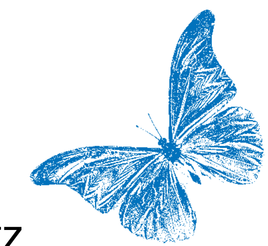
Introduction to the theory of $\mathscr{D}$-modules

Claude Sabbah



Centre de Mathématiques Laurent Schwartz

UMR 7640 du CNRS

École polytechnique, Palaiseau, France

Lecture 1

Basic constructions

1.1. The sheaf of holom. diff. operators

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- P local section of $F_{k+1}\mathscr{D}_X \iff [P, \varphi]$ local section of $F_k\mathscr{D}_X\ \forall \varphi \in \mathscr{O}_X$.

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- The filtration $F_\bullet \mathscr{D}_X$ is exhaustive ($\mathscr{D}_X = \bigcup_k F_k \mathscr{D}_X$) and

$$F_k \mathscr{D}_X \cdot F_\ell \mathscr{D}_X = F_{k+\ell} \mathscr{D}_X.$$

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  - $\Rightarrow \exists\, h \in \mathscr{I}_x$ s.t. $h(x) \neq 0$, $\Rightarrow \mathscr{I}_x = \mathscr{D}_x$.

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  - $\Rightarrow \exists\, h \in \mathscr{I}_x$ s.t. $h(x) \neq 0$, $\Rightarrow \mathscr{I}_x = \mathscr{D}_x$.

$\Rightarrow$ No interesting two-sided $\mathscr{D}_X$-mod.

1.1. The sheaf of holom. diff. operators

DEFINITION 1.1.3 (Integrable connection):

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$\mathscr{M}$: a $\mathscr{O}_X$-module.
A holomorphic connection: a $\mathbb{C}$-linear morphism
$\nabla : \mathscr{M} \to \Omega^1_X \otimes \mathscr{M}$ which satisfies the ***Leibniz rule***:
$\nabla(fm) = df \otimes m + f\nabla m$, $\forall\, m \in \mathscr{M}$ and $\forall\, f \in \mathscr{O}_X$.

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EXAMPLE (EX. E.1.4):
$\exists$ a global section $\nabla(1)$ of $\Omega^1_X \otimes \Theta_X$ loc. defined as $\sum_i dx_i \otimes \partial_{x_i}$.

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$\exists$ a global section $\nabla(1)$ of $\Omega^1_X \otimes \Theta_X$ loc. defined as
$\sum_i dx_i \otimes \partial_{x_i}$.
$\Rightarrow \exists\, \nabla : \mathscr{D}_X \to \Omega^1_X \otimes \mathscr{D}_X$ integrable,

$$P \longmapsto \nabla(1) \cdot P \overset{\text{loc.}}{=} \sum_i dx_i \otimes (\partial_{x_i} \cdot P).$$

1.1. The sheaf of holom. diff. operators

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Giving a ***left*** $\mathscr{D}_X$-module $\mathscr{M}$ $\iff$
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PROOF: $\mathscr{M}$: left $\mathscr{D}_X$-module.

$$\begin{array}{ccccccc} \mathscr{M} & \longrightarrow & \mathscr{D}_X \otimes \mathscr{M} & \longrightarrow & \Omega^1_X \otimes \mathscr{D}_X \otimes \mathscr{M} & \longrightarrow & \Omega^1_X \otimes \mathscr{M} \\ m & \longmapsto & 1 \otimes m & \longmapsto & \nabla(1) \otimes m & \longmapsto & \nabla(1) \cdot m \end{array}$$

1.1. The sheaf of holom. diff. operators

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Conversely: Given $(\mathscr{M}, \nabla)$. Define $\xi \cdot m := \nabla_\xi m$.

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$$\mathscr{M} \to \mathscr{D}_X \otimes \mathscr{M} \to \Omega^1_X \otimes \mathscr{D}_X \otimes \mathscr{M} \to \Omega^1_X \otimes \mathscr{M}$$
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Conversely: Given $(\mathscr{M}, \nabla)$. Define $\xi \cdot m := \nabla_\xi m$.

EXAMPLE:
A holom. vector bdle with integrable holom. connect. (V, ∇) is a left $\mathscr{D}_X$-module.

1.2. Left and right

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LEMMA 1.2.1 (Generating left $\mathscr{D}_X$-modules):

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$\mathscr{M}$: a $\mathscr{O}_X$-module.
$\varphi : \Theta_X \otimes_{\mathbb{C}_X} \mathscr{M} \longrightarrow \mathscr{M}$: a $\mathbb{C}$-linear morphism s.t.,
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Then $\exists$ unique struct. of left $\mathscr{D}_X$-mod. on $\mathscr{M}$ s.t.
$\xi m = \varphi(\xi \otimes m)\ \forall\, \xi, m$.

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E.g., $(\mathscr{M}, \nabla)$, $\quad \varphi(\xi, m) = \nabla_\xi m$.

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EXAMPLE 1.2.3 (Most basic examples):

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$$\boxed{(n \otimes m) \cdot \xi := (n \cdot \xi) \otimes m - n \otimes (\xi \cdot m)}$$

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- $\mathscr{M}$, $\mathscr{N}$ ***right*** $\mathscr{D}_X$-modules.
  Then $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ is a ***left*** $\mathscr{D}_X$-module:

1.2. Left and right

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$$\boxed{(\xi \cdot \varphi)(m) = \varphi(m \cdot \xi) - \varphi(m) \cdot \xi}$$

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DEFINITION 1.2.4 (right$\leftrightarrow$left transformation):

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DEFINITION 1.2.4 (right↔left transformation):

- $\mathscr{M}$ a ***left*** $\mathscr{D}_X$-module. Then $\omega_X \otimes_{\mathscr{O}_X} \mathscr{M}$ is a ***right*** $\mathscr{D}_X$-module.

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- $\mathscr{M}$ a ***left*** $\mathscr{D}_X$-module. Then $\omega_X \otimes_{\mathscr{O}_X} \mathscr{M}$ is a ***right*** $\mathscr{D}_X$-module.
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- left $\longrightarrow$ right $\longrightarrow$ left $= \mathrm{Id}$, right $\longrightarrow$ left $\longrightarrow$ right $= \mathrm{Id}$.

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EXAMPLE (EX. E.1.13):

1.2. Left and right

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- left $\longrightarrow$ right $\longrightarrow$ left $= \mathrm{Id}$, right $\longrightarrow$ left $\longrightarrow$ right $= \mathrm{Id}$.

EXAMPLE (EX. E.1.13):
$\mathscr{N}$ a ***right*** $\mathscr{D}_X$-module. Then $\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ has two commuting right $\mathscr{D}_X$-module structures.

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DEFINITION 1.2.4 (right↔left transformation):

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LEMMA: Both are isomorphic.

1.3. Examples of $\mathscr{D}_X$-modules

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- $X = \mathbb{C}^n, \quad P_j = \sum_\alpha a_{j,\alpha}(x)\partial^\alpha, \quad a_{j,\alpha} \in \mathscr{O}(\mathbb{C}^n)$.
  Left ideal $\mathscr{I} = \sum_j \mathscr{D}_X \cdot P_j \rightsquigarrow$ left $\mathscr{D}_X$-mod. $\mathscr{D}_X/\mathscr{I}$.

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- $\mathscr{L}$: a $\mathscr{O}_X$-mod. Then $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{L}$ is a left $\mathscr{D}_X$-mod.

1.3. Examples of $\mathscr{D}_X$-modules

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- $\mathscr{L}$: a $\mathscr{O}_X$-mod. Then $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{L}$ is a left $\mathscr{D}_X$-mod.
- D: a divisor in X. $\mathscr{O}_X(*D)$: merom. functs with poles along D.
  Then $\mathscr{O}_X(*D)$ is a left $\mathscr{D}_X$-mod.

1.3. Examples of $\mathscr{D}_X$-modules

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- $\mathscr{M}$ a locally free $\mathscr{O}_X(*D)$-module with ***integrable*** connection $\nabla : \mathscr{M} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{M}$ is a left $\mathscr{D}_X$-mod.

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- Twist of the previous: ω a ***closed*** merom. 1-form.
  Then $(\mathscr{M}, \nabla + \omega \otimes \mathrm{Id})$ is a left $\mathscr{D}_X$-module.

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- Twist of the previous: ω a ***closed*** merom. 1-form.
  Then $(\mathscr{M}, \nabla + \omega \otimes \mathrm{Id})$ is a left $\mathscr{D}_X$-module.
- $\mathfrak{Db}_X$: sheaf of distributions on X. This is a left $\mathscr{D}_X$-module and a left $\mathscr{D}_{\overline{X}}$-module.

1.3.g. Solutions of $\mathscr{D}_X$-modules

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DEFINITION 1.3.1: $\mathscr{M}, \mathscr{N}$ left $\mathscr{D}_X$-mod.

1.3.g. Solutions of $\mathscr{D}_X$-modules

DEFINITION 1.3.1: $\mathscr{M}, \mathscr{N}$ left $\mathscr{D}_X$-mod.

Solutions of $\mathscr{M}$ in $\mathscr{N}$: sheaf $\boxed{\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})}$.

1.3.g. Solutions of $\mathscr{D}_X$-modules

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Filtration $F_\bullet\mathscr{N}$ of a right $\mathscr{D}_X$-module $\mathscr{N}$:
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Lecture 2

Coherence and characteristic varieties

2.0. Reminder on coherence

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PROOF: Choose a pres. $\mathscr{D}^q_{X|U} \to \mathscr{D}^p_{X|U} \to \mathscr{M}_{|U} \to 0$.
Show that $F_\bullet \mathscr{M}_{|U} := \mathrm{Im}\, F_\bullet \mathscr{D}^p_{X|U}$ is a good filtr. $\square$

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PROPOSITION 2.2.2:
$\mathscr{M}$ is $\mathscr{D}_X$-coherent $\Rightarrow$ ***locally*** $\exists$ good filtr.

PROOF: Choose a pres. $\mathscr{D}^q_{X|U} \to \mathscr{D}^p_{X|U} \to \mathscr{M}_{|U} \to 0$.
Show that $F_\bullet \mathscr{M}_{|U} := \mathrm{Im}\, F_\bullet \mathscr{D}^p_{X|U}$ is a good filtr. $\square$

ARTIN-REES THEOREM (Cor. 2.2.8 and Ex. E.2.6):
Short exact seq. $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$,
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$$F_\bullet\mathscr{M}' := \mathscr{M}' \cap F_\bullet\mathscr{M} \quad \text{and} \quad F_\bullet\mathscr{M}'' := \mathrm{Im}\, F_\bullet\mathscr{M}$$

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$\Rightarrow m = m_1 + \partial_{x_n} m_2$ with $x_n^\ell m_1 = x_n^\ell m_2 = 0$. □

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$\rightsquigarrow \forall\,(x,\xi) \in T^*X$, nondeg. sympl. form on $T_{(x,\xi)}(T^*X)$.

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- $\Rightarrow i^+\mathscr{M}$ is $\mathscr{D}_Y$-coherent.
- With this good filtr., $\mathrm{Char}\, i^+\mathscr{M} \subset \varpi(\mathrm{Char}\,\mathscr{M}_{|Y})$.

Lecture 3

Direct images of $\mathscr{D}$-modules

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PROPOSITION 3.1.1:

$$\exists\, 0 \neq q(t, \partial_t) \in \mathbb{C}[t]\langle \partial_t \rangle, \quad \boxed{q(t, \partial_t) \cdot I(t) = 0}$$

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3.1. Comput. of a Gauss-Manin diff. eq.

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3.1. Comput. of a Gauss-Manin diff. eq.

LEMMA 3.1.4: $\mathrm{GM}^k(f) := H^k(\Omega^\bullet[\tau], d - \tau df\wedge)$:
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PROOF: $p = \sum_{i=1}^{d} \partial_t^i a_i(t)$.

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PROOF: $p = \sum_{i=1}^d \partial_t^i a_i(t)$. $\qquad p\cdot[\omega] = 0 \iff$

$$\exists\,\eta = \sum_{j=0}^{d+k}\eta_j\tau^j \in \Omega^{n-1}[\tau], \quad \sum_i (a_i\circ f)\omega\tau^i = (d - \tau df\wedge)\eta.$$

3.1. Comput. of a Gauss-Manin diff. eq.

$$
\begin{aligned}
0 &= \phantom{d\eta_{d+k}} - df \wedge \eta_{d+k} \\
0 &= d\eta_{d+k} - df \wedge \eta_{d+k-1} \\
&\vdots \\
0 &= d\eta_{d+1} - df \wedge \eta_d \\
a_d \circ f \cdot \omega &= d\eta_d - df \wedge \eta_{d-1} \\
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3.1. Comput. of a Gauss-Manin diff. eq.

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Stokes $\Rightarrow$ $\qquad \equiv - \displaystyle\int_{\mathbb{T}^n} \frac{d\eta_{d-1}}{f-t}$

$$\left.\begin{array}{c} a_{d-1} \circ f \cdot \omega \\ = \\ d\eta_{d-1} - df \wedge \eta_{d-2} \end{array}\right\} \Rightarrow \equiv -a_{d-1}(t) \int_{\mathbb{T}^n} \frac{\omega}{f-t} - \int_{\mathbb{T}^n} \frac{df \wedge \eta_{d-2}}{f-t}$$

3.1. Comput. of a Gauss-Manin diff. eq.

$$\Longrightarrow (\partial_t a_d(t) + a_{d-1}(t)) \int_{\mathbb{T}^n} \frac{\omega}{f - t} \equiv - \int_{\mathbb{T}^n} \frac{df \wedge \eta_{d-2}}{f - t}$$

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and iterate... to get $p(t, \partial_t) \int_{\mathbb{T}^n} \frac{\omega}{f-t} \in \mathbb{C}[t]$.

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3.2. Transfer modules

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$$\xi \cdot (m \otimes n) = (\xi m) \otimes n + m \otimes Tf(\xi)n, \quad \forall \xi \in \Theta_X.$$

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DEFINITION 3.2.1: $\mathscr{D}_{X\to Y} := \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$:
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EXAMPLE 3.2.2: $f : X \to \mathbb{C}$, t: coord. on $\mathbb{C}$. Then
$\mathscr{D}_{X \to \mathbb{C}} = \mathscr{O}_X[\partial_t]$. $\quad \partial_{x_i} \cdot 1 = \dfrac{\partial f}{\partial x_i}, \quad 1 \cdot g(t) = g(f)$.

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LEMMA 3.3.1: The complex

$$\mathrm{Sp}^{\bullet}_{X\to Y}(\mathscr{D}_X) := \mathrm{Sp}(\mathscr{D}_X) \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y}$$

is a resolution of $\mathscr{D}_{X\to Y}$ as a bimodule by locally free left $\mathscr{D}_X$-modules.

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$$\mathrm{Sp}(\mathscr{D}_X) = \{0 \to \mathscr{D}_X \otimes \wedge^n \Theta_X \xrightarrow{\delta} \cdots \mathscr{D}_X \otimes \Theta_X \to \mathscr{D}_X \to 0\}$$

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$$f_+\mathscr{N} = Rf_*\Big(\mathscr{N} \otimes_{\mathscr{D}_X} \mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)\Big).$$

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EXAMPLE: In the alg. setting,
$(\mathbb{C}^*)^n = \mathrm{Spec}\,\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $f : (\mathbb{C}^*)^n \to \mathbb{C}$.

$$f_+\mathscr{O}_{(\mathbb{C}^*)^n} = (\Omega^\bullet_X[\partial_t], d - \partial_t \cdot df\wedge)$$

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EXAMPLE:

- $\mathscr{M}$ holonomic (see later) $\Rightarrow$ $\mathscr{N}$ f-good for any f.

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  Artin-Rees $\Rightarrow$ can continue $\Rightarrow$ infinite resol.
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  - Set $\mathscr{N}' = \mathrm{Ker}[\mathscr{L}_{-\ell} \otimes \mathscr{D}_X \to \mathscr{L}_{-\ell+1} \otimes \mathscr{D}_X]$.

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  - $\Rightarrow$ long exact seq.

$$\cdots \longrightarrow \mathscr{H}^{j+\ell}(f_+\mathscr{N}') \longrightarrow \mathscr{H}^j(f_+\mathscr{N}_\bullet) \longrightarrow \mathscr{H}^j(f_+\mathscr{N}) \longrightarrow \mathscr{H}^{j+\ell+1}(f_+\mathscr{N}') \longrightarrow \cdots$$

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  - $\mathscr{H}^j(f_+\mathscr{N}) \neq 0 \Rightarrow \mathscr{H}^j(f_+\mathscr{N}_\bullet) \xrightarrow{\sim} \mathscr{H}^j(f_+\mathscr{N})$.

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  - $\mathscr{H}^j(f_+\mathscr{N}) \neq 0 \Rightarrow \mathscr{H}^j(f_+\mathscr{N}_\bullet) \xrightarrow{\sim} \mathscr{H}^j(f_+\mathscr{N})$.
  - Apply Step 2. □

3.5. Direct images and char. varieties

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X, Y cplx mfld, $f : X \to Y$ holom. map. $\mathscr{M}$: f-good $\mathscr{D}_X$-mod., f proper on $\operatorname{Supp} \mathscr{M}$.

PROBLEM: To compute (or estimate) $\operatorname{Char} \mathscr{H}^j(f_+\mathscr{M})$ from $\operatorname{Char} \mathscr{M}$.

3.5. Direct images and char. varieties

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$$\Rightarrow \mathrm{Supp}\,\mathscr{H}^j f_+\,\mathrm{gr}^F\mathscr{M} \subset \widetilde{f}\big((T^*f)^{-1}(\mathrm{Supp}\,\mathrm{gr}^F\mathscr{M})\big)$$
$$= \widetilde{f}\big((T^*f)^{-1}(\mathrm{Char}\,\mathscr{M})\big).$$

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- $\Rightarrow \mathrm{Supp}\, \mathrm{gr}^F_{[\ell]} \mathscr{H}^j(f_+\mathscr{M}) \subset \mathrm{Supp}\, \mathscr{H}^j(f_+ \mathrm{gr}^F_{[\ell]} \mathscr{M})$.
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- $\Rightarrow$ $\mathrm{Supp}\,\mathrm{gr}^F_{[\ell]}\,\mathscr{H}^j(f_+\mathscr{M}) \subset \mathrm{Supp}\,\mathscr{H}^j(f_+\,\mathrm{gr}^F_{[\ell]}\,\mathscr{M})$.
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Lecture 4

Holonomic $\mathscr{D}$-modules

4.1. Motivation: division of distributions

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- Assume: $\exists\, P \in \mathbb{C}[s][x]\langle \partial_x \rangle, Q \in \mathbb{C}[s][\overline{x}]\langle \partial_{\overline{x}} \rangle$,
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  - (Set $\psi = Q^* P^* \varphi$) $\Rightarrow$

$$b''(s)b'(s) \int |f|^{2s}\varphi = b''(s) \int |f|^{2s} f P^* \varphi$$

$$= \int (Q|f|^{2s}\overline{f}) f P^* \varphi = \int |f|^{2(s+1)} \psi$$

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- Iterate this process $\Rightarrow$ merom. ext.

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SOLUTION (contd): $\exists\, P, b'$ s.t. $b'(s)|f|^{2s} = P(|f|^{2s} f)$?

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Holonomy related to ***finite length***.

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COROLLARY 4.2.6: Decreasing seq. $\mathscr{M}_1 \supset \mathscr{M}_2 \supset \cdots$ of holon. $\mathscr{D}_X$-mod is ***locally stationary***.

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$$0 \longrightarrow \mathscr{M}' \longrightarrow \mathscr{M} \longrightarrow \mathscr{M}'' \longrightarrow 0$$

$\Rightarrow$ $\mathscr{M}$ holon. $\iff$ $\mathscr{M}'$ and $\mathscr{M}''$ holon.

PROOF: Loc. good filtr. $F_\bullet\mathscr{M}$. Induces good filtr. $F_\bullet\mathscr{M}', F_\bullet\mathscr{M}''$ $\Rightarrow$ Exact. seq.

$$0 \longrightarrow \mathrm{gr}^F\mathscr{M}' \longrightarrow \mathrm{gr}^F\mathscr{M} \longrightarrow \mathrm{gr}^F\mathscr{M}'' \longrightarrow 0$$

$$\Longrightarrow \mathrm{Char}\,\mathscr{M} = \mathrm{Char}\,\mathscr{M}' \cup \mathrm{Char}\,\mathscr{M}''.$$

RMK: Can define char. ***cycle***: $\mathrm{CChar}\,\mathscr{M} = \sum_\alpha \nu_\alpha T^*_{Y_\alpha} X$.

$$\Longrightarrow \mathrm{CChar}\,\mathscr{M} = \mathrm{CChar}\,\mathscr{M}' + \mathrm{CChar}\,\mathscr{M}''.$$

COROLLARY 4.2.6: Decreasing seq. $\mathscr{M}_1 \supset \mathscr{M}_2 \supset \cdots$ of holon. $\mathscr{D}_X$-mod is ***locally stationary***.
$\Rightarrow \exists$ ***Jordan-Hölder*** seq. for each holon. $\mathscr{M}$.

4.3. Vector bdles with integr. connection

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$\mathscr{E}$: locally free $\mathscr{O}_X$-mod., finite rank r,

$+$ ***integrable*** connect. $\nabla : \mathscr{E} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{E} \quad (\nabla^2 = 0)$.

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LEMMA 4.3.1 (Cauchy-Kowalevski): $\forall\, x \in X,\ \exists\, \mathrm{nb}(x)$ and local basis $e_1, \dots, e_r \in \mathscr{E}(\mathrm{nb}(x))$ s.t. $\nabla(e_i) = 0$:

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Converse:

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$\Rightarrow \mathrm{Char}(\mathscr{E}, \nabla) = T^*_X X \quad \Rightarrow (\mathscr{E}, \nabla)$ holon.
Converse: $\mathscr{M}$ coh. and $\mathrm{Char}\,\mathscr{M} = T^*_X X \Rightarrow \mathscr{M} = (\mathscr{E}, \nabla)$.

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Converse: $\mathscr{M}$ coh. and $\mathrm{Char}\,\mathscr{M} = T^*_X X \Rightarrow \mathscr{M} = (\mathscr{E}, \nabla)$.

$D \subset X$, $\mathrm{codim}\, D = 1$. $\nabla : \mathscr{E} \longrightarrow \Omega^1_X(*D) \otimes \mathscr{E}$ merom. integr. conn. $\Rightarrow (\mathscr{E}(*D), \nabla)$ left $\mathscr{D}_X$-mod.

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PROBLEM: Is $(\mathscr{E}(*D), \nabla)$ coh.? holon.? Char. variety?

4.3. Vector bdles with integr. connection

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RECALL: ***OK*** if D non-singular.

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PROBLEM: Is $(\mathscr{E}(*D), \nabla)$ coh.? holon.? Char. variety?

THEOREM 4.3.2: $\mathscr{M}$: coherent $\mathscr{D}_X$-mod.

$$\boxed{\mathscr{M}_{|X \smallsetminus D} \text{ holon.} \Rightarrow \mathscr{M}(*D) \text{ holon. (hence } \mathscr{D}_X\text{-coh.)}}$$

4.3. Vector bdles with integr. connection

COROLLARY 4.3.3: $(\mathscr{E}(*D), \nabla)$ is holonomic.

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PROOF:

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COROLLARY 4.3.3: $(\mathscr{E}(*D), \nabla)$ is holonomic.

PROOF: $\mathscr{E} \subset \mathscr{E}(*D)$.

- $\mathscr{M} := \mathscr{D}_X \cdot \mathscr{E} \subset \mathscr{E}(*D)$ with filtr. $F_k\mathscr{M} = F_k\mathscr{D}_X \cdot \mathscr{E}$.

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COROLLARY 4.3.3: $(\mathscr{E}(*D), \nabla)$ is holonomic.

PROOF: $\mathscr{E} \subset \mathscr{E}(*D)$.

- $\mathscr{M} := \mathscr{D}_X \cdot \mathscr{E} \subset \mathscr{E}(*D)$ with filtr. $F_k\mathscr{M} = F_k\mathscr{D}_X \cdot \mathscr{E}$.
- $F_\bullet\mathscr{M}$ ***good***

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COROLLARY 4.3.3: $(\mathscr{E}(*D), \nabla)$ is holonomic.

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- $F_\bullet\mathscr{M}$ ***good***
- $\Rightarrow \mathscr{M}$ coh. $\mathscr{D}_X$-mod.

4.3. Vector bdles with integr. connection

COROLLARY 4.3.3: $(\mathscr{E}(*D),\nabla)$ is holonomic.

PROOF: $\mathscr{E}\subset\mathscr{E}(*D)$.

- $\mathscr{M}:=\mathscr{D}_X\cdot\mathscr{E}\subset\mathscr{E}(*D)$ with filtr. $F_k\mathscr{M}=F_k\mathscr{D}_X\cdot\mathscr{E}$.
- $F_\bullet\mathscr{M}$ ***good***
- $\Rightarrow\mathscr{M}$ coh. $\mathscr{D}_X$-mod.
- $\mathscr{M}_{|X\smallsetminus D}=(\mathscr{E}_{|X\smallsetminus D},\nabla)$ holonomic.

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COROLLARY 4.3.3: $(\mathscr{E}(*D), \nabla)$ is holonomic.

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- $F_\bullet\mathscr{M}$ ***good***
- $\Rightarrow \mathscr{M}$ coh. $\mathscr{D}_X$-mod.
- $\mathscr{M}_{|X \smallsetminus D} = (\mathscr{E}_{|X \smallsetminus D}, \nabla)$ holonomic.
- Thm 4.3.2 $\Rightarrow \mathscr{M}(*D)$ holonomic

4.3. Vector bdles with integr. connection

COROLLARY 4.3.3: $(\mathscr{E}(*D), \nabla)$ is holonomic.

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- $\Rightarrow \mathscr{M}$ coh. $\mathscr{D}_X$-mod.
- $\mathscr{M}_{|X \setminus D} = (\mathscr{E}_{|X \setminus D}, \nabla)$ holonomic.
- Thm 4.3.2 $\Rightarrow \mathscr{M}(*D)$ holonomic
- $\mathscr{E} \subset \mathscr{M} \subset \mathscr{E}(*D) \Rightarrow \mathscr{M}(*D) = \mathscr{E}(*D)$. $\square$

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- $\mathscr{E} \subset \mathscr{M} \subset \mathscr{E}(*D) \Rightarrow \mathscr{M}(*D) = \mathscr{E}(*D)$. $\square$

CONVERSE (Th. 4.3.4): $\mathscr{M}$: holon. $\mathscr{D}_X$-mod.

4.3. Vector bdles with integr. connection

COROLLARY 4.3.3: $(\mathscr{E}(*D), \nabla)$ is holonomic.

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- $\mathscr{M} := \mathscr{D}_X \cdot \mathscr{E} \subset \mathscr{E}(*D)$ with filtr. $F_k\mathscr{M} = F_k\mathscr{D}_X \cdot \mathscr{E}$.
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CONVERSE (Th. 4.3.4): $\mathscr{M}$: holon. $\mathscr{D}_X$-mod.

- $\mathrm{Char}\,\mathscr{M}_{|X \smallsetminus D} = T^*_{X \smallsetminus D}(X \smallsetminus D) \Rightarrow$
  $\exists\,(\mathscr{E}, \nabla)$ coh. $\mathscr{O}_X$-mod. s.t. $\mathscr{M}(*D) = (\mathscr{E}(*D), \nabla)$.

4.3. Vector bdles with integr. connection

COROLLARY 4.3.3: $(\mathscr{E}(*D), \nabla)$ is holonomic.

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- $\operatorname{Char} \mathscr{M}_{|X \setminus D} = T^*_{X \setminus D}(X \setminus D) \Rightarrow$
  $\exists\, (\mathscr{E}, \nabla)$ coh. $\mathscr{O}_X$-mod. s.t. $\mathscr{M}(*D) = (\mathscr{E}(*D), \nabla)$.
- In general, $\mathscr{M}$ has a globally defined good filtration.

4.4. Direct images of holon. $\mathscr{D}_X$-mod.

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THEOREM 4.4.1: X, Y: cplx mflds, $f : X \to Y$: proper, $\mathscr{M}$ holonomic $\Rightarrow \mathscr{H}^j(f_+\mathscr{M})$ holonomic $\forall j$.

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PROOF:

- $\exists$ global good filtr. $F_\bullet\mathscr{M} \Rightarrow \mathscr{M}$ f-good.
- $\Rightarrow \mathscr{H}^j(f_+\mathscr{M})$ coh. and $\mathrm{Char}(\mathscr{H}^j(f_+\mathscr{M})) \subset \widetilde{f}\big((T^*f)^{-1}\,\mathrm{Char}\,\mathscr{M}\big)$.

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- LEMMA 4.4.2: $Z \subset X$: closed anal. set, $\Lambda := T^*_Z X$. Then each irreducible component of $\widetilde{f}\big((T^*f)^{-1}\Lambda\big)$ is isotropic in T^*Y.

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- $\exists$ global good filtr. $F_\bullet\mathscr{M} \Rightarrow \mathscr{M}$ f-good.
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- $\exists$ global good filtr. $F_\bullet\mathscr{M} \Rightarrow \mathscr{M}$ f-good.
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- $\Rightarrow$ each irred. compon. of $\mathrm{Char}(\mathscr{H}^j(f_+\mathscr{M}))$ isotropic.
- $\Rightarrow \dim \mathrm{Char}(\mathscr{H}^j(f_+\mathscr{M})) \leqslant n$ (hence $= n$). $\square$

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PROOF: $\underset{x}{f : X} \overset{i_f}{\hookrightarrow} \underset{(x, f(x))}{Z := X \times Y} \overset{p}{\longrightarrow} \underset{f(x)}{Y}$.

4.4. Direct images of holon. $\mathscr{D}_X$-mod.

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PROOF: $f : \underset{x}{X} \xrightarrow{i_f} Z := \underset{(x,f(x))}{X \times Y} \xrightarrow{p} \underset{f(x)}{Y}$.

Lemma for $f =$ inclusion: easy. Consider $f = p$.

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- $\Lambda' \subset \Lambda$ closed anal. in T^*X.
  Λ isotropic $\Rightarrow$ Λ' isotropic (Whitney's thm).

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- $\Lambda' \subset \Lambda$ closed anal. in T^*X.
  Λ isotropic $\Rightarrow \Lambda'$ isotropic (Whitney's thm).
- Basic diagram:

$$\begin{array}{ccccc} T^*Z & \xleftarrow{T^*p =: \rho} & X \times T^*Y & \xrightarrow{\widetilde{p}} & T^*Y \\ \omega_Z & & \rho^*\omega_Z = \widetilde{p}^*\omega_Y & & \omega_Y \end{array}$$

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  Λ isotropic $\Rightarrow \Lambda'$ isotropic (Whitney's thm).
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$$\begin{array}{ccccc} T^*Z & \xleftarrow{T^*p =: \rho} & X \times T^*Y & \xrightarrow{\widetilde{p}} & T^*Y \\ \omega_Z & & \rho^*\omega_Z = \widetilde{p}^*\omega_Y & & \omega_Y \end{array}$$

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LEMMA 4.4.2: $Z \subset X$: closed anal. set, $\Lambda := T^*_Z X$.
Then each irreducible component of $\widetilde{f}\big((T^*f)^{-1}\Lambda\big)$ is isotropic in T^*Y.

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  - Corollary: Any holon. $\mathscr{M}$ loc. $\subset \mathfrak{Db}_X$.

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  - $\rightsquigarrow$ Wild pure twistor $\mathscr{D}$-modules (Mochizuki, C.S.).