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INTRODUCTION TO MIXED HODGE MODULES

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Abstract. These notes intend to be an introduction to the theory of mixed Hodge modules, as developed by M. Saito in the fundamental articles [**Sai88, Sai90**] and in his various subsequent publications. It is no question to give proofs of the main results, since the theorems come in an intricate construction. Nevertheless, we intend to explain the main objectives of the theory, and some of the features of mixed Hodge modules. Pure Hodge modules are of special interest, in particular concerning the decomposition theorem. The theory of mixed Hodge modules has applications to classical questions of algebraic geometry, singularity theory, and mirror symmetry.

Since these notes are addressed to an audience with few knowledge in D-module theory, we devote Lecture 1 to a very short introduction to the subject, and we emphasize the main notions needed for the sequel. In Lecture 2, we focus on the problem of interaction between Hodge filtration and Kashiwara-Malgrange filtration, which is at the heart of the definition of Hodge modules, and we also consider the behaviour of the Hodge filtration with respect to localization along a divisor, which is one of the main questions for passing from pure to mixed objects. An overview of the theory of pure and mixed Hodge modules is given in Lecture 3. Lecture 4 is an introduction to the question of extending a variation of Hodge structure and the properties of the objects obtained by such an extension: these are the pure Hodge modules. In Lecture 5, we come back to pure Hodge modules: the decomposition theorem is one of the main reasons for developing the theory.

The interested reader can consult the introductory articles [**Pop16, Sai91, Sai94, Sai17, Sch14**] for more details.

CONTENTS

LECTURE 1

TRAINING ON $\mathcal{D}$-MODULES

1.1. Basics on $\mathcal{D}_X$-modules

X: complex manifold, $\mathcal{O}_X$ sheaf of holomorphic functions, $(x_1, \dots, x_n)$ local coordinates on X.

1.1.a. $\mathcal{D}_X$-modules

$\mathcal{O}_X$-modules with integrable connection.

• Holomorphic vector bundle $V \iff$ locally free sheaf $\mathcal{V}$ of $\mathcal{O}_X$-modules of finite rank.

• Connection $\nabla : \mathcal{V} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{V}$ satisfying Leibniz rule $\nabla(fv) = \mathrm{d}f \otimes v + f\nabla v$. In local coordinates: $\nabla_{\partial_{x_i}}(fv) = (\partial f/\partial x_i)v + f\nabla_{\partial_{x_i}}(v)$

• Integrability (i.e., curvature equal to zero): $\nabla^2 : \mathcal{V} \to \Omega^2_X \otimes \mathcal{V}$ should be zero. In local coordinates, $\nabla_{\partial_{x_i}}\nabla_{\partial_{x_j}}(v) = \nabla_{\partial_{x_j}}\nabla_{\partial_{x_i}}(v)$.

• Can define the notion of integrable connection on *any* $\mathcal{O}_X$-module $\mathcal{M}$. An $\mathcal{O}_X$-module with integrable holomorphic connection $\iff$ a left $\mathcal{D}_X$-module.

Left $\mathcal{D}_X$-modules. The sheaf of rings $\mathcal{D}_X$ is locally defined as $\mathcal{O}_X\langle\partial_{x_1}, \dots, \partial_{x_n}\rangle$, with the commutation relations $[\partial_{x_i}, \partial_{x_j}] = 0$ and $[\partial_{x_i}, f(x)] = \partial f/\partial x_i$. An $\mathcal{O}_X$-module $\mathcal{M}$ with an integrable connection ∇ is nothing but a left $\mathcal{D}_X$-module, with the action defined by $\partial_{x_i}m := \nabla_{\partial_{x_i}}m$.

Examples:

• $\mathcal{O}_X$ endowed with d.

• A holomorphic vector bundle with integrable connection.

• If D is a divisor in X and $\mathcal{O}_X(*D)$ is the sheaf of meromorphic functions on X with poles along D of any order, then $(\mathcal{O}_X(*D), \mathrm{d})$ is a left $\mathcal{D}_X$-module.

Side-changing. Because $\mathcal{D}_X$ is not commutative, one has to distinguish between left and right $\mathcal{D}_X$-modules. The main example of a right $\mathcal{D}_X$-module is ω_X, with the right $\mathcal{D}_X$-module structure given by the local formula

$$(f(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) \cdot \partial_{x_i} = -(\partial f/\partial x_i) \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

Lemma 1.1. *Equivalence "left $\mathcal{D}_X$-modules $\longleftrightarrow$ right $\mathcal{D}_X$-modules" by*

$$\mathcal{M}^{\text{left}} \longmapsto \mathcal{M}^{\text{right}} := \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}^{\text{left}},$$
$$((\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) \otimes m) \cdot \partial_{x_i} := -(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) \otimes \partial_{x_i} m.$$

The filtration by the order. The sheaf $\mathcal{D}_X$ is endowed with the increasing filtration: $F_p\mathcal{D}_X := \{\text{operators of order } \leqslant p\}$.

- $\rightsquigarrow$ sheaf of graded rings $\mathscr{R}_X := \bigoplus_p F_p\mathcal{D}_X z^p$ (z = grading variable).
- $\mathscr{R}_X/z\mathscr{R}_X \simeq \mathrm{gr}^F\mathcal{D}_X$ is a sheaf of commutative rings that can be identified with a sheaf of functions on the cotangent space T^*X.
- $\rightsquigarrow$ flat $\mathcal{O}_X$-module with filtration satisfying Griffiths transversality $\nabla F^p\mathcal{M} \subset \Omega^1_X \otimes F^{p-1}\mathcal{M} \longleftrightarrow$ graded $\mathscr{R}_X$-module $\mathscr{M}$ without $\mathbf{C}[z]$-torsion.
- Then $\mathscr{M}/z\mathscr{M} \simeq \mathrm{gr}_F\mathcal{M}$ is a $\mathrm{gr}^F\mathcal{D}_X$-module.
- $\rightsquigarrow$ Abelian category of graded $\mathscr{R}_X$-modules and graded morphisms of degree zero contains the full *non-abelian* subcategory of *strict* objects (i.e., without $\mathbf{C}[z]$-torsion).
- A morphism between graded $\mathscr{R}_X$-modules is *strict* if its kernel and cokernel are strict.

Main property of Hodge modules. *All operations on pure or mixed Hodge modules preserve strictness with respect to the Hodge filtration.*

1.1.b. Coherence and Kashiwara's equivalence.

Coherence. One needs of course some finiteness properties on the $\mathcal{O}_X$-modules with integrable connection one considers. Remark that $\mathcal{O}_X$-coherence is too strong since it implies local freeness of finite rank.

- The sheaves of rings $\mathcal{D}_X$ and $\mathscr{R}_X$ are coherent, and there is a good notion of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ or a coherent $\mathscr{R}_X$-module $\mathscr{M}$.
- Strict coherent $\mathscr{R}_X$-module $\mathscr{M} \iff$ coherent $\mathcal{D}_X$-module $\mathcal{M}$ with a coherent filtration $F^\bullet\mathcal{M}$, i.e., such that $\mathrm{gr}_F\mathcal{M}$ is a coherent $\mathrm{gr}^F\mathcal{D}_X$-module. Its support is a closed analytic subset of T^*X called *the characteristic variety of* $\mathcal{M}$ and does not depend on the choice of the coherent filtration on $\mathcal{M}$.

Kashiwara's equivalence. Let $Y \subset X$ be a closed submanifold. The category of coherent $\mathcal{D}_X$-modules supported on Y is equivalent to that of $\mathcal{D}_Y$-modules. The similar statement is not true for coherent $\mathcal{O}_X$-modules or for coherent $\mathscr{R}_X$-modules.

Example 1.2. Let $f : X \to \mathbf{C}$ be a holomorphic function and let $\iota_f : X \hookrightarrow X \times \mathbf{C}_t$ be the inclusion of the graph of f, i.e., $\iota_f(x) = (x, f(x))$. Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. We associate with it a $\mathcal{D}_{X\times\mathbf{C}}$-module supported on the graph of f by the following formula:

- $\iota_{f+}\mathcal{M} = \iota_{f*}\mathcal{M} \otimes_{\mathbf{C}} \mathbf{C}[\partial_t] = \bigoplus_{k\geqslant 0} \iota_{f*}\mathcal{M} \otimes \partial_t^k$,
- $\varphi(x,t)(m \otimes 1) = (\varphi(x, f(x))m) \otimes 1$,
- $\partial_t(m \otimes \partial_t^k) = m \otimes \partial_t^{k+1}$,
- $\partial_{x_i}(m \otimes \partial_t^k) = (\partial_{x_i}m) \otimes \partial_t^k - (\partial f/\partial x_i)m \otimes \partial_t^{k+1}$,
- the action of a general element of $\mathcal{D}_{X\times\mathbf{C}}$ is deduced from these formulas and the commutation relations in the ring $\mathcal{D}_{X\times\mathbf{C}}$.

1.2. Bernstein-Sato polynomials and the Kashiwara-Malgrange filtration

1.2.a. Bernstein-Sato polynomials. Let $f : X \to \mathbf{C}$ be a holomorphic function. A natural question is to ask for which values of $\alpha < 0$ is the function $|f|^{2\alpha}$ locally integrable.

Theorem 1.3. *In any coordinate chart, there exists a nonzero polynomial $b_f(s)$ and a holomorphic differential operator $P(x, \partial_x, s)$ in $\mathcal{D}_X[s]$ such that, for* $\mathrm{Re}(s)$ *large enough,*

$$b_f(s)|f(x)|^{2s} = P(x, \partial_x, s)f(x)|f(x)|^{2s}, \quad P(x, \partial_x, s) := \sum_{i=0}^{m} P_i(x, \partial_x)s^i.$$

Moreover, the roots of $b_f(s)$ are in $\mathbf{Q}_{<0}$, and $b_f(s) = \overline{b}_f(s)$.

Let η be a C^∞ form of maximal degree with compact support, which is $\equiv$ vol near x_o (center of the coordinate chart). Write, for $\mathrm{Re}(s)$ big enough,

$$b_f(s) \int |f|^{2s}\eta = \int P(x, \partial_x, s)f|f|^{2s}\eta = \int f|f|^{2s}\eta^{(1)}(s), \quad \eta^{(1)}(s) := \eta\cdot P^*(x, \partial_x, s).$$

Applying this to $\overline{f}$ and iterating k-times, we find

$$\int |f|^{2s}\eta = \frac{1}{\prod_{j=0}^{k-1} b_f(s+j)\overline{b}_f(s+j)} \int |f|^{2(s+k)}\eta^{(2k)}(s) \quad \mathrm{Re}(s) \gg 0.$$

The LHS is holomorphic in s for $\mathrm{Re}(s) > 0$, the RHS is meromorphic in s for $\mathrm{Re}(s) > -k$, and both coincide for $\mathrm{Re}(s) \gg 0$, hence the LHS extends meromorphically in s for $\mathrm{Re}(s) > -k$ for any $k \geqslant 1$, hence it extends on $\mathbf{C}_s$.

Corollary 1.4. *The function $|f|^{2\alpha}$ is integrable for any $\alpha >$ biggest root of b_f.*

1.2.b. Specialization of $\mathcal{D}_X$-modules. We re-interpret this result in terms of the *Kashiwara-Malgrange* filtration. Kashiwara's equivalence is used to simplify the function but making more complicated the $\mathcal{D}$-module, by the procedure of Example 1.2.

• V-filtration of $\mathcal{D}_{X\times\mathbf{C}_t}$

$$\begin{aligned}V^0\mathcal{D}_{X\times\mathbf{C}_t} &= \mathcal{O}_{X\times\mathbf{C}_t}\langle\partial_{x_1},\dots,\partial_{x_n},t\partial_t\rangle,\\ V^k\mathcal{D}_{X\times\mathbf{C}_t} &= t^kV^0\mathcal{D}_{X\times\mathbf{C}_t},\quad k\geqslant 0,\\ V^{-k}\mathcal{D}_{X\times\mathbf{C}_t} &= \sum_{j=1}^k \partial_t^jV^0\mathcal{D}_{X\times\mathbf{C}_t},\quad k\geqslant 0.\end{aligned}$$

• $\rightsquigarrow$ Notion of coherent V-filtration for a $\mathcal{D}_{X\times\mathbf{C}_t}$-module $\mathcal{M}$.

• For $\mathcal{M} = \iota_{f+}\mathcal{O}_X \simeq \mathcal{O}_X[\partial_t]$ with the structure as in Example 1.2, set $U^k(\iota_{f+}\mathcal{O}_X) = V^k\mathcal{D}_{X\times\mathbf{C}_t}\cdot(\mathcal{O}_X\otimes 1)$. Theorem 1.3 can be read as follows:

(1.5) *There exists a non-zero polynomial $b_U(s)$ such that $b_U(t\partial_t - k)$ vanishes on $U^k(\iota_{f+}\mathcal{O}_X)/U^{k+1}(\iota_{f+}\mathcal{O}_X)$ for each $k\in\mathbf{Z}$.*

Theorem-Definition 1.6 **(R-specializability).**

(1) *Let $\mathcal{M}$ be a coherent $\mathcal{D}_{X\times\mathbf{C}}$-module. We say that $\mathcal{M}$ is* $\mathbf{R}$-specializable *along $X\times\{0\}$ if for some coherent V-filtration $U^\bullet\mathcal{M}$ (1.5) holds with $b_U(s)$ having real roots.*

(2) *Assume $\mathcal{M}$ is $\mathbf{R}$-specializable along $X\times\{0\}$. Any coherent V-filtration satisfies the above property. Moreover, there exists a unique coherent V-filtration $V^\bullet\mathcal{M}$ such that the set A of roots of b_V is a finite subset of $(-1,0]$. It is called the* Kashiwara-Malgrange *filtration of $\mathcal{M}$.*

One can index the filtration by $A+\mathbf{Z}$ in such a way that $t\partial_t - a$ is nilpotent on $\mathrm{gr}_V^a\mathcal{M} = V^a\mathcal{M}/V^{>a}\mathcal{M}$.

Proposition 1.7. *Assume $\mathcal{M}$ is $\mathbf{R}$-specializable along $X\times\{0\}$. Then the V-filtration satisfies the following properties:*

• $t: V^a\mathcal{M}\xrightarrow{\sim}V^{a+1}\mathcal{M}$, *for all $a>-1$,*

• $-\partial_t:\mathrm{gr}_V^a\mathcal{M}\xrightarrow{\sim}\mathrm{gr}_V^{a-1}\mathcal{M}$, *for all $a<0$.*

Proof. Let us check the second point. If we compose on the left with $t:\mathrm{gr}_V^{a-1}\mathcal{M}\to\mathrm{gr}_V^a\mathcal{M}$, we obtain $-t\partial_t:\mathrm{gr}_V^a\mathcal{M}\to\mathrm{gr}_V^a\mathcal{M}$, which is an isomorphism because $-t\partial_t+a$ is nilpotent and $a<0$. If we compose on the right with t, we obtain $-\partial_tt:\mathrm{gr}_V^{a-1}\mathcal{M}\to\mathrm{gr}_V^{a-1}\mathcal{M}$ is an isomorphism since

$$-\partial_tt = -(t\partial_t+1) = -(t\partial_t-(a-1))-a,$$

$t\partial_t-(a-1)$ is nilpotent and $a\neq 0$. $\square$

LECTURE 2

HODGE FILTRATION AND KASHIWARA-MALGRANGE FILTRATION

One of the main properties of Hodge modules is the good interaction between the Hodge filtration and the Kashiwara-Malgrange filtration attached to any holomorphic function. In this chapter, we explain what "good interaction" means, under the name of **R**-specializability of a filtered $\mathcal{D}_X$-module, or strict **R**-specializability of the associated Rees module.

This good interaction is also instrumental when considering the notion of localization, which is fundamental in mixed Hodge theory. We will restrict to the case of localization with respect to a hypersurface.

The setting is then as follows. We consider a complex manifold X and a reduced divisor $D \subset X$ with complement $U \xhookrightarrow{j} X$. If D has a global defining equation $f : X \to \mathbf{C}$, then we consider the graph inclusion $\iota_f : X \hookrightarrow X \times \mathbf{C}_t$, and the divisor is $X \times \{0\}$. For an open set $V \subset X$, we set $V^* = V \cap U$.

2.0. The Riemann-Hilbert correspondence

For a left $\mathcal{D}_X$-module (i.e., $\mathscr{O}_X$-module with integrable connection), the de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is the complex

$$0 \longrightarrow \mathcal{M} \xrightarrow{\nabla} \Omega^1_X \otimes \mathcal{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^n_X \underset{\bullet}{\otimes} \mathcal{M} \longrightarrow 0.$$

If $\mathcal{M}$ is holonomic, ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ has constructible cohomology (Kashiwara) and is a perverse complex. The de Rham functor behaves well with respect to various functors in $\mathcal{D}$-module theory on the one hand and in sheaf theory on the other hand.

2.1. Specialization of a filtered $\mathcal{D}_X$-module

2.1.a. R-specializability of filtered $\mathcal{D}_X$-modules. The following properties are justified by Schmid's theorem on the limit of a polarized variation of Hodge structure (see Lecture 4).

- $\mathcal{M}$: a coherent $\mathcal{D}_{X\times\mathbf{C}}$-module which is $\mathbf{R}$-specializable along $X\times\{0\}$ with V-filtration $V^\bullet\mathcal{M}$. The properties of Proposition 1.7 hold, as a consequence of the definition of the Bernstein-Sato polynomial.
- Assume now we are given $F^\bullet\mathcal{M}$: a coherent filtration on $\mathcal{M}$. We define the compatibility property between $F^\bullet\mathcal{M}$ and $V^\bullet\mathcal{M}$.
- $(\mathcal{M},F^\bullet\mathcal{M})$ is $\mathbf{R}$-*specializable along* $X\times\{0\}$ if
  - $t: F^pV^a\mathcal{M}\xrightarrow{\sim}F^pV^{a+1}\mathcal{M}$, for all $a>-1$,
  - $-\partial_t: F^p\mathrm{gr}_V^a\mathcal{M}\xrightarrow{\sim}F^{p-1}\mathrm{gr}_V^{a-1}\mathcal{M}$, for all $a<0$.

2.1.b. Nearby cycles. For an $\mathbf{R}$-specializable module $\mathcal{M}$, the construction $\mathrm{gr}_V^a\mathcal{M}$ will appear to be fundamental. It can be put in a good framework as follows. Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module and let $f: X\to\mathbf{C}$ be a holomorphic function. We say that $\mathcal{M}$ is $\mathbf{R}$-specializable along (f) if $\iota_{f+}\mathcal{M}$ is $\mathbf{R}$-specializable along $X\times\{0\}$ in $X\times\mathbf{C}_t$. For $\lambda\in\mathbf{C}^*$ with $|\lambda|=1$, we then set $\psi_{f,\lambda}\mathcal{M}=\mathrm{gr}_V^a\mathcal{M}$ with $a\in(-1,0]$ such that $\lambda=\exp(-2\pi\mathrm{i}\,a)$. It is known that any holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is $\mathbf{R}$-specializable along any f, and that $\psi_{f,\lambda}\mathcal{M}$ remains holonomic, supported on $f^{-1}(0)$, and endowed with a nilpotent endomorphism N.
What about the perverse sheaf $\mathcal{F}:={}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$? We apply Deligne's topological construction of nearby cycles ${}^{\mathrm{p}}\psi_{f,\lambda}\mathcal{F}$, which is known to produce a perverse sheaf on X supported on $f^{-1}(0)$.

Theorem 2.1 **(Kashiwara-Malgrange, M. Saito).** *If* $\mathcal{M}$ *is holonomic and*
- *regular along* $f^{-1}(0)$,
- *or underlies* $(\mathcal{M},F^\bullet\mathcal{M})$ *which is* $\mathbf{R}$-*specializable along* (f),

then there is a functorial isomorphism ${}^{\mathrm{p}}\mathrm{DR}\,\psi_{f,\lambda}\mathcal{M}\simeq{}^{\mathrm{p}}\psi_{f,\lambda}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$.

2.2. Localization and dual localization of perverse sheaves and $\mathcal{D}$-modules

2.2.a. Localization and dual localization of constructible complexes. Let $\mathcal{F}$ be a sheaf of vector spaces on X. The sheaf $j_*j^{-1}\mathcal{F}$ satisfies
- $(j_*j^{-1}\mathcal{F})_x=\mathcal{F}_x$ if $x\in U$,
- $(j_*j^{-1}\mathcal{F})_x=\varinjlim_{V\ni x}\Gamma(V^*,\mathcal{F})$.

Similarly, the extension by zero (proper pushforward) is defined by
- $(j_!j^{-1}\mathcal{F})_x=\mathcal{F}_x$ if $x\in U$,
- $(j_!j^{-1}\mathcal{F})_x=0$.

These functors can be derived into $\boldsymbol{R}j_*j^{-1}$ and $\boldsymbol{R}j_!j^{-1}$. They are called the localization and dual localization functor. It is known that they preserve the property of having constructible cohomology, and they also preserve perversity (because D has codimension one in X).

2.2.b. Localization of holonomic $\mathcal{D}_X$-modules

Given an $\mathcal{O}_X$-module $\mathcal{M}$, the localization functor is $\mathcal{O}_X(*D)\otimes_{\mathcal{O}_X}\bullet$, where $\mathcal{O}_X(*D)$ is the sheaf of meromorphic functions on X with poles along D at most. The sheaf $\mathcal{O}_X(*D)$ is known to be flat over $\mathcal{O}_X$ (but not faithfully flat: If $\mathcal{M}$ is $\mathcal{O}_X$-coherent and supported on D, then $\mathcal{M}(*D)=0$). Since $\mathcal{O}_X(*D)$ is a left $\mathcal{D}_X$-module in a natural way, the localization functor acts on the category of $\mathcal{D}_X$-modules.
The fundamental result based on the existence of a Bernstein-Sato polynomial for the defining equation of D is Kashiwara's theorem:

Theorem 2.2. *Assume that $\mathcal{M}$ is holonomic. Then $\mathcal{M}(*D)$ is holonomic (hence coherent).*

Example 2.3.

- If D is locally defined by a coordinate x_1, it is easy to check that $\mathcal{O}_X(*D)$ is locally generated by $1/x_1$, which belongs to $V^{-1}\mathcal{O}_X(*D)$ since $(x_1\partial_{x_1}+1)\cdot(1/x_1)=\partial_{x_1}x_1\cdot(1/x_1)=0$.
- If $D=(f)$ for some holomorphic function f, we derive for each $k\geqslant 1$ a functional equation

$$b_f(s-k)\cdots b_f(s-1)f^{s-k}=P_k(x,\partial_x,s)f^s.$$

If $p\geqslant 1$ is an integer such that $-p<$ roots of b_f,then $-p-j$ is not a root of b_f for any $j\geqslant 1$, and we can fix $s=-p$ in the previous equation for each $k\geqslant 1$, showing that f^{-p} generates $\mathscr{O}_X(*D)$ as a $\mathcal{D}_X$-module.

2.2.c. V-filtration and localization. The Kashiwara-Malgrange filtration enables one to define localization and dual localization in a way suitable for extending the definition to the filtered case. For that purpose, we assume that D is smooth and $X=D\times\mathbf{C}_t$ (by using the trick of the pushforward by the inclusion of the graph of a defining equation of D). In such a case, the Kashiwara-Malgrange filtration along D is well-defined for any holonomic $\mathcal{D}_X$-module $\mathcal{M}$.

Proposition 2.4. *We have $\mathcal{M}(*D)=\mathcal{D}_X\otimes_{V^0\mathcal{D}_X}V^{-1}\mathcal{M}$.*

It is then natural to define

- $\mathcal{M}(!D)=\mathcal{D}_X\otimes_{V^0\mathcal{D}_X}V^{>-1}\mathcal{M}$,
- $\mathcal{M}(!*D)=\operatorname{image}[\mathcal{M}(!D)\to\mathcal{M}(*D)]$.

Proposition 2.5. *The kernel and cokernel of the morphism $\mathcal{M}(!D) \to \mathcal{M}(*D)$ are those of the nilpotent morphism*

$$\mathrm{gr}_V^0 \mathcal{M} \xrightarrow{\mathrm{N} = -t\partial_t} \mathrm{gr}_V^0 \mathcal{M}.$$

This nilpotent morphism N is the composition

$$\mathrm{gr}_V^0 \mathcal{M} \xrightarrow{\mathrm{can} = -\partial_t} \mathrm{gr}_V^{-1} \mathcal{M} \xrightarrow{\mathrm{var} = t} \mathrm{gr}_V^0 \mathcal{M}.$$

Remark 2.6. For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we have natural morphisms

$$\begin{tikzcd}
 & \mathcal{M} \arrow[dr] & \\
\mathcal{M}(!D) \arrow[ur] \arrow[dr, two heads] & & \mathcal{M}(*D) \\
 & \mathcal{M}(!*D) \arrow[ur, hook] &
\end{tikzcd}$$

which induce isomorphisms when restricted to $V^{>-1}$ of each module. On the other hand,

- can : $\mathrm{gr}_V^0 \mathcal{M}(!D) \to \mathrm{gr}_V^{-1} \mathcal{M}(!D)$ is an isomorphism,
- var : $\mathrm{gr}_V^{-1} \mathcal{M}(*D) \to \mathrm{gr}_V^0 \mathcal{M}(*D)$ is an isomorphism,
- for $\mathcal{M}(!*D)$, can is onto and var is injective.

***Theorem 2.7* (Beilinson).** *The category of holonomic $\mathcal{D}_X$-modules is equivalent to the category whose objects consist of quadruples $(\mathcal{M}_*, \mathcal{N}, \mathrm{c}, \mathrm{v})$, where $\mathcal{M}_*, \mathcal{N}$ are holonomic $\mathcal{D}_X$-modules, $\mathcal{M}_* = \mathcal{M}_*(*D)$ and $\mathcal{N}$ is supported on D, and morphisms* $\mathrm{c} : \mathrm{gr}_V^0 \mathcal{M}_* \to \mathcal{N}$, $\mathrm{v} : \mathcal{N} \to \mathrm{gr}_V^0 \mathcal{M}_*$ *such that* $\mathrm{v} \circ \mathrm{c} = \mathrm{N}$.

2.3. Localization and dual localization with filtration

The previous framework can be adapted to holonomic filtered $\mathcal{D}_X$-modules, and is convenient to explain the way of extending the Hodge filtration across a divisor.

Example 2.8. Let D be any divisor in X. Given the filtered $\mathcal{D}_X$-module $(\mathcal{O}_X, F^\bullet \mathcal{O}_X)$ (recall this is the trivial filtration jumping at 0 only), how to define a *coherent* filtration on $\mathcal{O}_X(*D)$? A candidate is the *filtration by the order of the pole*. When D is smooth or has only normal crossings as singularities, this filtration, suitably defined, is indeed the filtration to be used for Hodge theory. For other singularities, the situation is much more complicated.

We use the language of coherent $\mathscr{R}_X$-modules (Lecture 1) in order to treat $\mathcal{D}_X$-modules with a coherent filtration. We say that a coherent $\mathscr{R}_X$-module $\mathscr{M}$ is *strictly* $\mathbf{R}$*-specializable* along D if the corresponding $(\mathcal{M}, F^\bullet \mathcal{M})$ is $\mathbf{R}$-specializable along D. One can then define the notion of Kashiwara-Malgrange filtration $V^\bullet \mathscr{M}$ in the case $X = D \times \mathbf{C}_t$ with D smooth.

Definition 2.9. We *define* the localization and dual localization of $\mathscr{M}$ along D as

$$\begin{aligned}\mathscr{M}[*D] &:= \mathscr{R}_X \otimes_{V^0\mathscr{R}_X} V^{-1}\mathscr{M},\\ \mathscr{M}[!D] &:= \mathscr{R}_X \otimes_{V^0\mathscr{R}_X} V^{>-1}\mathscr{M}.\end{aligned}$$

In other words, for example,

$$F^p\mathcal{M}(*D) = \sum_{r-q\geqslant p} \partial_t^q t^{-1}(F^r\mathcal{M}\cap V^0\mathcal{M}).$$

Theorem 2.10 **(Beilinson correspondence with filtration)**
The category of coherent $\mathscr{R}_X$-modules which are strictly $\mathbf{R}$-specializable along D is equivalent to the category whose objects consist of quadruples $(\mathscr{M}_*, \mathscr{N}, \mathrm{c}, \mathrm{v})$, where $\mathscr{M}_*, \mathscr{N}$ are coherent strictly $\mathbf{R}$-specializable $\mathcal{D}_X$-modules, $\mathscr{M}_* = \mathscr{M}_*[*D]$ and $\mathscr{N}$ is supported on D, and morphisms $\mathrm{c} : \mathrm{gr}_V^0\mathscr{M}_* \to \mathscr{N}$, $\mathrm{v} : \mathscr{N} \to \mathrm{gr}_V^0\mathscr{M}_*$ such that $\mathrm{v}\circ\mathrm{c} = \mathrm{N}$.

LECTURE 3

OVERVIEW OF THE THEORY OF MIXED HODGE MODULES

3.1. Some results of classical Hodge theory

3.1.a. Hodge structure. Pure polarizable Hodge structure of weight k on the cohomology of a smooth complex variety or compact Kähler manifold X:

$$H^k(X,\mathbf{Q}),\quad H^k(X,\mathbf{Q})\otimes\mathbf{C}\simeq H^k(X,\mathbf{C})\simeq\bigoplus_{p+q=k}H^{p,q}_{\overline{\partial}}(X)$$
$$\simeq\bigoplus_{p+q=k}H^q(X,\Omega^p_X).$$

• Hodge filtration $F^pH^k(X,\mathbf{C}):=\bigoplus_{p'\geqslant p}H^{p',k-p'}_{\overline{\partial}}(X)$.

• Polarization on $H^k(X,\mathbf{C})$: uses Poincaré duality and the first Chern class of an ample line bundle, or the class of the Kähler form. $\Rightarrow$ Many polarizations.

• "Polarizable": there exists a polarization, but one does not need to make precise which one is used.

3.1.b. Variation of Hodge structure and the Hodge-Deligne theorem

The first step to *localize* the definition of a Hodge structure is that of variation of Hodge structure.
Data:

• $\mathcal{H}_{\mathbf{Q}}$: locally constant sheaf of $\mathbf{Q}$-vector spaces on X,

• $(\mathcal{V},\nabla)$: holomorphic vector bundle $\mathcal{V}$ with flat holomorphic connection $\nabla:\mathcal{V}\to\Omega^1_X\otimes\mathcal{V}$,

• Filtration $F^\bullet\mathcal{V}\subset\mathcal{V}$ by holomorphic subbundles,

• Duality pairing $\mathcal{H}_{\mathbf{C}}\otimes\mathcal{H}_{\mathbf{C}}\to\mathbf{C}$,

subject to the following compatibility properties:

• $\mathcal{V}^\nabla\simeq\mathbf{C}\otimes\mathcal{H}_{\mathbf{Q}}$,

• (Griffiths transversality) $\forall\,p$, $\nabla F^p\mathcal{V}\subset\Omega^1_X\otimes F^{p-1}\mathcal{V}$,

• The restriction to each $x\in X$ is a polarized pure Hodge structure of some weight.

De Rham complex:

$$\mathrm{DR}(\mathcal{V},\nabla)=\{\mathcal{V}\xrightarrow{\nabla}\Omega^1_X\otimes\mathcal{V}\xrightarrow{\nabla}\cdots\xrightarrow{\nabla}\Omega^n_X\otimes\mathcal{V}\}$$

and

$$\boldsymbol{H}^k(X,\mathrm{DR}(\mathcal{V},\nabla))\simeq H^k(X,\mathcal{V}^\nabla)=H^k(X,\underline{\mathcal{V}}_{\mathbf{C}}).$$

Filtered de Rham complex:

$$F^p\,\mathrm{DR}(\mathcal{V},\nabla)=\{F^p\mathcal{V}\xrightarrow{\nabla}\Omega^1_X\otimes F^{p-1}\mathcal{V}\xrightarrow{\nabla}\cdots\}$$

Theorem 3.1 **(Hodge-Deligne theorem).** *Let X be smooth projective or Kähler, let $(\mathcal{V},\nabla,F^\bullet\mathcal{V},\mathcal{H}_{\mathbf{Q}})$ be a pVHS of weight w. Then*

- $\forall p$, $\boldsymbol{H}^k(X,F^p\,\mathrm{DR}(\mathcal{V},\nabla))\to\boldsymbol{H}^k(X,\mathrm{DR}(\mathcal{V},\nabla))$ *is* injective.
  Image $:=F^p\boldsymbol{H}^k(X,\mathrm{DR}(\mathcal{V},\nabla))$.
- $\big(H^k(X,\mathcal{H}_{\mathbf{C}}),F^\bullet H^k(X,\mathcal{H}_{\mathbf{C}}),H^k(X,\mathcal{H}_{\mathbf{Q}})\big)$ *is a pHS of weight $w+k$.*

3.1.c. Open varieties. Assume now that U is smooth and quasi-projective and let X be a smooth projective variety containing U as a Zariski dense subset. It is often convenient to choose X such that $D:=X\smallsetminus U$ is a divisor with normal crossings.

Deligne has introduced the notion of a mixed Hodge structure, by adding to the data of a Hodge structure that of an increasing filtration $W_\bullet\mathcal{H}_{\mathbf{Q}}$ (the weight filtration) such that the graded piece gr^W_ℓ with induced F-filtration is a pure Hodge structure of weight ℓ. A remarkable property is:

Theorem 3.2 **(Deligne).** *The category of mixed Hodge structures is abelian and any morphism is bi-strict with respect to the Hodge and the weight filtrations.*

More precisely, given mixed Hodge srtuctures $(H_i,F^\bullet H_i,W_\bullet H_i)$ $(i=1,2,3)$ and an exact sequence

$$0\longrightarrow H_1\longrightarrow H_2\longrightarrow H_3\longrightarrow 0$$

of $\mathbf{Q}$-vector spaces for which the morphisms are compatible with the filtrations, then the sequence of bi-graded objects is still exact and moreover, the filtrations on H_1 and H_3 are the filtrations naturally induced by those of H_2.

Deligne has proved that $H^k(U,\mathbf{C})$ can be endowed with a Hodge filtration $F^\bullet H^k(U,\mathbf{C})$ and a weight filtration $W_\bullet H^k(U,\mathbf{Q})$, giving rise to a canonical mixed Hodge structure. These have an explicit expression on X if D has normal crossings. For example, $H^*(U,\mathbf{C})$ is realized as the hypercohomology on X of the *logarithmic de Rham complex* $\Omega^\bullet_X(\log D)$ and we have

$$H^k(U,\mathbf{C})\simeq\bigoplus_{p+q=k}H^q(X,\Omega^p_X(\log D)).$$

Moreover, the restriction morphism

$$H^k(X, \mathbf{Q}) \longrightarrow H^k(U, \mathbf{Q})$$

is a morphism of mixed Hodge structure (the left-hand side is pure, however).
Let now $(\mathcal{H}_{\mathbf{Q}}, \dots)$ be a pVHS on U. How to extend the previous properties? Forgetting the Hodge and weight filtrations, the good replacement for $H^k(X, \mathbf{Q})$ is $\mathrm{IH}^k(X, \mathcal{H}_{\mathbf{Q}})$ (intersection cohomology with coefficients in the locally constant sheaf $\mathcal{H}_{\mathbf{Q}}$) and there is a natural restriction morphism

$$\mathrm{IH}^k(X, \mathcal{H}_{\mathbf{Q}}) \longrightarrow H^k(U, \mathcal{H}_{\mathbf{Q}}).$$

We can regard $\mathrm{IH}^k(X, \mathcal{H}_{\mathbf{Q}})$ as the kth hypercohomology of the *intersection complex* $\mathrm{IC}_X(\mathcal{H}_{\mathbf{Q}})$, while $H^k(U, \mathcal{H}_{\mathbf{Q}})$ is the hypercohomology of the complex $\boldsymbol{R}j_*\mathcal{H}_{\mathbf{Q}}$, and the morphism comes from the natural morphism

$$\mathrm{IC}_X(\mathcal{H}_{\mathbf{Q}}) \longrightarrow \boldsymbol{R}j_*j^{-1}\,\mathrm{IC}_X(\mathcal{H}_{\mathbf{Q}}) = \boldsymbol{R}j_*\mathcal{H}_{\mathbf{Q}}.$$

De Rham realization:

- Deligne's meromorphic extension $(\mathcal{M}_*, \nabla)$ of $(\mathcal{V}, \nabla)$ is a locally free $\mathcal{O}_X(*D)$-module of finite rank endowed with an integrable connection, which satisfies $\mathrm{DR}(\mathcal{M}_*, \nabla) \simeq \boldsymbol{R}j_*\mathcal{H}_{\mathbf{C}}$.
- In order to define the de Rham realization of $\mathrm{IC}_X(\mathcal{H}_{\mathbf{Q}})$, one needs to use the notion of *holonomic* $\mathcal{D}_X$*-module* instead of that of vector bundle with integrable connection.

3.2. The purpose of the theory of mixed Hodge modules

3.2.a. The objects (pure case). The objects we consider take the form

$$((\mathcal{M}, \nabla, F^\bullet\mathcal{M}), \mathcal{F}_{\mathbf{Q}}, \mathrm{Iso}),$$

where

(1) $(\mathcal{M}, \nabla)$ is an $\mathcal{O}_X$-module with integrable connection ∇. We do not impose $\mathcal{O}_X$-coherence, otherwise it would be locally free of finite rank. The finiteness property is coherence over the ring $\mathcal{D}_X$ of differential operators, for the structure given by the integrable connection. More precisely, one requires holonomicity.

(2) $F^\bullet\mathcal{M}$ is a decreasing filtration bounded from above, satisfying Griffiths transversality with respect to ∇, and also a coherence property with respect to the $\mathcal{D}_X$-structure (in particular, there exists locally on X a number p_o such that, for each $p \geqslant 0$, $F^{p_o-p}\mathcal{M}$ is locally generated by the coherent $\mathcal{O}_X$-module $F^{p_o}\mathcal{M}$ and its images by powers of ∇. In general, the filtration is not indexed by a finite set, as in the case of pVHS.

(3) $\mathcal{F}_{\mathbf{Q}}$ is a $\mathbf{Q}$-perverse sheaf on X.

(4) Iso is an isomorphism ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} \simeq \mathbf{C} \otimes_{\mathbf{Q}} \mathcal{F}_{\mathbf{Q}}$.

The morphisms are the natural ones which respect this structure. The category $\mathsf{pHM}(X, w)$ defined in Lecture 4 is a full subcategory of the category consisting of such objects and morphisms.

Remark 3.3. When X is reduced to a point, the category $\mathsf{pHM}(X, w)$ is nothing but the category of polarizable pure Hodge structures of weight w.

3.2.b. The main theorems (pure case)

***Theorem 3.4* (Abelianity).** *The category* $\mathsf{pHM}(X, w)$ *of polarizable pure Hodge modules of some weight* $w \in \mathbf{Z}$ *is an abelian category and any morphism is strict with respect to Hodge filtration.*

***Theorem 3.5* (Structure theorem of pure Hodge modules)**

(1) *A pure Hodge module with support a closed analytic subset* Z *of* X *decomposes as the direct sum of pure Hodge modules, each of which supported on an irreducible component* Z_i *of* Z.

(2) *A pure Hodge module with support an irreducible closed analytic subset* Z *of* X *is uniquely determined by its restriction to the smooth part* Z° *of* Z, *which is a pVHS.*

***Theorem 3.6* (Hodge-Saito theorem).**

- X: *smooth projective,*
- $(\mathcal{M}, \nabla, F^\bullet\mathcal{M}, \mathcal{F}_{\mathbf{Q}})$: *an object of* $\mathsf{pHM}(X, w)$.

Then

- $\forall p$, $\boldsymbol{H}^k(X, F^p\, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}, \nabla)) \to \boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}, \nabla))$ *is* injective.
  Image $:= F^p\boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}, \nabla))$.
- $\big(\boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}, \nabla)), F^\bullet\boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}, \nabla)), H^k(X, \mathcal{F}_{\mathbf{Q}})\big)$ *is a pHS of weight* $w + k$.

Example 3.7. Consider the trivial filtration of $\mathcal{O}_X$ defined by $F^0\mathcal{O}_X = \mathcal{O}_X$ and $F^1\mathcal{O}_X = 0$. A particular case of the theorem, which is *not* a trivial statement in this framework, is:
The data $((\mathcal{O}_X, \mathrm{d}), F^\bullet\mathcal{O}_X, \mathbf{Q}_X[\dim X], \mathrm{Iso})$ *form a polarizable pure Hodge module of weight* $\dim X$.

3.2.c. The objects (mixed case). The objects are endowed with a supplementary increasing filtration, of the form

$$(W_\bullet(\mathcal{M}, \nabla), F^\bullet\mathcal{M}, W_\bullet\mathcal{F}_{\mathbf{Q}}, \mathrm{Iso}),$$

where $W_\bullet\mathcal{F}_{\mathbf{Q}}$ is an increasing filtration of the perverse sheaf $\mathcal{F}_{\mathbf{Q}}$ in the abelian category of perverse sheaves and $W_\bullet(\mathcal{M}, \nabla)$ is an increasing filtration in the category

of $\mathcal{O}_X$-modules with integrable connection. The isomorphism Iso should respect the $W_\bullet$ filtrations. Morphisms should be compatible with all data.
The category $\mathsf{MHM}(X)$ of mixed Hodge modules on X is a full subcategory of the above very big category whose objects satisfy in particular the property that gr_ℓ^W is an object of $\mathsf{pHM}(X, \ell)$.

Example 3.8. For any divisor D in X, the $\mathcal{D}_X$-module $\mathcal{O}_X(*D)$ underlies a mixed Hodge module. Difficult question: to make explicit the Hodge and the weight filtration in terms of the singularities of D. Simple answer (Deligne) if D has normal crossings.

3.2.d. The main theorems (mixed case)

Theorem 3.9. *The category $\mathsf{MHM}(X)$ of mixed pure Hodge modules is an abelian category and any morphism is bi-strict with respect to Hodge filtration and the weight filtration.*

***Theorem 3.10* (Six operations).** *In the algebraic setting (we restrict to quasi-projective varieties and morphisms), the category $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(X))$ is endowed with a formalism of six operations of Grothendieck.*

LECTURE 4

PURE AND MIXED HODGE MODULES

4.1. The case of curves

4.1.a. Schmid's theorems. $X :=$ compact Riemann surface, $D :=$ finite set of points, $j : U = X \smallsetminus D \hookrightarrow X$, $(\mathcal{V}, \nabla, F^\bullet\mathcal{V}, \mathcal{H}_{\mathbf{Q}})$ a pVHS of weight w on U. How to extend such an object on X? Various possibilities for $\mathcal{H}_{\mathbf{Q}}$: $j_*\mathcal{H}_{\mathbf{Q}}$, $\boldsymbol{R}j_*\mathcal{H}_{\mathbf{Q}}$, $j_!\mathcal{H}_{\mathbf{Q}}$. The questions are local, so one can assume $X = \Delta_t \subset \mathbf{C}_t$ and $D = \{t = 0\}$.

• Deligne's meromorphic extension $(\mathcal{V}_*, \nabla)$ only depends on $(\mathcal{V}, \nabla)$: it produces a $\mathcal{O}_X(*D)$-locally free module with connection having a regular singularity at every point of D. We have $\mathrm{DR}(\mathcal{V}_*, \nabla) \simeq \boldsymbol{R}j_*\mathcal{H}_{\mathbf{C}}$. In fact Deligne constructs the locally free $\mathcal{O}_X$-modules with connection having a simple pole $(\mathcal{V}^a, \nabla)$ $(a \in \mathbf{R})$ such that $\operatorname{Res}\nabla$ on $\mathcal{V}^a$ has eigenvalues with real part in $[a, a+1)$, and $\mathcal{V}_* = \bigcup_a \mathcal{V}^a$.

• From the point of view of $\mathcal{D}_X$-modules, $\mathcal{V}_*$ is generated by $\mathcal{V}^{-1}$. Define $\mathcal{V}_{!*}$ as the $\mathcal{D}_X$-submodule of $\mathcal{V}_*$ generated by $\mathcal{V}^{>-1}$. One can show $\mathrm{DR}(\mathcal{V}_{!*}, \nabla) \simeq j_*\mathcal{H}_{\mathbf{C}}$.

• Each $\mathcal{V}^a$ is a $\mathcal{D}_X(\log D)$-module (in local coordinate at a point of D, $\mathbf{C}\{t\}\langle t\partial_t\rangle$, and $\operatorname{Res}\nabla$ is the endomorphism of $\mathcal{V}^a/t\mathcal{V}^a$ induced by $t\partial_t$). Set $\mathcal{V}_! := \mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{V}^{>-1}$. Then one can show $\mathrm{DR}(\mathcal{V}_!, \nabla) \simeq j_!\mathcal{H}_{\mathbf{C}}$.

Theorem 4.1 **(Schmid).** *For a pVHS, the metric fits with Deligne's construction, i.e., $\mathcal{V}^a$ can be defined only in terms of order of growth of the norms of sections.*

How to extend the filtration $F^\bullet\mathcal{V}$?

Theorem 4.2 **(Schmid).** *$j_*F^p\mathcal{V} \cap \mathcal{V}^{>-1}$ is a locally free $\mathcal{O}_X$-module.*

Define locally

$$F^p\mathcal{V}_{!*} := j_*F^p\mathcal{V} \cap \mathcal{V}^{>-1} + \partial_t(j_*F^{p+1}\mathcal{V} \cap \mathcal{V}^{>-1}) + \partial_t^2(j_*F^{p+2}\mathcal{V} \cap \mathcal{V}^{>-1}) + \cdots$$

We obtain a coherent filtered $\mathcal{D}_X$-module $(\mathcal{V}_{!*}, F^\bullet\mathcal{V}_{!*})$.

4.1.b. Monodromy filtration. Each $\mathrm{gr}^a\mathcal{V} := \mathcal{V}^a/\mathcal{V}^{>a}$ is a finite dimensional vector space endowed with the endomorphism induced by $t\partial_t - a$. By a lemma of Borel, it is *nilpotent*, and denoted by N. Moreoever, Deligne's construction fits with topology, namely, $\mathrm{gr}^a\mathcal{V}$ is identified with the nearby cycle space $\psi_{t,\lambda}\mathcal{H}_{\mathbf{C}}$ (generalized eigenspace of the nearby cycle space $\psi_t\mathcal{H}_{\mathbf{C}}$ with respect to

the eigenvalue λ of the monodromy T), and we have $\mathrm{T} = \exp(-2\pi\mathsf{i}\, t\partial_t)$, so $\lambda = \exp -2\pi\mathsf{i}\, a$.
A variant of the Jordan decomposition of N produces a unique increasing filtration $\mathrm{M}_\bullet \mathrm{gr}^a\mathcal{V}$ such that

- $\mathrm{N}\cdot\mathrm{M}_\ell \subset \mathrm{M}_{\ell-2}$,
- for any $\ell \geqslant 1$, $\mathrm{N}^\ell : \mathrm{gr}^{\mathrm{M}}_\ell \to \mathrm{gr}^{\mathrm{M}}_{-\ell}$ is an isomorphism.

Example 4.3. $\mathbf{C}^3$ with basis e_1, e_2, e_3 such that the matrix of N in this basis is the lower Jordan block of size 3: $\mathrm{N}e_1 = e_2$, $\mathrm{N}e_2 = e_3$ and $\mathrm{N}e_3 = 0$. Then

$$\mathbf{C}^3 = \mathrm{M}_2 \supset \mathrm{M}_1 = \mathrm{M}_0 = \langle e_2, e_3\rangle \supset \mathrm{M}_{-1} = \mathrm{M}_{-2} = \langle e_3\rangle \supset 0$$

and the basis should better be denoted as e_2, e_0, e_{-2} with respect to the *weight*.

For the pVHS as above and $a \in (-1, 0]$, the filtration $F^p\mathcal{V}^a := j_*F^p\mathcal{V}\cap\mathcal{V}^a$ induces a filtration $F^p\mathrm{gr}^a\mathcal{V}$, and then a filtration

$$F^p\mathrm{gr}^{\mathrm{M}}_\ell\mathrm{gr}^a\mathcal{V} := F^p \cap \mathrm{M}_\ell / F^p \cap \mathrm{M}_{\ell-1}.$$

***Theorem 4.4* (Schmid).** *For each $\ell \in \mathbf{Z}$, $(\bigoplus_a \mathrm{gr}^{\mathrm{M}}_\ell\mathrm{gr}^a\mathcal{V}, \bigoplus_a F^\bullet\mathrm{gr}^{\mathrm{M}}_\ell\mathrm{gr}^a\mathcal{V}, \mathrm{gr}^{\mathrm{M}}_\ell\psi_t\mathcal{H}_{\mathbf{Q}})$ is a pHS of weight $w+\ell$.*

4.2. Pure Hodge modules

4.2.a. Support-decomposability. X smooth projective (or compact Kähler). The category $\mathsf{pVHS}(X, w)$ is semi-simple, i.e., every object in this category decomposes as the direct sum of simple objects.

Remark 4.5. A $\mathbf{Q}$-local system on a topological space Z is semi-simple iff the associated $\mathbf{C}$-local system is semi-simple (but the simple components are not the same in general!).

Deligne has proved that the $\mathbf{Q}$-local system underlying a pVHS on X is semi-simple (this is stronger than the semi-simplicity of $\mathsf{pVHS}(X, w)$). This can also be obtained by using properties of the Hodge metric (it is a harmonic metric).
Semi-simplicity of a local system is a *global property*, i.e., it does not localize in general.

Example 4.6. Choose two automorphisms T, T' of $\mathbf{C}^d$ which do not leave invariant any nontrivial subspace (this can be achieved with a suitable choice of upper and lower triangular matrices). For a curve C of genus two with

$$\pi_1(C,\star) = \big\langle\gamma_1,\gamma_2,\gamma_3,\gamma_4 \mid (\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1})(\gamma_3\gamma_4\gamma_3^{-1}\gamma_4^{-1}) = 1\big\rangle,$$

consider the representation

$$\gamma_1 \longmapsto T, \quad \gamma_2 \longmapsto T', \quad \gamma_3 \longmapsto T'^{-1}, \quad \gamma_4 \longmapsto T^{-1}.$$

The representation is irreducible, but the restriction of the representation to $\mathrm{nb}(\gamma_1)$ is not irreducible if T is not diagonalizable.

For a coherent $\mathcal{D}_X$-module, semi-simplicity may be broken because of support properties.

Example 4.7. D a divisor in X smooth. We have an exact sequence of $\mathcal{D}_X$-modules

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(*D) \longrightarrow \mathcal{O}_X(*D)/\mathcal{O}_X \longrightarrow 0.$$

It is clear that $\mathcal{O}_X$ is irreducible as a $\mathcal{D}_X$-module, while $\mathcal{O}_X(*D)$ is not, even though the constant local system on $X \smallsetminus D$ is irreducible.

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. If it is irreducible, then its support Z must be irreducible (otherwise, $Z = Z_1 \cup Z_2$ and the submodule of $\mathcal{M}$ consisting of sections supported on Z_1 is a proper submodule). If $\mathcal{M}$ is semi-simple, then it must decomposes as a direct sum with respect to the irreducible components of the support.
But If Z is irreducible, it may happen that for some $x_o \in Z$, the space $Z \cap \mathrm{nb}(x_o)$ is not irreducible. We wish however to keep the decomposability with respect to the support by localization. This leads to the notion of S-decomposable (filtered) $\mathcal{D}_X$-module.

***Definition 4.8* (S-decomposability).** Let $(\mathcal{M}, F^\bullet\mathcal{M})$ be a filtered $\mathcal{D}_X$-module which is $\mathbf{R}$-specializable along any germ of holomorphic function $f : (X, x_o) \to \mathbf{C}$. We say that it is S-decomposable if for any x_o and any such f, $(\mathcal{M}, F^\bullet\mathcal{M})$ decomposes as the direct sum $(\mathcal{M}', F^\bullet\mathcal{M}') \oplus (\mathcal{M}'', F^\bullet\mathcal{M}'')$ such that $(\mathcal{M}'', F^\bullet\mathcal{M}'')$ is supported on $f^{-1}(0)$ and $(\mathcal{M}', F^\bullet\mathcal{M}')$ has no subobject nor quotient object supported on $f^{-1}(0)$.

4.2.b. Definition of pure Hodge modules. The category $\mathsf{HM}(X, w)$ of pure Hodge modules of weight w on X is defined in an inductive way with respect to the dimension d of the support. We thus define the category $\mathsf{HM}_{\leqslant d}(X, w)$. The objects take the form $M = ((\mathcal{M}, F^\bullet\mathcal{M}), \mathcal{F}_{\mathbf{Q}}, \mathrm{Iso})$, where $(\mathcal{M}, F^\bullet\mathcal{M})$ is a holonomic $\mathcal{D}_X$-module with a coherent filtration, $\mathcal{F}_{\mathbf{Q}}$ is a perverse sheaf on X, and $\mathrm{Iso} : {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} \xrightarrow{\sim} \mathcal{F}_{\mathbf{C}}$ is an isomorphism, and the morphisms are the natural ones.

Definition 4.9.

• The category $\mathsf{HM}_{\{x_o\}}(X,w)$ of pure Hodge modules of weight w supported in $\{x_o\}$ is equivalent to the category of pure Hodge structures of weight w via the pushforward $(i_+,\boldsymbol{R}i_*)$.

• For $d\geqslant 1$, $M=((\mathcal{M},F^\bullet\mathcal{M}),\mathcal{F}_{\mathbf{Q}},\mathrm{Iso})$ is in the category $\mathsf{HM}_{\leqslant d}(X,w)$ if it satisfies the following properties:

  • $(\mathcal{M},F^\bullet\mathcal{M})$ is $\mathbf{R}$-specializable along any germ (f,x_o) and it is S-decomposable,

  • for any holomorphic $f:X\supset U\to\mathbf{C}$, and any $\ell\in\mathbf{Z}$, the object $\mathrm{gr}_\ell^{\mathrm{M}}\psi_f M$ is an object of $\mathsf{HM}_{\leqslant d-1}(X,\ell+w-1)$.

The notion of *polarization* is defined in a way similar to this, by induction on the dimension of the support. This leads to the definition of the category $\mathsf{pHM}(X,w)$ of polarizable pure Hodge modules of weight w.

Remark 4.10. Recall the two important theorems: Theorem 3.4 (Abelianity) and the structure theorem 3.5. That $\mathbf{Q}^{\mathrm{H}}:=((\mathcal{O}_X,F^\bullet\mathcal{O}_X),\mathbf{Q}_X[\dim X])$ (resp. a VHS of weight w) is a pure Hodge module of weight $\dim X$ (resp. $w+\dim X$) is non-trivial.

4.3. Mixed Hodge modules

A naive approach in order to define the category of mixed Hodge modules would be to mimic the definition of mixed Hodge structures from that of pure Hodge structures. Namely, we define the category $\mathsf{WHM}(X)$ of W-filtered Hodge modules: objects consist of

• a filtered holonomic $\mathcal{D}_X$-module $(\mathcal{M},F_\bullet\mathcal{M})$ endowed with an increasing finite exhaustive filtration $W_\ell\mathcal{M}$ by holonomic $\mathcal{D}_X$-submodules,

• a filtration $W_\bullet\mathcal{F}_{\mathbf{Q}}$ of $\mathcal{F}_{\mathbf{Q}}$ in the abelain category of perverse sheaves,

• both W-filtrations corresponding by Iso,

such that each graded object $(\mathrm{gr}_\ell^W\mathcal{M},F_\bullet\mathrm{gr}_\ell^W\mathcal{M},\mathrm{gr}_\ell^W\mathcal{F}_{\mathbf{Q}},\mathrm{gr}_\ell^W\mathrm{Iso})$ is an object of $\mathsf{pHM}(X,\ell)$.

Although this category $\mathsf{WHM}(X)$ has good properties (it is abelian and behaves well by projective pushforward), it is not fine enough, as the localization functor $[*D]$ is possibly not defined in it. The category $\mathsf{MHM}(X)$ of mixed Hodge modules on X is a full subcategory of $\mathsf{WHM}(X)$ so that

• The localization functor $[*D]$ is well defined for any divisor,

• A subtler property, called *admissibility*, is satisfied.

LECTURE 5

THE DECOMPOSITION THEOREM

The decomposition theorem is an essential result in algebraic geometry. Roughly speaking, it means that "semi-simplicity is a property that is preserved by projective pushforward". At the heart of the decomposition theorem is the Hard Lefschetz theorem [**Del68**] and the preservation of polarizability by projective pushforward.

5.1. The topological decomposition theorem [BBDG82]

5.1.a. The case of a smooth morphism. Let us start with the simplest case, that of a *smooth projective* morphism $f : X \to Y$ between smooth quasi-projective varieties. In particular, each sheaf $R^k f_* \mathbf{C}_X$ is a locally constant sheaf on Y with stalk $H^k(f^{-1}(y), \mathbf{C})$ at y. Each fibre $f^{-1}(y)$ being smooth projective, one can apply to it the Hodge Lefschetz package. Let us choose a relatively ample line bundle L on X. Then the relative Hard Lefschetz theorem holds, so that for each $\ell \geqslant 0$, the ℓth power L_L^ℓ of the associated Lefschetz operator $\mathrm{L}_L := c_1(L) \cup \bullet$ induces an isomorphism $R^{n-\ell} f_* \mathbf{C}_X \xrightarrow{\sim} R^{n+\ell} f_* \mathbf{C}_X$ ($n = \dim X$). Deligne [**Del68**] has shown that this implies a (possibly non-canonical) decomposition in $\mathsf{D}^{\mathrm{b}}(\mathbf{C}_Y)$:

$$\boldsymbol{R} f_* \mathbf{C}_X \simeq \bigoplus_k R^k f_* \mathbf{C}_X[-k].$$

This result is in fact local (in the analytic topology) on Y, i.e., we do not need to assume that Y is quasi-projective and any complex manifold would satisfy the same property. Moreover, the local systems $R^k f_* \mathbf{C}_X$ underlie a pVHS of weight k. We now use the quasi-projectivity of Y, which implies that each $R^k f_* \mathbf{C}_X$ is a semi-simple local system, hence can be furthermore decomposed as the direct sum of irreducible local systems.

5.1.b. The case of more general local systems. If instead of the constant sheaf $\mathbf{C}_X$ we start from a locally constant sheaf $\mathcal{L}$ on X, what kind of properties are enough to ensure a similar decomposition theorem?

- If $\mathcal{L}$ underlies a pVHS on X of some weight w, then a similar argument applies.
- If X and Y are projective, and if $\mathcal{L}$ is a semi-simple local system on X, then Simpson has shown that a similar decomposition theorem holds. It follows from the work of T. Mochizuki one the one hand, and Drinfeld on the other hand (by arithmetic arguments) that it is enough to assume that X and Y are quasi-projective (and f is still smooth).

5.1.c. The case of perverse sheaves of geometric origin. We assume some familiarity with the notion of perverse sheaf. Since we will be mainly interested in the behaviour of the Hodge filtration, the reader may think of a perverse sheaf over $\mathbf{C}$ as a holonomic $\mathcal{D}_X$-module.
Assume that X, Y are smooth quasi-projective varieties, and $f : X \to Y$ is projective but not necessarily smooth. Since we will work in the realm of perverse sheaves, we adopt the perverse convention and denote ${}^{\mathrm{p}}\mathbf{C}_X := \mathbf{C}_X[\dim X]$. The complex $\boldsymbol{R}f_*{}^{\mathrm{p}}\mathbf{C}_X$ is $\mathbf{C}$-constructible, and it is more convenient to consider its perverse cohomology sheaves ${}^{\mathrm{p}}R^jf_*{}^{\mathrm{p}}\mathbf{C}_X$. It follows from BBDG [**BBDG82**, Th. 6.2.5] that there is a non-canonical decomposition in $\mathsf{D}^{\mathrm{b}}(\mathbf{C}_Y)$:

$$\boldsymbol{R}f_*{}^{\mathrm{p}}\mathbf{C}_X \simeq \bigoplus_j {}^{\mathrm{p}}R^jf_*{}^{\mathrm{p}}\mathbf{C}_X[-j],$$

and each perverse sheaf ${}^{\mathrm{p}}R^jf_*{}^{\mathrm{p}}\mathbf{C}_X$ is semi-simple.
Another proof, relying on classical Hodge theory, i.e., Deligne's theory, has been given by de Cataldo and Migliorini [**dCM05**]. In [**BW19**], Budur and Wang propose a new approach to this result.

5.1.d. The general case of a semi-simple perverse sheaf. Let us mention that the decomposition theorem has been much generalized by many authors by replacing the constant sheaf with any semi-simple perverse sheaf. We will not discuss these generalizations (see [**dC16**] and the references therein).

5.2. M. Saito's decomposition theorem

We will se how the decomposition theorem of M. Saito [**Sai88**] for pure Hodge modules unifies the previous results.

5.2.a. Pushforward of $\mathcal{D}_X$-modules. Let $f : X \to Y$ be a holomorphic map between complex manifolds and let $\mathcal{M}$ be a left $\mathcal{D}_X$-module. Equip $\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ with the structure of left $\mathcal{D}_X$-module as follows: holomorphic functions act on $\mathcal{M}$, and vector fields ξ on X act following Lebniz' rule:

$$\xi \cdot (m \otimes P) = (\xi m) \otimes P + m \otimes Tf(\xi) \cdot P,$$

where $Tf : \Theta_X \to f^*\Theta_Y$ is the tangent map to f. This module is also equipped with a right $f^{-1}\mathcal{D}_Y$-module structure in a trivial way.

Example 5.1. If $\mathcal{M} = \mathcal{O}_X$, the bimodule thus obtained is usually denoted by $\mathcal{D}_{X\to Y}$ and is called the transfer $\mathcal{D}_X$-module. The above bimodule is then isomorphic to $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}$.

We can apply the de Rham functor on X to the left $\mathcal{D}_X$-module $\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$ and get the complex

$$ {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) = (\Omega_X^{n+\bullet} \otimes (\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)). $$

Its direct image $\boldsymbol{R}f_*\, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)$ is a bounded complex in the category of right $\mathcal{D}_Y$-modules. This is $f_+\mathcal{M}$ up to side-changing. The cohomology modules are denoted by $f_+^j\mathcal{M}$. If Y is reduced to a point, we recover the de Rham cohomology of $\mathcal{M}$.

5.2.b. Pushforward of filtered $\mathcal{D}_X$-modules and E_1 degeneration

Assume now that $(\mathcal{M}, F^\bullet\mathcal{M})$ is a coherent $\mathcal{D}_X$-module with a coherent filtration. By using the filtration by the order of differential operators on $\mathcal{D}_Y$ one can define a filtration

$$ F^p(\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) := \sum_{q+r=p} F^q\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}F^r\mathcal{D}_Y, $$

and one can filter the de Rham complex

$$ \begin{aligned} &F^p\, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \\ &\qquad = \{F^p(\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \xrightarrow{\nabla} F^{p-1}(\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \longrightarrow \cdots\} \end{aligned} $$

Let us emphasize a consequence of the Hodge-Saito theorem stated later.

Theorem 5.2 **(Degeneration at E_1).** *Assume that $(\mathcal{M}, F^\bullet\mathcal{M})$ underlies an object of $\mathsf{pHM}(X, w)$ and that f is projective. Then for each p and j, the natural morphism*

$$ R^jf_*F^p\, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \longrightarrow R^jf_*\, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) = f_+^j\mathcal{M} $$

is injective, and its image is denoted $F^pf_+^j\mathcal{M}$.

5.2.c. The decomposition theorem for $\mathsf{pHM}(X, w)$

Theorem 5.3 **(Hodge-Saito theorem).** *Let f be a projective morphism of complex manifolds and let $M = ((\mathcal{M}, F^\bullet\mathcal{M}), \mathcal{F}_{\mathbf{Q}}, \mathrm{Iso})$ be an object of $\mathsf{pHM}(X, w)$. Then,*

- *We have decompositions compatible with* p *in* $\mathsf{D}^{\mathrm{b}}(\mathcal{O}_Y)$*:*

$$\boldsymbol{R}f_* F^p\,{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \simeq \bigoplus_j R^j f_* F^p\,{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)[-j]$$

giving rise at the limit $p = -\infty$ *to a decomposition in* $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_Y)$*:*

$$f_+\mathcal{M} \simeq \bigoplus_j f_+^j\mathcal{M}[-j]$$

which is compatible via ${}^{\mathrm{p}}\mathrm{DR}$ *and* Iso *with a decomposition in* $\mathsf{D}^{\mathrm{b}}(\mathbf{C}_Y)$*:*

$$\boldsymbol{R}f_*\mathcal{F}_{\mathbf{Q}} \simeq \bigoplus_j {}^{\mathrm{p}}R^j f_*\mathcal{F}_{\mathbf{Q}}[-j],$$

- *for each* $j \in \mathbf{Z}$*, the object* $((f_+^j\mathcal{M}, F^{\delta+\bullet}f_+^j\mathcal{M}), {}^{\mathrm{p}}R^j f_*\mathcal{F}_{\mathbf{Q}}, {}^{\mathrm{p}}R^j f_*\,\mathrm{Iso})$ *belongs to* $\mathsf{pHM}(Y, w+j)$*, where* δ *is a suitable normalization depending on* $\dim X$ *and* $\dim Y$*.*

The first point is obtained, as expected, by proving a relative version of the Hard Lefschetz theorem for the cohomology pushforward objects. Since the Lefschetz operator is defined over $\mathbf{Q}$ in this projective setting, it acts on the triples occurring as pushforwards of objects of $\mathsf{pHM}(X, w)$.

Remark 5.4. (1) If Y is assumed quasi-projective, then by the structure theorem 3.5 and Deligne's semi-simplicity theorem mentioned after Remark 4.5, each perverse cohomology sheaf ${}^{\mathrm{p}}R^j f_*\mathcal{F}_{\mathbf{Q}}$ is semi-simple.

(2) If X is smooth projective and f is the constant map, then one recovers the Hodge-Saito theorem 3.6.

5.2.d. Kollár's theorems.

In [**Kol86a, Kol86b**], Kollár has proved a decomposition theorem in the holomorphic setting, together with vanishing theorems.

Theorem 5.5 **(Kollár).** *Let* $f : X \to Z$ *be a surjective map* $X \to Z$*,* X, Z *projective,* X *smooth and* Z *possibly singular. Then the following holds:*

(Decomposition) $\boldsymbol{R}f_*\omega_X \simeq \bigoplus_j R^j f_*\omega_X$*,*

(Torsion freeness) $R^j f_*\omega_X$ *are torsion-free* $\mathcal{O}_Z$*-modules,*

(Vanishing) *If* L *is an ample line bundle on* Z*, then* $H^k(Z, L \otimes R^j f_*\omega_X) = 0$ *for* $k > 0$ *and any* j*.*

For the first and third properties, one can embed Z into a smooth projective variety Y and it is not restrictive to assume Z smooth.

5.2.e. Generalization of Kollár's theorems. Let $f : X \to Y$ be a projective morphism between smooth quasi-projective varieties. For $M \in \mathsf{pHM}(X, w)$, let $F^{p_o}\mathcal{M}$ be the first nonzero step of the Hodge filtration. Then the first nonzero step in $F^\bullet\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is $F^{p_o+n}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} = \omega_X \otimes_{\mathcal{O}_X} F^{p_o}\mathcal{M}[-n]$. The filtered decomposition theorem implies the decomposition

$$\boldsymbol{R}f_*(\omega_X \otimes_{\mathcal{O}_X} F^{p_o}\mathcal{M}) \simeq \bigoplus_j R^j f_*(\omega_X \otimes_{\mathcal{O}_X} F^{p_o}\mathcal{M})[-j].$$

The two other properties also extend to the case of pure Hodge modules and f as above, and $\operatorname{Supp} f_+^j\mathcal{M} = Z$ possibly singular.

BIBLIOGRAPHY

[BBDG82] A.A. Beilinson, J.N. Bernstein, P. Deligne & O. Gabber – Faisceaux pervers, in *Analyse et topologie sur les espaces singuliers*, Astérisque, vol. 100, Société Mathématique de France, Paris, 1982, ed. 2018, p. 1–180.

[BW19] N. Budur & B. Wang – Absolute sets and the Decomposition Theorem, *Ann. Sci. École Norm. Sup. (4)* **52** (2019), to appear, `arXiv:1702.06267`.

[dC16] M.A. de Cataldo – Decomposition theorem for semi-simples, *J. Singul.* **14** (2016), p. 194–197.

[dCM05] M.A. de Cataldo & L. Migliorini – The Hodge theory of algebraic maps, *Ann. Sci. École Norm. Sup. (4)* **38** (2005), no. 5, p. 693–750, `arXiv:math/0306030`.

[Del68] P. Deligne – Théorème de Lefschetz et critères de dégénérescence de suites spectrales, *Publ. Math. Inst. Hautes Études Sci.* **35** (1968), p. 107–126.

[Kol86a] J. Kollár – Higher direct images of dualizing sheaves. I, *Ann. of Math. (2)* **123** (1986), no. 1, p. 11–42.

[Kol86b] ______, Higher direct images of dualizing sheaves. II, *Ann. of Math. (2)* **124** (1986), no. 1, p. 171–202.

[Pop16] M. Popa – Kodaira-Saito vanishing and applications, *Enseign. Math.* **62** (2016), no. 1-2, p. 49–89.

[Sai88] M. Saito – Modules de Hodge polarisables, *Publ. RIMS, Kyoto Univ.* **24** (1988), p. 849–995.

[Sai90] ______, Mixed Hodge Modules, *Publ. RIMS, Kyoto Univ.* **26** (1990), p. 221–333.

[Sai91] ______, Mixed Hodge modules and applications, in *Proceedings of the ICM (Kyoto, 1990)*, Math. Soc. Japan, Tokyo, 1991, p. 725–734.

[Sai94] ______, On the theory of mixed Hodge modules, in *Selected papers on number theory, algebraic geometry, and differential geometry*, Amer. Math. Soc. Transl. Ser. 2, vol. 160, Amer. Math. Soc., Providence, R.I., 1994, p. 47–61.

[Sai17] ______, A young person's guide to mixed Hodge modules, in *Hodge theory and L^2-analysis*, Adv. Lect. Math. (ALM), vol. 39, Int. Press, Somerville, MA, 2017, p. 517–553, `arXiv:1605.00435`.

[Sch14] C. Schnell – An overview of M. Saito's theory of mixed Hodge modules, `arXiv:1405.3096`, 2014.