Claude Sabbah

FOURIER TRANSFORMATION AND STOKES STRUCTURES

CHICAGO, MAY 2012

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C. Sabbah
UMR 7640 du CNRS, Centre de Mathématiques Laurent Schwartz,
École polytechnique, F–91128 Palaiseau cedex, France.
E-mail : sabbah@math.polytechnique.fr
Url : http://www.math.polytechnique.fr/~sabbah

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INTRODUCTION

In his introductory text to [**DMR07**], Deligne explains his understanding of some analogies between the ℓ-adic theory in characteristic p and the irregular singularities of holonomic $\mathscr{D}$-modules. Some of these analogies will be our guides for these lectures.

Much work has been realized in order to prove analogues in the theory of holonomic $\mathscr{D}$-modules to the fundamental results proved by Katz and Laumon in [**KL85**] and by Laumon in [**Lau87**], as well as subsequent work of Katz [**Kat90a, Kat96**], all in the ℓ-adic theory. Let us mention in particular [**Kat90b**], and [**BE04, GL04**] for the product formula of [**Lau87**].

Although our point of view in these notes is mainly from the $\mathscr{D}$-module theory, we will keep the link with the ℓ-adic theory.

The Fourier transformation from the $\mathscr{D}$-module theoretic point of view has been considered with details in [**Mal91**]. Here, we will also focus on questions firstly raised in the ℓ-adic setting, like the question of purity.

LECTURE 1

FOURIER TRANSFORMATION: SOME ANALOGIES BETWEEN ARITHMETIC AND COMPLEX ALGEBRAIC GEOMETRY

1.1. Exponential sums and exponential integrals

1.1.a. Exponential sums (see [Kat90b]). Given a polynomial $f \in \mathbb{Z}[x_1, \dots, x_n]$ with integral coefficients and a prime number p, we regard the reduction of f mod p as a morphism $\mathbb{A}^n_{\mathbb{F}_p} \to \mathbb{A}^1_{\mathbb{F}_p}$ and we consider the counting function

$$\begin{aligned} \operatorname{Sol}(f, p, \bullet) : \mathbb{A}^1_{\mathbb{F}_p} &\longrightarrow \mathbb{Z} \\ t &\longmapsto \# f^{-1}(t). \end{aligned}$$

Let $\psi_p : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}^\times$ be the additive character defined by $\psi_p(t) = \exp(2\pi i t/p)$, so that any additive character $\psi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}^\times$ takes the form $\psi(t) = \psi_p(\tau t)$ for a unique $\tau \in \mathbb{Z}/p\mathbb{Z}$.

Giving the function $\operatorname{Sol}(f, p, \bullet)$ is equivalent to giving its Fourier transform

$$\psi \longmapsto \sum_{t \in \mathbb{A}^1_{\mathbb{F}_p}} \psi(t) \operatorname{Sol}(f, p, t) = \sum_{\boldsymbol{x} \in \mathbb{A}^n_{\mathbb{F}_p}} \psi(f(\boldsymbol{x})),$$

and if we regard ψ as corresponding to $\tau \in \widehat{\mathbb{A}}^1_{\mathbb{F}_p}$, this is the function

$$\begin{aligned} \widehat{\operatorname{Sol}}(f, p, \bullet) : \widehat{\mathbb{A}}^1_{\mathbb{F}_p} &\longrightarrow \mathbb{C}^\times \\ \tau &\longmapsto \sum_{\boldsymbol{x} \in \mathbb{A}^n_{\mathbb{F}_p}} \exp(2\pi i \tau f(\boldsymbol{x})/p). \end{aligned}$$

1.1.b. Exponential integrals. In order to avoid convergence questions, I will use a slightly different setting. Let Y be a smooth complex projective variety of dimension n. Assume that Y is Calabi-Yau, e.g. Y is the quintic Fermat hypersurface in $\mathbb{P}^4$. Then there exists a global holomorphic volume form ω. Let $f : X \to \mathbb{P}^1$ be a rational function defined on some blow-up of X of Y. I will also denote by ω the pull-back of ω to X. I will consider the integration along the fibres of f of the form $\omega \wedge \overline{\omega}$, which has degree (n, n). This is the push-forward $f_*(\omega \wedge \overline{\omega})$ regarded as a $(1,1)$-current on $\mathbb{P}^1$:

$$C^\infty(\mathbb{P}^1) \ni \varphi \longmapsto \int_X \varphi \circ f \cdot \omega \wedge \overline{\omega}.$$

Let us fix an affine chart $\mathbb{A}^1$ of $\mathbb{P}^1$ with coordinate t. Then this current defines a temperate distribution on $\mathbb{A}^1$, i.e., belongs to the Schwartz space $\mathscr{S}'(\mathbb{A}^1)$ (by fixing the $(1,1)$-form $\frac{i}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar{t}$). It is then completely determined by its Fourier transform. In order to keep clear the holomorphic aspect, I will write the Fourier kernel as

$$\exp(t\tau-\overline{t\tau})=\exp 2i\,\mathrm{Im}(t\tau)=\exp 2i(x\eta+y\xi),\quad t=x+iy,\ \tau=\xi+i\eta.$$

The Fourier transform $F_t(f_*(\omega\wedge\overline{\omega}))$ is a temperate distribution on the affine line $\widehat{\mathbb{A}^1}$ with coordinate τ. In the present setting, it is the continuous function defined as

$$\tau\longmapsto\int_{X_0}e^{\tau f-\overline{\tau f}}\cdot\omega\wedge\overline{\omega},\qquad X_0=f^{-1}(\mathbb{A}^1).$$

The following relations hods in the sense of temperate distributions:

$$\partial_\tau F_t(f_*(\omega\wedge\overline{\omega}))=F_t(t\cdot f_*(\omega\wedge\overline{\omega})),\quad \tau\cdot F_t(f_*(\omega\wedge\overline{\omega}))=-F_t(\partial_t f_*(\omega\wedge\overline{\omega})).$$

1.2. $\overline{\mathbb{Q}}_\ell$ sheaves and holonomic $\mathscr{D}_X$-modules

1.2.a. Constructible $\overline{\mathbb{Q}}_\ell$ sheaves. Let ℓ be a prime number $\neq p$. We now work with $\overline{\mathbb{Q}}_\ell$-valued functions instead of $\mathbb{Z}$-valued or $\mathbb{C}^\times$-valued functions. We have a push-forward morphism $f_!$ (see [**Lau87**, (1.1.1)]) for such functions. The function $\mathrm{Sol}(f,p,t)$ is now replaced by $f_!(1_{\mathbb{A}^n_{\mathbb{F}p}})$.

Given a $\overline{\mathbb{Q}}_\ell$-sheaf K on $\mathbb{A}^n_{\mathbb{F}p}$, the function t_K (trace of Frobenius on the germs of K at geometric points of $\mathbb{A}^n_{\mathbb{F}p}$) satisfies

$$f_!t_K=t_{\boldsymbol{R}f_!K}.$$

Therefore, $f_!(1_{\mathbb{A}^n_{\mathbb{F}p}})$ can be regarded as the Frobenius trace function of $\boldsymbol{R}f_!\overline{\mathbb{Q}}_\ell$.

Now, the $\overline{\mathbb{Q}}_\ell$-analogue of the Fourier transform of $f_!(1_{\mathbb{A}^n_{\mathbb{F}p}})$, that is, of $\widehat{\mathrm{Sol}}(f,p,\bullet)$, is, up to sign (see [**Lau87**, Th. 1.2.1.2]) the Frobenius trace function of the Fourier-Deligne transform of $\boldsymbol{R}f_!\overline{\mathbb{Q}}_\ell$ with respect to the character ψ_p.

For this reason, one focuses on the properties of $\overline{\mathbb{Q}}_\ell$-sheaves on the affine line $\mathbb{A}^1_{\mathbb{F}_p}$, and their Fourier-Deligne transform.

1.2.b. Holonomic $\mathscr{D}$-modules. Let us come back to the setting of §1.1.b. We denote by $\mathbb{C}[t]\langle\partial_t\rangle$ the *Weyl algebra* in one variable, consisting of linear differential operators in one variable t with polynomial coefficients.

Proposition 1. *The distribution $f_*(\omega\wedge\overline{\omega})$ satisfies a differential equation (in the distributional sense) $P(t,\partial_t)f_*(\omega\wedge\overline{\omega})=0$ for some nonzero $P\in\mathbb{C}[t]\langle\partial_t\rangle$.*

Proof. Let us recall the definition of Gauss-Manin system of $f:X_0\to\mathbb{A}^1$ and the Picard-Fuchs differential equation for ω. Let τ be a new variable. Consider the twisted de Rham complex $(\Omega^\bullet_{X_0}[\tau],\mathrm{d}-\tau\mathrm{d}f)$, where $\Omega^k_{X_0}[\tau]$ is the sheaf of polynomials in the new variable τ with coefficients in $\Omega^k_{X_0}$. The differential $\mathrm{d} : \Omega^k_{X_0} \to \Omega^{k+1}_{X_0}$ gives rise to a twisted differential

$$\nabla = \mathrm{d} - \tau \mathrm{d}f\wedge : \Omega^k_{X_0}[\tau] \longrightarrow \Omega^{k+1}_{X_0}[\tau],$$

and it is easily checked that $(\mathrm{d} - \tau\mathrm{d}f\wedge)^2 = 0$, that is, $(\Omega^\bullet[\tau], \mathrm{d} - \tau\mathrm{d}f\wedge)$ is a complex.

Lemma 2. *The following operations*

$$\begin{aligned}\partial_t \cdot \left(\textstyle\sum \eta_i \tau^i\right) &= \textstyle\sum \eta_i \tau^{i+1}\\ t \cdot \left(\textstyle\sum \eta_i \tau^i\right) &= \textstyle\sum (f\eta_i - (i+1)\eta_{i+1})\tau^i\end{aligned}$$

define an action of the Weyl algebra $\mathbb{C}[t]\langle\partial_t\rangle$ on each $\Omega^k_{X_0}[\tau]$ which commutes with ∇. □

As a consequence, the cohomology modules $\mathscr{H}^k\big(\boldsymbol{R}f_*(\Omega^\bullet_{X_0}[\tau], \mathrm{d} - \tau\mathrm{d}f\wedge)\big)$ are equipped with the structure of a $\mathbb{C}[t]\langle\partial_t\rangle$-module. We denote them $\mathrm{GM}^k(f)$. The following is well-known (see [**Bor87**]).

Bernstein's theorem. *Each non-zero element of $\mathrm{GM}^k(f)$ is annihilated by a non-zero element of $\mathbb{C}[t]\langle\partial_t\rangle$, i.e., each $\mathrm{GM}^k(f)$ is a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module.* □

Lemma 3. *The algebraic form $\omega \in \Omega^n(X_0)$ defines a class $[\omega]$ in $\mathrm{GM}^n(f)$.*

Proof. Let us choose an affine covering $(U_i)_{i\in I}$ of X_0 and realize $\boldsymbol{R}f_*(\Omega^\bullet_{X_0}[\tau], \mathrm{d} - \tau\mathrm{d}f\wedge)$ as the simple complex attached to the double complex $(\Omega^\bullet_{X_0}[\tau](U_{J\subset I}), \mathrm{d} - \tau\mathrm{d}f\wedge, \delta)$, where δ is the Čech differential and $U_J = \bigcap_{i\in J} U_i$. Then ω has type $(n, 0)$ in this double complex, and both differentials vanish on ω. □

Let $P(t, \partial_t)$ the Picard-Fuchs operator of ω, i.e., the nonzero operator of minimal degree in ∂_t annihilating $[\omega]$, which exists according to Bernstein's theorem. The assertion of the proposition follows from the following claim: we have $P(t, \partial_t) f_*(\omega \wedge \overline{\omega}) = 0$.

We write $P = \sum_{i=0}^d \partial_t^i a_i(t)$ with $a_i \in \mathbb{C}[t]$. We will now work with the analytic Gauss-Manin system $\mathrm{GM}^n_{\mathrm{an}}(f) = \mathscr{H}^n\big(\boldsymbol{R}f_*(\Omega^\bullet_{X_0^{\mathrm{an}}}[\tau], \mathrm{d} - \tau\mathrm{d}f\wedge)\big)$. It is computed by using the soft resolution $(\mathscr{A}^\bullet_{X_0^{\mathrm{an}}}[\tau], \mathrm{d} - \tau\mathrm{d}f)$ of $(\Omega^\bullet_{X_0^{\mathrm{an}}}[\tau], \mathrm{d} - \tau\mathrm{d}f)$, where $\mathscr{A}^\bullet_{X_0}$ denote the C^∞ differential forms on X_0^{an}. We have $\mathrm{GM}^n_{\mathrm{an}}(f) = \mathscr{O}^{\mathrm{an}}_{\mathbb{A}^1} \otimes_{\mathscr{O}_{\mathbb{A}^1}} \mathrm{GM}^n(f)$. Hence ω also defines a class in $\mathrm{GM}^n_{\mathrm{an}}(f)$ and this class satisfies $P \cdot [\omega] = 0$.

Firstly, given $(n-1, 0)$-form η and a $(0, n)$-form ψ (both C^∞), Stokes formula

$$\int_{X_0} d(\varphi \circ f \cdot \eta \wedge \psi) = 0, \quad \forall\, \varphi \in C^\infty_{\mathrm{c}}(\mathbb{A}^1)$$

reads

$$-\int (\partial_t \varphi \circ f)\, \mathrm{d}f \wedge \eta \wedge \overline{\psi} = \int (\varphi \circ f)\, \mathrm{d}\eta \wedge \psi, \quad \forall\, \varphi \in C^\infty_{\mathrm{c}}(\mathbb{A}^1),$$

that is,

$$\partial_t f_*(df \wedge \eta \wedge \overline{\psi}) = f_*(d\eta \wedge \overline{\psi}). \tag{4}$$

The relation $P\cdot[\omega]=0$ means that $\sum_{i=0}^d a_i\circ f\,\omega\tau^i\in(\mathrm{d}-\tau\mathrm{d}f)\mathscr{A}^{n-1}_{X_0^{\mathrm{an}}}[\tau]$, i.e., there exists $k\geqslant 0$ and $\eta_0,\dots,\eta_{d+k}\in\mathscr{A}^{n-1}_{X_0^{\mathrm{an}}}$ such that

$$\begin{cases} a_0\circ f\cdot\omega=\mathrm{d}\eta_0\\ \qquad\vdots\\ a_d\circ f\cdot\omega=\mathrm{d}\eta_d-\mathrm{d}f\wedge\eta_{d-1}\\ \qquad 0=\mathrm{d}\eta_{d+1}-\mathrm{d}f\wedge\eta_d\\ \qquad\vdots\\ \qquad 0=\mathrm{d}\eta_{d+k}-\mathrm{d}f\wedge\eta_{d+k-1}\\ \qquad 0=\qquad -\mathrm{d}f\wedge\eta_{d+k}.\end{cases}\tag{5}$$

Then, by using (4), the equations $d+1,\dots,d+k+1$ in (5) give $f_*(\mathrm{d}\eta_d\wedge\overline{\omega})=0$, and the dth one gives $a_d(t)f_*(\omega\wedge\overline{\omega})+f_*(\mathrm{d}f\wedge\eta_{d-1}\wedge\overline{\omega})=0$. Applying ∂_t on the left and using the $(d-1)$st equation gives $(\partial_t a_d(t)+a_{d-1}(t))f_*(\omega\wedge\overline{\omega})+f_*(\mathrm{d}f\wedge\eta_{d-2}\wedge\overline{\omega})=0$. Continuing the use of (5) actually gives $P(t,\partial_t)f_*(\omega\wedge\overline{\omega})=0$. $\square$

The Fourier transform $F_t(f_*(\omega\wedge\overline{\omega}))$ satisfies therefore the differential equation

$${}^F\!P(\tau,\partial_\tau)F_t(f_*(\omega\wedge\overline{\omega}))=0,$$

with ${}^F\!P(\tau,\partial_\tau)=\sum_i(-\tau)^i a_i(\partial_\tau)$. In other words, $f_*(\omega\wedge\overline{\omega})$ is a distribution solution of the holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module $\mathbb{C}[t]\langle\partial_t\rangle/(P)$ and $F_tf_*(\omega\wedge\overline{\omega})$ is a distribution solution of the holonomic $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module $\mathbb{C}[\tau]\langle\partial_\tau\rangle/({}^F\!P)$.

We will now focus on the properties of holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-modules.

1.3. Fourier-Deligne transformation of $\overline{\mathbb{Q}}_\ell$-sheaves and Fourier-Laplace transformation of holonomic $\mathscr{D}_X$-modules

1.3.a. Fourier-Deligne transformation of $\overline{\mathbb{Q}}_\ell$-sheaves. In his introduction to [**DMR07**], Deligne writes (approximate translation from the French):

> Around 1970, I was considering vector bundles with irregular singularity as being "pathological". I changed my mind only after having assimilated the analogy between the bundle $(\mathscr{O},\mathrm{d}+\mathrm{d}x)$ on the line, with horizontal section $\exp(-x)$, and the ℓ-adic sheaves $\mathscr{L}(\psi)$ deduced from the Artin-Schreier coverings, on the line in characteristic p. The natural question became: What does the ℓ-adic theory suggests for holonomic $\mathscr{D}$-modules with possibly irregular singularities – or some of them.

The Fourier-Deligne transformation follows this analogy between exp and Artin-Schreier coverings. The Artin-Schreier covering $x^p-x=t$ of the affine line $\mathbb{A}^1_{\mathbb{F}_p}$ with coordinate t produces a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{L}_\psi$ for each non-trivial additive character

$\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell$. On $\mathbb{A}^1_{\mathbb{F}_p} \times \widehat{\mathbb{A}}^1_{\mathbb{F}_p}$ consider the morphism $\mu(t,\tau) = t\tau$ and the pull-back $\overline{\mathbb{Q}}_\ell$-sheaf $\mu^*\mathscr{L}_\psi$. Consider then the diagram

$$\begin{tikzcd} & \mathbb{A}^1_{\mathbb{F}_p} \times \widehat{\mathbb{A}}^1_{\mathbb{F}_p} \arrow[dl, "\pi"'] \arrow[dr, "\widehat{\pi}"] & \\ \mathbb{A}^1_{\mathbb{F}_p} & & \widehat{\mathbb{A}}^1_{\mathbb{F}_p} \end{tikzcd}$$

The ψ-Fourier-Deligne transform of a $\overline{\mathbb{Q}}_\ell$-sheaf F on $\mathbb{A}^1_{\mathbb{F}_p}$ is

$$\mathscr{F}_{!,\psi}(F) = \boldsymbol{R}\widehat{\pi}_!(\boldsymbol{L}\pi^*F \otimes \mu^*\mathscr{L}_\psi)[1].$$

There is also a variant $\mathscr{F}_{*,\psi}$. Various properties are proved in [**Lau87**], in particular $\mathscr{F}_{*,\psi} = \mathscr{F}_{!,\psi}$, a Fourier inversion formula and a formula for the commutation with duality.

1.3.b. Fourier-Laplace transformation of holonomic $\mathscr{D}_X$-modules. Let M be a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module (i.e., M is a left $\mathbb{C}[t]\langle\partial_t\rangle$-module such that each element is annihilated by some nonzero operator in $\mathbb{C}[t]\langle\partial_t\rangle$), that we also regard as a sheaf of holonomic modules over the sheaf $\mathscr{D}_{\mathbb{A}^1}$ of algebraic differential operators on the affine line $\mathbb{A}^1$ (with its Zariski topology). Its *Laplace transform* ${}^F\!M$ (also called the Fourier transform) is a holonomic $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module, where τ is a new variable. Recall that ${}^F\!M$ can be defined in various equivalent ways. We consider below the Laplace transform with kernel $e^{t\tau}$, and a similar description can be made for the inverse Laplace transform, which has kernel $e^{-t\tau}$.

(1) The simplest way to define ${}^F\!M$ is to set ${}^F\!M = M$ as a $\mathbb{C}$-vector space and to define the action of $\mathbb{C}[\tau]\langle\partial_\tau\rangle$ in such a way that τ acts as $-\partial_t$ and ∂_τ as t (this is modeled on the behaviour of the action of differential operators under Fourier transform of temperate distributions). As already remarked, if $M = \mathbb{C}[t]\langle\partial_t\rangle/(P)$, then ${}^F\!M = \mathbb{C}[\tau]\langle\partial_\tau\rangle/({}^F\!P)$.

(2) One can mimic the Laplace integral formula, replacing the integral by the direct image of $\mathscr{D}$-modules. We consider the diagram

$$\begin{tikzcd} & \mathbb{A}^1 \times \widehat{\mathbb{A}}^1 \arrow[dl, "\pi"'] \arrow[dr, "\widehat{\pi}"] & \\ \mathbb{A}^1 & & \widehat{\mathbb{A}}^1 \end{tikzcd}$$

where t is the coordinate on $\mathbb{A}^1$ and τ that on $\widehat{\mathbb{A}}^1$. Then ${}^F\!M = \widehat{\pi}_+(\pi^+M \otimes E^{t\tau})$, where $E^{t\tau}$ is $\mathbb{C}[t,\tau]$ equipped with the connection $\mathrm{d} + \mathrm{d}(t\tau)$, and π^+M is $\mathbb{C}[\tau] \otimes_{\mathbb{C}} M$ equipped with its natural connection. Recall also that $\widehat{\pi}_+$ is the *direct image* of $\mathscr{D}$-modules, which is defined here in a simple way: $\widehat{\pi}_+(\pi^+M \otimes E^{t\tau})$ is the complex

$$0 \longrightarrow (\pi^+M \otimes E^{t\tau}) \xrightarrow{\partial_t} (\pi^+M \otimes E^{t\tau}) \longrightarrow 0$$

where the source of ∂_t is in degree -1 and the target in degree 0. More concretely, this complex is written

$$0 \longrightarrow \mathbb{C}[\tau]\otimes_{\mathbb{C}} M \xrightarrow{\;1\otimes\partial_t+\tau\otimes 1\;} \mathbb{C}[\tau]\otimes_{\mathbb{C}} M \longrightarrow 0$$

and one checks that the differential is injective, so that the complex is quasi-isomorphic to its cokernel. This complex is in the category of $\mathbb{C}[\tau]$-modules, and is equipped with an action of $\mathbb{C}[\tau]\langle\partial_\tau\rangle$, if ∂_τ acts as $\partial_\tau\otimes 1+1\otimes t$. The map $\mathbb{C}[\tau]\otimes_{\mathbb{C}} M\to M$ sending $\tau^k\otimes m$ to $(-\partial_t)^k m$ identifies the cokernel with ${}^F\!M$ as defined in (1).

(3) It may be useful to work with proper maps. In order to do so, one has to consider the projective completion $\mathbb{P}^1$ of $\mathbb{A}^1$ (resp. $\widehat{\mathbb{P}}^1$ of $\widehat{\mathbb{A}}^1$) obtained by adding the point ∞ to $\mathbb{A}^1$ (resp. the point $\widehat{\infty}$ to $\widehat{\mathbb{A}}^1$). In the following, we shall denote by t' (resp. τ') the coordinate centered at ∞ (resp. $\widehat{\infty}$), so that $t'=1/t$ (resp. $\tau'=1/\tau$) on $\mathbb{A}^1\smallsetminus\{0\}$ (resp. on $\widehat{\mathbb{A}}^1\smallsetminus\{\widehat{0}\}$). We consider the diagram

$$\begin{tikzcd} & \mathbb{P}^1\times\widehat{\mathbb{P}}^1 \arrow[dl, "\pi"'] \arrow[dr, "\widehat{\pi}"] & \\ \mathbb{P}^1 & & \widehat{\mathbb{P}}^1 \end{tikzcd} \tag{6}$$

Let $\mathscr{M}$ (resp. ${}^F\!\mathscr{M}$) be the algebraic $\mathscr{D}_{\mathbb{P}^1}$-module (resp. $\mathscr{D}_{\widehat{\mathbb{P}}^1}$-module) determined by M (resp. ${}^F\!M$): it satisfies by definition $\mathscr{M}=\mathscr{O}_{\mathbb{P}^1}(*\infty)\otimes_{\mathscr{O}_{\mathbb{P}^1}}\mathscr{M}$ and $M=\Gamma(\mathbb{P}^1,\mathscr{M})$ (and similarly for ${}^F\!M$). It is known that $\mathscr{M}$ (resp. ${}^F\!\mathscr{M}$) is still holonomic. Applying a similar construction to $E^{t\tau}$ we get $\mathscr{E}^{t\tau}$ on $\mathbb{P}^1\times\widehat{\mathbb{P}}^1$, which is a free $\mathscr{O}_{\mathbb{P}^1\times\widehat{\mathbb{P}}^1}(*(D_\infty\cup D_{\widehat{\infty}}))$-module of rank one, with $D_\infty\cup D_{\widehat{\infty}}=\mathbb{P}^1\times\widehat{\mathbb{P}}^1\smallsetminus(\mathbb{A}^1\times\widehat{\mathbb{A}}^1)$. Then we have

$${}^F\!\mathscr{M}=\mathscr{H}^0\widehat{\pi}_+(\pi^+\mathscr{M}\otimes\mathscr{E}^{t\tau}). \tag{7}$$

Remark 8. The identification $\mathscr{F}_{!,\psi}=\mathscr{F}_{*,\psi}$ for $\overline{\mathbb{Q}}_\ell$-sheaves is translated here by replacing $\widehat{\pi}_+$ with the direct image of $\mathscr{D}$-modules with proper support $\pi_\dagger$, or equivalently, the conjugate of π_+ by duality. The identification comes then from the observation that, if $\mathscr{N}$ is any holonomic $\mathscr{D}_{\mathbb{P}^1}$-module supported at ∞, then ${}^F\!\mathscr{N}=0$. This is because already $\pi^+\mathscr{N}\otimes\mathscr{E}^{t\tau}=0$, since $\pi^+\mathscr{N}\otimes\mathscr{O}_{\mathbb{P}^1\times\widehat{\mathbb{P}}^1}(*D_\infty)=0$.

LECTURE 2

MICROLOCAL ASPECTS OF THE FOURIER TRANSFORMATION

2.1. Introduction

2.1.a. The stationary phase formula. Let us consider the setting of §1.1.b. The Fourier transform $F_t f_*(\omega\wedge\overline{\omega})$ is a continuous function on $\widehat{\mathbb{A}^1}$, which is at the same time a temperate distribution, hence has moderate growth as well as all its derivatives when $|\tau| \to \infty$. However, it is rapidly oscillating and the stationary phase formula gives information on the speed of oscillation. It expresses this function as a superposition of functions $e^{2i\,\mathrm{Im}(c\tau)}$ with coefficients having non-oscillating phases, where $c \in \mathbb{C}$ varies among the critical values of f. In other words, the *stationary points of the phase* f, i.e., the critical points of f, in the Fourier integral $\int e^{\tau f - \overline{\tau f}} \omega \wedge \overline{\omega}$ give the speed of oscillation of the integral.

This stationary phase formula can be obtained by analyzing the differential equation satisfied by the Fourier integral when $|\tau| \to \infty$. Our point of view will be to describe the formal structure at $\tau = \infty$ of the Fourier transform ${}^F\!M$ of the Gauss-Manin system M attached to f. For P as in Proposition 1, the $\mathbb{C}[t]\langle\partial_t\rangle$-module $M = \mathbb{C}[t]\langle\partial_t\rangle/\mathbb{C}[t]\langle\partial_t\rangle \cdot P$ is regular, by the regularity theorem of Griffiths. Assume more generally that M is any regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module, i.e., regular at finite distance *and* at infinity.

Under this assumption, the following is well-known (see e.g. [**Mal91**] for the results and the definition of the *vanishing cycle* functor):

(a) The Laplace transform ${}^F\!M$ is holonomic, has a regular singularity at the origin $\tau = 0$, no other singularity at finite distance, and possibly irregular at infinity.

(b) The formal structure of ${}^F\!M$ at infinity can be described exactly from the vanishing cycles of M (or of $\mathrm{DR}^{\mathrm{an}} M$) at its critical points at finite distance. More precisely, denoting by $\widehat{{}^F\!\mathscr{M}}$ the formalized connection at $\widehat{\infty}$, we have a decomposition $\widehat{{}^F\!\mathscr{M}} \simeq \bigoplus_c (R_c, \nabla + \mathrm{d}(c/\tau'))$, where the sum is taken over the singular points $c \in \mathbb{A}^1$ of M, and (R_c, ∇) is a regular formal meromorphic connection corresponding in a one-to-one way to the data of the vanishing cycles of the perverse sheaf $\mathrm{DR}^{\mathrm{an}} M$ at c.

The purpose of this lecture is to generalize this stationary phase formula to arbitrary singularities of $\mathbb{C}[t]\langle\partial_t\rangle$-modules. This is justified by the following observation. Although the Gauss-Manin system M as above has regular singularities only, so that the only rapidly oscillating phenomena are created by $|\tau|\to\infty$, other examples may produce such rapid oscillation for finite values of τ. For example, if g is another meromorphic function on X, one can consider integrals like

$$\int_{X_1} e^{(\tau f+g)-\overline{(\tau f+g)}}\omega\wedge\overline{\omega},\qquad X_1=f^{-1}(\mathbb{A}^1)\cap g^{-1}(\mathbb{A}^1).$$

2.1.b. Local Fourier transformation. In [**Lau87**], Laumon defines the local Fourier transform of $\overline{\mathbb{Q}}_\ell$-sheaves by using the vanishing cycle functor for the purpose of proving a product formula for the determinant of the cohomology of such a sheaf in terms of local data.

The questions considered in [**Lau87**] can also be considered in the frame of holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-modules as follows. Let M be a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module with arbitrary singularities. Let ${}^F\!M$ be its Laplace transform. For each $\widehat{c}\in\widehat{\mathbb{A}}^1\cup\{\widehat{\infty}\}$, set $\tau_{\widehat{c}}=\tau-\widehat{c}$ ($\widehat{c}\neq\widehat{\infty}$) and $\tau_{\widehat{\infty}}=1/\tau$. We ask the following questions:

(1) To determine the singular points $\widehat{c}\in\widehat{\mathbb{A}^1}\cup\{\widehat{\infty}\}$ of ${}^F\!M$.

(2) To show that $\mathbb{C}[\![\tau_{\widehat{c}}]\!]\otimes_{\mathbb{C}[\tau_{\widehat{c}}]}{}^F\!M$ as a $\mathbb{C}[\![\tau_{\widehat{c}}]\!]\langle\partial_{\tau_{\widehat{c}}}\rangle$-module only depends on the various $\mathbb{C}[\![t_c]\!]\otimes_{\mathbb{C}[t_c]}M$, when c varies in the finite set of singular points of M (including ∞).

A natural question is then to make precise (2) above:

(3) To compute $\mathbb{C}(\!(\tau_{\widehat{c}})\!)\otimes{}^F\!M$ as a $\mathbb{C}(\!(\tau_{\widehat{c}})\!)$-vector space with a connection, or equivalently as a $\mathbb{C}[\![\tau_{\widehat{c}}]\!]\langle\partial_{\tau_{\widehat{c}}}\rangle$-module, in terms of the various $\mathbb{C}[\![t_c]\!]\otimes_{\mathbb{C}[t_c]}M$.

2.2. Local Fourier transformation and microlocalization

2.2.a. Definition of the Local Fourier transformation. Let M be a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module. It defines a $\mathscr{D}_{\mathbb{P}^1}$-module j_+M ($j:\mathbb{A}^1_t\hookrightarrow\mathbb{P}^1$). For each $c\in\mathbb{P}^1$, we denote by $\widehat{\mathscr{M}_c}$ the formalization $\mathbb{C}[\![t_c]\!]\otimes_{\mathscr{O}_{\mathbb{P}^1,c}}j_+M$ (with the above convention for t_∞).

The ring $\mathscr{E}^{(c,\infty)}$ of formal microdifferential operators at c. Let $c\in\mathbb{A}^1$. The ring $\mathscr{E}^{(c,\infty)}$ is the vector space of formal sums

$$\sum_{i\leqslant r}a_i(t_c)\eta^i,\quad a_i(t_c)\in\mathbb{C}[\![t_c]\!],\ r\in\mathbb{Z}.$$

For $P,Q\in\mathscr{E}^{(c,\infty)}$, the product is defined by the formula

$$P\cdot Q=\sum_{\alpha\geqslant 0}\frac{1}{\alpha!}\,\partial_\eta^\alpha P\,\partial_{t_c}^\alpha Q\in\mathscr{E}^{(c,\infty)}.$$

It makes $\mathscr{E}^{(c,\infty)}$ a ring, filtered with respect to the degree in η. One has morphism of $\mathbb{C}$-algebras given by

$$\mathbb{C}[t]\langle\partial_t\rangle\longrightarrow\mathscr{E}^{(c,\infty)},\quad t\longmapsto t_c+c,\quad\partial_t\longmapsto\eta,$$

which endows $\mathscr{E}^{(c,\infty)}$ with a structure of left-right $\mathbb{C}[t]\langle\partial_t\rangle$-bimodule. Note that ∂_t acts in an invertible way on both sides on $\mathscr{E}^{(c,\infty)}$ and thus $\mathscr{E}^{(c,\infty)}$ is a left-right $\mathbb{C}[t_c]\langle\partial_t,\partial_t^{-1}\rangle$-module. It is also a left-right $\mathbb{C}[\![t_c]\!]\langle\partial_t,\partial_t^{-1}\rangle$-module

Let M be a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module. Its formal microlocalization at $c\in\mathbb{A}^1$ is $\mathscr{E}^{(c,\infty)}\otimes_{\mathbb{C}[t]\langle\partial_t\rangle}M$. It comes equipped with an action of $\mathbb{C}(\!(\tau_{\widehat{\infty}})\!)$: this is the left action by $\mathbb{C}(\!(\eta^{-1})\!)$. As such, it is known that it is a finite dimensional $\mathbb{C}(\!(\tau_{\widehat{\infty}})\!)$-vector space, the dimension of which is the dimension of the *moderate vanishing cycles* of M at c (see [**Mal91**] and Lecture 4). On the other hand, it also has an action of $\partial_{\tau_{\widehat{\infty}}}$ defined so that

$$\partial_{\tau_{\widehat{\infty}}}(P\otimes m)=\partial_{\eta^{-1}}P\otimes m+\eta^2P\otimes tm. \tag{1}$$

This makes $\mathscr{E}^{(c,\infty)}\otimes_{\mathbb{C}[t]\langle\partial_t\rangle}M$ a finite dimensional $\mathbb{C}(\!(\tau_{\widehat{\infty}})\!)$-vector space with connection. Note that $\mathscr{E}^{(c,\infty)}\otimes_{\mathbb{C}[t]\langle\partial_t\rangle}M=\mathscr{E}^{(c,\infty)}\otimes_{\mathbb{C}[\![t_c]\!]\langle\partial_{t_c}\rangle}\widehat{\mathscr{M}_c}$ only depends on the formalized module $\widehat{\mathscr{M}_c}:=\mathbb{C}[\![t_c]\!]\langle\partial_{t_c}\rangle\otimes_{\mathbb{C}[t]\langle\partial_t\rangle}M$. This is $\mathscr{F}^{(c,\infty)}(\widehat{\mathscr{M}_c})$.

The ring $\mathscr{E}^{(\infty,\infty)}$. We denote by $\mathscr{E}^{(\infty,\infty)}$ the set of formal sums

$$\sum_{i\leqslant r}a_i(t_\infty)\eta^i,\quad a_i(t_\infty)\in\mathbb{C}[\![t_\infty]\!],\ r\in\mathbb{Z}.$$

If $P,Q\in\mathscr{E}^{(\infty,\infty)}$, the product is given by

$$P*Q=\sum_{\alpha\geqslant0}\frac{1}{\alpha!}\,\partial_\eta^\alpha P\cdot\partial_t^\alpha Q,$$

where ∂_t acts on $\mathbb{C}[\![t_\infty]\!]$ as $-t_\infty^2\partial_{t_\infty}$. Similarly, $\mathscr{E}^{(\infty,\infty)}$ is a filtered ring. One has a morphism of $\mathbb{C}$-algebras

$$\mathbb{C}[t_\infty]\langle\partial_t\rangle\longrightarrow\mathscr{E}^{(\infty,\infty)}\quad t_\infty\longmapsto t_\infty\quad\partial_t\longmapsto\eta$$

(notice that on the ring $\mathbb{C}[t_\infty]\langle\partial_t\rangle$, one has the relation $[\partial_t,t_\infty]=-t_\infty^2$). This morphism endows $\mathscr{E}^{(\infty,\infty)}$ with a structure of left-right $\mathbb{C}[t_\infty]\langle\partial_t\rangle$-module and of left-right $\mathbb{C}[\![t_\infty]\!]\langle\partial_t\rangle$-module. The ring $\mathscr{E}^{(\infty,\infty)}$ is a subring of the ring of formal microdifferential operators on $\mathbb{P}^1$ at ∞.

Given a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module M, the module $\mathscr{E}^{(\infty,\infty)}\otimes_{\mathbb{C}[t]\langle\partial_t\rangle}j_+M=\mathscr{E}^{(\infty,\infty)}\otimes_{\mathbb{C}[\![t_\infty]\!]\langle\partial_t\rangle}\widehat{\mathscr{M}_\infty}$ is a left $\mathscr{E}^{(\infty,\infty)}$-module. It has an action of $\mathbb{C}(\!(\tau_{\widehat{\infty}})\!)$ defined as above, and is a finite dimensional $\mathbb{C}(\!(\tau_{\widehat{\infty}})\!)$-vector space as such. It has a connection defined by Formula (1). This is $\mathscr{F}^{(\infty,\infty)}(\widehat{\mathscr{M}_\infty})$.

Theorem 2 **(R. García López [GL04]).** *Let M be a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module, and let ${}^F\!M$ be its Laplace transform, with associated formalized module $\widehat{{}^F\!\mathscr{M}}_{\widehat{\infty}}$ at $\tau=\widehat{\infty}$. Then the natural morphism*

$$\begin{aligned}\widehat{{}^F\!\mathscr{M}}_{\widehat{\infty}}=\mathbb{C}(\!(\tau_{\widehat{\infty}})\!)\otimes_{\mathbb{C}[\tau]}{}^F\!M&\longrightarrow\bigoplus_{c\in\mathbb{P}^1}\mathscr{E}^{(c,\infty)}\otimes_{\mathbb{C}[t]\langle\partial_t\rangle}j_+M=\bigoplus_{c\in\mathbb{P}^1}\mathscr{F}^{(c,\infty)}(\widehat{\mathscr{M}_c})\\ \varphi(\tau_{\widehat{\infty}})\otimes m&\longmapsto\bigoplus_c\varphi(\eta^{-1})\otimes m\end{aligned}$$

is an isomorphism of $\mathbb{C}(\!(\tau_{\widehat{\infty}})\!)$-vector spaces with connection.

2.2.b. Local formal structure of a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module. We will apply the next result to $\widehat{\mathscr{M}}_c$ for each $c \in \mathbb{P}^1$. Ten the coordinate is t_c. It can also be applied to ${}^F\widehat{\mathscr{M}}_{\widehat{c}}$ for $\widehat{c} \in \widehat{\mathbb{P}}^1$, and then $x = \tau_{\widehat{c}}$.

***Theorem 3* (Levelt-Turrittin).** *Let $\widehat{\mathscr{M}}$ be a holonomic $\mathbb{C}[\![x]\!]\langle\partial_x\rangle$-module. Then:*

(1) *$\widehat{\mathscr{M}}$ decomposes as the direct sum $\widehat{\mathscr{M}} = \bigoplus_{\lambda\in\mathbb{Q}_+} \widehat{\mathscr{M}}^{(\lambda)}$, where $\widehat{\mathscr{M}}^{(\lambda)}$ has pure slope λ (pure slope 0 means regular singularity).*

(2) *This decomposition can be refined, for $\lambda = q/p \neq 0$, as*

$$\widehat{\mathscr{M}}^{(\lambda)} = \bigoplus_{\substack{\rho\in u\mathbb{C}[\![u]\!],\, \mathrm{val}_u(\rho)=p \\ \varphi\in\mathbb{C}(\!(u)\!)/\mathbb{C}[\![u]\!],\, \mathrm{val}_u(\varphi)=-q \\ (R,\nabla)\ \textit{reg. sing.}}} \mathrm{El}(\rho,\varphi,(R,\nabla)),$$

where $\mathrm{El}(\rho,\varphi,(R,\nabla)) = \rho_+(R,\nabla + d\varphi)$.

2.2.c. Slope correspondence.

$$\begin{tikzcd}[column sep=small]
\widehat{\mathscr{M}}_0 \arrow[d, bend right, "\mathscr{F}_-^{(0,\infty)}"'] & \oplus_{c\neq 0}\widehat{\mathscr{M}}_c \arrow[d, bend right, "\oplus\mathscr{F}_-^{(c,\infty)}"'] & \widehat{\mathscr{M}}_\infty^{>1} \arrow[d, bend right, "\mathscr{F}_-^{(\infty,\infty)}"'] & \widehat{\mathscr{M}}_\infty^{=1} \arrow[d, bend right, "\oplus\mathscr{F}_-^{(\infty,\widehat{c})}"'] & \widehat{\mathscr{M}}_\infty^{<1} \arrow[d, bend right, "\mathscr{F}_-^{(\infty,0)}"'] \\
{}^F\widehat{\mathscr{M}}_{\widehat{\infty}}^{<1} \arrow[u, bend right, "\mathscr{F}_+^{(\infty,0)}"'] & {}^F\widehat{\mathscr{M}}_{\widehat{\infty}}^{=1} \arrow[u, bend right, "\oplus\mathscr{F}_+^{(\infty,c)}"'] & {}^F\widehat{\mathscr{M}}_{\widehat{\infty}}^{>1} \arrow[u, bend right, "\mathscr{F}_+^{(\infty,\infty)}"'] & \oplus_{\widehat{c}\neq 0}{}^F\widehat{\mathscr{M}}_{\widehat{c}} \arrow[u, bend right, "\oplus\mathscr{F}_+^{(\widehat{c},\infty)}"'] & {}^F\widehat{\mathscr{M}}_{\widehat{0}} \arrow[u, bend right, "\mathscr{F}_+^{(0,\infty)}"']
\end{tikzcd}$$

F_- F_+

2.2.d. Stationary phase formula.

***Theorem 4* (Fang [Fan09], C.S. [Sab08], Graham-Squire).** *For any elementary $\mathbb{C}(\!(x)\!)$-vector space $\mathrm{El}(\rho,\varphi,(R,\nabla))$ with irregular connection (i.e., such that $\varphi \neq 0$), the local Fourier transform $\mathscr{F}_\pm^{(0,\infty)}\,\mathrm{El}(\rho,\varphi,(R,\nabla))$ is isomorphic to the elementary finite dimensional $\mathbb{C}(\!(x)\!)$-vector space with connection $\mathrm{El}(\widehat{\rho}_\pm,\widehat{\varphi},(\widehat{R},\widehat{\nabla}))$ with (setting $L_q = (\mathbb{C}(\!(u)\!), d - \frac{q}{2}\frac{du}{u})$)*

$$\widehat{\rho}_\pm(u) = \mp\frac{\rho'(u)}{\varphi'(u)}, \quad \widehat{\varphi}(u) = \varphi(u) - \frac{\rho(u)}{\rho'(u)}\,\varphi'(u), \quad \widehat{R} \simeq R\otimes L_q.$$

Remark 5. There are similar formulas for $\mathscr{F}^{(c,\infty)}$, $c \in \mathbb{P}^1$.

2.3. Slopes and characteristic varieties (Abbes-Saito)

The theorem of Levelt-Turrittin does not have an analogue in the ℓ-adic setting. Nevertheless, for a $\overline{\mathbb{Q}}_\ell$-sheaf on the affine line $\mathbb{A}^1_{\mathbb{F}_p}$ (for example), Abbes and Saito propose a construction to produce the analogues of the principal parts of the polar parts φ occurring in the Levelt-Turrittin decomposition at $x = 0$ (say).

In a work in progress, J.-B. Teyssier (École polytechnique) intends to produce a similar construction for holonomic $\mathscr{D}$-modules and to prove that it produces exactly the principal parts of the polar parts φ occurring in the Levelt-Turrittin decomposition. In this simplified setting, the information given is that of the micro-characteristic varieties at the origin, as defined by Y. Laurent (see [**Lau04**]).

LECTURE 3

PURITY (1): PUNCTUAL PURITY

3.1. Punctual purity in arithmetic

Recall ([**Del80**]) that an number in $\overline{\mathbb{Q}}_\ell$ is pure of weight w relative to $q = p^k$ if it is algebraic and all its complex conjugates have absolute value equal to $q^{w/2}$.

Given a $\overline{\mathbb{Q}}_\ell$-sheaf F on $\mathbb{A}^1_{\mathbb{F}_p}$, one says that it is punctually pure of weight w if the eigenvalues of the Frobenius automorphism acting on the geometric fibre of F at each $x \in \mathbb{A}^1_{\mathbb{F}_p}$ are pure of weight w relative to $q = p^{\deg x}$.

There is a similar notion of purity for complexes and for perverse sheaves. The fundamental theorem which will serve us as a guide is stated and proved in [**KL85**] (Th. 2.2.1 and Scholie 2.3.1), that we only consider in the present simple setting.

***Theorem 1* (Katz-Laumon).** *Let K be a perverse object of $D^b_c(\overline{\mathbb{Q}}_\ell)$ and let $\widehat{U} \subset \widehat{\mathbb{A}}^1_{\mathbb{F}_p}$ be an open set over which the cohomology sheaves of $\mathscr{F}_\psi K$ are lisses. Then $\mathscr{H}^j \mathscr{F}_\psi K = 0$ for $j \neq -1$ and, if K is pure of weight w, then $\mathscr{H}^{-1}(\mathscr{F}_\psi K)_{|\widehat{U}}$ is pure of weight w.*

In loc. cit., (7.3.3.3), Katz and Laumon add the following:

> We would like to be able to formulate and to prove a similar statement for $\mathscr{D}$-modules, the notion of purity being understood in the sense of Hodge theory.

3.2. Purity in the sense of Hodge theory

In complex algebraic geometry, the standard analogue of the notion of pure lisse sheaf is that of a variation of polarized $\mathbb{Q}$-Hodge structure. Below, we will only consider variations of *complex* Hodge structures, due to the lack of time.

3.2.a. Variations of Hodge structures and their limits. Let V be a holomorphic vector bundle on a complex manifold X, equipped with a *flat holomorphic connection* $\nabla : V \to \Omega^1_X \otimes V$. Let $\mathscr{V} = \ker \nabla$ the associated locally constant sheaf of $\mathbb{C}$-vector spaces, so that $V \simeq \mathscr{O}_X \otimes_{\mathbb{C}} \mathscr{V}$, and let $(H, \nabla + \overline{\partial})$ the associated C^∞-bundle, so that $V = \ker\overline{\partial} : H \to \mathscr{A}_X^{(0,1)} \otimes H$. A *variation of complex Hodge structure* of weight w consists of a grading $H = \bigoplus_p H^{p,w-p}$ by C^∞ sub-bundles such that, for each p,

(1) $F^pH := \bigoplus_{p'\geqslant p} H^{p,w-p}$ is stable by $\overline{\partial}$ and defines a holomorphic sub-bundle $F^pV \subset V$,

(2) (Griffiths transversality) $\nabla F^pV \subset \Omega^1_X \otimes F^{p-1}V$.

Giving the decomposition is equivalent to giving the semi-simple operator $\mathscr{Q}$ on H with constant half integral eigenvalues, whose eigenspace corresponding to $p - w/2$ is $H^{p,w-p}$.

A *polarization* of a variation of complex Hodge structure is a $(\nabla + \overline{\partial})$*-horizontal* non-degenerate $(-1)^w$-Hermitian pairing k on H, or equivalently a non-degenerate $(-1)^w$-Hermitian pairing k on $\mathscr{V}$, so that $h(u, v) := k(e^{\pi i\mathscr{Q}}u, v)$ is a polarization at each $x \in X$, that is, h is a Hermitian metric on H.

Example 2. Let $f : X \to Y$ be a smooth projective morphism between complex varieties equipped with a relative ample line bundle. Then the primitive cohomology $H^k_{\mathrm{prim}}(f^{-1}(y), \mathbb{C})$ ($k \leqslant \dim X/Y$) defines a variation of complex Hodge structure for which the polarization is defined from the Poincaré duality pairing made Hermitian and the cup product by the Chern class of the ample line bundle.

Assume that X is a disc, X^* is the punctured disc, and set $j : X^* \hookrightarrow X$. Let $(V, \nabla, F^\bullet V, k)$ be a variation of polarized complex Hodge structure of weight w on X^*.

***Theorem 3* (Regularity theorem, Griffiths-Schmid).** *Under these assumptions, the subsheaf $(j_*V)^{\mathrm{lb}}$ of j_*V, consisting of local sections whose h-norm is locally bounded near the origin, is a locally free sheaf on which the connection ∇ has at most a simple pole. Moreover, for each p, $j_*F^pV \cap (j_*V)^{\mathrm{lb}}$ (intersection taken in j_*V) is a locally free sheaf.*

3.2.b. Construction of a family of vector bundles on $\mathbb{P}^1$. Let $(V, \nabla, F^\bullet V, k)$ be a variation of polarized Hodge structure of weight w on $X = \mathbb{A}^{1\mathrm{an}} \smallsetminus \{c_1, \dots, c_r\}$. In order to have simple formulas, we will assume that the weight w is equal to 0.

To $(V, F^\bullet V)$ we associate a holomorphic vector bundle on $X \times \mathbb{C}$, where $\mathbb{C}$ comes equipped with a fixed variable z. This is the holomorphic bundle associate to the Rees module $R_FV := \bigoplus_p F^pV \cdot z^{-p}$, which is a locally free $\mathscr{O}_X[z]$-module.

We will regard $\mathbb{P}^1$ as the union of the chart $\mathbb{C}_z$ with coordinate z and the chart $\mathbb{C}_{z'}$ with coordinate z', with $z' = 1/z$ on $\mathbb{C}^*$. Let $\sigma : \mathbb{P}^1 \to \overline{\mathbb{P}}^1$ be the anti-holomorphic involution defined by $z \mapsto -1/\overline{z}$. It sends the firs chart to the conjugate of the second chart.

Given a holomorphic vector bundle $\mathscr{H}$ on $X \times \mathbb{C}_z$, we obtain a holomorphic vector bundle $\sigma^*\overline{\mathscr{H}}$ on $\overline{X} \times \mathbb{C}_{z'}$ (where $\overline{X}$ is the conjugate complex manifold).

The ∇-horizontal pairing k induces a pairing

$$R_F k\Big(\sum_p v_p z^{-p}, \sum_q \sigma^* \overline{v_q z^{-q}}\Big) = \sum_{p,q} (-1)^q k(v_p, \overline{v}_q) z^{q-p},$$

that we restrict to $\mathbf{S} = \{|z| = 1\}$ and we regard as taking values in the sheaf-theoretic restriction to $X \times \mathbf{S}$ of $\mathscr{C}^{\infty,\mathrm{an}}_{X\times\mathbb{C}^*}$. We will set $\mathscr{H} = \mathscr{O}_{X\times\mathbb{C}} \otimes_{\mathscr{O}_X[z]} R_F V$.

Lemma 4. *The pairing* $\mathscr{C} : \mathscr{H}_{|\mathbf{S}} \otimes \sigma^* \overline{\mathscr{H}}_{|\mathbf{S}} \to \mathscr{C}^{\infty,\mathrm{an}}_{X\times\mathbf{S}}$ *induces a gluing* $\mathscr{H}^\vee_{|\mathbf{S}} \simeq \sigma^* \overline{\mathscr{H}}_{|\mathbf{S}}$. *The corresponding* C^∞ *family of holomorphic vector bundles on* $\mathbb{P}^1$ *parametrized by* X *is a family of trivializable vector bundles.*

Proof. This follows from the opposedness condition for the Hodge filtration and its conjugate, and the k-orthogonality of the Hodge decomposition. ☐

3.2.c. Extension of the Hodge data. Let (V, ∇) be a holomorphic bundle with connection on $\mathbb{A}^{1\mathrm{an}} \smallsetminus \{c_1, \dots, c_r\}$. Setting $D = \{c_1, \dots, c_r, \infty\}$, it extends in a unique way (Deligne's meromorphic extension) as a locally free $\mathscr{O}_{\mathbb{P}^1}(*D)$-module with a connection having a regular singularity at each point of D. It defines therefore an algebraic holonomic $\mathscr{D}_{\mathbb{P}^1}$-module $\widetilde{\mathscr{M}}$ and thus a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module $\widetilde{M}$ with regular singularities. We will mainly consider the submodule M of $\widetilde{M}$ which has no quotient module supported on $\{c_1, \dots, c_r\}$ and which coincides with $\widetilde{M}$ when restricted to $\mathbb{A}^1 \smallsetminus \{c_1, \dots, c_r\}$ (M is called the minimal or intermediate extension of $\widetilde{M}$).

The theorem of Griffiths-Schmid allows one to extend $F^\bullet V$ as a good filtration $F^\bullet M$ of M (taken in a decreasing way). Moreover, the analysis mad by Schmid (see [**Sch73**]) of the behaviour of the metric h in the neighbourhood of each singular point implies that the sesquilinear pairing k extends as a $\mathscr{D}_{\mathbb{P}^1} \otimes_{\mathbb{C}} \mathscr{D}_{\overline{\mathbb{P}}^1}$-linear pairing on $\widetilde{M}$ with values in moderate distributions on $\mathbb{P}^1 \smallsetminus D$, and then as a $\mathbb{C}[t]\langle\partial_t\rangle \otimes \overline{\mathbb{C}[t]\langle\partial_t\rangle}$-linear pairing $k : M \otimes_{\mathbb{C}} \overline{M} \to \mathscr{S}'(\mathbb{A}^{1\mathrm{an}})$.

3.3. Purity of the Fourier-Laplace transform of a variation of polarizable Hodge structure

3.3.a. The need for an extension of the notion of Hodge structure. Assume that (V, ∇) as in §3.2.c underlies a variation of polarized Hodge structure and let M be its minimal extension. Its Laplace transform ${}^F\!M$ is a vector bundle with connection once restricted to $\widehat{U} = \widehat{\mathbb{A}}^1 \smallsetminus \{\widehat{0}\}$. However, the connection has an irregular singularity at $\widehat{\infty}$, in general (as shown by the stationary phase formula). The regularity theorem of Griffiths and Schmid prevents then ${}^F\!M_{|\widehat{U}}$ to underlie in a natural way a variation of polarized Hodge structure, because the connection has an irregular singularity at $\widehat{\infty}$. What kind of a structure does ${}^F\!M_{|\widehat{U}}$, or its fibre at $\tau = 1$, say, underlie?

We will use the twistor approach of §3.2.b in order to define purity. More precisely, we will define, from the extended data $(M, F^\bullet M)$ a holomorphic bundle on $\widehat{U}\times\mathbb{C}_z$, and from the extended $k: M\otimes_{\mathbb{C}}\overline{M}\to\mathscr{S}'(\mathbb{A}^1)$ a family of gluings.

3.3.b. Construction of a family of vector bundles on $\mathbb{P}^1$. We wish to construct a holomorphic bundle $\mathscr{H}$ on $\mathbb{C}_z$ and a pairing $\mathscr{C}:\mathscr{H}_{|\mathbf{S}}\otimes\sigma^*\overline{\mathscr{H}}_{|\mathbf{S}}\to\mathscr{O}_{\mathbf{S}}$, giving rise to a gluing $\mathscr{H}_{|\mathbf{S}}^\vee\simeq\sigma^*\overline{\mathscr{H}}_{|\mathbf{S}}$. In the general construction, the variation parameter is τ, and the construction we give being understood at $\tau=1$. However, in the construction below, the variable τ^{-1} will be identified with z, and one should think of this construction as an analogue with filtration and sesquilinear pairing of the Laplace transform with kernel $e^{t/z}$, while the family construction would correspond to the kernel $e^{t\tau/z}$.

The holomorphic bundle $\mathscr{H}$ on $\mathbb{C}_z$. We say that p_o is a generating index for $F^\bullet M$ if, for each $\ell\geqslant 0$, $F^{p_o-\ell}M=F^{p_o}M+\cdots+\partial_t^\ell F^{p_o}M$.

Let $\widehat{\mathrm{loc}}: M\to G:=\mathbb{C}[t]\langle\partial_t,\partial_t^{-1}\rangle\otimes_{\mathbb{C}[t]\langle\partial_t\rangle}M$ be the localization morphism with respect to ∂_t. For each $\mathbb{C}[t]$-submodule L of M, we set $G_0^{(L)}=\sum_{j\geqslant 0}\partial_t^{-j}\widehat{\mathrm{loc}}(L)$ and $G_p^{(L)}=\partial_t^pG_0^{(L)}$.

Because of the regular singularity of M at ∞, we have a finiteness result.

Lemma 5.

(1) *For any L and any p, $G_p^{(L)}$ is a free $\mathbb{C}[\partial_t^{-1}]$-module of finite rank.*

(2) *For a good filtration $F^\bullet M$, the free $\mathbb{C}[\partial_t^{-1}]$-module $G_{p_o}^{(F^{p_o}M)}$ does not depend on the generating index p_o and generates G, that is, $G=\mathbb{C}[\partial_t,\partial_t^{-1}]\otimes_{\mathbb{C}[\partial_t^{-1}]}G_{p_o}^{(F^{p_o}M)}$.*

We will regard $G_{p_o}^{(F^{p_o}M)}$ as a free $\mathbb{C}[z]$-module by setting $z=\partial_t^{-1}$. We will denote by $\mathscr{H}$ the corresponding holomorphic bundle.

The pairing $\mathscr{C}$. The pairing $k: M\otimes\overline{M}\to\mathscr{S}'(\mathbb{A}^1)$ can be composed with the Fourier transformation $F_t:\mathscr{S}'(\mathbb{A}^1)\to\mathscr{S}'(\widehat{\mathbb{A}}^1)$. Then F_tk defines a $\mathbb{C}[\tau]\langle\partial_\tau\rangle\otimes\overline{\mathbb{C}[\tau]\langle\partial_\tau\rangle}$-linear morphism ${}^F\!M\otimes_{\mathbb{C}}\overline{\iota^+{}^F\!M}\to\mathscr{S}'(\widehat{\mathbb{A}}^1)$, where ι is the involution $\tau\mapsto-\tau$. That we need the involution ι is seen on the kernel $e^{t\tau-\overline{t\tau}}$. Restricted to $\widehat{U}$, F_tk takes values in $\mathscr{C}^\infty_{\widehat{U}}$: indeed, ${}^F\!M_{|\widehat{U}}$ is a free $\mathscr{O}(\widehat{U})$-module, and if one fixes a $\mathscr{O}(\widehat{U})$-basis m_j of ${}^F\!M_{|\widehat{U}}$, then the matrix $F_tk(m_i,\overline{m}_j)$ is horizontal with respect to a flat connection (in the C^∞ sense, i.e., with respect to the derivations ∂_τ and $\overline{\partial}_\tau$). We set $z=\tau^{-1}$ and restrict to $\mathbf{S}$. Let $\mathscr{L}$ denote the local system defined by ${}^F\!M$ on $\widehat{U}^{\mathrm{an}}$, that we restrict to $\mathbf{S}$. Since F_tk is $\mathbb{C}[\tau]\langle\partial_\tau\rangle\otimes\overline{\mathbb{C}[\tau]\langle\partial_\tau\rangle}$-linear, it defines a pairing, still denoted by $F_tk:\mathscr{L}\otimes_{\mathbb{C}}\iota^{-1}\overline{\mathscr{L}}\to\mathbb{C}_{\mathbf{S}}$.

We now remark that, on $\mathbf{S}$, we have $\iota=\sigma$ so that, tensoring by $\mathscr{O}_{\mathbf{S}}$ we regard F_tk as defining a pairing

$$\mathscr{C}:\mathscr{H}_{|\mathbf{S}}\otimes_{\mathscr{O}_{\mathbf{S}}}\sigma^*\overline{\mathscr{H}}_{|\mathbf{S}}\longrightarrow\mathscr{O}_{\mathbf{S}}.$$

From the non-degeneracy of k one deduces the non-degeneracy of $\mathscr{C}$, hence a gluing $\mathscr{H}^\vee_{|\mathbf{S}} \simeq \sigma^*\overline{\mathscr{H}}_{|\mathbf{S}}$.

3.3.c. The purity theorem.

Theorem 6. *Assume that* $(V, F^\bullet V, \nabla, k)$ *is a variation of complex Hodge structure of weight* 0. *In particular, it defines a* C^∞ *family, parametrized by* $\mathbb{A}^{1\mathrm{an}} \smallsetminus \{c_1, \dots, c_r\}$*, of trivial holomorphic bundles on* $\mathbb{P}^1$ *(Lemma* 4*). Then the holomorphic bundle* $\widetilde{\mathscr{H}}$ *obtained by gluing* $\mathscr{H}^\vee$ *with* $\sigma^*\overline{\mathscr{H}}$ *through* $\mathscr{C}$*, as defined in* §3.3.b*, is trivial.*

If we had defined the family instead of the fibre at $\tau = 1$ only, we would have obtained a family of trivial bundles, as in Lemma 4, but which do not come from a filtered holomorphic bundle in general.

Sketch of proof. □

3.3.d. Integrability and the "new supersymmetric index". In the construction of §3.3.b, the connection played an essential role in order to define $\mathscr{C}$, as we had to first restrict $F_t k$ to the local system $\mathscr{L}$ in order to replace ι with σ and $\mathscr{C}^\infty_{\mathbf{S}}$ with $\mathscr{O}_{\mathbf{S}}$.

By construction, $\mathscr{H}$ comes equipped with a connection having a pole of order at most two at the origin, and the gluing induced by $\mathscr{C}$ is compatible with the connection on $\mathscr{H}^\vee$ and on $\sigma^*\overline{\mathscr{H}}$, defining therefore a connection on the trivial bundle $\widehat{\mathscr{H}}$, with a pole of order at most two at $z = 0$ and at $z = \infty$.

Let us set $H = \Gamma(\mathbb{P}^1, \mathscr{H})$, so that $\mathscr{H} \simeq \mathscr{O}_{\mathbb{P}^1} \otimes_{\mathbb{C}} H$. Then the connection on $\mathscr{H}$ can be written as

$$\nabla = \mathrm{d} + \big(z^{-1}\mathscr{U} - \mathscr{Q} - z\mathscr{U}^\dagger\big)\frac{\mathrm{d}z}{z}$$

for some endomorphisms $\mathscr{U}, \mathscr{U}^\dagger, \mathscr{Q}$ of H. In the case considered in §3.2.a, we have $\mathscr{U} = \mathscr{U}^\dagger = 0$ and $\mathscr{Q}$ is as defined there.

In the case provided by Theorem 6, $\mathscr{Q}$ is a semi-simple endomorphism (being self-adjoint with respect to a metric that we did not make explicit), but its eigenvalues need not be half-integers. Moreover, in the family construction, the eigenvalues of $\mathscr{Q}$ may vary with the family parameter τ in a real-analytic way.

The operator $\mathscr{Q}$ already appeared in [**CV91, CFIV92**], as noticed by C. Hertling [**Her03**], under the name of a "new supersymmetric index".

LECTURE 4

PURITY (2): COUPLING THE STATIONARY PHASE FORMULA WITH HODGE THEORY

4.1. Introduction

4.1.a. Is the vanishing cycle space a "limit object"? In this lecture, we will mainly consider the Laplace transform of a regular holonomic $\mathscr{D}$-module on the affine line, i.e., of a regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module M. In such a case, the Laplace transform ${}^F\!M$ is a holonomic $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module with a regular singularity at $\tau = 0$, a possibly irregular singularity at $\tau = \infty$ of exponential type, and no other singularity.

At each singular point c of M, the nearby cycle space represents the "limit" in a precise sense of the fibers of M at neighbouring points t of c. In particular it has the same dimension as M_t. This limit construction has been extended by W. Schmid [**Sch73**] in order to take into account a Hodge filtration, in the case $M^{\mathrm{an}}_{|\mathbb{A}^1\smallsetminus \mathrm{Sing}\, M} = (V, \nabla)$ underlies a variation of polarized Hodge structure. Note that the nearby cycle space does not depend on the fibre of M at c.

The vanishing cycle space, on the other hand, measures the difference between the nearby cycle space and the fibre of M at c. It is not a "limit" space. However, it shares a lot of properties of a limit, even from the Hodge point of view. It is therefore natural to ask whether it can be realized as a limit.

In §2.1.a we have recalled the meaning of the stationary phase formula in the case of a regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module, and how the vanishing cycles of M at c enter in a *formal* limiting expression for ${}^F\!M$ at $\widehat{\infty}$.

From the Hodge point of view, the irregular singularity of ${}^F\!M$ at $\widehat{\infty}$ prevents us to consider the Hodge structure on the vanishing cycle space as a limit of a variation of Hodge structure when $\tau \to \infty$, because of the regularity theorem of Griffiths-Schmid.

The notion of integrable twistor structure and of the variation of such an object gives a solution to this question and allows one to understand the space of vanishing cycles, together with its Hodge structure, as a limiting object.

4.1.b. Moderate nearby and vanishing cycles. Let me quickly recall theses notions. Let M be a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module and let $c \in \mathbb{C}$. Let t_c denote the local coordinate at c. Replacing t with t_c in what follows allows one to define the moderate nearby/vanishing cycles at $t = c$, by using the definition at $t = 0$.

We consider the increasing filtration $V_\bullet\mathbb{C}[t]\langle\partial_t\rangle$ indexed by $\mathbb{Z}$ such that t has order -1 and ∂_t has order 1, and $V_0\mathbb{C}[t]\langle\partial_t\rangle = \mathbb{C}[t]\langle t\partial_t\rangle$. For any holonomic module M there exists a unique increasing filtration $V_\bullet M$ indexed $\mathbb{Z}$ such that

- each V_kM is a $V_0\mathbb{C}[t]\langle\partial_t\rangle$-module of finite type,
- there exists a nonzero polynomial $b(s)$ with roots having real part in $(-1, 0]$ such that $b(t\partial_t + k)$ acts by 0 on $\mathrm{gr}_k^V M$ for each $k \in \mathbb{Z}$.

In the following I will assume that M is such that the roots of b are real, and we write $b(s) = \prod_{\alpha\in(-1,0]}(s+\alpha)^{\nu_\alpha}$. For each $\alpha \in [0, 1]$ one sets

$$\psi_{t,\alpha}^{\mathrm{mod}\,0} M := \begin{cases} \ker\left[(t\partial_t)^{\nu_\alpha} : \mathrm{gr}_0^V M \to \mathrm{gr}_0^V M\right] & \text{for } \alpha = 0, \\ \ker\left[(t\partial_t+\alpha)^{\nu_\alpha} : \mathrm{gr}_1^V M \to \mathrm{gr}_1^V M\right] & \text{for } \alpha \in (0, 1]. \end{cases}$$

One now defines

$$\begin{aligned} \psi_t^{\mathrm{mod}\,0} M &:= \bigoplus_{\alpha\in[0,1)} \psi_{t,\alpha}^{\mathrm{mod}\,0} M \quad \text{(nearby cycles)}, \\ \phi_t^{\mathrm{mod}\,0} M &:= \bigoplus_{\alpha\in(0,1]} \psi_{t,\alpha}^{\mathrm{mod}\,0} M \quad \text{(vanishing cycles)}. \end{aligned}$$

These are known to be finite dimensional $\mathbb{C}$-vector spaces equipped with the automorphism $\mathrm{T} = \exp(-2\pi i t\partial_t)$. More precisely, if M has a regular singularity at $t = 0$, $\psi_t^{\mathrm{mod}\,0} M$ has dimension equal to the generic rank of M.

This construction can also be done at ∞ by using the coordinate t_∞. Since t' acts in an invertible way on M at ∞, there is no distinction between nearby and vanishing cycles, which are then isomorphic.

This construction can also be applied to ${}^F\!M$. If M has only regular singularities, which will be the case in this lecture, then the only singular points of ${}^F\!M$ are $\widehat{c} = \widehat{0}$ and $\widehat{c} = \widehat{\infty}$. The moderate vanishing cycles of ${}^F\!M$ at $\tau = \widehat{\infty}$ also coincide with the nearby cycles. However, due to the possible irregular singularity of ${}^F\!M$ at $\widehat{\infty}$, the dimension of $\psi_{\tau'}^{\mathrm{mod}\,\widehat{\infty}}({}^F\!M)$ may be strictly smaller than the generic rank of ${}^F\!M$ (it may even be zero). On the other hand, due to the regularity of ${}^F\!M$ at $\tau = 0$, the dimension of $\psi_\tau^{\mathrm{mod}\,\widehat{0}}({}^F\!M)$ is equal to the generic rank of ${}^F\!M$.

4.2. Witten's complex

In the beginning of the eighties, Witten used Hodge theory for a twisted Laplace operator in order to recover the Morse inequalities. This motivated G. Laumon for using the Fourier transformation in the ℓ-adic setting and to exploit its purity properties. Up to recently, the ideas of Witten had not been extended (except some special cases) in the complex setting because of the non-compactness of the spaces on which the analogue of a Morse function is defined.

Witten considers a Morse function $f : X \to \mathbb{R}$ on a compact C^∞ manifold (equipped with some Riemannian metric). He considers the twisted de Rham complex $(\mathscr{A}^\bullet(X), \mathrm{d} + \tau \mathrm{d}f)$ for some real number τ and analyzes the behaviour of the corresponding Laplace operator Δ_τ (with respect to the chosen metric) when $|\tau| \to \infty$. For a fixed τ, this operator is an elliptic self-adjoint (unbounded) operator on the spaces of L^2-forms $\mathscr{L}_2^\bullet(X)$ and is positive with discrete spectrum.

When $\tau \to +\infty$, Witten shows that the eigenvectors of Δ_τ concentrate around the critical points of f in the following sense: for any closed set $K \subset X$ containing no critical point, there exists positive constants C, A such that any eigenvector η with eigenvalue λ such that $|\lambda| \leqslant A$ satisfies

$$\|\eta\|_{L^2(K)} \leqslant \frac{C}{\tau} \|\eta\|_{L^2(X)}.$$

In some sense, Witten introduces Hodge theory in the classical stationary phase method.

4.3. Local Fourier transformation and Hodge theory

We will assume that M is a regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module and, in §4.3.b that M is equipped with extended Hodge data as in §3.2.c, so that in particular $M_{|\mathbb{A}^{1\,\mathrm{an}} \smallsetminus \{c_1,\dots,c_r\}} = (V, \nabla)$ underlies a polarized variation of Hodge structure.

4.3.a. Local Fourier transformation from the point of view of nearby/vanishing cycles. We only need to consider the local Fourier transforms $\mathscr{F}^{(c,\infty)}M$ ($c \in \mathbb{C}$) and $\mathscr{F}^{(\infty,0)}M$, according to the slope correspondence of §2.2.c, since M has the only slope 0 at ∞.

Notice that the second one produces a regular holonomic $\mathbb{C}[\![\tau]\!]\langle\partial_\tau\rangle$-module, an object which is completely determined by $\psi_\tau^{\mathrm{mod}\,\widehat{0}}$ and $\phi_\tau^{\mathrm{mod}\,\widehat{0}}$ together with the "canonical" and the "variation" morphisms between both, according to the classification theorem of regular holonomic $\mathbb{C}[\![\tau]\!]\langle\partial_\tau\rangle$-modules.

The idea to express the behaviour of the various local Fourier transformations on nearby/vanishing cycle is to use the integral formula (7) of Lecture 1 and the good behaviour of the functors $\psi_\tau^{\mathrm{mod}}, \phi_\tau^{\mathrm{mod}}$ with respect to the proper push-forward by $\widehat{\pi}$.

Let us set ${}^{\mathscr{F}}\!\mathscr{M} = \pi^+\mathscr{M} \otimes \mathscr{E}^{t\tau}$. Then

$$\begin{aligned}
\psi_\tau^{\mathrm{mod}\,\widehat{0}}({}^{F}\!M) &= \boldsymbol{R}\Gamma(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}\,\psi_\tau^{\mathrm{mod}\,\widehat{0}}({}^{\mathscr{F}}\!\mathscr{M})),\\
\phi_\tau^{\mathrm{mod}\,\widehat{0}}({}^{F}\!M) &= \boldsymbol{R}\Gamma(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}\,\phi_\tau^{\mathrm{mod}\,\widehat{0}}({}^{\mathscr{F}}\!\mathscr{M})),\\
\psi_{\tau'}^{\mathrm{mod}\,\widehat{\infty}}({}^{F}\!M) &= \boldsymbol{R}\Gamma(\mathbb{P}^1, {}^{\mathrm{p}}\mathrm{DR}\,\psi_{\tau'}^{\mathrm{mod}\,\widehat{\infty}}({}^{\mathscr{F}}\!\mathscr{M})),
\end{aligned}$$

where here ${}^{\mathrm{p}}\mathrm{DR}(\bullet)$ means $\mathrm{DR}(\bullet)[1]$.

4.3.b. The local Fourier transformation $\mathscr{F}^{(\infty,0)}$. The V-filtration of ${}^{\mathscr{F}}\!\mathscr{M}$ along $\tau=0$ can be computed explicitly and we will not explain the computation here which can be found in [**Sab06**, proof of Prop. 4.1]. A priori, $\psi_\tau^{\mathrm{mod}\,\widehat{0}}({}^{\mathscr{F}}\!\mathscr{M}), \phi_\tau^{\mathrm{mod}\,\widehat{0}}({}^{\mathscr{F}}\!\mathscr{M})$ are holonomic $\mathscr{D}_{\mathbb{P}^1}$-modules equipped with an automorphism T_τ.

Proposition 1. *The $\mathscr{D}_{\mathbb{P}^1}$-module $\phi_\tau^{\mathrm{mod}\,\widehat{0}}({}^{\mathscr{F}}\!\mathscr{M})$ is supported at ∞ and the corresponding vector space with automorphism T_τ is isomorphic (canonically and functorially) to the nearby cycle space $(\psi_{t'}^{\mathrm{mod}\,\infty}\mathscr{M}, \mathrm{T}_{t'})$.*

It remains to understand $\psi_{\tau,0}^{\mathrm{mod}\,\widehat{0}}({}^{\mathscr{F}}\!\mathscr{M})$, which does not have punctual support, and is a global object on $\mathbb{P}^1$. In order to state the result, let us denote by $\mathscr{M}_{\min}$ the minimal (also called intermediate) extension of $\mathscr{M}$ at ∞. It is the unique submodule $\mathscr{N}$ of $\mathscr{M}$ with no quotient supported at ∞ and such that $\mathscr{O}_{\mathbb{P}^1}(*\infty)\otimes\mathscr{N}=\mathscr{M}$. We then have the following complement to Proposition 1

Proposition 2. *The holonomic $\mathscr{D}_{\mathbb{P}^1\times\mathbb{A}^1}$-module ${}^{\mathscr{F}}\!\mathscr{M}$ is a minimal extension of its localization along $\mathbb{P}^1\times\{\widehat{0}\}$, so that the holonomic $\mathscr{D}_{\mathbb{P}^1}$-module with automorphism $(\psi_\tau^{\mathrm{mod}\,\widehat{0}}({}^{\mathscr{F}}\!\mathscr{M}), \mathrm{T}_\tau)$ is completely determined from $(\psi_\tau^{\mathrm{mod}\,\widehat{0}}({}^{\mathscr{F}}\!\mathscr{M}), \mathrm{T}_\tau)$ and the primitive part $P_0\psi_{\tau,0}^{\mathrm{mod}\,\widehat{0}}({}^{\mathscr{F}}\!\mathscr{M})=\ker\mathrm{N}_\tau\cap\operatorname{Im}\mathrm{N}_\tau$ in $\psi_{\tau,0}^{\mathrm{mod}\,\widehat{0}}({}^{\mathscr{F}}\!\mathscr{M})$.*

There is a canonical and functorial isomorphism

$$P_0\psi_{\tau,0}^{\mathrm{mod}\,\widehat{0}}({}^{\mathscr{F}}\!\mathscr{M})\simeq\mathscr{M}_{\min}.$$

Let us now go back to ${}^{F}\!M$. As a consequence of the compatibility of taking proper direct images by $\widehat{\pi}$ and nearby/vanishing cycles we obtain

***Corollary 3* (of Proposition 1).** *For any regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module M, we have a canonical and functorial isomorphism*

$$(\phi_\tau^{\mathrm{mod}\,\widehat{0}}({}^{F}\!M), \mathrm{T}_\tau)\simeq(\psi_{t'}^{\mathrm{mod}\,\infty}\mathscr{M}, \mathrm{T}_{t'}).$$

On the other hand, although $\psi_{\tau,0}^{\mathrm{mod}\,\widehat{0}}$ is compatible with taking direct image by $\widehat{\pi}$, the operation of taking P_0 need not be so. In the present setting the assumption that M is a semi-simple regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module is enough to ensure this compatibility. In particular, with such an assumption, ${}^{F}\!M$ is also (obviously) semi-simple as a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module, and this implies that ${}^{F}\!M$ is the minimal extension of its localization at $\tau=0$. As a consequence, $(\psi_\tau^{\mathrm{mod}\,\widehat{0}}({}^{F}\!M), \mathrm{T}_\tau)$ is completely determined from $(\psi_\tau^{\mathrm{mod}\,\widehat{0}}({}^{F}\!M), \mathrm{T}_\tau)$ and the primitive part $P_0\psi_{\tau,0}^{\mathrm{mod}\,\widehat{0}}({}^{F}\!M)=\ker\mathrm{N}_\tau\cap\operatorname{Im}\mathrm{N}_\tau$ in $\psi_{\tau,0}^{\mathrm{mod}\,\widehat{0}}({}^{F}\!M)$.

***Corollary 4* (of Proposition 2).** *If M is a semi-simple regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module, then we have a canonical and functorial isomorphism*

$$P_0\psi_{\tau,0}^{\mathrm{mod}\,\widehat{0}}({}^{F}\!M)\simeq\boldsymbol{H}^1(\mathbb{P}^1, \mathrm{DR}\,\mathscr{M}_{\min}).$$

Let us now consider the case where $(M, F^\bullet M)$ is a filtered Hodge module, as obtained from a variation of polarized Hodge structure in §3.2.c. In order to mimick the previous results, it is convenient to introduce the ring $R_F\mathbb{C}[t]\langle\partial_t\rangle = \bigoplus_{k\in\mathbb{Z}} F^k\mathbb{C}[t]\langle\partial_t\rangle z^{-k}$, where $F^k\mathbb{C}[t]\langle\partial_t\rangle \subset \mathbb{C}[t]\langle\partial_t\rangle$ consists of operators of degree $\leqslant -k$ with respect to ∂_t. It is identified with the ring $\mathbb{C}[t,z]\langle z\partial_t\rangle$. The filtered $\mathbb{C}[t]\langle\partial_t\rangle$-module $(M, F^\bullet M)$ gives rise to a $R_F\mathbb{C}[t]\langle\partial_t\rangle$-module $R_F M = \bigoplus_k F^k M z^{-k}$, which has finite type because $F^\bullet M$ is a good filtration. One can define the filtered Laplace transform by using the kernel $e^{-t\tau/z}$, that is, by replacing t with $-z\partial_\tau$ and $z\partial_t$ with τ. This produces a $R_F\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module ${}^F\!R_F M$, which is not graded however, and therefore does cannot be realized as $R_F{}^F\!M$ for some filtration on ${}^F\!M$, in general.

Nevertheless, the formalism of the V-filtration can be adapted to the ring $R_F\mathbb{C}[\tau]\langle\partial_\tau\rangle$.

4.3.c. The local Fourier transformations $\mathscr{F}^{(c,\infty)}$.

4.4. Limits of the new supersymmetric index

LECTURE 5

APPLICATION TO e^f

5.1. The twisted de Rham complex

Let U be a smooth complex quasi-projective variety and let $f : U \to \mathbb{A}^1$ be a regular function on U. What kind of a structure do we have on the twisted de Rham cohomology $H^*(U, f; \mathbb{C}) := \boldsymbol{H}^*\big(U, (\Omega_U^\bullet, \mathrm{d} - \mathrm{d}f)\big)$? This is related to the properties of exponential periods, i.e., integrals $\int_\Gamma e^f \omega$, ω an algebraic form on U, Γ a locally closed cycle. The $\mathbb{Q}$-structure is understood: $H^*(U^{\mathrm{an}}, f; \mathbb{Q}) = H^*(U^{\mathrm{an}}, \mathrm{Re}(f) \gg 0; \mathbb{Q})$.

Deligne remarked in 1984 that the classical formula

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$

suggests that for $f(x) = -x^2 : \mathbb{A}^1 \to \mathbb{A}^1$, $H^*(U, f; \mathbb{C})$ has a Hodge structure of type $(1/2, 1/2)$. In the case where U is a curve, Deligne defines a filtration of the complex $(\Omega_U^\bullet, \mathrm{d} - \mathrm{d}f)$ indexed by rational numbers, and shows a E_1-degeneracy property, looking like the standard Hodge $\Rightarrow$ de Rham degeneracy.

5.1.a. Reduction to dimension one.

Let us introduce a parameter τ and consider the complex $(\Omega_U^\bullet[\tau], \mathrm{d} - \tau\mathrm{d}f)$. Its hypercohomology modules, regarded as $\mathbb{C}[t]\langle\partial_t\rangle$-modules as in §1.2.b, are nothing but the Gauss-Manin systems $\mathrm{GM}^i(f) := \mathscr{H}^i f_+ \mathscr{O}_U$ of f. According to Bernstein's theorem (see §1.2.b), they are holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-modules. When regarded as $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-modules, they are the Laplace transforms ${}^F\mathrm{GM}^i(f)$, with respect to the Laplace transformation with kernel $e^{-t\tau}$.

We can regard $(\Omega_U^\bullet, \mathrm{d} - \mathrm{d}f)$ as the cone of the morphism

$$(\Omega_U^\bullet[\tau], \mathrm{d} - \tau\mathrm{d}f) \xrightarrow{\ \tau - 1\ } (\Omega_U^\bullet[\tau], \mathrm{d} - \tau\mathrm{d}f).$$

Let us consider the associated long exact sequence

$$\cdots \longrightarrow {}^F\mathrm{GM}^i(f) \xrightarrow{\ \tau - 1\ } {}^F\mathrm{GM}^i(f) \longrightarrow \boldsymbol{H}^i\big(U, (\Omega_U^\bullet, \mathrm{d} - \mathrm{d}f)\big) \longrightarrow \cdots$$

We note that, since $\mathrm{GM}^i(f)$ has regular singularities, $\mathbb{C}[\tau, \tau^{-1}] \otimes_{\mathbb{C}[\tau]} {}^F\mathrm{GM}^i(f)$ is $\mathbb{C}[\tau, \tau^{-1}]$-free, hence $\tau - 1$ is injective on each ${}^F\mathrm{GM}^i(f)$. It follows that, for each i,

$$\boldsymbol{H}^i\big(U, (\Omega_U^\bullet, \mathrm{d} - \mathrm{d}f)\big) = {}^F\mathrm{GM}^i(f)/(\tau - 1){}^F\mathrm{GM}^i(f).$$

In other words, for each i, one can compute $\boldsymbol{H}^i\big(U,(\Omega_U^\bullet,\mathrm{d}-\mathrm{d}f)\big)$ as the hypercohomology $\boldsymbol{H}^1(\mathbb{A}^1,\mathrm{DR}(\mathrm{GM}^i(f),\nabla-dt))$ of the de Rham complex of $\mathrm{GM}^i(f)$ twisted by the function $\mathrm{Id}:\mathbb{A}^1\to\mathbb{A}^1$ (and all other $\boldsymbol{H}^j$, $j\neq 1$, are zero).

5.1.b. Topological computation of the twisted de Rham cohomology. Let $F:X\to\mathbb{A}^1$ be a projective completion of $f:U\to\mathbb{A}^1$ and let us denote by $j:U\hookrightarrow X$ the open inclusion. For each complex number $c\in\mathbb{C}$, let $\phi_{F-c}(\boldsymbol{R}j_*\mathbb{C}_{U^{\mathrm{an}}})$ denote the vanishing cycle complex along $f=c$ of the complex $\boldsymbol{R}j_*\mathbb{C}_{U^{\mathrm{an}}}$.

Proposition 1. *Assume that X is smooth and $X\smallsetminus U$ is a divisor in X. Then for each i, we have*

$$\dim\boldsymbol{H}^i\big(U,(\Omega_U^\bullet,\mathrm{d}-\mathrm{d}f)\big)=\sum_{c\in\mathbb{C}}\dim\boldsymbol{H}^{i-1}\big(F^{-1}(c),\phi_{F-c}(\boldsymbol{R}j_*\mathbb{C}_{U^{\mathrm{an}}})\big).$$

Sketch or proof. Since X is smooth and $X\smallsetminus U$ is a divisor, $\boldsymbol{R}j_*\mathbb{C}_{U^{\mathrm{an}}}$ is perverse on X (up to a shift, depending on the convention). Then $\phi_{F-c}(\boldsymbol{R}j_*\mathbb{C}_{U^{\mathrm{an}}})$ is perverse up to a shift. On the other hand, using the commutation of ϕ_{F-c} with proper direct images and taking perverse cohomology of direct images reduces the result to an analogous result for any regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module M (proved e.g. in [**Mal91**]):

$$\dim{}^F\!M/(\tau-1){}^F\!M\;(=\mathrm{rk}\,\mathbb{C}[\tau,\tau^{-1}]\otimes_{\mathbb{C}[\tau]}{}^F\!M)=\sum_{c\in\mathbb{C}}\dim\phi_{t-c}\,\mathrm{DR}^{\mathrm{an}}M. \tag{2}$$ $\square$

5.2. The stationary phase formula for a function

A particular case of the stationary phase formula proved in Lecture 2 reads as follows for a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module M with only regular singularities. Let ${}^F\!M$ be the Laplace transform of M. Then $G:=\mathbb{C}[\tau,\tau^{-1}]\otimes_{\mathbb{C}[\tau]}{}^F\!M$ is a free $\mathbb{C}[\tau,\tau^{-1}]$-module whose rank is equal to $\mu:=\sum_{c\in\mathbb{C}}\dim\phi_{t-c}\,\mathrm{DR}^{\mathrm{an}}M$. Set $z=\tau^{-1}$ and $\widehat{G}=\mathbb{C}(\!(z)\!)\otimes_{\mathbb{C}[z,z^{-1}]}G$. This is a $\mathbb{C}(\!(z)\!)$-vector space of dimension μ equipped with a connection ∇_{∂_z} (induced by by the action of $-\tau^2\partial_\tau$ on G).

On the other hand, for each $c\in\mathbb{C}$, the $\mathbb{C}$-vector space $\phi_{t-c}\,\mathrm{DR}^{\mathrm{an}}M$ comes equipped with a monodromy operator T. In general, let E be a finite dimensional $\mathbb{C}$-vector space equipped with an automorphism T. Given a choice of a logarithm of T, that is, writing $\mathrm{T}=\exp(-2\pi i\mathrm{M})$ for some $\mathrm{M}:E\to E$, we denote by $\widehat{\mathrm{RH}}^{-1}(E,\mathrm{T})$ the $\mathbb{C}(\!(z)\!)$-vector space $E(\!(z)\!)$ equipped with the connection $\mathrm{d}+\mathrm{M}\mathrm{d}z/z$. Then Formula (2) above can be refined as

$$(\widehat{G},\nabla_{\partial_z})\simeq\bigoplus_{c\in\mathbb{C}}\Big(\widehat{\mathrm{RH}}^{-1}(\phi_{t-c}\,\mathrm{DR}^{\mathrm{an}}M,\mathrm{T}),\nabla-\mathrm{d}(c/z)\Big).$$

If we set for short

$$\begin{aligned}G^i(f)=\mathbb{C}[\tau,\tau^{-1}]\otimes_{\mathbb{C}[\tau]}{}^F\mathrm{GM}^i(f)&=\boldsymbol{H}^i(\Omega_U^\bullet[\tau,\tau^{-1}],\mathrm{d}-\tau\mathrm{d}f)\\&=\boldsymbol{H}^i(\Omega_U^\bullet[z,z^{-1}],\mathrm{d}-\mathrm{d}f/z),\end{aligned}$$

we obtain, by an argument similar to that of Proposition 1,

$$(\widehat{G}^i(f),\nabla_{\partial_z})\simeq\bigoplus_{c\in\mathbb{C}}\Big(\widehat{\mathrm{RH}}^{-1}\big(\boldsymbol{H}^{i-1}\big(f^{-1}(c),\phi_{f-c}(\boldsymbol{R}j_*\mathbb{C}_{U^{\mathrm{an}}})\big),\mathrm{T}\big),\nabla-\mathrm{d}(c/z)\Big).$$

This formula can be refined in order to only use $\phi_{f-c}(\mathbb{C}_{U^{\mathrm{an}}})$, i.e., the vanishing cycles at finite distance only. For a sheaf $\mathscr{F}$ on U, $\mathscr{F}(\!(z)\!)$ denotes the sheaf associated to the presheaf $V\mapsto\mathscr{F}(V)(\!(z)\!)$. The sheaf $\mathscr{O}_U(\!(z)\!)$ is $\mathscr{O}_U$-flat.

***Theorem 3* (conjectured by M. Kontsevich).** *We have, for each i,*

$$\Big(\boldsymbol{H}^i\big(U,(\Omega_U^\bullet(\!(z)\!),\mathrm{d}-\mathrm{d}f/z)\big),\nabla_{\partial_z}\Big)\simeq\bigoplus_{c\in\mathbb{C}}\Big(\widehat{\mathrm{RH}}^{-1}\Big[\boldsymbol{H}^{i-1}\big(f^{-1}(c),\phi_{f-c}\mathbb{C}_{U^{\mathrm{an}}}\big),\mathrm{T}\Big],\nabla-\mathrm{d}(c/z)\Big).$$

Sketch of proof. □

5.3. The Barannikov-Kontsevich theorem

It may be instructive to recall here the Barannikov-Kontsevich theorem. It applies if f is projective, that is, so that $F=f$.

***Theorem 4* (see [Sab99, OV07]).** *Assume that f is projective. Then, for each i, the $\mathbb{C}[z]$-module $\boldsymbol{H}^i\big(U,(\Omega_U^\bullet[z],z\mathrm{d}-\mathrm{d}f)$ is free.*

As a consequence, setting $z=0$, $z=1$ and using Proposition 1, we obtain (under our assumption, $F=f$ and $j=\mathrm{Id}$, and $U=X$):

$$\dim\boldsymbol{H}^i\big(X,(\Omega_X^\bullet,\mathrm{d}f)\big)=\sum_{c\in\mathbb{C}}\dim\boldsymbol{H}^{i-1}\big(f^{-1}(c),\phi_{f-c}(\mathbb{C}_{X^{\mathrm{an}}})\big).$$

5.4. The Deligne filtration

LECTURE 6

STOKES STRUCTURES IN DIMENSION ONE

Let X be a Riemann surface. The functor which associates to any holomorphic vector bundle with connection (V, ∇) the locally constant sheaf of its local sections is an equivalence of categories. The Riemann-Hilbert correspondence extends as a correspondence between regular holonomic $\mathscr{D}_X$-modules and perverse sheaves of $\mathbb{C}$-vector spaces on X. In 1978 and later (see [**DMR07**]), Deligne proposed to extend the previous correspondence, starting from arbitrary holonomic $\mathscr{D}_X$-modules, with values in a category of perverse sheaves with Stokes structure. The notion of a Stokes filtration is intended to express in a topological way the classical notion of Stokes data at an irregular singular point.

6.1. The Riemann-Hilbert correspondence for germs of meromorphic connections

Let Δ be a small disc with coordinate x and let M be a $\mathscr{O}_\Delta(*0)$-free module of finite rank with connection ∇. We will only consider its germ at 0, which is a $\mathbb{C}(\{x\})$-vector space of finite dimension with a connection. It can also be regarded as a germ at $x = 0$ of a $\mathscr{D}_\Delta = \mathscr{O}_\Delta\langle\partial_x\rangle$-module on which left multiplication by t is bijective.

Let $(\widehat{M}, \widehat{\nabla}) = (\mathbb{C}(\!(x)\!) \otimes_{\mathbb{C}(\{x\})} M, \text{induced } \nabla)$ be the associated formal meromorphic connection. A classical theorem called Levelt-Turrittin theorem classifies such objects: after a ramification $x = y^m$ for some $m \in \mathbb{N}$, $\widehat{M}' := \mathbb{C}(\!(y)\!) \otimes_{\mathbb{C}(\!(x)\!)} \widehat{M}$, then $\widehat{M}'$ is isomorphic to the direct sum of the elementary objects $(\mathbb{C}(\!(y)\!)^n, \mathrm{d} + \mathrm{d}\varphi \,\mathrm{Id} + C\mathrm{d}y/y)$, $\varphi \in \mathbb{C}(\!(y)\!)$ (modulo $\mathbb{C}[\![y]\!]$) and C is a constant matrix. One says that M has a regular singularity if only $\varphi = 0$ (or $\varphi \in \mathbb{C}[\![y]\!]$ can occur, and in such a case the ramification is not needed. Moreover, the isomorphism already exists at the convergent level.

On the other hand, when some nonzero $\varphi \in \mathbb{C}(\!(y)\!)/\mathbb{C}[\![y]\!]$ occur, the isomorphism may not exist at the convergent level.

For the sake of simplicity, I will assume below that the ramification is not needed for the M that I will consider, and I will say in such a case that M *has no ramification.*

Let $\widetilde{\Delta} = S^1 \times [0, \varepsilon)$ be the space of polar coordinates $x = re^{i\theta}$. The operator $x\partial_x$ lifts as $\frac{1}{2}(r\partial_r - i\partial_\theta)$. Let $\mathscr{A}_{\widetilde{\Delta}}^{\mathrm{mod}}$ be the sheaf on $\widetilde{\Delta}$ of holomorphic functions on Δ^* which have moderate growth in the neighbourhood of any compact set of $S^1 \times \{0\}$. For any $\varphi \in x^{-1}\mathbb{C}[x^{-1}]$, the sheaf $\ker \nabla : e^{\varphi}\mathscr{A}_{\widetilde{\Delta}}^{\mathrm{mod}} \otimes M \to e^{\varphi}\mathscr{A}_{\widetilde{\Delta}}^{\mathrm{mod}} \otimes M$ is a local system $\mathscr{L}$ of rank $\dim M$ on Δ^* and is a subsheaf $\mathscr{L}_{\leqslant\varphi} \subset \mathscr{L}$ on $S^1 \times \{0\}$.

Example 1. If $M = (\mathbb{C}(\!(x)\!), \mathrm{d} + \mathrm{d}\varphi_o)$, then $\mathscr{L}_{\leqslant\varphi,\theta} = \mathbb{C}$ if $\mathrm{Re}(\varphi - \varphi_o) < 0$ near θ, and $\mathscr{L}_{\leqslant\varphi,\theta} = 0$ otherwise, while $\mathscr{L} = \mathbb{C}_{S^1}$.

***Theorem 2* (Riemann-Hilbert correspondence).** *The functor $M \mapsto (\mathscr{L}, \mathscr{L}_\bullet)$ is an equivalence between the category of (non-ramified) germs of meromorphic connections and the category of Stokes-filtered local systems on S^1.*

6.2. Stokes filtered local systems on S^1 (the non-ramified case)

We consider the circle of directions S^1 equipped with the constant sheaf $\mathcal{I}_1$ with fibre $\mathcal{P} = \mathbb{C}(\{x\})/\mathbb{C}\{x\}$ consisting of polar parts of Laurent series. This sheaf is a sheaf of ordered groups: the order depends on the point $e^{i\theta} = x/|x| \in S^1$ as follows. Let $\eta \in \mathcal{P}$ and let us set $\eta = u_n(x)x^{-n}$ with $n \geqslant 1$ and $u_n(0) \neq 0$ if $\eta \neq 0$. Then

$$\eta \leqslant_\theta 0 \iff \eta = 0 \text{ or } \arg u_n(0) - n\theta \in (\pi/2, 3\pi/2) \mod 2\pi, \tag{3}$$

and $\eta <_\theta 0 \Leftrightarrow (\eta \leqslant_\theta 0$ and $\eta \neq 0)$. The order is supposed to be compatible with addition, namely, $\varphi \leqslant_\theta \psi \Leftrightarrow \varphi - \psi \leqslant_\theta 0$ and similarly for $<_\theta$. We also have $\varphi \leqslant_\theta \psi \Leftrightarrow e^{\varphi-\psi}$ has moderate growth in some neighbourhood of $(e^{i\theta}, 0)$ in $\widetilde{\Delta}$.

Definition 4. Let $\boldsymbol{k}$ be a field. A *non-ramified pre-Stokes filtration* on a local system $\mathscr{L}$ of finite dimensional $\boldsymbol{k}$-vector spaces on S^1 consists of the data of a family of subsheaves $\mathscr{L}_{\leqslant\varphi}$ indexed by $\mathcal{P}$ such that, for any $\theta \in S^1$, $\varphi \leqslant_\theta \psi \Rightarrow \mathscr{L}_{\leqslant\varphi,\theta} \subset \mathscr{L}_{\leqslant\psi,\theta}$.

Let us set, for any $\varphi \in \mathcal{P}$ and any $\theta \in S^1$,

$$\mathscr{L}_{<\varphi,\theta} = \sum_{\psi <_\theta \varphi} \mathscr{L}_{\leqslant\psi,\theta}. \tag{5}$$

This defines a subsheaf $\mathscr{L}_{<\varphi}$ of $\mathscr{L}_{\leqslant\varphi}$, and we set $\mathrm{gr}_\varphi \mathscr{L} = \mathscr{L}_{\leqslant\varphi}/\mathscr{L}_{<\varphi}$.

Definition 6. A pre-Stokes filtration of $\mathscr{L}$ is called a *Stokes filtration* if

(1) each graded sheaf $\mathrm{gr}_\varphi \mathscr{L}$ is a local system on S^1,
(2) $\mathrm{rk}\,\mathscr{L} = \sum_\varphi \mathrm{rk}\,\mathrm{gr}_\varphi \mathscr{L}$.

Note that when 6(1) is satisfied, 6(2) is equivalent to one of the following:

(2′) for any $\theta \in S^1$ and any $\varphi \in \mathcal{P}$, $\dim \mathscr{L}_{\leqslant\varphi,\theta} = \sum_{\psi \leqslant_\theta \varphi} \dim \mathrm{gr}_\psi \mathscr{L}_\theta$.
(2″) For any $\theta \in S^1$ and any $\varphi \in \mathcal{P}$, $\dim \mathscr{L}_{<\varphi,\theta} = \sum_{\psi <_\theta \varphi} \dim \mathrm{gr}_\psi \mathscr{L}_\theta$.

The finite subset $\Phi \subset \mathcal{P}$ such that $\mathrm{gr}_\varphi \mathscr{L} \neq 0 \Rightarrow \varphi \in \Phi$ is called the *set of exponential factors* of the non-ramified Stokes filtration. The following proposition is easily checked, showing more precisely exhaustivity.

Proposition 7. *Let $\mathscr{L}_\bullet$ be a non-ramified $\boldsymbol{k}$-Stokes filtration on $\mathscr{L}$. Then, for any $\theta \in S^1$, and any $\varphi \in \mathcal{P}$,*

- *if $\varphi <_\theta \Phi$, then $\mathscr{L}_{\leqslant\varphi,\theta} = 0$,*
- *if $\Phi <_\theta \varphi$, then $\mathscr{L}_{<\varphi,\theta} = \mathscr{L}_{\leqslant\varphi,\theta} = \mathscr{L}_\theta$.* □

6.3. Abelianity and strictness (the non-ramified case)

Definition 8. A *morphism* $\lambda : (\mathscr{L}, \mathscr{L}_\bullet) \to (\mathscr{L}', \mathscr{L}'_\bullet)$ of non-ramified $\boldsymbol{k}$-Stokes-filtered local systems is a morphism of local systems $\mathscr{L} \to \mathscr{L}'$ on S^1 such that, for any $\varphi \in \mathcal{P}$, $\lambda(\mathscr{L}_{\leqslant\varphi}) \subset \mathscr{L}'_{\leqslant\varphi}$. According to (5), a morphism also satisfies $\lambda(\mathscr{L}_{<\varphi}) \subset \mathscr{L}'_{<\varphi}$. A morphism λ is said to be *strict* if, for any φ, $\lambda(\mathscr{L}_{\leqslant\varphi}) = \lambda(\mathscr{L}) \cap \mathscr{L}'_{\leqslant\varphi}$.

Theorem 9. *The category of $\boldsymbol{k}$-Stokes-filtered local systems on S^1 is abelian and every morphism is strict. Moreover, it is stable by extension in the category of pre-Stokes-filtered sheaves.*

Of course, when $\boldsymbol{k} \subset \mathbb{C}$, one can obtained this result by using the Riemann-Hilbert correspondence and the abelianity of the category of germs of meromorphic connection. It is however instructive to give a proof in the setting of Stokes-filtered local systems in order to understand better their structure. One proves more precisely:

Theorem 10. *Given two non-ramified Stokes-filtered local systems $(\mathscr{L}, \mathscr{L}_\bullet)$ and $(\mathscr{L}', \mathscr{L}'_\bullet)$, there exist trivializations of them in the neighbourhood of any point of S^1 such that any morphism λ between them is diagonal with respect to these local trivializations, hence is strict. In particular, such a morphism satisfies* (**??**)*, and the natural pre-Stokes filtrations on the local systems* $\ker\lambda$*,* $\mathrm{Im}\,\lambda$ *and* $\mathrm{coker}\,\lambda$ *are Stokes filtrations. Their sets of exponential factors satisfy*

$$\Phi(\ker\lambda) \subset \Phi, \quad \Phi(\mathrm{coker}\,\lambda) \subset \Phi', \quad \Phi(\mathrm{Im}\,\lambda) \subset \Phi \cap \Phi'.$$

The proof is done by using the notion of level structure, which will not be developed here.

6.4. Stokes-perverse sheaves on a Riemann surface

Let Δ be a disc as in §6.1 and let $\mathscr{D} = \mathscr{D}_{\Delta,0}$ be the germ at the origin of the sheaf of holomorphic differential operators on Δ. We have $\mathscr{D} = \mathbb{C}\{x\}\langle\partial_x\rangle$. We will also set $\mathscr{O} = \mathbb{C}\{x\}$, $\mathscr{O}(*0) = \mathbb{C}(\{x\})$, $\widehat{\mathscr{O}} = \mathbb{C}[\![x]\!]$, $\widehat{\mathscr{D}} = \mathbb{C}[\![x]\!]\langle\partial_x\rangle$. Let M be a holonomic $\mathscr{D}$-module. We will consider the following objects obtained from M

- $M(*0)$ is the localization $\mathscr{O}_(*0)\otimes_{\mathscr{O}}M=\mathscr{D}(*0)\otimes_{\mathscr{D}}M$; this is a free $\mathscr{O}(*0)$-module of finite rank with connection, as considered in §6.1.
- $\widehat{M}$ is the formalization $\widehat{\mathscr{O}}\otimes_{\mathscr{O}}M=\widehat{\mathscr{D}}\otimes_{\mathscr{D}}M$.
- $\widehat{M}_{\mathrm{reg}}$ is the regular component, i.e., the component corresponding to $\varphi=0$ in the Levelt-Turrittin decomposition of $\widehat{M}$ (a decomposition which also exists for holonomic $\widehat{\mathscr{D}}$-modules).
- $\widehat{\mu}$ is the natural isomorphism $\widehat{M(*0)}\xrightarrow{\sim}(\widehat{M})(*0)$ that we restrict to the regular parts: $\widehat{M(*0)}_{\mathrm{reg}}\xrightarrow{\sim}(\widehat{M}_{\mathrm{reg}})(*0)$.

Lemma 11. *The category of holonomic $\mathscr{D}$-modules is equivalent, via this correspondence, to the category of triples $(\widetilde{M},N,\widehat{\nu})$, where $\widetilde{M}$ is a germ of meromorphic connection, N is a germ of regular holonomic $\mathscr{D}$-module, and $\widehat{\nu}$ is an isomorphism $\widehat{(\widetilde{M})}_{\mathrm{reg}}\xrightarrow{\sim}\widehat{N}$.*

Definition 12. A germ of Stokes-perverse sheaf on Δ at the origin consists of a triple $\big((\mathscr{L},\mathscr{L}_\bullet),\mathscr{F},\widehat{\nu}\big)$, where $(\mathscr{L},\mathscr{L}_\bullet)$ is a Stokes-filtered local system, $\mathscr{F}$ is a germ of perverse sheaf at the origin and $\widehat{\nu}$ is an isomorphism $\mathrm{gr}_0\mathscr{L}\xrightarrow{\sim}(\psi_x\mathscr{F},\mathrm{T})$, where the nearby cycle space $\psi_x\mathscr{F}$ equipped with its monodromy is regarded in a canonical way as a local system on S^1.

***Theorem 13* (R-H for holonomic $\mathscr{D}$-modules).** *The functor $(\widetilde{M},N,\widehat{\nu})\mapsto\big((\mathscr{L},\mathscr{L}_\bullet),\mathscr{F},\widehat{\nu}\big)$, whose first component is the R-H functor of Theorem 2, and the second one is the R-H correspondence for regular holonomic $\mathscr{D}$-modules, is an equivalence of categories.*

We can now give the definition of a Stokes-perverse sheaf on a Riemann surface X:

Definition 14. A Stokes-perverse sheaf on a Riemann surface X with singular set contained in a divisor D consists of the data $\big((\mathscr{L},\mathscr{L}_\bullet),\mathscr{F},\widehat{\nu}\big)$, where $\mathscr{L}$ is a local system on $X\smallsetminus D$ (equivalently on the real blown-up space $\widetilde{X}(D)$, which is a Riemann surface with boundary), $\mathscr{L}_\bullet$ consists, for each $p\in D$, of a Stokes filtration of $\mathscr{L}_{|S^1_p}$, $\mathscr{F}$ is a germ of perverse sheaf at each point of D and $\widehat{\nu}$ is as in Definition 12.

Theorem 13 extends as an equivalence between the category of holonomic $\mathscr{D}_X$-modules and the category of Stokes-perverse sheaves on X.

LECTURE 7

STOKES STRUCTURES IN HIGHER DIMENSIONS

Due to the work of Kashiwara and Mebkhout, it is now well-known that the de Rham functor induces an equivalence between the category of regular holonomic $\mathscr{D}$-modules on a complex manifold X and the category of perverse sheaves on X with complex coefficients. In particular, this allows one to speak of a rational structure on a holonomic $\mathscr{D}_X$-module M, by considering a rational structure on the perverse sheaf $\mathrm{DR}\, M$.

Is it possible to extend this correspondence to arbitrary holonomic $\mathscr{D}_X$-modules, and to extend the target category, in order to keep an equivalence? In Lecture 6 have indicated how to solve this question in dimension one. In dimension $\geqslant 2$, the construction of such a target category (Stokes-perverse sheaves) is not yet achieved. We will consider a sub-problem, by restricting to the category of meromorphic bundles with flat connection with poles along a divisor D having normal crossings.

7.1. The structure of meromorphic connections

Let X be a complex manifold equipped with a divisor D and let V be a coherent $\mathscr{O}_X(*D)$-module equipped with a flat connection $\nabla : V \to \Omega^1_X \otimes V$. Given $x_o \in D$, what can be said about the germ V_{x_o}, or the formal germ $\widehat{V}_{\widehat{x}_o} := \widehat{\mathscr{O}}_{X,x_o} \otimes_{\mathscr{O}_{X,x_o}} V_{x_o}$? In dimension one, an essential tool is the Levelt-Turrittin theorem mentioned in §6.1.

The basic result is as follows.

Theorem 1 **([Moc09, Moc11], [Ked10, Ked11]).** *Let X' be a smooth complex algebraic variety (resp. a germ of complex analytic manifold) and let (V, ∇) be a meromorphic bundle on X' with a flat connection, holomorphic on a Zariski open setX'^o in X'. Then there exists a projective modification $\pi : X \to X'$ with X smooth, which is an isomorphism above X'^o, such that $D := X \smallsetminus X'^o$ is a normal crossing divisor, and that, for each point $x \in D$, the formalized bundle $(\widehat{\mathscr{O}}_{X,x} \otimes_{\mathscr{O}_x} V, \widehat{\nabla})$ decomposes, after* *a possible ramification around local components of D, as*

$$(\widehat{\mathscr{O}}_{X,x} \otimes_{\mathscr{O}_x} V, \widehat{\nabla}) = \bigoplus_{\varphi} (\widehat{V}_\varphi, \widehat{\nabla}_\varphi),$$

*where φ runs over a finite set (depending on x) of polar parts of meromorphic functions and, for each φ, $\widehat{\nabla}^{\mathrm{reg}}_\varphi := \widehat{\nabla}_\varphi - \mathrm{d}\varphi \otimes \mathrm{Id}_{\widehat{V}_\varphi}$ has regular singularities ([**Del70**]).*

The following complement is important, and in fact is included in the proof of the theorem above. Let us fix local coordinates $(x_1, \dots, x_n)$ centered at $x_o \in D$ so that $D = \{x_1 \cdots x_\ell = 0\}$.

Definition 2. We say that a finite family $\Phi_{x_o} \subset \mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}$ of polar parts is *good* if $\#\Phi_{x_o} = 1$ or, for any $\varphi \neq \psi$ in Φ_{x_o}, the Newton polyhedron $\mathrm{NP}(\varphi - \psi)$ is an octant with vertex in $-\mathbb{N}^\ell \times \{0_{n-\ell}\}$, i.e., $\varphi - \psi = x^{-\boldsymbol{m}_\ell} \cdot u(x)$ with $\boldsymbol{m}_\ell \in \mathbb{N}^\ell \smallsetminus \{0\}$ (or $\varphi - \psi \equiv 0$) and $u \in \mathscr{O}_{X,x_o}$ with $u(0) \neq 0$.

***Complement to Theorem* 1.** *In Theorem* 1, *one can moreover achieve that, at each point of $x \in D$, the family Φ_x is good.*

7.2. Working on the real blow-up space

7.2.a. Real blow-up. Recall that the oriented *real blow-up* space $\widetilde{\mathbb{C}}^\ell$ of $\mathbb{C}^\ell$ along $t_1, \dots, t_\ell$ is the space of polar coordinates in each variable t_j, that is, the product $(S^1 \times \mathbb{R}_+)^\ell$ with coordinates $(e^{i\theta_j}, \rho_j)_{j=1,\dots,\ell}$ and $t_j = \rho_j e^{i\theta_j}$. The *oriented real blowing-up* map $\varpi : \widetilde{\mathbb{C}}^\ell \to \mathbb{C}^\ell$ induces a diffeomorphism $\{\rho_1 \cdots \rho_\ell \neq 0\} =: (\widetilde{\mathbb{C}}^\ell)^* \xrightarrow{\sim} (\mathbb{C}^\ell)^* := \{t_1 \cdots t_\ell \neq 0\}$. In the following, we just call them real blow-up space or real blowing-up map.

Let X be a reduced complex analytic space (e.g. a complex manifold) and let $f : X \to \mathbb{C}$ be a holomorphic function on X with zero set $X_0 = X_0(f)$. The oriented real blow-up space of X along f, denoted by $\widetilde{X}(f)$, is the closure in $X \times S^1$ of the graph of the map $f/|f| : X^* = X \smallsetminus X_0 \to S^1$. The real blowing-up map $\varpi : \widetilde{X} \to X$ is the map induced by the first projection. The inverse image $\varpi^{-1}(X_0)$, that we denote by $\partial\widetilde{X}$, is a priori contained in $X_0 \times S^1$.

Let now D be a locally principal divisor in X and let $(U_\alpha)_{\alpha \in A}$ be a locally finite covering of X by open sets U_α such that in each U_α, the divisor D is defined by a holomorphic function $f^{(\alpha)}$. The data $[U_\alpha, f^{(\alpha)}]_{\alpha \in A}$ allow one to define, by gluing the real blow-up spaces $\widetilde{U}_\alpha(f^{(\alpha)})$, a space $\widetilde{X}(D)$. Set $f^{(\alpha)} = u^{(\alpha,\beta)} f^{(\beta)}$ on $U_\alpha \cap U_\beta$. The gluing map is induced by

$$\begin{aligned}(U_\alpha \cap U_\beta) \times (\mathbb{C}^*/\mathbb{R}^*_+) &\longrightarrow (U_\alpha \cap U_\beta) \times (\mathbb{C}^*/\mathbb{R}^*_+)\\ \big(x, (e^{i\theta})\big) &\longmapsto \big(x, (u^{(\alpha,\beta)} e^{i\theta} \bmod \mathbb{R}^*_+)\big).\end{aligned}$$

One checks that the space $\widetilde{X}(D)$ does not depend on the choices made (up to a unique homeomorphism compatible with the projection to X).

Real blow-up along a family of divisors. Let now $(D_j)_{j\in J}$ be a locally finite family of locally principal divisors in X and let f_j be sections $\mathscr{O}_X \to \mathscr{O}_X(D_j)$. The fibre product over X of the $\widetilde{X}(D_j)$ (each defined with f_j), when restricted over $X_J^* := X \smallsetminus \bigcup_j D_j$, is isomorphic to X_J^*. We then define the real blow-up $\widetilde{X}(D_{j\in J})$ as the closure of X_J^* in this fibre product. It is defined up to unique homeomorphism compatible with the projection to X. We usually regard X_J^* as an open analytic submanifold in $\widetilde{X}(D_{j\in J})$.

7.2.b. Sheaves of holomorphic functions on the real blow-up space. We consider a locally finite family $(D_j)_{j\in J}$ of smooth (and reduced) divisors in a smooth complex manifold X, and we set $D = \bigcup_j D_j$. We will assume that D has only normal crossings. We will work on the real blow-ups space on $\widetilde{X} := \widetilde{X}(D_{j\in J})$.

The sheaf of holomorphic functions. The space $\widetilde{X}$ is a C^∞ manifold with corners, and the operators $\overline{x_j\partial_{x_j}}$ $(j = 1, \dots, \ell)$, as well as $\overline{\partial}_{x_k}$ $(k > \ell)$ act on it, by using the usual formula in polar coordinates. The kernel $\mathscr{A}_{\widetilde{X}}$ is a sheaf on $\widetilde{X}$ which coincides with $\mathscr{O}_{X^*}$ on $X^* := X \smallsetminus D$. Moreover, let $\varpi : \widetilde{X} \to X$ denote the real blowing-up map. Then $\mathscr{A}_{\widetilde{X}}$ is naturally a left $\varpi^{-1}\mathscr{D}_X$-module (i.e., one can differentiate a function in $\mathscr{A}_{\widetilde{X}}$ with respect to all x_j).

Given a meromorphic bundle with integrable connection (V, ∇), we associate with it the locally free $\mathscr{A}_{\widetilde{X}}(*D)$-module $\mathscr{A}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathscr{O}_X} V$, equipped with the integrable connection (left $\varpi^{-1}\mathscr{D}_X$-module structure) $\widetilde{\nabla}$.

Results in asymptotic analysis (Sibuya, Majima, C.S., T. Mochizuki) allow one to extend to higher dimensions the classical results in dimension one (see [**Was65, Mal91**]).

Theorem 3. *Let (V, ∇) be a meromorphic flat bundle with integrable connection having poles along a normal crossing divisor D. Let $x_o \in D$ and assume that the formal decomposition (after possibly a ramification) as in Theorem 1 holds at x_o, and that Φ_{x_o} satisfies the goodness property. Then this decomposition can be lifted locally on $\varpi^{-1}(x_o)$ as a decomposition of $\mathscr{A}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathscr{O}_X} V$.*

The sheaf of holomorphic functions with moderate growth. Let $\mathscr{O}_X(*D)$ denotes the sheaf of meromorphic functions on X with poles along D at most. It can also be defined as the subsheaf of $j_*\mathscr{O}_{X^*}$ (with $j : X^* = X \smallsetminus D \hookrightarrow X$ the open inclusion) consisting of holomorphic functions having moderate growth along D.

We define a similar sheaf on $\widetilde{X} := \widetilde{X}(D_{j\in J})$, that we denote by $\mathscr{A}_{\widetilde{X}}^{\mathrm{mod}\,D}$: Given an open set $\widetilde{U}$ of $\widetilde{X}$, a section f of $\mathscr{A}_{\widetilde{X}}^{\mathrm{mod}\,D}$ on $\widetilde{U}$ is a holomorphic function on $U^* := \widetilde{U} \cap X^*$ such that, for any compact set K in $\widetilde{U}$, in the neighbourhood of which D is defined by $g_K \in \mathscr{O}_X(K)$, there exists constants $C_K > 0$ and $N_K \geqslant 0$ such that $|f| \leqslant C_K |g_K|^{-N_K}$ on K.

The moderate de Rham complex. The sheaf $\mathscr{A}_{\widetilde{X}}^{\mathrm{mod}\,D}$ is stable by derivations of X (in local coordinates) and there is a natural de Rham complex on $\widetilde{X}(D_{j\in J})$:

$$\mathrm{DR}^{\mathrm{mod}\,D}(\mathscr{O}_X) := \{\mathscr{A}_{\widetilde{X}}^{\mathrm{mod}\,D} \xrightarrow{d} \mathscr{A}_{\widetilde{X}}^{\mathrm{mod}\,D} \otimes \varpi^{-1}\Omega_X^1 \longrightarrow \cdots\}$$

When restricted to X^*, this complex is nothing but the usual holomorphic de Rham complex.

Let (V, ∇) be a meromorphic bundle with integrable connection having poles along D. We associate with (V, ∇) the *moderate de Rham complex*

$$\mathrm{DR}^{\mathrm{mod}\,D}(V, \nabla) := \{\mathscr{A}_{\widetilde{X}}^{\mathrm{mod}\,D} \otimes \varpi^{-1}(V) \xrightarrow{\nabla} \mathscr{A}_{\widetilde{X}}^{\mathrm{mod}\,D} \otimes \varpi^{-1}(\Omega^1 \otimes V_X) \to \cdots\}$$

which coincides with $\mathrm{DR}(V, \nabla)$ on X^*.

Set $\mathscr{V} = \ker \nabla : V_{|X^*} \to \Omega^1_{X^*} \otimes V_{|X^*}$ (this is a locally constant sheaf of finite dimensional $\mathbb{C}$-vector spaces on X^*) and let $\widetilde{\jmath}$ denote the inclusion $X^* \hookrightarrow \widetilde{X}$.

Theorem 4. *Assume that (V, ∇) is good along D (Definition 2). Then*

$$\mathscr{H}^j \,\mathrm{DR}^{\mathrm{mod}\,D}(V, \nabla) = \begin{cases} 0 & \text{if } j > 0, \\ \widetilde{\jmath}_*\mathscr{V} & \text{if } j = 0 \quad \text{(local system on } \widetilde{X}). \end{cases}$$

7.3. The notion of a Stokes filtration

7.3.a. The sheaf $\mathfrak{I}$ in the case of normal crossings. Let us consider a family $(D_{j\in J})$ of smooth divisors of X whose union D has only normal crossings, and the corresponding real blowing-up map $\varpi : \widetilde{X}(D_{j\in J}) \to X$. We will consider multi-integers $\boldsymbol{d} \in (\mathbb{N}^*)^J$. The definition of the sheaves $\widetilde{\mathfrak{I}}_{\boldsymbol{d}}$ and $\mathfrak{I}_{\boldsymbol{d}}$ is similar to that in dimension one.

Let us set $\boldsymbol{1} = (1, \ldots, 1)$ ($\#J$ terms) and $\widetilde{\mathfrak{I}}_{\boldsymbol{1}} = \varpi^{-1}\mathscr{O}_X(*D) \subset \widetilde{\jmath}_*\mathscr{O}_{X^*}$. Let us fix $x_o \in D$, let us denote by $D_1, \ldots, D_\ell$ the components of D going through x_o, and set $\widetilde{x}_o \in \varpi^{-1}(x_o) \simeq (S^1)^\ell$. Then a local section of $\varpi^{-1}\mathscr{O}_X(*D)$ near $\widetilde{x}_o$ is locally bounded in the neighbourhood of $\widetilde{x}_o$ if and only if it is holomorphic in the neighbourhood of x_o. In other words, as in the smooth case, $\varpi^{-1}\mathscr{O}_X(*D) \cap (\widetilde{\jmath}_*\mathscr{O}_{X^*})^{\mathrm{lb}} = \varpi^{-1}(\mathscr{O}_X)$.

We locally define $\widetilde{\mathfrak{I}}_{\boldsymbol{d}}$ near x_o, by using a ramified covering $\rho_{\boldsymbol{d}}$ of (X, x_o) along (D, x_o) of order $\boldsymbol{d} = (d_1, \ldots, d_\ell)$, by the formula $\widetilde{\mathfrak{I}}_{\boldsymbol{d}} := \widetilde{\rho}_{\boldsymbol{d},*}[\varpi_{\boldsymbol{d},*}\mathscr{O}_{X_{\boldsymbol{d}}}(*D)] \cap \widetilde{\jmath}_*\mathscr{O}_{X^*}$, and $\mathfrak{I}_{\boldsymbol{d}}$ by $\mathfrak{I}_{\boldsymbol{d}} := \widetilde{\mathfrak{I}}_{\boldsymbol{d}}/\widetilde{\mathfrak{I}}_{\boldsymbol{d}} \cap (\widetilde{\jmath}_*\mathscr{O}_{X^*})^{\mathrm{lb}}$.

The locally defined subsheaves $\widetilde{\mathfrak{I}}_{\boldsymbol{d}}$ glue together all over D as a subsheaf $\widetilde{\mathfrak{I}}_{\boldsymbol{d}}$ of $\widetilde{\jmath}_*\mathscr{O}_{X^*}$. We also set globally $\mathfrak{I}_{\boldsymbol{d}} = \widetilde{\mathfrak{I}}_{\boldsymbol{d}}/\widetilde{\mathfrak{I}}_{\boldsymbol{d}} \cap (\widetilde{\jmath}_*\mathscr{O}_{X^*})^{\mathrm{lb}}$.

Definition 5. The subsheaf $\widetilde{\mathfrak{I}} \subset \widetilde{\jmath}_*\mathscr{O}_{X^*}$ is the union of the subsheaves $\widetilde{\mathfrak{I}}_{\boldsymbol{d}}$ for $\boldsymbol{d} \in (\mathbb{N}^*)^J$. The sheaf $\mathfrak{I}$ is the subsheaf $\widetilde{\mathfrak{I}}/\widetilde{\mathfrak{I}} \cap (\widetilde{\jmath}_*\mathscr{O}_{X^*})^{\mathrm{lb}}$ of $\widetilde{\jmath}_*\mathscr{O}_{X^*}/(\widetilde{\jmath}_*\mathscr{O}_{X^*})^{\mathrm{lb}}$.

Definition 6. The order on $\widetilde{\mathfrak{I}}$ is given by $\widetilde{\mathfrak{I}}_{\leqslant 0} := \widetilde{\mathfrak{I}} \cap \log \mathscr{A}_{\widetilde{X}(D_{j\in J})}^{\mathrm{mod}\,D}$. It is stable by the addition of an element of $(\widetilde{\jmath}_*\mathscr{O}_{X^*})^{\mathrm{lb}}$ and defines an order on $\mathfrak{I}$.

7.3.b. Stokes-filtered local systems. A Stokes-filtered local system on $\partial\widetilde{X}$ is intended to be a local system $\mathscr{L}$ on $\partial\widetilde{X}$ together with a "filtration by subsheaves indexed by the sheaf of ordered groups $\mathcal{I}$". It is useful to consider the étalé space $\mathcal{I}^{\text{ét}}$ of the sheaf $\mathcal{I}$, together with the map (local homeomorphism) $\mu : \mathcal{I}^{\text{ét}} \to \widetilde{X}$, so that by definition $\Gamma(U, \mathcal{I})$ is the space of continuous sections $s : U \to \mathcal{I}^{\text{ét}}$ of μ. The topological space $\mathcal{I}^{\text{ét}}$ may be quite awful (in particular, not Hausdorff).

A Stokes filtration of $\mathscr{L}$ is therefore a subsheaf $\mathscr{L}_{\leqslant}$ of $\mu^{-1}\mathscr{L}$ (i.e., consists of the data of subsheaves $\mathscr{L}_{\leqslant\varphi}$ of $\mathscr{L}_{|U}$ for any open set $U \subset \partial\widetilde{X}$ and any section $\varphi \in \Gamma(U, \mathcal{I})$, in a way compatible with restrictions $V \subset U$ and such that, if $\psi \leqslant_U \varphi$, then $\mathscr{L}_{\leqslant\psi} \subset \mathscr{L}_{\leqslant\varphi}$. In order to get the full strength of a Stokes filtration, one adds a local splitting property as follows.

For each $I \subset J$, set $D_I = \bigcap_{j\in I} D_i$, and let $D_I^o = D_I \smallsetminus \bigcup_{j\notin I} D_j$. This is a smooth locally closed submanifold in X. When restricted above D_I^o, $\mathcal{I}^{\text{ét}}$ is Hausdorff, and for each φ, $\mathscr{L}_{<\varphi,\theta} := \sum_{\psi <_\theta \varphi,\theta} \mathscr{L}_{\leqslant\psi,\theta}$, for $\theta \in \varpi^{-1}(D_I^o)$, defines a subsheaf of $\mathscr{L}_{\leqslant\varphi|\varpi^{-1}(D_I^o)}$. Therefore, above each stratum D_I^o one can define a graded sheaf $\operatorname{gr}_\varphi\mathscr{L}$. However, the gluing between various strata needs some care.

The splitting property means that, above each stratum D_I^o, the local system $\mathscr{L}$ is locally isomorphic to the direct sum of its graded pieces $\operatorname{gr}_\varphi\mathscr{L}$.

In a more intrinsic way, there exists on each $\mathcal{I}^{\text{ét}}_{|D_I^o}$ a subsheaf $\mathscr{L}_{<}$ of $\mathscr{L}_{\leqslant}$ (the latter exists on the whole $\mathcal{I}^{\text{ét}}$) with suitable gluing properties between strata, such that $\operatorname{gr}\mathscr{L} := \mathscr{L}_{\leqslant}/\mathscr{L}_{<}$ is a local system supported on a finite covering $\widetilde{\Sigma}_I^o \subset \mathcal{I}^{\text{ét}}_{|D_I^o}$ of $\partial\widetilde{X}_{|D_I^o}$. The union $\widetilde{\Sigma} := \widetilde{\Sigma}_I^o$ is a *stratified covering* of $\partial\widetilde{X}$. The set Σ takes into account

- the non-Hausdorff property of $\mathcal{I}^{\text{ét}}$,
- the fact that the exponential factors φ may be multivalued since the fibres of ϖ are homeomorphic to tori $(S^1)^k$,
- the fact that the exponential factors φ may be multivalued since D_I^o may be non simply connected.

Therefore, the notion of Stokes-filtered local system takes into account the global (along D) aspects of the local Stokes phenomena.

7.3.c. The Riemann-Hilbert correspondence.

Definition 7. We say that a stratified covering $\widetilde{\Sigma}$ is good if its fibers above each point of D are good families of polar parts. A Stokes-filtered local system $(\mathscr{L}, \mathscr{L}_\bullet)$ is said to be good if the associated stratifed covering $\widetilde{\Sigma}$ is good.

Theorem 8. *Let $\widetilde{\Sigma}$ be a good stratified $\mathcal{I}$-covering with respect to the (pull-back to $\partial\widetilde{X}(D)$ of the) natural stratification of D. The Riemann-Hilbert functor induces an equivalence between the category of germs of good meromorphic connections along D with stratified $\mathcal{I}$-covering contained in $\widetilde{\Sigma}$ and the category of good Stokes-filtered $\mathbb{C}$-local systems on $\partial\widetilde{X}$ with stratified $\mathcal{I}$-covering contained in $\widetilde{\Sigma}$.*

LECTURE 8

PUSH-FORWARD OF STOKES-FILTERED LOCAL SYSTEMS. APPLICATIONS TO THE FOURIER-LAPLACE TRANSFORMATION

8.1. Push-forward of Stokes-filtered local systems

Let $f : X \to \mathbb{C}$ be a proper holomorphic function on a complex manifold and let D be a divisor with normal crossings in X. Let (V, ∇) be a flat meromorphic bundle with poles along D. Assume that (V, ∇) is good (i.e., satisfies Theorem 1 of Lecture 7 and its complement). We can regard (V, ∇) as a left $\mathscr{D}_X$-module, and consider its direct image (Gauss-Manin system) with respect to f. Each cohomology module $\mathscr{H}^j f_+(V, \nabla)$ is a holonomic $\mathscr{D}_{\mathbb{C}}$-module, which corresponds therefore to a Stokes-perverse sheaf on $\mathbb{C}$. The main question we consider in this lecture is: how to compute this Stokes-perverse sheaf directly from the Stokes-filtered local system attached to (V, ∇)?

We will consider a simpler variant of the previous question, by localizing at some point in $\mathbb{C}$, the origin say, and by assuming that $D = f^{-1}(0)$. We regard now f as a proper holomorphic map $f : X \to \Delta$, where Δ is a small disc centered at 0. The direct image $\mathscr{H}^j f_+(V, \nabla)$ is then a meromorphic connection on Δ, with poles at 0.

Let $\mathscr{V}$ be the local system induced by (V, ∇) on $X^* := X \smallsetminus D$. Setting $n+1 = \dim X$ and assuming f non constant, the local system on $\Delta \smallsetminus \{0\}$ associated with $\mathscr{H}^j f_+(V, \nabla)$ is $R^{n+j} f_* \mathscr{V}$.

The map f induces a map $\widetilde{f} : \widetilde{X} \to \widetilde{\Delta}$, where $\widetilde{X}$ is the real-blow-up space of X along the components of D. It also induces a map $\widetilde{f} : \partial\widetilde{X} \to \widetilde{\Delta} = S^1$. We now denote by $\widetilde{\mathscr{V}}$ the local system on $\partial\widetilde{X}$ obtained by pushing-forward $\mathscr{V}$ by $\widetilde{\jmath} : X^* \hookrightarrow \widetilde{X}$ and then restricting to $\partial\widetilde{X}$. Then $R^{n+j}\widetilde{f}_*\widetilde{\mathscr{V}}$ is a local system on S^1: still denoting by $\widetilde{\jmath} : \Delta^* \hookrightarrow \widetilde{\Delta}$ the inclusion, it is nothing but the restriction to S^1 of $\widetilde{\jmath}_* R^{n+j} f_* \mathscr{V}$.

Let $(\widetilde{\mathscr{V}}, \widetilde{\mathscr{V}}_{\bullet})$ be a good Stokes-filtered local system on $\partial\widetilde{X}$ (e.g. $(\widetilde{\mathscr{V}}, \widetilde{\mathscr{V}}_{\bullet})$ is the Stokes-filtered local system associated with the good meromorphic connection (V, ∇) by the Riemann-Hilbert correspondence). Let $\varphi \in \mathscr{O}_{\Delta,0}(*0)/\mathscr{O}_{\Delta,0}$ be a polar part of one variable, and let $f^*\varphi$ be its pull-back on X. Then $\widetilde{\mathscr{V}}_{f^*\varphi}$ is a subsheaf of $\widetilde{\mathscr{V}}$.

Theorem 1. *For each such φ (and the ramified analogue), the natural morphism $R^{n+j}\widetilde{f}_*\widetilde{\mathscr{V}}_{\leqslant f^*\varphi} \to R^{n+j}\widetilde{f}_*\widetilde{\mathscr{V}}$ is injective.*

This theorem, which is a kind of E_1-degeneracy property of the Stokes filtration, is proved through the Riemann-Hilbert correspondence, by using the following general result.

Let $\pi : X \to X'$ be a holomorphic map between complex manifolds X and X'. We assume that X and X' are equipped with normal crossing divisors D and D' with smooth components $D_{j\in J}$ and $D'_{j'\in J'}$, and that

(1) $D = \pi^{-1}(D')$,

(2) $\pi : X \smallsetminus D \to X' \smallsetminus D'$ is smooth.

Let $\varpi : \widetilde{X}(D_{j\in J}) \to X$ (resp. $\varpi' : \widetilde{X}'(D'_{j'\in J'}) \to X'$) be the real blowing-up of the components $D_{j\in J}$ in X (resp. $D'_{j'\in J'}$ in X'). There exists a lifting $\widetilde{\pi} : \widetilde{X} \to \widetilde{X}'$ of π (see §7.2.a) such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\varpi"] \arrow[d, "\widetilde{\pi}"'] & X \arrow[d, "\pi"] \\ \widetilde{X}' \arrow[r, "\varpi'"] & X' \end{tikzcd}$$

Notice that $\partial\widetilde{X} = \widetilde{\pi}^{-1}(\partial\widetilde{X}')$.

***Theorem 2* (T. Mochizuki).** *Let $\pi : (X, D) \to (X', D')$ be as above. Let $\mathscr{M}$ be a meromorphic connection with poles along D at most. Let $\pi_+\mathscr{M}$ the direct image of $\mathscr{M}$ (as a $\mathscr{D}_X(*D)$-module). Then*

$$\mathrm{DR}^{\mathrm{mod}\, D'}(\pi_+\mathscr{M}) \simeq \boldsymbol{R}\widetilde{\pi}_*\,\mathrm{DR}^{\mathrm{mod}\, D}(\mathscr{M}).$$

8.2. Stokes filtration and Fourier-Laplace transformation

Let M be a regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module. The purpose of this section is to give an explicit formula for the Stokes filtration of its Laplace transform ${}^F\!M$ at infinity, in terms of topological data obtained from M. More precisely, let $\mathscr{F} = {}^{\mathrm{p}}\mathrm{DR}^{\mathrm{an}}\, M$ be the analytic de Rham complex of M (shifted according to the usual perverse convention). The question we address is a formula for the Stokes filtration of ${}^F\!M$ at $\tau = \infty$ in terms of $\mathscr{F}$ only. In other words, we will define a topological Laplace transform of $\mathscr{F}$ as being a perverse sheaf on $\widehat{\mathbb{A}}^1$ with a Stokes filtration at infinity, in such a way that the topological Laplace transform of $\mathrm{DR}^{\mathrm{an}}\, M$ is $\mathrm{DR}^{\mathrm{an}}\, {}^F\!M$ together with its Stokes filtration at infinity (i.e., $\mathrm{DR}^{\mathrm{an}}\, {}^F\!M$ as a Stokes-perverse sheaf on $\widehat{\mathbb{P}}^1$, see Definition 14 of Lecture 6).

The more general case where M has arbitrary singularities has been considered in [**Mal91**] and [**Moc10**].

A consequence of this result is that, if the perverse sheaf $\mathscr{F}$ is defined over $\mathbb{Q}$ (say), then the Stokes filtration of ${}^{F}\!M$ at infinity is also defined over $\mathbb{Q}$.

We note that the formal structure of ${}^{F}\!M$ at ∞ is described through the vanishing cycles of $\mathrm{DR}\,M$ (or the moderate vanishing cycles of M) at each of the singular points of M in $\mathbb{A}^1$. Therefore, according to the Riemann-Hilbert correspondence (Theorem 2 of Lecture 6), the set of exponential factors of ${}^{F}\!M$ at $\widehat{\infty}$ are the c/τ', where τ' is the coordinate centered at $\widehat{\infty}$ in $\widehat{\mathbb{A}}^1$ and c runs among the singular points of M. Similarly, the graded sheaf $\mathrm{gr}_{c/\tau'}\mathscr{L}$ is obtained from the vanishing cycle space $\phi_{t-c}^{\mathrm{mod}}\,M$ together with its monodromy.

We will try to recover this Stokes-filtered local system $(\mathscr{L},\mathscr{L}_\bullet)$ from the perverse sheaf $\mathscr{F}={}^{\mathrm{p}}\mathrm{DR}^{\mathrm{an}}\,M$.

8.2.a. The topological Laplace transformation. Since we a priori know that the set of exponential factors of ${}^{F}\!M$ at $\widehat{\infty}$ is non-ramified and of the form $\{c/\tau'\mid c\in C\subset\mathbb{C}\}$, it is enough to give the topological description of $\mathscr{L}_{\leqslant c/\tau'}$ and $\mathscr{L}_{<c/\tau'}$ for any $c\in\mathbb{C}$ (and jumps will only occur for $c\in C$).

Let $\mathscr{F}$ be any perverse sheaf (of $\mathbb{Q}$-vector spaces) on $\mathbb{A}^1$. We will define a Stokes-filtered local system $(\mathscr{L},\mathscr{L}_\bullet)$ indexed by C/τ' on $S^1_{\widehat{\infty}}$.

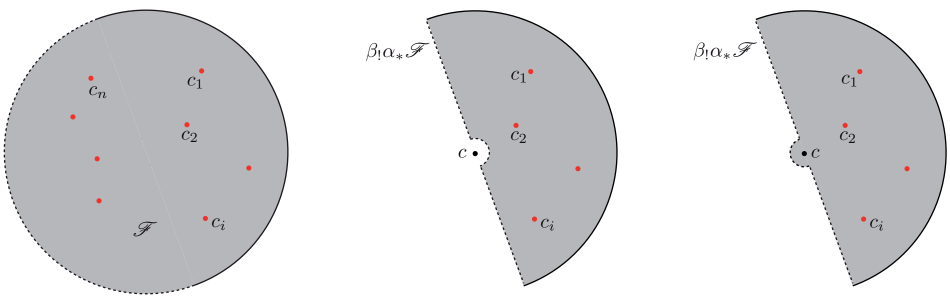
For $\widehat{\theta}'_o\in S^1_{\widehat{\infty}}$, let us denote by $B_{\widehat{\theta}'_o}\subset S^1_\infty$ the closed interval $\{(\infty,e^{i\theta})\mid \mathrm{Re}(e^{i\theta-\widehat{\theta}'_o})\geqslant 0\}$, and let $A_{\widehat{\theta}'_o}\subset S^1_\infty$ be the complementary open interval. Let us denote by $\Phi_{\widehat{\theta}'_o}$ the family of closed sets $S\subset\mathbb{A}^1$ such that $\overline{S}\cap B_{\widehat{\theta}'_o}=\varnothing$. We then define

$$\mathscr{L}_{\widehat{\theta}'_o}=H^1_{\Phi_{\widehat{\theta}'_o}}(\mathbb{A}^1,\mathscr{F}).$$

If $\alpha:\mathbb{A}^1\hookrightarrow(\mathbb{A}^1\cup A_{\widehat{\theta}'_o})$ and $\beta:(\mathbb{A}^1\cup A_{\widehat{\theta}'_o})\hookrightarrow(\mathbb{A}^1\cup S^1_\infty)$ denote the open inclusions, we can also write

$$\mathscr{L}_{\widehat{\theta}'_o}=H^1(\widetilde{\mathbb{P}}^1,\mathscr{F}).$$

In the following pictures, the set $A_{\widehat{\theta}'_o}$ is drawn with a full line on the boundary (it is open) and the set $B_{\widehat{\theta}'_o}$ (or the corresponding set for computing $\mathscr{L}_{\widehat{\theta}'_o,<c/\tau'}$ and $\mathscr{L}_{\widehat{\theta}'_o,\leqslant c/\tau'}$ is drawn with a dashed line (it is closed).



$\mathscr{L}_{\widehat{\theta}'_o}$ $\mathscr{L}_{\widehat{\theta}'_o,<c/\tau'}$ $\mathscr{L}_{\widehat{\theta}'_o,\leqslant c/\tau'}$

8.2.b. Compatibility with Riemann-Hilbert.

Theorem 3. *The Stokes filtration of* ${}^{F}\!M$ *at* $\widehat{\infty}$ *is obtained from* $\mathscr{F} := {}^{\mathrm{p}}\mathrm{DR}^{\mathrm{an}}\, M$ *by topological Laplace transformation.*

LECTURE 9

SOME EXERCISES

9.1. Twistor construction

We regard $\mathbb{P}^1$ as the union of two affine charts $\mathbb{C}_z$ and $\mathbb{C}_{z'}$, with $z' = 1/z$ on the intersection, and we set $S^1 = \{|z = 1\} = \{|z' = 1|\}$. Let $\sigma : \mathbb{P}^1 \to \overline{\mathbb{P}}^1$ be the anti-holomorphic involution $z \mapsto -1/\overline{z}$.

(1) Let $\mathscr{H}$ be a holomorphic vector bundle on $\mathbb{C}_z$.

- Show that $\sigma^*\overline{\mathscr{H}}$ is a holomorphic vector bundle on $\mathbb{C}_{z'}$.

(2) Let $\mathscr{C} : \mathscr{H}_{|S^1} \otimes_{\mathscr{O}_{S^1}} \sigma^*\overline{\mathscr{H}}_{|S^1} \to \mathscr{O}_{S^1}$ be $\mathscr{O}_{S^1}$-linear inducing an isomorphism $\mathscr{H}^\vee_{|S^1} \simeq \sigma^*\overline{\mathscr{H}}_{|S^1}$. Then $\mathscr{C}$ defines a holomorphic bundle $\widetilde{\mathscr{H}}$ on $\mathbb{P}^1$ by gluing $\mathscr{H}^\vee$ and $\sigma^*\overline{\mathscr{H}}$ along the previous isomorphism. Assume that $\mathscr{H}$ is equipped with a meromorphic connection ∇ having a pole at $z = 0$ only.

- Show that $\sigma^*\overline{\mathscr{H}}$ has a meromorphic connection having a pole at $z' = 0$ only.
- Show that if $\mathscr{C}$ is compatible with the connections, then the connection ∇ on $\mathscr{H}^\vee$ and that on $\sigma^*\overline{\mathscr{H}}$ are compatible and define a meromorphic connection ∇ on $\widetilde{\mathscr{H}}$ with pole at $0, \infty$ only.
- In such a case, show that $\mathscr{C}$ is uniquely determined from its restriction to the local system $\mathscr{L} = \ker\nabla$, which is a non-degenerate pairing $C : \mathscr{L}_{|S^1} \otimes_{\mathbb{C}_{S^1}} \iota^{-1}\overline{\mathscr{L}}_{|S^1} \to \mathbb{C}_{S^1}$, where ι is the involution $z \mapsto -z$ (note that, for $z \in S^1$, $\sigma(z) = \iota(z)$).

Remark. Given $(\mathscr{H}, \nabla)$ and a non-degenerate pairing $C : \mathscr{L}_{|S^1} \otimes_{\mathbb{C}_{S^1}} \iota^{-1}\overline{\mathscr{L}}_{|S^1} \to \mathbb{C}_{S^1}$ as above, it is difficult to check whether $\widetilde{\mathscr{H}}$ is trivial, or to compute the Birkhoff-Grothendieck decomposition of $\widetilde{\mathscr{H}}$, as this reduces to a transcendental question.

(3) Assume that we are given $(\mathscr{H}, \mathscr{C})$ as above. Show that $\widetilde{\mathscr{H}} \simeq \sigma^*\overline{\widetilde{\mathscr{H}}}$. Conclude that, if $(\mathscr{H}, \mathscr{C})$ is a *pure twistor of weight* 0, that is, if $\widetilde{\mathscr{H}}$ is the trivial bundle, then $H := \Gamma(\mathbb{P}^1, \widetilde{\mathscr{H}})$ is equipped with a nondegenerate sesquilinear form.

9.2. Elementary $\mathbb{C}(\!(z)\!)$-vector spaces with connection

Let R be a finite dimensional $\mathbb{C}(\!(z)\!)$-vector space equipped with a connection ∇ having a *regular singularity*, i.e., there exists a basis of R in which $\nabla = d + A\mathrm{d}z/z$, A a constant matrix.

(1) Let $\varphi \in \mathbb{C}(\!(z)\!)$. Show that $\nabla + \mathrm{d}\varphi\,\mathrm{Id}$ is a connection which only depends on $\varphi \bmod \mathbb{C}[\![z]\!]$, that is, if $\varphi, \psi \in \mathbb{C}(\!(z)\!)$ are such that $\varphi - \psi \in \mathbb{C}[\![z]\!]$, then $(R, \nabla + \mathrm{d}\varphi\,\mathrm{Id}) \simeq (R, \nabla + \mathrm{d}\psi\,\mathrm{Id})$.

(2) Show that if $\varphi \neq 0$ in $\mathbb{C}(\!(z)\!)/\mathbb{C}[\![z]\!]$, then $\ker\nabla = 0$. Applying this to End, show the converse to the implication above.

(3) Let u be a new variable, let $\rho \in u\mathbb{C}[\![u]\!]$ with valuation $v_u(\rho) = p \geqslant 1$, and set $z = \rho(u)$. Show that $\mathbb{C}(\!(u)\!)$ is a $\mathbb{C}(\!(z)\!)$-vector space. Let R be a n-dimensional $\mathbb{C}(\!(u)\!)$-vector space. Show that R is a finite dimensional $\mathbb{C}(\!(z)\!)$-vector space and compute its dimension. It is denoted by $\rho_* R$.

(4) Assume R has a connection ∇ (w.r.t. to u). Show that $\nabla_{\partial_z} := \rho'(u)^{-1}\nabla_{\partial_u}$ defines a derivation of R as a $\mathbb{C}(\!(z)\!)$-vector space. Then (R, ∇_{∂_z}) is denoted $\rho_+(R, \nabla_{\partial_u})$.

(5) Let S be a m-dimensional $\mathbb{C}(\!(z)\!)$-vector space with a connection ∇ (w.r.t. z) and set $\rho^* S = \mathbb{C}(\!(u)\!) \otimes_{\mathbb{C}(\!(z)\!)} S$. Show that the formula $\nabla_{\partial_u}(1 \otimes s) = \rho'(u) \otimes \nabla_{\partial_z} s$ defines a connection on $\rho^* S$ (w.r.t. u). It is denoted $\rho^+(S, \nabla)$.

(6) Let $\lambda \in u\mathbb{C}[\![u]\!]$ with $v_u(\lambda) = 1$. Compute $\lambda^+(S, \mathrm{d} + \mathrm{d}\psi\,\mathrm{Id} + A\mathrm{d}z/z)$ and $\lambda_+(R, \mathrm{d} + \mathrm{d}\varphi\,\mathrm{Id} + A\mathrm{d}u/u)$, $\varphi \in \mathbb{C}(\!(u)\!)$, $\psi \in \mathbb{C}(\!(z)\!)$ and A a constant matrix.

(7) Let (R, ∇) and (R', ∇') be two $\mathbb{C}(\!(u)\!)$-vector spaces with *regular* connection, and let $\lambda \in u\mathbb{C}[\![u]\!]$ with $v_u(\lambda) = 1$. Show that $\lambda_+(R, \nabla + \mathrm{d}\varphi\,\mathrm{Id}) \simeq (R', \nabla' + \mathrm{d}\psi\,\mathrm{Id})$ iff $\psi \circ \lambda \equiv \varphi \bmod \mathbb{C}[\![u]\!]$ and $(R, \nabla) \simeq (R', \nabla')$. (Hint: use the series $\rho(u)$ such that $\lambda \circ \rho = 1$ and show that $\lambda_+ = \rho^+$.)

BIBLIOGRAPHY

[BE04] S. Bloch & H. Esnault – Local Fourier transforms and rigidity for $\mathscr{D}$-Modules, *Asian J. Math.* **8** (2004), no. 4, p. 587–606.

[Bor87] A. Borel (ed.) – *Algebraic $\mathscr{D}$-modules*, Perspectives in Math., vol. 2, Boston, Academic Press, 1987.

[CFIV92] S. Cecotti, P. Fendley, K. Intriligator & C. Vafa – A new supersymmetric index, *Nuclear Phys. B* **386** (1992), p. 405–452.

[CV91] S. Cecotti & C. Vafa – Topological-antitopological fusion, *Nuclear Phys. B* **367** (1991), p. 359–461.

[Del70] P. Deligne – *Équations diffÃ©rentielles Ã points singuliers rÃ©guliers*, Lect. Notes in Math., vol. 163, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[Del80] ———, La conjecture de Weil, II, *Publ. Math. Inst. Hautes Études Sci.* **52** (1980), p. 137–252.

[DMR07] P. Deligne, B. Malgrange & J.-P. Ramis – *SingularitÃ©s irrÃ©guliÃ¨res, Correspondance et documents*, Documents mathÃ©matiques, vol. 5, Société Mathématique de France, Paris, 2007.

[Fan09] J. Fang – Calculation of local Fourier transforms for formal connections, *Sci. China Ser. A* **52** (2009), no. 10, p. 2195–2206.

[GL04] R. García López – Microlocalization and stationary phase, *Asian J. Math.* **8** (2004), no. 4, p. 747–768.

[Her03] C. Hertling – tt^* geometry, Frobenius manifolds, their connections, and the construction for singularities, *J. reine angew. Math.* **555** (2003), p. 77–161.

[Kat90a] N. Katz – *Exponential sums and differential equations*, Ann. of Math. studies, vol. 124, Princeton University Press, Princeton, NJ, 1990.

[Kat90b] ———, Exponential sums over finite fields and differential equations over the complex numbers: some interactions, *Bull. Amer. Math. Soc. (N.S.)* **23** (1990), no. 2, p. 269–309.

[Kat96] ———, *Rigid local systems*, Ann. of Math. studies, vol. 139, Princeton University Press, Princeton, NJ, 1996.

[KL85] N. Katz & G. Laumon – Transformation de Fourier et majoration de sommes exponentielles, *Publ. Math. Inst. Hautes Études Sci.* **62** (1985), p. 145–202.

[Ked10] K. Kedlaya – Good formal structures for flat meromorphic connections, I: surfaces, *Duke Math. J.* **154** (2010), no. 2, p. 343–418.

[Ked11] ———, Good formal structures for flat meromorphic connections, II: excellent schemes, *J. Amer. Math. Soc.* **24** (2011), no. 1, p. 183–229.

[Lau87] G. Laumon – Transformation de Fourier, constantes d'Ã©quations fonctionnelles et conjectures de Weil, *Publ. Math. Inst. Hautes Études Sci.* **65** (1987), p. 131–210.

[Lau04] Y. Laurent – Geometric irregularity and $\mathscr{D}$-modules, in *ÉlÃ©ments de la thÃ©orie des systÃ¨mes diffÃ©rentiels gÃ©omÃ©triques*, SÃ©minaires & CongrÃ¨s, vol. 8, Société Mathématique de France, Paris, 2004, p. 411–430.

[Mal91] B. Malgrange – *Équations diffÃ©rentielles Ã coefficients polynomiaux*, Progress in Math., vol. 96, Birkhäuser, Basel, Boston, 1991.

[Moc09] T. Mochizuki – Good formal structure for meromorphic flat connections on smooth projective surfaces, in *Algebraic Analysis and Around (Kyoto, June 2007)*, Advanced Studies in Pure Math., vol. 54, Math. Soc. Japan, Tokyo, 2009, p. 223–253, arXiv:0803.1346.

[Moc10] ———, Note on the Stokes structure of the Fourier transform, *Acta Math. Vietnam.* **35** (2010), no. 1, p. 101–158.

[Moc11] ———, *Wild harmonic bundles and wild pure twistor D-modules*, Astérisque, vol. 340, Société Mathématique de France, Paris, 2011.

[OV07] A. Ogus & V. Vologodsky – Nonabelian Hodge theory in characteristic p, *Publ. Math. Inst. Hautes Études Sci.* **106** (2007), p. 1–138.

[Sab99] C. Sabbah – On a twisted de Rham complex, *Tôhoku Math. J.* **51** (1999), p. 125–140.

[Sab06] ______, Monodromy at infinity and Fourier transform II, *Publ. RIMS, Kyoto Univ.* **42** (2006), p. 803–835.

[Sab08] ______, An explicit stationary phase formula for the local formal Fourier-Laplace transform, in *Singularities, vol. 1*, Contemp. Math., American Mathematical Society, Providence, R.I., 2008, p. 300–330, `arXiv:0706.3570`.

[Sch73] W. Schmid – Variation of Hodge structure: the singularities of the period mapping, *Invent. Math.* **22** (1973), p. 211–319.

[Was65] W. Wasow – *Asymptotic expansions for ordinary differential equations*, Interscience, New York, 1965.