_U}\operatorname{gr}^F\mathscr{M}_{|U})$*.*

We denote by $\operatorname{Rad}(I)$ the radical of the ideal I and by ann the annihilator ideal of the corresponding module. Hence, for any $x \in U$, we have

$$\operatorname{Rad}(\operatorname{ann}_{\operatorname{gr}^F\mathscr{D}_{X,x}}\operatorname{gr}^F\mathscr{M}_x) = \{\varphi \in \operatorname{gr}^F\mathscr{D}_{X,x} \mid \exists\,\ell,\ \varphi^\ell\operatorname{gr}^F\mathscr{M}_x = 0\}.$$

Proof. It is a matter of showing that, if $F_\bullet\mathscr{M}_{|U}$ and $F'_\bullet\mathscr{M}_{|U}$ are two good filtrations, then the corresponding ideals coincide. Notice first that these ideals are homogeneous, i.e., if φ belongs to the ideal, then so does any homogeneous component of φ. Let φ be a homogeneous element of degree j in the ideal corresponding to $F_\bullet\mathscr{M}$. Then, locally, there exists ℓ such that, for any k, we have $\varphi^\ell F_k\mathscr{M} \subset F_{k+j\ell-1}\mathscr{M}$ and thus, for any $p \geqslant 0$,

$$\varphi^{(p+1)\ell}F_k\mathscr{M} \subset F_{k+j(p+1)\ell-p-1}\mathscr{M}.$$

Taking ℓ_1 as in Proposition 2.1.2 associated to $F_\bullet\mathscr{M}, F'_\bullet\mathscr{M}$, we have

$$\begin{aligned}\varphi^{(2\ell_1+1)\ell}F'_k\mathscr{M} \subset \varphi^{(2\ell_1+1)\ell}F_{k+\ell_1}\mathscr{M} &\subset F_{k+\ell_1+j(2\ell_1+1)\ell-2\ell_1-1}\mathscr{M}\\ &\subset F'_{k+2\ell_1+j(2\ell_1+1)\ell-2\ell_1-1}\mathscr{M} = F'_{k+j(2\ell_1+1)\ell-1}\mathscr{M}.\end{aligned}$$

This shows that φ is in the ideal corresponding to $F'_\bullet\mathscr{M}$. By a symmetric argument, we find that both ideals are identical. □

Notice that we consider the radicals of the annihilator ideals, and not these annihilator ideals themselves, because of the shift ℓ_1. In fact, the annihilator ideals may not be equal, as shown by Exercise E.2.1.

Definition 2.2.3 **(Characteristic variety).** The *characteristic variety* $\operatorname{Char}\mathscr{M}$ is the subset of the cotangent space T^*X defined by the ideal $\mathscr{I}(\mathscr{M})$.

Locally, given any good filtration of $\mathscr{M}$, the characteristic variety is defined as the set of common zeros of the elements of $\operatorname{ann}_{\mathrm{gr}^F\mathscr{D}_X}\mathrm{gr}^F\mathscr{M}$.

Assume that $\mathscr{M}$ is the quotient of $\mathscr{D}_X$ by the left ideal $\mathscr{I}$. Then one may choose for $F_\bullet\mathscr{M}$ the filtration induced by $F_\bullet\mathscr{D}_X$, so that $\operatorname{Char}\mathscr{M}$ is the locus of common zeros of the elements of $\mathrm{gr}^F\mathscr{I}$. In general, finding generators of $\mathrm{gr}^F\mathscr{I}$ from generators of $\mathscr{I}$ needs the use of Gröbner bases.

In local coordinates $x_1,\dots,x_n$, denote by $\xi_1,\dots,\xi_n$ the complementary symplectic coordinates in the cotangent space. Then $\mathrm{gr}^F\mathscr{I}$ is generated by a finite set of homogeneous elements $a_\alpha(x)\xi^\alpha$, where α belongs to a finite set of multi-indices. Hence the homogeneity of the ideal $\mathscr{I}(\mathscr{M})$ implies that

$$\operatorname{Supp}\mathscr{M}=\pi(\operatorname{Char}\mathscr{M})=\operatorname{Char}\mathscr{M}\cap T^*_XX, \tag{2.2.4}$$

where $\pi: T^*X\to X$ denotes the bundle projection and T^*_XX denotes the zero section of the cotangent bundle.

Examples 2.2.5

(1) $\mathscr{M}=\mathscr{O}_X$. Let us use the stupid filtration of $\mathscr{O}_X$ (cf. Example 2.1.3). The associated graded sheaf is

$$\mathrm{gr}\mathscr{M}=\mathscr{O}_X=\mathrm{gr}^0\mathscr{M}.$$

And we have $\operatorname{ann}\mathrm{gr}\mathscr{M}=\oplus_{k\geqslant1}\mathrm{gr}^F_k\mathscr{D}_X$. If $(x_1,\dots,x_n;\xi_1,\dots,\xi_n)$ is a local coordinate system of T^*X, $\operatorname{Char}\mathscr{O}_X$ is the subvariety of T^*X defined by the equations $\xi_1=\cdots=\xi_n=0$; in other words we have $\operatorname{Char}\mathscr{O}_X=T^*_XX$.

(2) $X=\mathbb{A}^n$, $\mathscr{M}=\mathscr{O}_X[1/x_1]$. Let us set by $\mathscr{I}=\mathscr{D}_X\cdot(x_1\partial_{x_1}+1,\partial_{x_2},\dots,\partial_{x_n})$. We have a natural morphism of left $\mathscr{D}_X$-modules

$$\begin{aligned}\mathscr{D}_X/\mathscr{I}&\longrightarrow\mathscr{O}_X[1/x_1]\\ [P]&\longmapsto P\cdot(1/x_1).\end{aligned}$$

This morphism is an *isomorphism*: surjectivity is easy; to prove injectivity, we note that any $[P]$ has a representative P of the form $h+\sum_{k=1}^d p_k\partial_{x_1}^k$ where h is a section of $\mathscr{O}_X$ and the p_k are constant; then $P\cdot(1/x_1)=0$ implies $h=0$ and $p_k=0$ for all k.

Let $F_\bullet\mathscr{M}$ be the filtration on $\mathscr{M}$ induced by the canonical filtration of $\mathscr{D}_X$:

$$F_k\mathscr{M}=F_k\mathscr{D}_X\cdot(1/x_1)=\{g/x_1^k\mid g\in\mathscr{O}\}.$$

We have $\operatorname{ann}_{\mathrm{gr}\mathscr{D}}(\mathrm{gr}\mathscr{M})=\mathrm{gr}\mathscr{I}$. In order to compute $\mathrm{gr}\mathscr{I}$, we remark that any element of $\mathscr{I}$ can be written in a unique way as

$$A_1(x_1\partial_{x_1}+1)+A_2\partial_{x_2}+\cdots+A_n\partial_{x_n},$$

with $A_i\in\mathscr{D}$ independent of the $(n-i)$ last derivations $\partial_{x_{i+1}},\dots$ Computing the graded object can now be done on the coefficients A_i and we find that $\mathrm{gr}\mathscr{I}$ is the ideal generated by $(x_1\xi_1,\xi_2,\dots,\xi_n)$, so

$$\operatorname{Char}(\mathscr{O}_X[1/x_1])=T^*_{x_1=0}X\cup T^*_XX.$$

2.2.c. Coherent $\mathscr{D}$-modules with characteristic variety contained in the zero section.

Proposition 2.2.6. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then* $\operatorname{Char}\mathscr{M} \subset T^*_X X$ *if and only if $\mathscr{M}$ is locally free of finite rank over $\mathscr{O}_X$, i.e., is a vector bundle with a flat connection.*

Proof

($\Leftarrow$) We choose the stupid filtration on $\mathscr{M}$, which is good. Then $\operatorname{gr}^F\mathscr{M}$ is $\mathscr{M}$ with the trivial action of $\operatorname{Sym}^k\Theta_X$ for $k \geqslant 1$, hence the support of $\mathscr{M}$ as a $\operatorname{Sym}\Theta_X$-module is $T^*_X X$.

($\Rightarrow$) Let us choose (locally) a good filtration $F_\bullet\mathscr{M}$ such that $F_k\mathscr{M} = F_k\mathscr{D}_X \cdot F_0\mathscr{M}$ for all $k \geqslant 0$ and $F_k\mathscr{M} = 0$ for $k \leqslant -1$. After the Nullstellensatz, there exists k_0 such that $F_{k_0}\mathscr{D}_X F_0\mathscr{M} \subset F_{k_0-1}\mathscr{M}$, hence $F_{k_0}\mathscr{M} = F_{k_0-1}\mathscr{M}$, and the filtration $F_\bullet\mathscr{M}$ remains constant for $k \geqslant k_0 - 1$. Therefore, $\mathscr{M} = F_{k_0-1}\mathscr{M}$ is $\mathscr{O}_X$-coherent.

Let us now show that $\mathscr{M}$, being $\mathscr{O}_X$-coherent, is $\mathscr{O}_X$-locally free. Let $x^o \in X$ and let $m_1, \dots, m_p$ be a system of generators of $\mathscr{M}_{x^o}$ which induces a basis of $\mathscr{M}_{x^o}/\mathfrak{m}_{x^o}\mathscr{M}_{x^o}$, where $\mathfrak{m}_{x^o}$ is the maximal ideal of $\mathscr{O}_{X,x^o}$. Let us choose local coordinates at x^o. Assume that there exists a non trivial relation:

$$\sum_{i=1}^p u_i e_i = 0, \quad u_i \in \mathscr{O}_{X,x^o}$$

of finite order $\nu = \inf\{k \mid \forall\, i,\, u_i \in \mathfrak{m}^k_{x^o}\}$. Choose $v^{(j)}_{ik} \in \mathscr{O}_{X,x^o}$ such that $\partial_{x_j} e_i = \sum v^{(j)}_{ik} e_k$. Applying ∂_{x_j} to the relation above we obtain

$$0 = \sum_{i=1}^p u_i \partial_{x_j} e_i + \frac{\partial u_i}{\partial x_j} e_i = \sum_{i=1}^p \frac{\partial u_i}{\partial x_j} e_i + \sum_{i=1}^p u_i \Big(\sum_{k=1}^p v^{(j)}_{ik} e_k\Big),$$

or

$$\sum_{i=1}^p \Big(\frac{\partial u_i}{\partial x_j} + \sum_{\ell=1}^p u_\ell v^{(j)}_{\ell i}\Big) e_i = 0.$$

For a conveniently chosen index j, this appears as a relation of order $\nu - 1$. By repeating this process we would obtain a relation of order zero which contradicts the independence over $\boldsymbol{k}$ in $\mathscr{M}_{x^o}/\mathfrak{m}_{x^o}\mathscr{M}_{x^o}$. $\square$

2.2.d. Involutiveness of the characteristic variety. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, $\operatorname{Char}\mathscr{M} \subset T^*X$ its characteristic variety and $\operatorname{Supp}\mathscr{M}$ its support. For $(x, 0) \in T^*X$, we denote by $\dim_{(x,0)}\operatorname{Char}\mathscr{M}$ the dimension at $(x, 0)$ of the variety $\operatorname{Char}\mathscr{M}$.

Proposition 2.2.7. *Let $\mathscr{M}$ be a non-zero coherent $\mathscr{D}_X$-module. Then, for any $x \in X$,* $\dim_{(x,0)}\operatorname{Char}\mathscr{M} \geqslant \dim X$. $\square$

This inequality is called *Bernstein's inequality.* There exists a more precise result. In order to state it, consider on T^*X the fundamental 2-form ω. In local coordinates

$(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, it is written $\omega = \sum_{1=1}^{n} d\xi_i \wedge dx_i$. For any $(x, \xi) \in T^*X$, ω defines on $T_{(x,\xi)}(T^*X)$ a non-degenerate bilinear form. We denote by $E^\perp$ the orthogonal space in the sense of ω of the vector subspace E of $T_{(x,\xi)}(T^*X)$. Recall that if V is a subvariety of T^*X, with smooth part V_0,

– V is said to be *involutive* if, for any $a \in V_0$, we have $(T_aV)^\perp \subset T_aV$,
– V is said to be *isotropic* if, for any $a \in V_0$, we have $T_aV \subset (T_aV)^\perp$,
– V is said to be *Lagrangian* if, for any $a \in V_0$, we have $(T_aV)^\perp = T_aV$.

We observe that if V is involutive, the dimension of any irreducible components of V is bigger than $\dim X$.

Theorem 2.2.8. *Let $\mathscr{M}$ be a non-zero coherent $\mathscr{D}_X$-module. Then* $\operatorname{Char}\mathscr{M}$ *is an involutive set in T^*X.* □

The first proof has been given by Sato, Kawai, Kashiwara [26]. Next, Malgrange gave a very simple proof in a seminar Bourbaki talk (see [12, p. 165]). And finally, Gabber gave proof of a general algebraic version of this theorem (see [9], see also [2, p. 473]).

The first consequence is that any irreducible component of the characteristic variety of a coherent $\mathscr{D}_X$-module has a dimension $\geqslant \dim X$.

2.3. Holonomic $\mathscr{D}_X$-modules

Definition 2.3.1. A coherent $\mathscr{D}_X$-module is said to be holonomic if its characteristic variety $\operatorname{Char}\mathscr{M}$ has dimension $\dim X$.

Proposition 2.3.2. *If $\mathscr{M}$ is holonomic and has support X, then there exists a Zariski dense open set X^o in X such that $\mathscr{M}_{|X^o}$ is a vector bundle with flat connection.*

Proof. One component of the characteristic variety must be T^*_XX. We define X^o to be the complement in $X = T^*_XX$ of the intersection of T^*_XX with the other components of the characteristic variety. We can apply Proposition 2.2.6. □

Examples 2.3.3

(1) Any vector bundle with flat connection is a holonomic $\mathscr{D}_X$-module.

(2) For any smooth hypersurface H of X, $\mathscr{O}_X(*H)$ is holonomic, $\operatorname{Char}\mathscr{O}_X(*H) = T^*_XX \cup T^*_YX$.

(3) For $n \geqslant 2$, if P is a section of $\mathscr{D}_X$, the quotient $\mathscr{D}_X/\mathscr{D}_XP$ is never holonomic. Its characteristic variety is a hypersurface of T^*X. On the other hand, if $n \geqslant 3$, two operators may be enough to define a holonomic $\mathscr{D}_X$-module (cf. Exercise E.1.16).

From Exercise E.2.2 we get

Proposition 2.3.4. *In an exact sequence $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ of coherent left $\mathscr{D}_X$-modules, $\mathscr{M}$ is holonomic if and only if $\mathscr{M}'$ and $\mathscr{M}''$ are so.* □

Remark 2.3.5. The explicit computation of the characteristic variety of a given $\mathscr{D}_X$-module may be complicated. A useful tool in the case of holonomic $\mathscr{D}_X$-modules is Kashiwara's index theorem, which can reduce the computation to a topological problem (cf. Remark 3.3.3).

Example 2.3.6. Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module with support equal to X. Assume that, for X^o as in Proposition 2.3.2, the complement $X \smallsetminus X^o$ is a divisor D in X. A much refined version of Proposition 2.2.c says that $\mathscr{O}_X(*D)\otimes_{\mathscr{O}_X}\mathscr{M}$ is a locally free $\mathscr{O}_X(*D)$-module of finite rank equipped with a meromorphic connection and that it is also a holonomic $\mathscr{D}_X$-module. The proof uses Bernstein's relation, as explained in dimension one in Proposition 2.4.5.

The kernel and the cokernel of the localization morphism $\mathscr{M} \to \mathscr{O}_X(*D)\otimes_{\mathscr{O}_X}\mathscr{M}$ are holonomic $\mathscr{D}_X$-modules (after Proposition 2.3.4) supported on $\mathscr{D}$. Iterating in a convenient way this procedure is useful for reducing the proof of a given property of holonomic $\mathscr{D}_X$-modules to the proof for meromorphic bundles with connection.

Concerning duality, the behaviour of holonomic $\mathscr{D}_X$-modules is similar to that of vector bundles with flat connection (cf. §1.3.g):

***Theorem 2.3.7* (Homological characterization of holonomic $\mathscr{D}_X$-modules)**
Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}_X) = 0$ for $k > \dim X$. Moreover, $\mathscr{M}$ is holonomic if and only if $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}_X) = 0$ for $k \neq \dim X$. The left $\mathscr{D}_X$-module associated to the right $\mathscr{D}_X$-module $\mathscr{E}xt^{\dim X}_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}_X)$ is then also holonomic.

2.4. Holonomic modules over the one-variable Weyl algebra

General references for this section are [1, 4, 7, 8, 12, 15, 20, 23, 29].

In the following, when saying "module" over the Weyl algebra, we will usually mean *left module of finite type* over $\boldsymbol{k}[t]\langle\partial_t\rangle$.

Any module has thus a presentation

$$\boldsymbol{k}[t]\langle\partial_t\rangle^p \xrightarrow{\cdot A} \boldsymbol{k}[t]\langle\partial_t\rangle^q \longrightarrow M \longrightarrow 0, \tag{2.4.1}$$

where A is a $p\times q$ matrix with entries in $\boldsymbol{k}[t]\langle\partial_t\rangle$. The vectors are here written as line vectors, multiplication by A is done on the right and commutes thus with the left action of $\boldsymbol{k}[t]\langle\partial_t\rangle$.

2.4.a. Holonomic modules. We will say that a module M over $\boldsymbol{k}[t]\langle\partial_t\rangle$ is *holonomic* if any element of M is annihilated by some non-zero operator of $\boldsymbol{k}[t]\langle\partial_t\rangle$, i.e., satisfies a non-trivial differential equation. This equation can have degree 0: we will then say that the element is *torsion*. A holonomic module contains by definition no sub-module isomorphic to $\boldsymbol{k}[t]\langle\partial_t\rangle$. In an exact sequence of $\boldsymbol{k}[t]\langle\partial_t\rangle$-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

the middle term is holonomic if and only if the extreme ones are so.

Given $\delta = (\delta_1, \dots, \delta_q) \in \mathbb{Z}^q$, let us call δ-degree of an element $(P_1, \dots, P_q)$ of $\boldsymbol{k}[t]\langle\partial_t\rangle^q$ the integer $\max_{k=1,\dots,q}(\deg P_k - \delta_k)$. Let us call *good filtration* any filtration of M obtained as the image, in some presentation like (2.4.1), of the filtration of $\boldsymbol{k}[t]\langle\partial_t\rangle^q$ by the δ-degree, for some $\delta \in \mathbb{Z}^q$.

The graded module of M with respect to some good filtration is a module of finite type over the ring of polynomials of two variables.

Proposition 2.4.2

(1) *A module is holonomic if and only if its graded module with respect to some (or any) good filtration has a non-trivial annihilator.*

(2) *Any holonomic module can be generated by one element.* □

We deduce that a holonomic module has a presentation (2.4.1) with $q = 1$. That any holonomic module is cyclic can be compared with the "lemma of the cyclic vector" for connections of one variable. We will not give the proof, as it is not essential and knowing that a module is finitely generated is usually enough.

Examples 2.4.3

(1) If $P \notin \boldsymbol{k}$ is an operator in $\boldsymbol{k}[t]\langle\partial_t\rangle$, the quotient module M of $\boldsymbol{k}[t]\langle\partial_t\rangle$ by the left ideal over $\boldsymbol{k}[t]\langle\partial_t\rangle \cdot P$ is a holonomic module: it has finite type, as it admits the presentation

$$0 \longrightarrow \boldsymbol{k}[t]\langle\partial_t\rangle \xrightarrow{\cdot P} \boldsymbol{k}[t]\langle\partial_t\rangle \longrightarrow M \longrightarrow 0.$$

The graded module with respect to the filtration induced by the degree is equal to the quotient of the ring of polynomials by the ideal generated by the symbol of P; its annihilator is non-trivial, hence the module is holonomic.

(2) If δ_0 denotes the Dirac distribution at 0 on $\boldsymbol{k}$, the sub-module of temperate distributions generated by δ_0 is holonomic: it has finite type by definition; moreover we have $t\delta_0 = 0$, whence an isomorphism of $\boldsymbol{k}$-vector spaces

$$\boldsymbol{k}[\partial_t] \xrightarrow{\sim} \boldsymbol{k}[t]\langle\partial_t\rangle \cdot \delta_0;$$

last, it is easy to check that $t^\ell \partial_t^k \delta_0 = 0$ as soon as $\ell > k$. This module is thus a torsion module. It takes the form given in Example (1) by taking for P the degree 0 operator equal to t.

(3) More generally, any torsion module is a direct sum of modules isomorphic to modules of the kind $\boldsymbol{k}[t]\langle\partial_t\rangle/(t - t^o)$ (the readers should check this).

(4) Any holonomic module M can be included in a short exact sequence

$$0 \longrightarrow K \longrightarrow \boldsymbol{k}[t]\langle\partial_t\rangle/(P) \longrightarrow M \longrightarrow 0$$

where K is torsion: in order to do so, we choose a generator e of M (Proposition 2.4.2(2)) and we take for P an operator of minimal degree (say d) which kills e; it is a matter of seeing that the kernel of the surjective morphism $\boldsymbol{k}[t]\langle\partial_t\rangle/(P) \to M$ which sends the class of 1 to e is torsion; if we work in the ring of fractions $\boldsymbol{k}[t, a_d^{-1}]$, in which we can invert the dominant coefficient a_d of P, we can use the Euclidean division algorithm [1] of any operator by P, and the minimality of the degree of P shows that $\boldsymbol{k}[t, a_d^{-1}]\langle\partial_t\rangle/(P) \to M[a_d^{-1}]$ is bijective; we deduce that K has support in the finite set of zeroes of a_d. $\square$

Proposition 2.4.4. *Let X be a Zariski open set of $\mathbb{A}^1$ and let M_X be a $\mathscr{O}(X)\langle\partial_t\rangle$-module. The following properties are equivalent:*

(1) *M_X is the restriction to X of some holonomic $\boldsymbol{k}[t]\langle\partial_t\rangle$-module M,*

(2) *there exists a Zariski dense open set $U \subset X$ such that $M_{X|U}$ is the restriction to U of some holonomic $\boldsymbol{k}[t]\langle\partial_t\rangle$-module M,*

(3) *there exists a Zariski dense open set $U \subset X$ such that $M_{X|U}$ is a (possibly zero) vector bundle with connection.* $\square$

The set of *singular points* of M_X is the minimal set $\Sigma \subset X$ such that $M_{X|X\smallsetminus\Sigma}$ is a vector bundle. We will call *rank* of M_X that of the associated meromorphic bundle. If $X = \mathbb{A}^1$,

$$\operatorname{rk} M = \dim_{\boldsymbol{k}(t)} \boldsymbol{k}(t) \underset{\boldsymbol{k}[t]}{\otimes} M.$$

Lattices. A *lattice* of the $\boldsymbol{k}[t]\langle\partial_t\rangle$-module M is a $\boldsymbol{k}[t]$ sub-module which generates it over $\boldsymbol{k}[t]\langle\partial_t\rangle$, that is, such that

$$M = \sum_{k=0}^{\infty} \partial_t^k E.$$

Proposition 2.4.5. *Let U be a Zariski open set of $\mathbb{A}^1$ such that $M_{|U}$ is a vector bundle and let $E \subset M$ be any vector bundle on $\mathbb{A}^1$ such that $E_{|U} = M_{|U}$. Then E is a lattice of the $\boldsymbol{k}[t]\langle\partial_t\rangle$-module M.*

Sketch of proof

Let $m_1, \dots, m_r$ be generators of E as a $\boldsymbol{k}[t]$-module. By assumption, there exists a polynomial $p(t)$ such that $M = \bigcup_{k\in-\mathbb{N}} p(t)^k E$. It is then enough to show that, for any $m \in M$ (for instance, one of the m_i), there exists an integer $j \leqslant 0$ such that, for all $k < j$, $p(t)^k m$ belongs to the $\boldsymbol{k}[t]\langle\partial_t\rangle$-module generated by the $p(t)^i m$, with $i \in [j, 0]$.

This result can be shown by using *Bernstein's relation*: there exists a non-zero polynomial $b(s)$ such that [2] $b(s)p(t)^s m = Q(t, \partial_t, s)p(t)^{s+1}m$. One chooses for j an integer less than the real part of any root of b. Then, for $k < j$, Bernstein's relation for $s = k$ shows that $p(t)^k m$ belongs to $\boldsymbol{k}[t]\langle\partial_t\rangle \cdot p(t)^{k+1}m$, hence, by induction, to $\boldsymbol{k}[t]\langle\partial_t\rangle \cdot p(t)^j m$. $\square$

1. although the non-commutativity of the Weyl algebra! What is important for the Euclid algorithm, is that the degree of the commutator $[Q_1, Q_2]$ of two operators is *strictly* smaller than the sum of the degrees of Q_1 and Q_2.

2. This relation has to be understood as an algebraic relation, obtained by formally differentiating $p(t)^{s+1}$, and then dividing both sides by $p(t)^s$, where s is a new variable.

2.4.b. Duality. If M is a left $\boldsymbol{k}[t]\langle\partial_t\rangle$-module, then $\mathrm{Hom}_{\boldsymbol{k}[t]\langle\partial_t\rangle}(M,\boldsymbol{k}[t]\langle\partial_t\rangle)$ is equipped with a structure of right $\boldsymbol{k}[t]\langle\partial_t\rangle$-module (by $(\varphi\cdot P)(m)=\varphi(m)P$). More generally, the spaces $\mathrm{Ext}^i_{\boldsymbol{k}[t]\langle\partial_t\rangle}(M,\boldsymbol{k}[t]\langle\partial_t\rangle)$ are right $\boldsymbol{k}[t]\langle\partial_t\rangle$-modules.

***Proposition 2.4.6* (Duality preserves holonomy).** *If M is holonomic,*

(1) *the right modules $\mathrm{Ext}^i_{\boldsymbol{k}[t]\langle\partial_t\rangle}(M,\boldsymbol{k}[t]\langle\partial_t\rangle)$ are zero for $i\neq 1$;*

(2) *the left module DM associated to the right module $\mathrm{Ext}^1_{\boldsymbol{k}[t]\langle\partial_t\rangle}(M,\boldsymbol{k}[t]\langle\partial_t\rangle)$ is holonomic;*

(3) *we have $D(DM)\simeq M$.*

Proof. By using the long Ext exact sequence associated to the short exact sequence 2.4.3-(4), we are reduced to showing the proposition for $M=\boldsymbol{k}[t]\langle\partial_t\rangle/(P)$. This module has the resolution

$$0\longrightarrow \boldsymbol{k}[t]\langle\partial_t\rangle \xrightarrow{\;\cdot P\;} \boldsymbol{k}[t]\langle\partial_t\rangle \longrightarrow M\longrightarrow 0$$

and, applying the functor $\mathrm{Hom}_{\boldsymbol{k}[t]\langle\partial_t\rangle}(\bullet,\boldsymbol{k}[t]\langle\partial_t\rangle)$ to this resolution, we get the exact sequence

$$\begin{aligned}0\longrightarrow \mathrm{Hom}_{\boldsymbol{k}[t]\langle\partial_t\rangle}(M,\boldsymbol{k}[t]\langle\partial_t\rangle)\longrightarrow \boldsymbol{k}[t]\langle\partial_t\rangle \xrightarrow{\;P\cdot\;} \boldsymbol{k}[t]\langle\partial_t\rangle&\\ \longrightarrow \mathrm{Ext}^1_{\boldsymbol{k}[t]\langle\partial_t\rangle}(M,\boldsymbol{k}[t]\langle\partial_t\rangle)\longrightarrow 0&\end{aligned}$$

on which we see that $\mathrm{Ext}^1_{\boldsymbol{k}[t]\langle\partial_t\rangle}(M,\boldsymbol{k}[t]\langle\partial_t\rangle)=\boldsymbol{k}[t]\langle\partial_t\rangle/(P)$ (here, (P) denotes the *right* ideal generated by P) and that all other Ext vanish. We thus have here

$$DM=\boldsymbol{k}[t]\langle\partial_t\rangle/({}^tP).\qquad\square$$

Exercises for Lecture 2

Exercise E.2.1. Let t be a coordinate on $\mathbb{A}^1$ and set $\mathscr{M}=\mathscr{O}_{\mathbb{A}^1}(*0)/\mathscr{O}_{\mathbb{A}^1}$. Consider the two elements $m_1=[1/t]$ and $m_2=[1/t^2]$, where $[\bullet]$ denotes the class modulo $\mathscr{O}_{\mathbb{A}^1}$. Show that the good filtrations generated respectively by m_1 and m_2 do not give rise to the same annihilator ideals.

Exercise E.2.2. Let $0\to\mathscr{M}'\to\mathscr{M}\to\mathscr{M}''\to 0$ be an exact sequence of $\mathscr{D}_X$-modules. Show that $\mathrm{Char}\,\mathscr{M}=\mathrm{Char}\,\mathscr{M}'\cup\mathrm{Char}\,\mathscr{M}''$. (Hint: take a good filtration on $\mathscr{M}$ and induce it on $\mathscr{M}'$ and $\mathscr{M}''$.)

***Exercise E.2.3* (Coherent $\mathscr{D}_X$-modules with characteristic variety contained in T^*_YX)**

Let $i:Y\hookrightarrow X$ be the inclusion of a smooth codimension p closed submanifold. Define the p-th algebraic local cohomology with support in Y by $R^p\Gamma_{[Y]}\mathscr{O}_X=\varinjlim_k \mathscr{E}xt^p(\mathscr{O}_X/\mathscr{I}_Y^k,\mathscr{O}_X)$, where $\mathscr{I}_Y$ is the ideal defining Y. $R^p\Gamma_{[Y]}\mathscr{O}_X$ has a natutal structure of $\mathscr{D}_X$-module. In local coordinates $(x_1,\dots,x_n)$ where Y is defined by $x_1=\cdots=x_p=0$, we have

$$R^p\Gamma_{[Y]}\mathscr{O}_X\simeq\frac{\mathscr{O}_{\mathbb{A}^n}[1/x_1\cdots x_n]}{\sum_{i=1}^p\mathscr{O}_{\mathbb{A}^n}(x_i/x_1\cdots x_n)}.$$

Denote this $\mathscr{D}_X$-module by $\mathscr{B}_Y X$.

(1) Show that $\mathscr{B}_Y X$ has support contained in Y and characteristic variety equal to $T^*_Y X$.

(2) Identify $\mathscr{B}_Y X$ with $i_+\mathscr{O}_Y$ (cf. Lecture 4).

(3) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module with characteristic variety equal to $T^*_Y X$. Show that $\mathscr{M}$ is locally isomorphic to $(\mathscr{B}_Y X)^d$ for some d.

***Exercise E.2.4* (Non-characteristic restriction).** Let $i : Y \hookrightarrow X$ denote the inclusion of a closed submanifold. The cotangent map to the inclusion defines a natural bundle morphism $\varpi : T^*X_{|Y} \to T^*Y$, the kernel of which is by definition the conormal bundle $T^*_Y X$ of Y in X. Let $\mathscr{M}$ be a coherent left $\mathscr{D}_X$-module.

(1) Define a natural left $\mathscr{D}_Y$-module structure on the $\mathscr{O}_Y$-module $i^*\mathscr{M} = \mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_X} i^{-1}\mathscr{M}$ and denote the corresponding $\mathscr{D}_Y$-module by $i^+\mathscr{M}$.

(2) Show that the following conditions are equivalent:

(a) $T^*_Y X \cap \operatorname{Char}\mathscr{M} \subset T^*_X X$,

(b) $\varpi : \operatorname{Char}\mathscr{M}_{|Y} \to T^*Y$ is finite, i.e., proper with finite fibres.

When one of these conditions is fulfilled, Y is said *non-characteristic with respect to $\mathscr{M}$*.

(3) Assume that $\mathscr{M}$ is $\mathscr{D}_X$-coherent and that Y is non-characteristic with respect to $\mathscr{M}$. Show that $i^+\mathscr{M}$ is $\mathscr{D}_Y$-coherent and $\operatorname{Char} i^+\mathscr{M} \subset \varpi(\operatorname{Char}\mathscr{M}_{|Y})$.

Remark first that the question is local near a point $x \in Y$, hence one can assume that $\mathscr{M}$ has a good filtration $F_\bullet\mathscr{M}$.

(a) Put $F_k i^*\mathscr{M} = \operatorname{image}[i^*F_k\mathscr{M} \to i^*\mathscr{M}]$. Show that $F_\bullet i^*\mathscr{M}$ is a good filtration with respect to $F_\bullet i^*\mathscr{D}_X$.

(b) Show that the module $\operatorname{gr}^F i^*\mathscr{M}$ is a quotient of $i^*\operatorname{gr}^F\mathscr{M}$ and that its support is contained in $\operatorname{Char}\mathscr{M}_{|Y}$. Using that coherence is preserved under finite morphisms, show that it is a coherent $\operatorname{gr}^F\mathscr{D}_Y$-module.

(c) Conclude that the filtration $F_\bullet i^*\mathscr{M}$ is a good filtration of the $\mathscr{D}_Y$-module $i^+\mathscr{M}$. Using the good filtration above, show that $\operatorname{Char} i^+\mathscr{M} \subset \varpi(\operatorname{Char}\mathscr{M}_{|Y})$.

LECTURE 3

DE RHAM AND DOLBEAULT COHOMOLOGY OF $\mathscr{D}$-MODULES

3.1. De Rham and Spencer

Let $\mathscr{M}^l$ be a left $\mathscr{D}_X$-module and let $\mathscr{M}^r$ be a right $\mathscr{D}_X$-module.

Definition 3.1.1 **(de Rham).** The *de Rham complex* ${}^p\mathrm{DR}\mathscr{M} = \Omega_X^{n+\bullet}(\mathscr{M}^l)$ of $\mathscr{M}^l$ is the complex having as terms the $\mathscr{O}_X$-modules $\Omega_X^{n+\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}^l$ and as differential the $\boldsymbol{k}$-linear morphism $(-1)^n\nabla$ defined in Exercise E.1.7.

Notice that the de Rham complex is shifted by $n = \dim X$ with respect to the usual convention (we denote by DR the unshifted de Rham complex; the left exponent p is here for the "perverse convention"). The shift produces, by definition, a sign change in the differential, which is then equal to $(-1)^n\nabla$. The need of such a shift is clear when considering the correspondence left$\leftrightarrow$right with the Spencer complex introduced below.

Definition 3.1.2 **(Spencer).** The *Spencer complex* $(\mathrm{Sp}_X^\bullet(\mathscr{M}^r), \delta)$ is the complex having as terms the $\mathscr{O}_X$-modules $\mathscr{M} \otimes_{\mathscr{O}_X} \wedge^{-\bullet}\Theta_X$ (with $\bullet \leqslant 0$) and as differential the $\boldsymbol{k}$-linear map δ given by

$$\begin{aligned} m \otimes \xi_1 \wedge \cdots \wedge \xi_k \stackrel{\delta}{\longmapsto} & \sum_{i=1}^{k} (-1)^{i-1} m\xi_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_k \\ & + \sum_{i<j} (-1)^{i+j} m \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_k. \end{aligned}$$

The relations between de Rham, Spencer, and the right-left transformation are analyzed in Exercises E.3.5–E.3.8.

Of special interest will be, of course, the de Rham or Spencer complex of the ring $\mathscr{D}_X$, considered as a left or right $\mathscr{D}_X$-module. Notice that, in $\Omega_X^{n+\bullet}(\mathscr{D}_X)$, the differentials are *right* $\mathscr{D}_X$-linear, and in $\mathrm{Sp}_X^\bullet(\mathscr{D}_X)$ they are *left* $\mathscr{D}_X$-linear.

Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module and let $\mathscr{M}^r$ the associated right module. We will now compare $\Omega_X^{n+\bullet}(\mathscr{M})$ and $\mathrm{Sp}_X^\bullet(\mathscr{M}^r)$.

***Proposition 3.1.3* (cf. Exercises E.3.6-E.3.8).** *For any left $\mathscr{D}_X$-module $\mathscr{M}$, there is a functorial isomorphism $\mathrm{Sp}_X^\bullet(\mathscr{M}^r) \xrightarrow{\sim} \Omega_X^{n+\bullet}(\mathscr{M})$ which is termwise $\mathscr{O}_X$-linear.*

3.2. Filtered objects: the Dolbeault complex

A F-filtration $F_\bullet\mathscr{M}$ of a $\mathscr{D}_X$-modules (cf. §2.1.a) gives rise to an increasing filtration of the de Rham complex:

$$F_p \operatorname{DR}\mathscr{M} = \{0 \longrightarrow F_p\mathscr{M} \longrightarrow \Omega_X^1 \otimes_{\mathscr{O}_X} F_{p+1}\mathscr{M} \longrightarrow \cdots \longrightarrow \Omega_X^n \otimes_{\mathscr{O}_X} F_{p+n}\mathscr{M} \longrightarrow 0\}$$

The *Dolbeault complex* of the filtered $\mathscr{D}_X$-module $(\mathscr{M}, F_\bullet\mathscr{M})$ is the graded complex $\mathrm{gr}^F \operatorname{DR}\mathscr{M}$. Let us note that, in Hodge theory, one uses decreasing filtrations. The way to go from one convention to the other one is to set $F_p = F^{-p}$.

The Dolbeault cohomology of the filtered $\mathscr{D}_X$-module $(\mathscr{M}, F_\bullet\mathscr{M})$ is the hypercohomology of the complex $\mathrm{gr}^F \operatorname{DR}\mathscr{M}$.

Example 3.2.1. We define the "stupid" (increasing) filtration on $\mathscr{O}_X$ by setting

$$F_p\mathscr{O}_X = \begin{cases} \mathscr{O}_X & \text{if } p \geqslant 0, \\ 0 & \text{if } p \leqslant -1. \end{cases}$$

The de Rham complex is filtered by

$$F^p(\Omega_X^\bullet, d) = \{0 \longrightarrow F_{-p}\mathscr{O}_X \xrightarrow{d} F_{-p+1}\mathscr{O}_X \otimes_{\mathscr{O}_X} \Omega_X^1 \xrightarrow{d} \cdots\}. \tag{3.2.2}$$

If $p \leqslant 0$, $F^p(\Omega_X^\bullet, d) = (\Omega_X^\bullet, d)$, although if $p \geqslant 1$,

$$F^p(\Omega_X^\bullet, d) = \{0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{p}{\Omega_X^p} \longrightarrow \cdots \longrightarrow \Omega_X^{\dim X} \longrightarrow 0\}.$$

Therefore, the p-th graded complex is 0 if $p \leqslant -1$ and, if $p \geqslant 0$, it is given by

$$\mathrm{gr}_F^p(\Omega_X^\bullet, d) = \{0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{p}{\Omega_X^p} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0\}.$$

In other words, the graded complex $\mathrm{gr}_F(\Omega_X^\bullet, d) = \bigoplus_p \mathrm{gr}_F^p(\Omega_X^\bullet, d)$, is the complex $(\Omega_X^\bullet, 0)$ (i.e., the same terms as for the de Rham complex, but with differential equal to 0).

From general results on filtered complexes, the filtration of the de Rham complex induces a (decreasing) filtration on the hypercohomology spaces (that is, on the de Rham cohomology of X) and there is a spectral sequence starting from $\mathbb{H}^*\big(X, \mathrm{gr}_F(\Omega_X^\bullet, d)\big)$ and abutting to $\mathrm{gr}_F H^*(X, \mathbb{C})$. Let us note that $\mathbb{H}^*\big(X, \mathrm{gr}_F(\Omega_X^\bullet, d)\big)$ is nothing but $\bigoplus_{p,q} H^q(X, \Omega_X^p)$.

3.3. The de Rham complex of a holonomic $\mathscr{D}_X$-module

Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. We analyze the local properties of the de Rham complex. We work over the field $\mathbb{C}$ of complex numbers. If X is an algebraic variety, we denote by X^{an} the corresponding complex analytic manifold. In particular, $\mathscr{O}_{X^{\mathrm{an}}}$ is the sheaf of holomorphic functions on X^{an} and $\mathscr{D}_{X^{\mathrm{an}}}$ is the sheaf of holomorphic differential operators. On X we use the Zariski topology and on X^{an} we use the analytic topology.

***Remark 3.3.1* (The case of $\mathscr{O}_X$).** The holomorphic Poincaré lemma asserts that $\mathrm{DR}\,\mathscr{O}_{X^{\mathrm{an}}}$ is a resolution of the constant sheaf $\mathbb{C}_{X^{\mathrm{an}}}$. *Warning*: $\mathrm{DR}\,\mathscr{O}_X$ *is not*, in general, a resolution of the constant sheaf $\mathbb{C}_X$.

3.3.a. Vector bundles with flat connection. Let (V,∇) be a vector bundle with a flat connection on X. The *theorem of Cauchy-Kowalevski* asserts that, locally, V^{an} has a local frame made of ∇-flat sections. In other words, $V^{\mathrm{an}} \simeq \mathscr{O}_{X^{\mathrm{an}}} \otimes_{\mathbb{C}} \operatorname{Ker}\nabla^{\mathrm{an}}$ and the connection ∇^{an} on V^{an} corresponds to the connection $d\otimes\mathrm{Id}$. From the computatoin of the holomorphic de Rham complex for $\mathscr{O}_X$ we conclude that the holomorphic de Rham complex

$$0 \longrightarrow V \xrightarrow{\ \nabla\ } \Omega^1_{X^{\mathrm{an}}}\otimes V \xrightarrow{\ \nabla\ } \Omega^2_{X^{\mathrm{an}}}\otimes V \longrightarrow \cdots \longrightarrow \Omega^{\dim X}_{X^{\mathrm{an}}}\otimes V \longrightarrow 0$$

is a resolution of the locally constant sheaf $\operatorname{Ker}\nabla^{\mathrm{an}}$.

3.3.b. The constructibility theorem of Kashiwara. To what extent can we apply the previous result for holonomic $\mathscr{D}_X$-modules?

Theorem 3.3.2. *If $\mathscr{M}$ is holonomic, each cohomology sheaf of $\mathrm{DR}\,\mathscr{M}$ is a constructible sheaf of $\mathbb{C}$-vector spaces.*

We say that a sheaf $\mathscr{F}$ is constructible if there exists a locally finite partition of X by locally closed Zariski subsets X_i such that, for any i, the sheaf-theoretical restriction $\mathscr{F}_{|X_i}$ is a locally constant sheaf of finite dimensional $\mathbb{C}$-vector spaces of finite rank on X_i.

***Remark 3.3.3* (The local index theorem).** Let $\mathscr{F}^\bullet$ be a bounded complex with constructible cohomology on X. Define the characteristic function $\chi_{\mathscr{F}}: X\to\mathbb{Z}$ by

$$\chi_{\mathscr{F}}(x)=\sum_k(-1)^k\dim_{\mathbb{C}}\mathscr{H}^k(\mathscr{F})_x.$$

This function is constant on the strata X_i of the partition adapted to all the $\mathscr{H}^k(\mathscr{F})$. The *local index theorem* of Kashiwara implies that, if $\mathscr{M}$ is holonomic, *the characteristic function of* $\mathrm{DR}\,\mathscr{M}$ *determines the characteristic variety of* $\mathscr{M}$.

***Corollary 3.3.4* (Finiteness of the de Rham cohomology).** *Let X be a compact complex analytic manifold (or a smooth projective variety) and let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then the cohomology spaces $\boldsymbol{H}^k(X,\mathrm{DR}\,\mathscr{M})$ are finite dimensional.*

3.4. The case of curves

We now make a detailed analysis of the de Rham complex for holonomic $\mathscr{D}_X$-modules when $\dim X=1$.

3.4.a. The constructibility theorem. If $\mathscr{M}$ is supported on a point $x^o \in X$, then $\mathscr{M} \simeq (\mathbb{C}[\partial_x]\delta)^d$ for some $d \in \mathbb{N}$, where t is a local coordinate at x^o and $\mathbb{C}[\partial_t]\delta = \mathscr{B}_{\{x^o\}|X}$ (cf. Exercise E.2.3). The generator δ of this $\mathscr{D}_X$-module satisfies $t\delta = 0$.

The de Rham complex of $\mathscr{B}_{\{x^o\}|X}$ is the complex

$$0 \longrightarrow \mathbb{C}[\partial_t]\delta \xrightarrow{\partial_t} \mathbb{C}[\partial_t]\delta \longrightarrow 0$$

where the convention is that the right hand term has degree 0. We thus have $\mathrm{DR}\,\mathscr{B}_{\{x^o\}|X} \simeq \mathbb{C}_{x^o}$.

If $\mathscr{M}$ is holonomic, with singular support S, then the constructibility of $\mathrm{DR}\,\mathscr{M}$ amounts to the following statement

Theorem 3.4.1. *Let $P \in \mathbb{C}\{t\}\langle\partial_t\rangle$ be a differential operator; then the kernel and the cokernel of $P : \mathbb{C}\{t\} \to \mathbb{C}\{t\}$ are finite dimensional vector spaces.*

Proof. We will prove the theorem when P can be written as $\sum_{i=0}^d a_i(t)(t\partial_t)^i$ where $a_d \not\equiv 0$ and one of the coefficients a_i is a unit. We will moreover assume that, if we write $P = b(t\partial_t) + t\sum_i c_i(t)(t\partial_t)^i$, where $b \in \mathbb{C}[s]$ is non-zero, then $b(s) \neq 0$ for $s \in \mathbb{N}$. The general case can be reduced to this one. In such a setting, we will prove more precisely that

(1) $\mathrm{Ker}[P : \mathbb{C}\{t\} \to \mathbb{C}\{t\}] = 0$,
(2) $\dim \mathrm{Coker}[P : \mathbb{C}\{t\} \to \mathbb{C}\{t\}] = v(a_d)$, where v denotes the valuation.

For the first assertion, we remark that $P \cdot (\sum_{n=n_0}^\infty f_n t^n) = b(n_0) f_{n_0} t^{n_0} + \cdots$.

Let Δ_r be a closed disk in $\mathbb{C}$ centered at zero and with radius r. Consider the space $B^m(\Delta_r)$ of functions which are C^m in some open neighborhood of Δ_r and which are holomorphic in the interior of Δ_r. This is a Banach space for the norm

$$\|f\|_m = \sup_{|\alpha| \leqslant m} \sup_{\Delta_r} \Big| \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x\, \partial^{\alpha_2} y} \Big|$$

where we have set $t = x + iy$, $\alpha = (\alpha_1, \alpha_2)$ and $|\alpha| = \alpha_1 + \alpha_2$. We shall use the following results:

Proposition 3.4.2. *For all $m \geqslant 0$ the injection $B^{m+1}(\Delta_r) \hookrightarrow B^m(\Delta_r)$ is compact.* □

Theorem 3.4.3. *Let $U, V : E \to F$ be continuous linear operators between two Banach spaces. If the index of U is defined (i.e., if the kernel and the cokernel of U are finite dimensional) and if V is compact then the index of $U + V$ is defined and is equal to the index of V.* □

Now P defines a continuous operator $B^d(\Delta_r) \to B^0(\Delta_r)$ for each $r > 0$ sufficiently small. We shall prove that this operator has an index, and that this index is equal to $-v(a_d)$. Remark that $P = a_d(t)(t\partial_t)^d + Q$ with $\deg Q < d$. Because of the previous proposition, Q induces a compact operator $B^d(\Delta_r) \to B^0(\Delta_r)$, and we are reduced to showing that the index of $a_d(t)(t\partial_t)^d$ is equal to $-v(a_d)$. This comes from the fact that ∂_t has index 1 and t has index -1. □

3.4.b. Example of computation of de Rham cohomology. Let X be a smooth projective curve and let ω be a meromorphic 1-form on X with poles on a finite set S. We consider the trivial rank-one bundle $\mathscr{O}_X(*S)$ with connection $d+\omega$. It defines a holonomic $\mathscr{D}_X$-module $\mathscr{M}$. We wish to give a topological expression of $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathscr{M})$.

The basic topological object attached to the situation is the locally constant sheaf $\mathscr{L}$ on $X \smallsetminus S$ defined as $\mathrm{Ker}[d+\omega : \mathscr{O}_{X\smallsetminus S} \to \mathscr{O}_{X\smallsetminus S}]$.

Question: Do we have $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathscr{M}) = H^k(X \smallsetminus S, \mathscr{L})$?

(1) If S is empty (i.e., ω is holomorphic) the answer is “yes”, since $\mathrm{DR}\,\mathscr{M} = \mathscr{L}$.

(2) If ω has only simple poles, the answer is “yes”. We will now show how to prove this result.

Remark 3.4.4. In higer rank and higher dimension, the analogous result is called the *Grothendieck-Deligne comparison theorem.*

Instead of making a global comparison, we make a local comparison. If $j : X\smallsetminus S \hookrightarrow S$ denotes the inclusion, the cohomology of $X \smallsetminus S$ with coefficients in $\mathscr{L}$ is the hypercohomology on X of a complex called $Rj_*\mathscr{L}$. So we want to compare $\mathrm{DR}\,\mathscr{M}$ and $Rj_*\mathscr{L}$. This is a local problem near each point of S, and we can assume that X is a disc with coordinate t and ω has a pole at the origin only. It is not difficult to reduce to the case where $\omega = \alpha dt/t$ with $\alpha \in \mathbb{C}$.

On the one hand, the germ at the origin of the de Rham complex is the complex

$$0 \longrightarrow \mathbb{C}\{t\}[t^{-1}] \xrightarrow{\ t\partial_t + \alpha\ } \mathbb{C}\{t\}[t^{-1}] \longrightarrow 0.$$

On the other hand, $Rj_*\mathscr{L}$ is isomorphic to the complex

$$0 \longrightarrow \mathbb{C}\{t\}\{t^{-1}\} \xrightarrow{\ t\partial_t + \alpha\ } \mathbb{C}\{t\}\{t^{-1}\} \longrightarrow 0,$$

where $\mathbb{C}\{t\}\{t^{-1}\}$ is the space of convergent Laurent series.

Now, we are left to proving that, setting $\tau = t^{-1}$ and if $\mathscr{O}(\mathbb{C})$ denotes the space of entire functions with respect to τ,

$$\tau\partial_\tau + \beta : \mathscr{O}(\mathbb{C})/\mathbb{C}[\tau] \longrightarrow \mathscr{O}(\mathbb{C})/\mathbb{C}[\tau]$$

is an isomorphism. The injectivity is easy. The surjectivity follows from the fact that, if $\sum_n g_n\tau^n$ is a series with infinite radius of convergence, the radius of convergence of the series $\sum_n f_n\tau^n$ defined for large n by $f_n = g_n/(n+\beta)$ is also infinite.

(3) Let us now assume that ω has a pole of order $\geqslant 2$ at one point of S at least. We will analyze the de Rham complex in a way which can be generalized.

Let us denote by $\rho : \widetilde{X} \to X$ the real oriented blow-up of X at the points of S (it amounts to working in polar coordinates near each point of S). Then $\widetilde{X}$ is a Riemann surface with boundary. Let $\mathscr{A}_{\widetilde{X}}$ be the sheaf on $\widetilde{X}$ of germs of functions on $\widetilde{X} \smallsetminus \rho^{-1}(S) = X \smallsetminus S$ which have moderate growth near $\rho^{-1}(S)$. In particular, $\mathscr{A}_{\widetilde{X}|\widetilde{X}\smallsetminus\rho^{-1}(S)} = \mathscr{O}_{X\smallsetminus S}$.

Proposition 3.4.5. *The complex*

$$0 \longrightarrow \mathscr{A}_{\widetilde{X}} \xrightarrow{\ d+\omega\ } \Omega^1_X \otimes_{\rho^{-1}\mathscr{O}_X} \mathscr{A}_{\widetilde{X}} \longrightarrow 0$$

has cohomology in degree 0 *at most. The sheaf* $\widetilde{\mathscr{L}} = \operatorname{Ker} d+\omega$ *is equal to* $\mathscr{L}$ *when restricted to* $X \smallsetminus S$ *and, for any* $x \in S$*, if* r_x *denotes the order of the pole of* ω *at* x*, there exists a subdivision of* S^1 *in* $\max(1, 2(r_x - 1))$ *intervals of the same length, alternatively closed and open, such that* $\widetilde{\mathscr{L}}$ *is zero on the closed intervals and is a (locally) constant sheaf of rank one on the open intervals.*

Moreover, $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathscr{M}) = H^k(\widetilde{X}, \widetilde{\mathscr{L}})$.

When $x \in S$ is a regular singularity (i.e., ω has a simple pole), then $\widetilde{\mathscr{L}}_{|\rho^{-1}(x)}$ is a locally constant sheaf on the circle. Computing Euler characteristics gives

$$\chi(X, \mathrm{DR}\,\mathscr{M}) = \chi(X \smallsetminus S) + \sum_{x\in S}(r_x - 1).$$

3.4.c. General results on a curve. The previous computation illustrates, in a simple case, a general result for meromorphic connections on a smooth complex projective curve.

Let X be a smooth complex projective curve, let S be a finite set of points and let $\mathscr{M}$ be a locally free $\mathscr{O}_X(*S)$-module of finite rank d with a connection.

Theorem 3.4.6 **(Analogue of the Grothendieck-Ogg-Shafarevitch formula)**

$$\chi(X, \mathrm{DR}\,\mathscr{M}) = d \cdot \chi(X \smallsetminus S) + \sum_{x\in S} \operatorname{irr}_x(\mathscr{M}, \nabla)$$

where $\operatorname{irr}_x(\mathscr{M}, \nabla)$ *is the irregularity number of* $(\mathscr{M}, \nabla)$ *at the point* x.

Theorem 3.4.7. *Let* $\rho : \widetilde{X} \to X$ *be the real blow-up of* X *at the points of* S*. Then the moderate de Rham complex* $\widetilde{\mathrm{DR}}\mathscr{M}$ *has cohomology in degree* 0 *at most and* $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathscr{M}) = H^k(\widetilde{X}, \widetilde{\mathscr{L}})$*, where* $\widetilde{\mathscr{L}} = \mathscr{H}^0(\widetilde{\mathrm{DR}}\mathscr{M})$*. Moreover,* $\widetilde{\mathscr{L}}_{|X\smallsetminus S} = \mathscr{L}$ *and, for any* $x \in S$*,* $\widetilde{\mathscr{L}}_{|\rho^{-1}(x)}$ *is a constructible sheaf on the circle* S^1.

Exercises for Lecture 3

Exercise E.3.1. Check that $(\mathrm{Sp}^{\bullet}_X(\mathscr{M}^r), \delta)$ is indeed a complex, i.e., $\delta \circ \delta = 0$.

Exercise E.3.2 **(The Spencer complex is a resolution of $\mathscr{O}_X$ as a left $\mathscr{D}_X$-module)**

Let $F_{\bullet}\mathscr{D}_X$ be the filtration of $\mathscr{D}_X$ by the order of differential operators. Filter the Spencer complex $\mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)$ by the subcomplexes $F_k(\mathrm{Sp}^{\bullet}_X(\mathscr{D}_X))$ defined as

$$\cdots \xrightarrow{\ \delta\ } F_{k+\ell}\mathscr{D}_X \otimes \wedge^{-\ell}\Theta_X \xrightarrow{\ \delta\ } F_{k+(\ell+1)}\mathscr{D}_X \otimes \wedge^{-(\ell+1)}\Theta_X \xrightarrow{\ \delta\ } \cdots$$

(1) Show that, locally on X, using coordinates $x_1, \dots, x_n$, the graded complex $\mathrm{gr}^F \mathrm{Sp}^{\bullet}_X(\mathscr{D}_X) \overset{\mathrm{def}}{=} \oplus_k \mathrm{gr}^F_k \mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)$ is equal to the Koszul complex of the ring $\mathscr{O}_X[\xi_1, \dots, \xi_n]$ with respect to the regular sequence $\xi_1, \dots, \xi_n$.

(2) Conclude that $\operatorname{gr}^F \operatorname{Sp}^\bullet_X(\mathscr{D}_X)$ is a resolution of $\mathscr{O}_X$.

(3) Show that the Spencer complex $\operatorname{Sp}^\bullet_X(\mathscr{D}_X)$ is a resolution of $\mathscr{O}_X$ as a left $\mathscr{D}_X$-module by locally free left $\mathscr{D}_X$-modules.

Exercise E.3.3. Similarly, show that the complex $\Omega_X^{n+\bullet}(\mathscr{D}_X)$ is a resolution of ω_X as a right $\mathscr{D}_X$-module by locally free right $\mathscr{D}_X$-modules.

Exercise E.3.4. Recall (holomorphic Poincaré Lemma) that, if X is a complex manifold, $(\Omega^\bullet_X, d)$ is a resolution of the constant sheaf. Therefore, the cohomology $H^k(X, \mathbb{C})$ is canonically identified with the hypercohomology $\mathbb{H}^k\big(X, (\Omega^\bullet_X, d)\big)$ of the de Rham complex.

(1) Let X be a smooth algebraic variety over $\mathbb{C}$ (equipped with the Zariski topology). Is the algebraic de Rham complex a resolution of the constant sheaf $\mathbb{C}_X$?

(2) Do we have $H^*(X, \mathbb{C}_X) = \mathbb{H}^*\big(X, (\Omega^\bullet_X, d)\big)$?

Exercise E.3.5. Let $\mathscr{M}^r$ be a right $\mathscr{D}_X$-module.

(1) Show that the natural morphism

$$\mathscr{M}^r \otimes_{\mathscr{D}_X} (\mathscr{D}_X \otimes_{\mathscr{O}_X} \wedge^k\Theta_X) \longrightarrow \mathscr{M}^r \otimes_{\mathscr{O}_X} \wedge^k\Theta_X$$

defined by $m \otimes P \otimes \xi \mapsto mP \otimes \xi$ induces an isomorphism of complexes

$$\mathscr{M}^r \otimes_{\mathscr{D}_X} \operatorname{Sp}^\bullet_X(\mathscr{D}_X) \xrightarrow{\sim} \operatorname{Sp}^\bullet(\mathscr{M}^r).$$

(2) Similar question for $\Omega_X^{n+\bullet}(\mathscr{D}_X) \otimes_{\mathscr{D}_X} \mathscr{M}^l \to \Omega_X^{n+\bullet}(\mathscr{M}^l)$.

***Exercise E.3.6* (de Rham, Spencer, left and right).** Consider the function

$$\mathbb{Z} \xrightarrow{\ \varepsilon\ } \{\pm 1\}, \quad a \longmapsto \varepsilon(a) = (-1)^{a(a-1)/2},$$

which satisfies in particular

$$\varepsilon(a+1) = \varepsilon(-a) = (-1)^a \varepsilon(a), \quad \varepsilon(a+b) = (-1)^{ab}\varepsilon(a)\varepsilon(b).$$

Given any $k \geqslant 0$, the *contraction* is the morphism

$$\begin{aligned} \omega_X \otimes_{\mathscr{O}_X} \wedge^k\Theta_X &\longrightarrow \Omega_X^{n-k} \\ \omega \otimes \xi &\longmapsto \varepsilon(n-k)\omega(\xi \wedge \bullet). \end{aligned}$$

Show that the isomorphism of right $\mathscr{D}_X$-modules

$$\begin{aligned} \omega_X \otimes_{\mathscr{O}_X} \big(\mathscr{D}_X \otimes_{\mathscr{O}_X} \wedge^k\Theta_X\big) &\xrightarrow[\sim]{\ \iota\ } \Omega_X^{n-k} \otimes_{\mathscr{O}_X} \mathscr{D}_X \\ \big[\omega \otimes (1 \otimes \xi)\big] \cdot P &\longmapsto \big(\varepsilon(n-k)\omega(\xi \wedge \bullet)\big) \otimes P \end{aligned}$$

(where the right structure of the right-hand term is the natural one and that of the left-hand term is nothing but that induced by the left structure after going from left to right) induces an isomorphism of complexes of right $\mathscr{D}_X$-modules

$$\iota : \omega_X \underset{\mathscr{O}_X}{\otimes} \big(\operatorname{Sp}^\bullet_X(\mathscr{D}_X), \delta\big) \xrightarrow{\sim} \big(\Omega_X^{n+\bullet} \underset{\mathscr{O}_X}{\otimes} \mathscr{D}_X, \nabla\big).$$

Exercise E.3.7. Similarly, if $\mathscr{M}$ is any left $\mathscr{D}_X$-module and $\mathscr{M}^r = \omega_X \otimes_{\mathscr{O}_X} \mathscr{M}$ is the associated right $\mathscr{D}_X$-module, show that there is an isomorphism

$$\begin{aligned}\mathscr{M}^r \otimes_{\mathscr{D}_X} \big(\mathrm{Sp}^\bullet_X(\mathscr{D}_X), \delta\big) &\simeq \big(\omega_X \otimes_{\mathscr{O}_X} \mathscr{M} \otimes_{\mathscr{O}_X} \wedge^{-\bullet}\Theta_X, \delta\big)\\ &\xrightarrow{\sim} \big(\Omega^{n+\bullet}_X \otimes_{\mathscr{O}_X} \mathscr{M}, \nabla\big) \simeq \big(\Omega^{n+\bullet}_X \underset{\mathscr{O}_X}{\otimes} \mathscr{D}_X, \nabla\big) \otimes_{\mathscr{D}_X} \mathscr{M}\end{aligned}$$

given on $\omega_X \otimes_{\mathscr{O}_X} \mathscr{M} \otimes_{\mathscr{O}_X} \wedge^k\Theta_X$ by

$$\omega \otimes m \otimes \xi \longmapsto \varepsilon(n-k)\omega(\xi \wedge \bullet) \otimes m.$$

Exercise E.3.8. Using Exercise E.3.7, show that there is a functorial isomorphism $\mathrm{Sp}^\bullet_X(\mathscr{M}^r) \xrightarrow{\sim} \Omega^{n+\bullet}_X(\mathscr{M})$ for any left $\mathscr{D}_X$-module $\mathscr{M}$, which is termwise $\mathscr{O}_X$-linear.

LECTURE 4

DIRECT IMAGES OF $\mathscr{D}$-MODULES

The notion of direct image of a $\mathscr{D}$-module answers the following problem: given a C^∞ differential form η of maximal degree on a complex manifold X, which satisfies a linear system of holomorphic differential equations (recall that $\mathscr{D}_X$ acts on the right on the sheaf $\mathscr{E}_X^{n,n}$ of forms of maximal degree), what can be said of the form (or more generally the current) obtained by integrating η along the fibres of a holomorphic map $f : X \to Y$? Does it satisfy a finite (i.e., coherent) system of holomorphic differential equations on Y? How can one define intrinsically this system?

Such a question arises in many domains of algebraic geometry. The system of differential equation is often called the “Picard-Fuchs system”, or the Gauss-Manin system. A way of “solving” a linear system of holomorphic or algebraic differential equations on a space Y consists in recognizing in this system the Gauss-Manin system attached to some holomorphic or algebraic function $f : X \to Y$. The geometric properties of f induce interesting properties of the system. Practically, this reduces to expressing solutions of the system as integrals over the fibers of f of some differential forms.

The definition of the direct image of a $\mathscr{D}$-module cannot be as simple as that of the direct image of a sheaf. One is faced to a problem which arises in differential geometry: the cotangent map of a holomorphic map $f : X \to Y$ is not a map from the cotangent space T^*Y of Y to that of X, but is a bundle map from the pull-back bundle f^*T^*Y to T^*X. In other words, a vector field on X does not act as a derivation on functions on Y. The transfer module $\mathscr{D}_{X\to Y}$ will give a reasonable solution to this problem.

We have seen that the notion of a left $\mathscr{D}_X$-module is equivalent to that of a $\mathscr{O}_X$-module equipped with a flat connection. Correspondingly, there are two notions of direct images.

– The direct image of a $\mathscr{O}_X$-module with a flat connection is known as the *Gauss-Manin* connection attached to te original one. This notion is only cohomological. Although many examples were given some centuries ago (related to the differential equations satisfied by the periods of a family of elliptic curves), the systematic construction was only achieved in [16]. The construction with a filtration is due to Griffiths [13, 14] (the main result is called *Griffiths' transversality theorem*). There is a strong constraint however: the map should be smooth (i.e., without critical points).

– The direct image of left $\mathscr{D}$-modules was constructed in [26]. This construction has the advantage of being very functorial, and defined at the level of derived categories, not only at the cohomology level as is the first one. It is very flexible. The filtered analogue is straightforward. It appears as a basic tool in various questions in algebraic geometry.

4.1. Direct images of right $\mathscr{D}$-modules

4.1.a. The transfer module. Let us begin with a preliminary remark. Let $\mathscr{M}^l$ be a left $\mathscr{D}_X$-module and let $\mathscr{N}$ be a left $f^{-1}\mathscr{D}_Y$-module. Then $\mathscr{M}^l \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{N}$ can be equipped with a left $\mathscr{D}_X$-module structure: if ξ is a local vector field on X, we set

$$\xi \cdot (m \otimes n) = (\xi m) \otimes n + Tf(\xi)(m \otimes n).$$

One can show that the conditions of Lemma 1.2.1 are fulfilled.

***Definition 4.1.1* (Transfer module).** The sheaf $\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$ is a left-right $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule when using the natural right $f^{-1}\mathscr{D}_Y$-module structure and the left $\mathscr{D}_X$-module introduced above.

(Cf. Exercises E.4.1, E.4.2 and E.4.3.)

4.1.b. The relative Spencer complex. Recall that the Spencer complex $\mathrm{Sp}^\bullet_X(\mathscr{D}_X)$, which was defined in 3.1.2, is a complex of left $\mathscr{D}_X$-modules. Denote by $\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ the complex $\mathrm{Sp}^\bullet_X(\mathscr{D}_X) \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y}$ (the left $\mathscr{O}_X$-structure on each factor is used for the tensor product). It is a complex of $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodules: the right $f^{-1}\mathscr{D}_Y$ structure is the trivial one; the left $\mathscr{D}_X$-structure is that defined by Exercise E.1.10(1).

Examples 4.1.2

(1) For $f = \mathrm{Id} : X \to X$, the complex $\mathrm{Sp}^\bullet_{X\to X}(\mathscr{D}_X) = \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathrm{Sp}^\bullet_X(\mathscr{D}_X)$ is a resolution of $\mathscr{D}_{X\to X} = \mathscr{D}_X$ as a left and right $\mathscr{D}_X$-module (notice that the left structure of $\mathscr{D}_X$ is used for the tensor product).

(2) For $f : X \to \mathrm{pt}$, the complex $\mathrm{Sp}^\bullet_{X\to \mathrm{pt}}(\mathscr{D}_X) = \mathrm{Sp}^\bullet_X(\mathscr{D}_X)$ is a resolution of $\mathscr{D}_{X\to\mathrm{pt}} = \mathscr{O}_X$.

(3) If $X = Y \times Z$ and f is the projection, denote by $\Theta_{X/Y}$ the sheaf of *relative* tangent vector fields, i.e., which do not contain ∂_{y_j} in their local expression in coordinates adapted to the product $Y \times Z$. The complex $\mathscr{D}_X \otimes_{\mathscr{O}_X} \wedge^{-\bullet}\Theta_{X/Y}$ is also a resolution of $\mathscr{D}_{X\to Y}$ as a bimodule by locally free left $\mathscr{D}_X$-modules (Exercise: describe the right

$f^{-1}\mathscr{D}_Y$-module structure). We moreover have a canonical quasi-isomorphism as bi-modules

$$\begin{aligned}\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X) &= \big(\mathscr{D}_X \otimes_{\mathscr{O}_X} \wedge^{-\bullet}\Theta_{X/Y}\big) \underset{f^{-1}\mathscr{O}_Y}{\otimes} f^{-1}\big(\wedge^{-\bullet}\Theta_Y \otimes_{\mathscr{O}_Y} \mathscr{D}_Y\big)\\ &= \big(\mathscr{D}_X \otimes_{\mathscr{O}_X} \wedge^{-\bullet}\Theta_{X/Y}\big) \underset{f^{-1}\mathscr{D}_Y}{\otimes} f^{-1}\big(\mathrm{Sp}^\bullet_Y(\mathscr{D}_Y) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y\big)\\ &\xrightarrow{\sim} \big(\mathscr{D}_X \otimes_{\mathscr{O}_X} \wedge^{-\bullet}\Theta_{X/Y}\big) \underset{f^{-1}\mathscr{D}_Y}{\otimes} f^{-1}\mathscr{D}_{Y\to Y}\\ &= \mathscr{D}_X \otimes_{\mathscr{O}_X} \wedge^{-\bullet}\Theta_{X/Y}.\end{aligned}$$

(4) In general, $\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X) = \mathrm{Sp}^\bullet_X(\mathscr{D}_X) \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$ is a resolution of $\mathscr{D}_{X\to Y}$ by locally free left $\mathscr{D}_X$-modules (cf. Exercise E.4.4).

Definition 4.1.3 **(Direct images of $\mathscr{D}$-modules).** The k-th *direct image* $\mathscr{H}^k f_+\mathscr{M}$ of the right $\mathscr{D}_X$-module $\mathscr{M}$ is the right $\mathscr{D}_Y$-module

$$\mathscr{H}^k f_+\mathscr{M} = R^k f_*\Big(\mathscr{M} \underset{\mathscr{D}_X}{\otimes} \mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)\Big).$$

Remarks 4.1.4

(1) If $\mathscr{F}$ is any sheaf on X, we have $R^j f_*\mathscr{F} = 0$ for $j \notin [0, 2\dim X]$. Therefore, taking into account the length $\dim X$ of the relative Spencer complex, we find that $\mathscr{H}^j f_+\mathscr{M}$ are zero for $j \notin [-\dim X, 2\dim X]$.

(2) If $\mathscr{M}$ is a left $\mathscr{D}_X$-module, one *defines* $\mathscr{H}^k f_+\mathscr{M}$ as $(\mathscr{H}^k f_+\mathscr{M}^r)^l$.

(3) In the case where $X = Y \times Z$ and f is the projection, and if $\mathscr{M}$ is a right $\mathscr{D}_X$-module, we have $\mathscr{M} \otimes_{\mathscr{D}_X} \mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X) = \mathscr{M} \otimes_{\mathscr{O}_X} \wedge^{-\bullet}\Theta_{X/Y}$. Similarly, if $\mathscr{M}$ is a left $\mathscr{D}_X$-module, we deduce that

$$\mathscr{H}^k f_+\mathscr{M} = R^k f_*(\Omega^{d+\bullet}_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{M}),$$

where $d = \dim Z$, the differential of the complex is the relative connection $\nabla_{X/Y}$ and the connection on $\mathscr{H}^k f_+\mathscr{M}$ is induced by ∇_Z on the complex.

Example 4.1.5 **(Direct image of induced $\mathscr{D}$-modules).** Let $\mathscr{L}$ be a $\mathscr{O}_X$-module and let $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ be the associated induced right $\mathscr{D}_X$-module. Let $f : X \to Y$ be a proper map. Then that $\mathscr{H}^k f_+(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \simeq (R^k f_*\mathscr{L}) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y$. Indeed, $\mathscr{L} \otimes_{\mathscr{O}_X} \mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X) \to \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y}$ is a quasi-isomorphism as $\mathscr{D}_X$ is $\mathscr{O}_X$-locally free. The result follows then from the projection formula.

4.2. Coherence of direct images

Let $f : X \to Y$ be a holomorphic map and $\mathscr{M}$ be a $\mathscr{D}_X$-module. We say that $\mathscr{M}$ is f-*good* if there exists a covering of Y by open sets V_j such that $\mathscr{M}$ is good on each $f^{-1}(V_j)$.

Theorem 4.2.1. *Let $\mathscr{M}$ be a f-good $\mathscr{D}_X$-module. Assume that f is proper on the support of $\mathscr{M}$. Then, for any k, $\mathscr{H}^k f_+\mathscr{M}$ has $\mathscr{D}_Y$-coherent cohomology.*

This theorem is an application of Grauert's coherence theorem for $\mathscr{O}_X$-modules, and this is why we restrict to f-good $\mathscr{D}_X$-modules. In general, it is not known whether the theorem holds for any coherent $\mathscr{D}_X$-module or not. Notice, however, that one may relax the geometric condition on $f_{|\operatorname{Supp}\mathscr{M}}$ (properness) by using more specific properties of $\mathscr{D}$-modules: as we have seen, the characteristic variety is a finer geometrical object attached to the $\mathscr{D}$-module, and one should expect that the right condition on f has to be related with the characteristic variety. The most general statement in this direction is the coherence theorem for *elliptic pairs*, due to P. Schapira and J.-P. Schneiders [28]. For instance, if X is an open set of X' and f is the restriction of $f' : X' \to Y$, and if the boundary of X is f-non-characteristic with respect to $\mathscr{M}$ then the direct image of $\mathscr{M}$ has $\mathscr{D}_Y$-coherent cohomology.

Proof of Theorem 4.2.1. As the coherence property is a local property on Y, the statement one proves is, more precisely, that the direct image of a good $\mathscr{D}_X$-module $\mathscr{M}$ is a good $\mathscr{D}_Y$-module when f is proper on $\operatorname{Supp}\mathscr{M}$. By an extension argument, it is even enough to assume that $\mathscr{M}$ has a good filtration and show that, locally on Y, the $\mathscr{D}_Y$-modules $\mathscr{H}^k f_\dagger\mathscr{M}$ have a good filtration.

First step: induced $\mathscr{D}$-modules. Assume that $\mathscr{M} = \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ and $\mathscr{L}$ is $\mathscr{O}_X$-coherent. By Example 4.1.5, it is enough to prove that $R^k f_*\mathscr{L}$ is $\mathscr{O}_Y$-coherent when f is proper on $\operatorname{Supp}\mathscr{L}$: this is Grauert's Theorem.

Second step: finite complexes of induced $\mathscr{D}$-modules. Let $\mathscr{L}_\bullet \otimes_{\mathscr{O}_X} \mathscr{D}_X$ be a *finite* complex of induced $\mathscr{D}_X$-modules. Assume that f restricted to the support of each term is proper. Using Artin-Rees (Corollary 2.1.5), one shows by induction on the length of the complex that the modules $\mathscr{H}^k f_+(\mathscr{L}_\bullet \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ have a good filtration.

Third step: general case. Fix a compact set K of Y. We will show that the $\mathscr{D}_Y$-modules $\mathscr{H}^k f_+\mathscr{M}$ have a good filtration in a neighbourhood of K. Fix a good filtration $F_\bullet\mathscr{M}$ of $\mathscr{M}$. As $f^{-1}(K) \cap \operatorname{Supp}\mathscr{M}$ is compact, there exists k such that $\mathscr{L}^0 \stackrel{\text{def}}{=} F_k\mathscr{M}$ generates $\mathscr{M}$ as a $\mathscr{D}_X$-module in some neighbourhood of $f^{-1}(K)$. Hence $\mathscr{L}^0$ is a coherent $\mathscr{O}_X$-module with support contained in $\operatorname{Supp}\mathscr{M}$ and we have a surjective morphism $\mathscr{L}^0 \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{M}$ in some neighbourhood of $f^{-1}(K)$ that we still call X. The kernel of this morphism is therefore $\mathscr{D}_X$-coherent, has support contained in $\operatorname{Supp}\mathscr{M}$ and, by Artin-Rees (Corollary 2.1.5), has a good filtration.

The process may therefore be continued and leads to the existence, in some neighbourhood of K, of a (maybe infinite) resolution $\mathscr{L}^{-\bullet} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ by coherent induced $\mathscr{D}_X$-modules with support contained in $\operatorname{Supp}\mathscr{M}$.

Fix some ℓ and stop the resolution at the ℓ-th step. Denote by $\mathscr{N}^{-\bullet}$ this bounded complex and by $\mathscr{M}'$ the kernel of $\mathscr{N}^{-\ell} \to \mathscr{N}^{-\ell+1}$. We have an exact sequence of complexes

$$0 \longrightarrow \mathscr{M}'[\ell] \longrightarrow \mathscr{N}^{-\bullet} \longrightarrow \mathscr{M} \longrightarrow 0,$$

where $\mathscr{M}$ is considered as a complex with only one term in degree 0, and $\mathscr{M}'[\ell]$ a complex with only one term in degree $-\ell$. This sequence induces a long exact sequence

$$\cdots \longrightarrow \mathscr{H}^{j+\ell} f_+ \mathscr{M}' \longrightarrow \mathscr{H}^j f_+ \mathscr{N}^{-\bullet} \longrightarrow \mathscr{H}^j f_+ \mathscr{M} \longrightarrow \mathscr{H}^{j+\ell+1} f_+ \mathscr{M}' \longrightarrow \cdots$$

Recall (cf. Remark 4.1.4(1)) that $\mathscr{H}^j(f_\dagger \mathscr{M}) = 0$ for $j \notin [-\dim X, 2\dim X]$. Choose then ℓ big enough so that, for any $j \in [-\dim X, 2\dim X]$, both numbers $j+\ell$ and $j+\ell+1$ do not belong to $[-\dim X, 2\dim X]$. With such a choice, we have $\mathscr{H}^j(f_\dagger \mathscr{M}) \simeq \mathscr{H}^j(f_\dagger \mathscr{N}^{-\bullet})$ for $j \in [-\dim X, 2\dim X]$ and $\mathscr{H}^j(f_\dagger \mathscr{M}) = 0$ otherwise. By the second step, $\mathscr{H}^j(f_\dagger \mathscr{M})$ has a good filtration in some neighbourhood of K. $\square$

4.3. Kashiwara's estimate for the behaviour of the characteristic variety

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module with characteristic variety $\operatorname{Char} \mathscr{M}$. Let $f : X \to Y$ be a holomorphic map and assume that the cohomology modules $\mathscr{H}^j(f_\dagger \mathscr{M})$ are $\mathscr{D}_Y$-coherent (for instance, assume that all conditions in Theorem 4.2.1 are fulfilled). Is it possible to give an upper bound of the characteristic variety of each $\mathscr{H}^j(f_\dagger \mathscr{M})$ in terms of that of $\mathscr{M}$? There is such an estimate which is known as *Kashiwara's estimate*.

The most natural approach to this question is to introduce the sheaf of microdifferential operators and to show that the characteristic variety is nothing but the support of the microlocalized module associated with $\mathscr{M}$. The behaviour of the support of a microdifferential module with respect to direct images is then easy to understand (see, e.g., [1, 2, 21] for such a proof, see [28] for a very general result and [17] for an algebraic approach).

Nevertheless, we will not introduce here microdifferential operators (see however [27] for a good introduction to the subject). Therefore, we will give a direct proof of Kashiwara's estimate.

This estimate may be understood as a weak version of a general Riemann-Roch theorem for $\mathscr{D}_X$-modules (see, e.g., [24] and the references given therein).

Let $f : X \to Y$ be a holomorphic map. We will consider the following associated cotangent diagram:

$$T^*X \xleftarrow{T^*f} f^*T^*Y \xrightarrow{f} T^*Y.$$

***Theorem 4.3.1* (Kashiwara's estimate for the characteristic variety)**

Let $\mathscr{M}$ be a f-good $\mathscr{D}_X$-module such that f is proper on $\operatorname{Supp} \mathscr{M}$. *Then, for any $j \in \mathbb{Z}$, we have*

$$\operatorname{Char} \mathscr{H}^j(f_\dagger \mathscr{M}) \subset f\big((T^*f)^{-1}(\operatorname{Char} \mathscr{M})\big).$$

Exercise 4.3.2. Explain more precisely this estimate when f is the inclusion of a closed submanifold.

Sketch of proof. As in the proof of Theorem 4.2.1, we first reduce to the case where $\mathscr{M}$ has a good filtration $F_\bullet \mathscr{M}$.

Notice first that it is possible to define a functor $\mathscr{H}^k f_+$ for $\mathrm{gr}^F\mathscr{D}_X$-modules, by the formula $\mathscr{H}^k f_+(\bullet) = R^k f_*(\boldsymbol{L}(T^*f)^*(\bullet))$. Moreover, the inverse image $(T^*f)^*$ is nothing but the tensor product $\otimes_{f^{-1}\mathscr{O}_Y}\mathrm{gr}^F\mathscr{D}_Y$. We therefore clearly have the inclusion

$$\operatorname{Supp}\mathscr{H}^k f_+\mathrm{gr}^F\mathscr{M} \subset f\big((T^*f)^{-1}(\operatorname{Supp}\mathrm{gr}^F\mathscr{M})\big) = f\big((T^*f)^{-1}(\operatorname{Char}\mathscr{M})\big).$$

The problem consists now in understanding the difference between $\mathscr{H}^k f_+\mathrm{gr}^F$ and $\mathrm{gr}^F\mathscr{H}^k f_+$. In order to analyse this difference, we will put $\mathscr{M}$ and $\mathrm{gr}^F\mathscr{M}$ in a one parameter family, i.e., we will consider the associated Rees module.

One then defines direct images of $R_F\mathscr{D}_X$-modules, still denoted by $f_\dagger$, and shows the R_F analogue of Theorem 4.2.1. Therefore, $\mathscr{H}^k f_+R_F\mathscr{M}$ is $R_F\mathscr{D}_Y$-coherent. One has to be aware that the cohomology of $\mathscr{H}^k f_+R_F\mathscr{M}$ can have z-torsion, hence does not take the form R_F of something. Nevertheless, as $\mathscr{H}^k f_+R_F\mathscr{M}$ is $R_F\mathscr{D}_Y$-coherent,

– the kernel sequence $\operatorname{Ker}\big[z^\ell : \mathscr{H}^k f_+R_F\mathscr{M} \to \mathscr{H}^k f_+R_F\mathscr{M}\big]$ is locally stationary,

– the quotient of $\mathscr{H}^k f_+R_F\mathscr{M}$ by its z-torsion (i.e., locally by $\operatorname{Ker} z^\ell$ for ℓ big enough) is $R_F\mathscr{D}_Y$-coherent, hence is the Rees module associated with some good filtration $F_\bullet$ on $\mathscr{H}^k f_+\mathscr{M}$. In other words, one has, for ℓ big enough,

$$\mathscr{H}^k f_+R_F\mathscr{M}/\operatorname{Ker} z^\ell = R_F\mathscr{H}^k f_+\mathscr{M}.$$

Consider the exact sequence

$$\cdots \longrightarrow \mathscr{H}^k f_+R_F\mathscr{M} \xrightarrow{z^\ell} \mathscr{H}^k f_+R_F\mathscr{M} \longrightarrow \mathscr{H}^k f_+(R_F\mathscr{M}/z^\ell R_F\mathscr{M}) \longrightarrow \cdots$$

Then,

– $\mathscr{H}^k f_+R_F\mathscr{M}/z^\ell\mathscr{H}^k f_+R_F\mathscr{M}$ is a submodule of $\mathscr{H}^k f_+(R_F\mathscr{M}/z^\ell R_F\mathscr{M})$,

– and, on the other hand, if ℓ is big enough, $R_F\mathscr{H}^k f_+\mathscr{M}/z^\ell R_F\mathscr{H}^k f_+\mathscr{M}$ is a quotient of $\mathscr{H}^k f_+R_F\mathscr{M}/z^\ell\mathscr{H}^k f_+R_F\mathscr{M}$.

Notice now that $\operatorname{Char}\mathscr{M}$ is the support of $R_F\mathscr{M}/z^\ell R_F\mathscr{M} = \oplus_k(F_k\mathscr{M}/F_{k-\ell}\mathscr{M})$ for any $\ell \geqslant 1$. Then, $f\big(T^*f^{-1}(\operatorname{Char}\mathscr{M})\big)$ contains the support of $\mathscr{H}^k f_+(R_F\mathscr{M}/z^\ell R_F\mathscr{M})$, hence that of $R_F\mathscr{H}^k f_+\mathscr{M}/z^\ell R_F\mathscr{H}^k f_+\mathscr{M}$, and therefore that of $\mathrm{gr}^F\mathscr{H}^k f_+\mathscr{M}$. □

4.4. The Gauss-Manin connection

Let $\mathscr{M}$ be a *left* $\mathscr{D}_X$-module and let $f : X \to Y$ be a morphism. On the one hand, one may define the direct images $\mathscr{H}^k f_+\mathscr{M}$ of $\mathscr{M}$ viewed as a $\mathscr{D}_X$-module. These are left $\mathscr{D}_Y$-modules. On the other hand, it is possible, *when f is a smooth morphism*, to define a flat connection, called the *Gauss-Manin connection* on the relative de Rham cohomology of $\mathscr{M}$. We will compare both constructions, when f is smooth. Such a comparison has yet been done *when f is the projection of a product $X = Y \times Z \to Z$* (cf. Example 4.1.2(3) and Remark 4.1.4(3)). The difficulty arises when f is not of this form. In the $\mathscr{D}$-module case, Definition 4.1.3 amounts to decomposing f as

$$\begin{tikzcd} X \arrow[r, hook, "i_f"] \arrow[dr, "f"'] & X \times Y \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where i_f is the graph embedding, and computing $\mathscr{H}^k p_+(i_{f,+}\mathscr{M})$. On the other hand, the Gauss-Manin connection is comuted directly from f.

Let us begin with the Gauss-Manin connection. We assume in this section that $f : X \to Y$ is a *smooth* morphism. We set $n = \dim X$, $m = \dim Y$ and $d = n - m$ (we assume that X and Y are pure dimensional, otherwise one works on each connected component of X and Y).

Consider the *Kozsul filtration* $\mathrm{L}^\bullet$ on the complex $(\Omega_X^{n+\bullet}, (-1)^n d)$, defined by

$$\mathrm{L}^p\,\Omega_X^{n+i} = \mathrm{Im}(f^*\Omega_Y^{m+p} \otimes_{\mathscr{O}_X} \Omega_X^{d+i-p} \longrightarrow \Omega_X^{n+i}).$$

One can check easily that the Kozsul filtration is a decreasing finite filtration and that it is compatible with the differential and that, locally, being in L^p means having at least $m + p$ factors dy_i in any summand.

Then, as f is smooth, we have (by computing with local coordinates adapted to f),

$$\mathrm{gr}_{\mathrm{L}}^p \Omega_X^{n+i} = f^*\Omega_Y^{m+p} \otimes_{\mathscr{O}_X} \Omega_{X/Y}^{d+i-p},$$

where $\Omega_{X/Y}^k$ is the sheaf of relative differential forms: $\Omega_{X/Y}^k = \wedge^k \Omega_{X/Y}^1$ and $\Omega_{X/Y}^1 = \Omega_X^1 \big/ f^*\Omega_Y^1$. Notice that $\Omega_{X/Y}^k$ is $\mathscr{O}_X$-locally free.

Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. We will moreover assume that f is proper when restricted to the support of $\mathscr{M}$. As f is smooth, the sheaf $\mathscr{D}_{X/Y}$ of relative differential operators is well defined and, composing the flat connection $\nabla : \mathscr{M} \to \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{M}$ with the projection $\Omega_X^1 \to \Omega_{X/Y}^1$, we get a relative flat connection $\nabla_{X/Y}$ on $\mathscr{M}$, and thus the structure of a left $\mathscr{D}_{X/Y}$-module on $\mathscr{M}$. In particular, the relative de Rham complex is defined as

$${}^p\mathrm{DR}_{X/Y}\mathscr{M} = (\Omega_{X/Y}^{d+\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}, \nabla_{X/Y}).$$

We have ${}^p\mathrm{DR}_X\mathscr{M} = (\Omega_X^{n+\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}, \nabla)$ (cf. Definition 3.1.1) and the Kozsul filtration $\mathrm{L}^p\,\Omega_X^{n+\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}$ is preserved by the differential ∇ (recall that being in L^p means having at least $m + p$ factors dy_i in any summand). We may therefore induce the filtration $\mathrm{L}^\bullet$ on the complex ${}^p\mathrm{DR}_X\mathscr{M}$. We then have an equality of complexes

$$\mathrm{gr}_{\mathrm{L}}^p {}^p\mathrm{DR}_X\mathscr{M} = f^*\Omega_Y^{m+p} \otimes_{\mathscr{O}_X} {}^p\mathrm{DR}_{X/Y}\mathscr{M}[-p].$$

Notice that the differential of these complexes are $f^{-1}\mathscr{O}_Y$-linear.

We get a spectral sequence (the Leray spectral sequence in the category of sheaves of $\mathbb{C}$-vector spaces, see, e.g., [11]). Using the projection formula for f_* (as f is propoer on $\operatorname{Supp}\mathscr{M}$) and the fact that Ω_Y^{m+p} is $\mathscr{O}_Y$-locally free, one obtains that the E_1 term is given by

$$E_1^{p,q} = \Omega_Y^{m+p} \otimes_{\mathscr{O}_Y} R^q f_* {}^p\mathrm{DR}_{X/Y}\mathscr{M},$$

and the spectral sequence converges to (a suitable graded object associated with) $R^{p+q} f_* {}^p\mathrm{DR}_X\mathscr{M}$.

By definition of the spectral sequence, the differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is the connecting morphism (see Exercise E.4.5) in the long exact sequence associated to the short exact sequence of complexes

$$0 \longrightarrow \mathrm{gr}_{\mathrm{L}}^{p+1}{}^p\mathrm{DR}_X\mathscr{M} \longrightarrow \mathrm{L}^p\, {}^p\mathrm{DR}_X\mathscr{M} \big/ \mathrm{L}^{p+2}\, {}^p\mathrm{DR}_X\mathscr{M} \longrightarrow \mathrm{gr}_{\mathrm{L}}^{p}{}^p\mathrm{DR}_X\mathscr{M} \longrightarrow 0$$

after applying $f_!\,\mathrm{God}^\bullet$ (or $f_*\,\mathrm{God}^\bullet$ if one of the previous properties is satisfied).

***Lemma 4.4.1* (The Gauss-Manin connection).** *The morphism*

$$\nabla^{\mathrm{GM}} \overset{\mathrm{def}}{=} d_1 : R^q f_* {}^p\mathrm{DR}_{X/Y}\mathscr{M} = E_1^{-m,q} \longrightarrow E_1^{-m+1,q} = \Omega_Y^1 \otimes_{\mathscr{O}_Y} R^q f_* {}^p\mathrm{DR}_{X/Y}\mathscr{M}$$

is a flat connection on $R^q f_* {}^p\mathrm{DR}_{X/Y}\mathscr{M}$, called the Gauss-Manin connection *and the complex $(E_1^{\bullet,q}, d_1)$ is equal to the de Rham complex ${}^p\mathrm{DR}_Y(R^q f_* {}^p\mathrm{DR}_{X/Y}\mathscr{M}, \nabla^{\mathrm{GM}})$.*

Sketch of proof of Lemma 4.4.1. We will give the proof in the complex analytic case. In the algebraic case, one should use Čech complexes. One can thus use a resolution with C^∞ differential forms $\mathscr{E}_X^\bullet$. One considers the complex $\mathscr{E}_X^{n+\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}$, with the differential D defined by

$$D(\varphi \otimes m) = (-1)^n d\varphi \otimes m + (-1)^k \varphi \wedge \nabla m,$$

if φ is a local section of $\mathscr{E}_X^{n+k}$ ($k \leqslant 0$). This C^∞ de Rham complex is quasi-isomorphic to the holomorphic one (this is not completely obvious), and is equipped with the Kozsul filtration. The quasi-isomorphism is strict with respect to $\mathrm{L}^\bullet$. One may therefore compute with the C^∞ de Rham complex.

Choose a partition of unity (χ_α) such that f is locally a product on a neighbourhood of $\operatorname{Supp}\chi_\alpha$ for any α.

Let $\eta \wedge (\varphi \otimes m)$ be a section of $\mathscr{E}_Y^{m+p} \otimes f_!(\mathscr{E}_{X/Y}^{d+q} \otimes \mathscr{M})$. In the neighbourhood of $\operatorname{Supp}\chi_\alpha$, we can choose a decomposition $D = D_Y^{(\alpha)} + D_{X/Y}^{(\alpha)}$. As $\sum_\alpha \chi_\alpha \equiv 1$, we have

$$\begin{aligned} d_1[\eta \wedge (\varphi \otimes m)] &= \sum_\alpha \chi_\alpha d_1[\eta \wedge (\varphi \otimes m)] = \sum_\alpha \chi_\alpha D_Y^{(\alpha)}[\eta \wedge (\varphi \otimes m)] \\ &= \sum_\alpha \chi_\alpha \Big[(-1)^m d\eta \wedge (\varphi \otimes m) + (-1)^r \eta \wedge (\varphi \nabla_Y^{(\alpha)} m)\Big], \end{aligned}$$

for a suitable r. One gets the desired result by a local computation. □

Theorem 4.4.2. *Let $f : X \to Y$ be a smooth morphism and let $\mathscr{M}$ be left $\mathscr{D}_X$-module. Assume that f is proper on* $\operatorname{Supp}\mathscr{M}$. *Then there is a functorial isomorphism of left $\mathscr{D}_Y$-modules*

$$R^k f_* {}^p\mathrm{DR}_{X/Y}\mathscr{M} \longrightarrow \mathscr{H}^k f_\dagger \mathscr{M}$$

when one endows the left-hand term with the Gauss-Manin connection ∇^{GM}.

Exercises for Lecture 4

Exercise E.4.1 **($\mathscr{D}_{X\to Y}$ for a closed embedding).** Assume that X is a complex submanifold of Y of codimension d, defined by $g_1 = \cdots = g_d = 0$, where the g_i are holomorphic functions on Y. Show that

$$\mathscr{D}_{X\to Y} = \mathscr{D}_Y \Big/ \textstyle\sum_{i=1}^d g_i \mathscr{D}_Y$$

with its natural right $\mathscr{D}_Y$ structure. In local coordinates $(x_1, \dots, x_n, y_1, \dots, y_d)$ such that $g_i = y_i$, show that $\mathscr{D}_{X\to Y} = \mathscr{D}_X[\partial_{y_1}, \dots, \partial_{y_d}]$.

Conclude that, if f is an embedding, the sheaves $\mathscr{D}_{X\to Y}$ and $\mathscr{D}_{Y\leftarrow X}$ are locally free over $\mathscr{D}_X$.

Exercise E.4.2 **(Filtration of $\mathscr{D}_{X\to Y}$).** Put $F_k\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}F_k\mathscr{D}_X$. Show that this defines a filtration (cf. Definition **??**) of $\mathscr{D}_{X\to Y}$ as a left $\mathscr{D}_X$-module and as a right $f^{-1}\mathscr{D}_Y$-module, and that $\mathrm{gr}^F\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathrm{gr}^F\mathscr{D}_Y$.

Exercise E.4.3 **(The chain rule).** Consider holomorphic maps $f : X \to Y$ and $g : Y \to Z$.

(1) Give an canonical isomorphism $\mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{D}_{Y\to Z} \xrightarrow{\sim} \mathscr{D}_{X\to Z}$ as right $(g\circ f)^{-1}\mathscr{D}_Z$-modules.

(2) Use the chain rule to show that this isomorphism is left $\mathscr{D}_X$-linear.

(3) Same question with filtrations $F_\bullet$.

Exercise E.4.4 **(The (filtered) relative Spencer complex)**

(1) Show that $\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ is a resolution of $\mathscr{D}_{X\to Y}$ as a bimodule.

(2) Show that the terms of the complex $\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ are locally free left $\mathscr{D}_X$-modules. (Hint: use Exercise E.1.12(4).)

(3) Define the filtration of $\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ by the formula

$$F_\ell \,\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X) = \sum_{j+k=\ell} F_j \,\mathrm{Sp}^\bullet_X(\mathscr{D}_X) \underset{f^{-1}\mathscr{O}_Y}{\otimes} F_k\mathscr{D}_{X\to Y},$$

where the filtration on the Spencer complex is defined in Exercise E.3.2. Show that, for any ℓ, $F_\ell \,\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ is a resolution of $F_\ell\mathscr{D}_{X\to Y}$.

Exercise E.4.5 **(The connecting morphism).** Let $0 \to C_1^\bullet \to C_2^\bullet \to C_3^\bullet \to 0$ be an exact sequence of complexes. Let $[\mu] \in H^kC_3^\bullet$ and choose a representative in C_3^k with $d\mu = 0$. Lift μ as $\widetilde{\mu} \in C_2^k$.

(1) Show that $d\widetilde{\mu} \in C_1^{k+1}$ and that its differential is zero, so that the class $[d\widetilde{\mu}] \in H^{k+1}C_1^\bullet$ is well defined.

(2) Show that $\delta : [\mu] \mapsto [d\widetilde{\mu}]$ is a well defined morphism $H^kC_3^\bullet \to H^{k+1}C_1^\bullet$.

(3) Deduce the existence of the cohomology long exact sequence, having δ as its connecting morphism.

LECTURE 5

HIGGS MODULES AND $\mathscr{D}$-MODULES

5.1. Higgs bundles and Higgs modules

5.1.a. Higgs fields. Let V be a vector bundle on X. Recall that a connection is a $\boldsymbol{k}$-linear morphism $\nabla : V \to \Omega_X^1 \otimes V$ satisfying Leibniz rule. A connection is flat if its curvature ∇^2 vanishes identically. Bundles with flat connection (V, ∇) are in one-to-one correspondence with $\mathscr{D}_X$-modules which are $\mathscr{O}_X$-coherent.

Definition 5.1.1. A *Higgs field* θ on V is a $\mathscr{O}_X$-linear morphism $V \to \Omega_X^1 \otimes V$ satisfying the "zero curvature condition" $\theta \wedge \theta = 0$.

A pair (V, θ) of a vector bundle with a Higgs field θ is called a Higgs bundle. For any local vector field ξ on X, we get a $\mathscr{O}_X$-linear morphism $\theta_\xi : V \to V$. The Higgs condition means that, for any two vector fields ξ, η, the endomorphisms θ_ξ and θ_η commute.

Corollary 5.1.2. *There is a one-to-one correspondence between Higgs bundles* (V, θ) *and* $\operatorname{Sym}\Theta_X$*-modules which are* $\mathscr{O}_X$*-locally free.*

Let $\pi : T^*X \to X$ be the cotangent bundle of X. In the algebraic setting, $\operatorname{Sym}\Theta_X = \pi_*\mathscr{O}_{T^*X}$. In the complex analytic setting, a local section of $\operatorname{Sym}\Theta_X$ is a polynomial in the cotangent variables with coefficients in $\mathscr{O}_X$.

Definition 5.1.3. A Higgs module $\mathscr{H}$ is a coherent $\operatorname{Sym}\Theta_X$-module.

The support Σ of a Higgs module is a closed subset of T^*X. If $\pi_{|\Sigma}$ is finite, then $\pi_*\mathscr{H}$ is a coherent $\mathscr{O}_X$-module with a Higgs field. There is a one-to-one correspondence between coherent $\mathscr{O}_X$-modules with a Higgs field and Higgs module whose support Σ is finite over X.

Examples 5.1.4

(1) Let (V, θ) be a Higgs bundle over a curve X. Choose a local coordinate x on the curve. Then the cotangent bundle T^*X is trivialised with the section dx. The characteristic polynomial of the endomorphism θ_{∂_x} defines a curve in T^*X which is finite over X. The local sections of this curve are the eigenvalues of θ_{∂_x}. The curve

is called the spectral curve of θ. It is the support of the Higgs module associated to the Higgs bundle.

(2) Let $\mathscr{M}$ be a $\mathscr{D}_X$-module equipped with a good filtration $F_\bullet\mathscr{M}$. Then $\mathrm{gr}_F\mathscr{M}$ is a Higgs module, its support is the characteritic variety of $\mathscr{M}$. As the characteristic variety is homogeneous with respect to π, it is finite over X if and only if it is equal to the zero section T^*_XX.

(3) Let (V,∇) be a vector bundle with a flat connection. Assume that V has a finite decreasing filtration by sub-bundles $\cdots\subset F^pV\subset F^{p-1}V\subset\cdots$ satisfying the Griffiths' transversality property $\nabla F^pV\subset\Omega^1_X\otimes F^{p-1}V$. The vector bundle $\mathrm{gr}_FV\overset{\mathrm{def}}{=}\oplus_p(F^pV/F^{p+1}V)$ is equipped with the Higgs field $\theta:\mathrm{gr}_FV\to\mathrm{gr}_FV$ induced by ∇. As it is homogeneous of degree -1, it is nilpotent and the support of the corresponding Higgs module is the zero section.

5.1.b. Holonomic Higgs modules. Recall that the cotangent bundle T^*X has a natural symplectic 2-form ω which is exact, being equal to $d\eta$ where η is the Liouville 1-form. Let $Z\subset T^*X$ be a closed irreducible subset. We say that Z is Lagrangian if its smooth part Z^o is Lagrangian, that is, the restriction of ω to Z^o vanishes identically (isotropy) and $\dim Z=\dim X$.

Definition 5.1.5. We say that a Higgs module is holonomic if each irreducible component of its support is Lagrangian.

Examples 5.1.6

(1) Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then the involutivity theorem (cf. §2.2.d) implies that each irreducible component of the characteristic variety $\mathrm{Char}\,\mathscr{M}$ is Lagrangian. In particular, for any good filtration of $\mathscr{M}$, the module $\mathrm{gr}^F\mathscr{M}$ is a holonomic Higgs module.

(2) Let (V,θ) be a Higgs bundle. Its support Σ (as a Higgs module) is finite over X, hence has dimension $\dim X$.

– If $\dim X=1$, then the smooth part Σ^o of Σ is tautologically Lagrangian.

– If $\dim X$ is arbitrary but $\mathrm{rk}\,V=1$, then θ can be any arbitrary 1-form. (V,θ) is holonomic if and only if θ is closed, i.e., $d\theta=1$.

– If X is projective and $\mathrm{rk}\,V$ is arbitrary, then the corresponding Higgs module is holonomic, i.e., Σ is Lagrangian. Indeed, in such a case, Σ is also projective, but possibly singular. Let $f:\widetilde{\Sigma}\to\Sigma$ be a resolution of singularities of Σ: $\widetilde{\Sigma}$ is smooth and projective, and f induces an isomorphism on Zariski dense open sets $\widetilde{\Sigma}^o\to\Sigma^o$. The pull-back $f^*\eta$ of the Liouville form is a holomorphic one-form on $\widetilde{\Sigma}$. By Hodge theory, it is closed, hence $f^*\omega=f^*d\eta=df^*\eta=0$. Restricting to $\widetilde{\Sigma}^o\simeq\Sigma^o$ gives the assertion.

5.2. z-connections and $\mathscr{R}$-modules

5.2.a. The Rees ring of $\mathscr{D}_X$. Given a Higgs module having support of dimension $\dim X$, a way to ensure that it is holonomic is to link it to a holonomic module. Let us consider the case of a $\mathscr{D}_X$-module $\mathscr{M}$ equipped with a good filtration $F_\bullet\mathscr{M}$. The Rees module $R_F\mathscr{M} = \oplus_k F_k\mathscr{M} z^k$ is a module over the Rees ring $R_F\mathscr{D}_X = \oplus_k F_k\mathscr{D}_X z^k$ and $\mathscr{M} = R_F\mathscr{M}/(z-1)R_F\mathscr{M}$, $\mathrm{gr}^F\mathscr{M} = R_F\mathscr{M}/zR_F\mathscr{M}$.

5.2.b. z-connections. Let (V, θ) be a Higgs bundle with $\mathrm{rk}\, V = 1$. The holonomy condition $d\theta = 0$ is equivalent to the integrability condition of the z-connection $zd+\theta$.

Exercises for Lecture 5

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