Philippe Maisonobe
Claude Sabbah

ASPECTS OF THE THEORY OF $\mathscr{D}$-MODULES

LECTURE NOTES (KAISERSLAUTERN 2002)

REVISED VERSION: SEPTEMBER 2022

Ph. Maisonobe

UMR 6621 du CNRS, Laboratoire J.A. Dieudonné, Université de Nice,
Parc Valrose, 06108 Nice cedex 2, France.

E-mail : `phm@math.unice.fr`

C. Sabbah

UMR 7640 du CNRS, Centre de Mathématiques, École polytechnique,
F–91128 Palaiseau cedex, France.

E-mail : `sabbah@math.polytechnique.fr`

Url : `http://www.math.polytechnique.fr/~sabbah/sabbah.html`

ASPECTS OF THE THEORY OF $\mathscr{D}$-MODULES

LECTURE NOTES (KAISERSLAUTERN 2002)

REVISED VERSION: SEPTEMBER 2022

Philippe Maisonobe, Claude Sabbah

CONTENTS

LECTURE 1

BASIC CONSTRUCTIONS

In this first lecture, we introduce the sheaf of differential operators and its (left or right) modules. Our main concern is to develop the relationship between two *a priori* different notions:

(1) the classical notion of an $\mathscr{O}_X$-module with a *flat* connection,

(2) the notion of a left $\mathscr{D}_X$-module.

Both notions are easily seen to be equivalent. However, the extension of the equivalence to complexes (or to the derived category) is less clear. Later on, we will introduce the notion of differential complex to express this equivalence. The main result in this direction will be Theorem 3.3.7.

The relationship between left and right $\mathscr{D}_X$-modules, although simple, is also somewhat subtle, and we insist on the basic isomorphisms.

We also develop the same theory with filtration, by explaining a recipe (Rees construction) to treat the filtered analogue along the same lines. The notion of *strictness* plays a major role here.

The results in this lecture are mainly algebraic, and do not involve any analytic property. They could be translated easily to the algebraic situation. One can find many of these notions in the classical books [**Kas95, Bjö79, Bor87, Meb89, MS93a, MS93b, Bjö93, Cou95, Kas03**]. Some of them are also directly inspired from the work of M. Saito [**Sai88, Sai89b, Sai89a**] about Hodge $\mathscr{D}$-modules.

In this lecture, we denote by X a complex manifold of dimension $\dim X = n$.

1.1. The sheaf of holomorphic differential operators

We will denote by Θ_X the sheaf of holomorphic vector fields on X. This is the $\mathscr{O}_X$-locally free sheaf generated in local coordinates by $\partial_{x_1}, \dots, \partial_{x_n}$. It is a sheaf of $\mathscr{O}_X$-Lie algebras which is locally free as an $\mathscr{O}_X$-module, and vector fields act (on the left) on functions by derivation, in a way compatible with the Lie algebra structure:

given a local vector field ξ acting on functions as a derivation $g \mapsto \xi(g)$, and a local holomorphic function f, $f\xi$ is the vector field acting as $f \cdot \xi(g)$, and given two vector fields ξ, η, their bracket as derivations $[\xi,\eta](g) := \xi(\eta(g)) - \eta(\xi(g))$ is still a derivation, hence defines a vector field.

Dually, we denote by Ω^1_X the sheaf of holomorphic 1-forms on X. We will set $\Omega^k_X = \wedge^k \Omega^1_X$. We denote by $d : \Omega^k_X \to \Omega^{k+1}_X$ the differential.

Exercise 1.1.1. Let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of rank d and let $\mathscr{E}^\vee$ be its dual. Show that, given any local basis $\boldsymbol{e} = (e_1, \dots, e_d)$ of $\mathscr{E}$ with dual basis $\boldsymbol{e}^\vee$, the section $\sum_{i=1}^d e_i \otimes e_i^\vee$ of $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{E}^\vee$ does not depend on the choice of the local basis $\boldsymbol{e}$ and extends as a global section of $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{E}^\vee$. Show that it defines, up to a constant, an $\mathscr{O}_X$-linear section $\mathscr{O}_X \to \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{E}^\vee$ of the natural duality pairing $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{E}^\vee \to \mathscr{O}_X$. Conclude that we have a natural global section of $\Omega^1_X \otimes_{\mathscr{O}_X} \Theta_X$ given, in local coordinates, by $\sum_i dx_i \otimes \partial_{x_i}$.

Let ω_X denote the sheaf $\Omega^{\dim X}_X$ of forms of maximal degree. Then there is a natural right action (in a compatible way with the Lie algebra structure) of Θ_X on ω_X: the action is given by $\omega \cdot \xi = -\mathcal{L}_\xi \omega$, where $\mathcal{L}_\xi$ denotes the Lie derivative, equal to the composition of the interior product ι_ξ by ξ with the differential d, as it acts on forms of maximal degree. The action is on the right since applying the vector field $f\xi$ (as defined above) to ω consists in multiplying first ω by f, and then applying ξ. The choice of the sign above makes this definition compatible with bracket.

***Exercise 1.1.2* (The sheaf $\mathscr{H}om$).** Let X be a topological space and let $\mathscr{F}$ and $\mathscr{G}$ be two sheaves of $\mathscr{A}$-modules on X, $\mathscr{A}$ being a sheaf of rings on X. We denote by $\mathrm{Hom}_{\mathscr{A}}(\mathscr{F},\mathscr{G})$ the $\Gamma(X,\mathscr{A})$-module of morphisms of sheaves of $\mathscr{A}$-modules from $\mathscr{F}$ to $\mathscr{G}$. An element ϕ of $\mathrm{Hom}_{\mathscr{A}}(\mathscr{F},\mathscr{G})$ is a collection of morphisms $\phi(U) \in \mathrm{Hom}_{\mathscr{A}(U)}(\mathscr{F}(U),\mathscr{G}(U))$, on open subsets U of X, compatible with the restrictions.

Show that the presheaf $\mathscr{H}om_{\mathscr{A}}(\mathscr{F},\mathscr{G})$ defined by

$$\Gamma(U, \mathscr{H}om_{\mathscr{A}}(\mathscr{F},\mathscr{G})) = \mathrm{Hom}_{\mathscr{A}_{|U}}(\mathscr{F}_{|U}, \mathscr{G}_{|U})$$

is a *sheaf* (notice that $U \mapsto \mathrm{Hom}_{\mathscr{A}(U)}(\mathscr{F}(U),\mathscr{G}(U))$ is not a presheaf, because there are no canonical morphisms of restriction).

***Definition 1.1.3* (The sheaf of holomorphic differential operators)**

For any open set U of X, the ring $\mathscr{D}_X(U)$ of *holomorphic differential operators on U* is the subring of $\mathrm{Hom}_{\mathbb{C}}(\mathscr{O}_U, \mathscr{O}_U)$ generated by

- multiplication by holomorphic functions on U,
- derivation by holomorphic vector fields on U.

The sheaf $\mathscr{D}_X$ is defined by $\Gamma(U, \mathscr{D}_X) = \mathscr{D}_X(U)$ for any open set U of X.

By construction, the sheaf $\mathscr{D}_X$ acts on the left on $\mathscr{O}_X$, i.e., $\mathscr{O}_X$ is a left $\mathscr{D}_X$-module.

***Definition 1.1.4* (The filtration of $\mathscr{D}_X$ by the order).** The increasing family of subsheaves $F_k\mathscr{D}_X \subset \mathscr{D}_X$ is defined inductively:

- $F_k\mathscr{D}_X = 0$ if $k \leqslant -1$,
- $F_0\mathscr{D}_X = \mathscr{O}_X$ (*via* the canonical injection $\mathscr{O}_X \hookrightarrow \mathscr{H}om_{\mathbb{C}}(\mathscr{O}_X, \mathscr{O}_X)$),
- the local sections P of $F_{k+1}\mathscr{D}_X$ are characterized by the fact that $[P, \varphi]$ is a local section of $F_k\mathscr{D}_X$ for any holomorphic function φ.

Exercise 1.1.5. Show that a differential operator P *of order* $\leqslant 1$ satisfying $P(1) = 0$ is a derivation of $\mathscr{O}_X$, i.e., a section of Θ_X.

***Exercise 1.1.6* (Local computations).** Let U be an open set of $\mathbb{C}^n$ with coordinates $x_1, \dots, x_n$. Denote by $\partial_{x_1}, \dots, \partial_{x_n}$ the corresponding vector fields.

(1) Show that the following relations are satisfied in $\mathscr{D}(U)$:

$$\begin{aligned}
[\partial_{x_i}, \varphi] &= \frac{\partial \varphi}{\partial x_i}, \quad \forall\, \varphi \in \mathscr{O}(U),\ \forall\, i \in \{1, \dots, n\},\\
[\partial_{x_i}, \partial_{x_j}] &= 0 \quad \forall\, i, j \in \{1, \dots, n\},\\
\partial_x^\alpha \cdot \varphi &= \sum_{0 \leqslant \beta \leqslant \alpha} \frac{\alpha!}{(\alpha-\beta)!\beta!} \partial_x^{\alpha-\beta}(\varphi)\partial_x^\beta,\\
\varphi \cdot \partial_x^\alpha &= \sum_{0 \leqslant \beta \leqslant \alpha} \frac{\alpha!}{(\alpha-\beta)!\beta!} (-1)^{|\alpha-\beta|}\partial_x^\beta \partial_x^{\alpha-\beta}(\varphi),
\end{aligned}$$

with standard notation concerning multi-indices α, β.

(2) Show that any element $P \in \mathscr{D}(U)$ can be written in a unique way as $\sum_\alpha a_\alpha \partial_x^\alpha$ or $\sum_\alpha \partial_x^\alpha b_\alpha$with $a_\alpha, b_\alpha \in \mathscr{O}(U)$. Conclude that $\mathscr{D}_X$ is a locally free left and right module over $\mathscr{O}_X$.

(3) Show that $\max\{|\alpha| \ ;\ a_\alpha \neq 0\} = \max\{|\alpha| \ ;\ b_\alpha \neq 0\}$. It is denoted by $\mathrm{ord}_x P$.

(4) Show that $\mathrm{ord}_x P$ does not depend on the coordinate system chosen on U.

(5) Show that $PQ = 0$ in $\mathscr{D}(U) \Rightarrow P = 0$ or $Q = 0$.

(6) Identify $F_k\mathscr{D}_X$ with the subsheaf of local sections of $\mathscr{D}_X$ having order $\leqslant k$ (in some or any local coordinate system). Show that it is a locally free $\mathscr{O}_X$-module of finite rank.

(7) Show that the filtration $F_\bullet\mathscr{D}_X$ is exhaustive (i.e., $\mathscr{D}_X = \bigcup_k F_k\mathscr{D}_X$) and that it satisfies

$$F_k\mathscr{D}_X \cdot F_\ell\mathscr{D}_X = F_{k+\ell}\mathscr{D}_X.$$

(The left-hand term consists by definition of all sums of products of a section of $F_k\mathscr{D}_X$ and a section of $F_\ell\mathscr{D}_X$.)

(8) Show that the bracket $[P, Q] := PQ - QP$ induces for each k, ℓ a $\mathbb{C}$-bilinear morphism $F_k\mathscr{D}_X \otimes_{\mathbb{C}} F_\ell\mathscr{D}_X \to F_{k+\ell-1}\mathscr{D}_X$.

(9) Conclude that the graded ring $\mathrm{gr}^F\mathscr{D}_X$ is commutative.

The sheaf $\mathscr{D}_X$ is not commutative. The lack of commutativity of $\mathscr{D}_X$ is analyzed in Exercise 1.1.7.

Exercise 1.1.7 **(The graded sheaf $\mathrm{gr}^F\mathscr{D}_X$).** The goal of this exercise is to show that the sheaf of graded rings $\mathrm{gr}^F\mathscr{D}_X$ may be canonically identified with the sheaf of graded rings $\mathrm{Sym}\,\Theta_X$. If one identifies Θ_X with the sheaf of functions on the cotangent space T^*X which are linear in the fibres, then $\mathrm{Sym}\,\Theta_X$ is the sheaf of functions on T^*X which are polynomial in the fibres. In particular, $\mathrm{gr}^F\mathscr{D}_X$ is a sheaf of commutative rings.

(1) Identify the $\mathscr{O}_X$-module $\mathrm{Sym}^k\,\Theta_X$ with the sheaf of symmetric $\mathbb{C}$-linear forms $\boldsymbol{\xi} : \mathscr{O}_X \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathscr{O}_X$ on the k-fold tensor product, which behave like a derivation with respect to each factor.

(2) Show that $\mathrm{Sym}\,\Theta_X := \oplus_k \mathrm{Sym}^k\,\Theta_X$ is a sheaf of graded $\mathscr{O}_X$-algebras on X and identify it with the sheaf of functions on T^*X which are polynomial in the fibres.

(3) Show that the map $F_k\mathscr{D}_X \to \mathscr{H}om_{\mathbb{C}}\big(\overset{k}{\otimes}_{\mathbb{C}}\mathscr{O}_X, \mathscr{O}_X\big)$ which sends any section P of $F_k\mathscr{D}_X$ to

$$\varphi_1 \otimes \cdots \otimes \varphi_k \longmapsto [\cdots[[P,\varphi_1]\varphi_2]\cdots\varphi_k]$$

induces an isomorphism of $\mathscr{O}_X$-modules $\mathrm{gr}^F_k\mathscr{D}_X \to \mathrm{Sym}^k\,\Theta_X$.

(4) Show that the induced morphism

$$\mathrm{gr}^F\mathscr{D}_X := \oplus_k \mathrm{gr}^F_k\mathscr{D}_X \longrightarrow \mathrm{Sym}\,\Theta_X$$

is an isomorphism of sheaves of $\mathscr{O}_X$-algebras.

On the other hand, it has no non-trivial two-sided ideals (see Exercise 1.1.8), hence it is simple.

Exercise 1.1.8 **(The sheaf of rings $\mathscr{D}_X$ has no non-trivial two-sided ideals)**
Let $\mathscr{I}$ be a non-zero two-sided ideal of $\mathscr{D}_X$.

(1) Let $x \in X$ and $0 \neq P \in \mathscr{I}_x$. Show that there exists $f \in \mathscr{O}_{X,x}$ such that $[P,f] \neq 0$. [Hint: use local coordinates to express P].

(2) Conclude by induction on the order that $\mathscr{I}_x$ contains a non-zero $g \in \mathscr{O}_{X,x}$.

(3) Show that $\mathscr{I}_x$ contains all iterated differentials of g, and conclude that $\mathscr{I}_x$ contains $h \in \mathscr{O}_{X,x}$ such that $h(x) \neq 0$.

(4) Conclude that $\mathscr{I}_x \ni 1$, hence $\mathscr{I}_x = \mathscr{D}_{X,x}$.

This leads us to consider left or right $\mathscr{D}_X$-modules (or ideals), and the theory of two-sided objects is empty.

Exercise 1.1.9 **(The universal connection)**

(1) Show that the natural left multiplication of Θ_X on $\mathscr{D}_X$ can be written as a *connection*
$$\nabla : \mathscr{D}_X \longrightarrow \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{D}_X,$$
i.e., as a $\mathbb{C}$-linear morphism satisfying the *Leibniz rule* $\nabla(fP) = df \otimes P + f\nabla P$, where f is any local section of $\mathscr{O}_X$ and P any local section of $\mathscr{D}_X$. [Hint: $\nabla(1)$ is the global section of $\Omega^1_X \otimes_{\mathscr{O}_X} \Theta_X$ considered in Exercise 1.1.1.]

(2) Extend this connection for any $k \geqslant 1$ as a $\mathbb{C}$-linear morphism
$${}^{(k)}\nabla : \Omega^k_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \Omega^{k+1}_X \otimes_{\mathscr{O}_X} \mathscr{D}_X$$
satisfying the Leibniz rule written as
$${}^{(k)}\nabla(\omega \otimes P) = d\omega \otimes P + (-1)^k \omega \wedge \nabla P.$$

(3) Show that ${}^{(k+1)}\nabla \circ {}^{(k)}\nabla = 0$ for any $k \geqslant 0$ (i.e., ∇ is *flat*).

(4) Show that the morphisms ${}^{(k)}\nabla$ are *right* $\mathscr{D}_X$-linear (but not left $\mathscr{O}_X$-linear).

Exercise 1.1.10. More generally, show that a left $\mathscr{D}_X$-module $\mathscr{M}$ is nothing but an $\mathscr{O}_X$-module with an *integrable* connection $\nabla : \mathscr{M} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{M}$. [Hint: to get the connection, tensor the left $\mathscr{D}_X$-action $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M} \to \mathscr{M}$ by Ω^1_X on the left and compose with the universal connection to get $\mathscr{D}_X \otimes \mathscr{M} \to \Omega^1_X \otimes \mathscr{M}$; compose it on the left with $\mathscr{M} \to \mathscr{D}_X \otimes \mathscr{M}$ given by $m \mapsto 1 \otimes m$.] Define similarly the iterated connections ${}^{(k)}\nabla : \Omega^k_X \otimes_{\mathscr{O}_X} \mathscr{M} \to \Omega^{k+1}_X \otimes_{\mathscr{O}_X} \mathscr{M}$. Show that ${}^{(k+1)}\nabla \circ {}^{(k)}\nabla = 0$.

In conclusion:

Proposition 1.1.11. *Giving a left $\mathscr{D}_X$-module $\mathscr{M}$ is equivalent to giving an $\mathscr{O}_X$-module $\mathscr{M}$ together with an integrable connection ∇.*

Proof. Exercises 1.1.1, 1.1.9 and 1.1.10. □

1.2. Left and right

The categories of left (resp. right) $\mathscr{D}_X$-modules are denoted by ${}^{\ell}\mathbf{M}(\mathscr{D}_X)$ (resp. ${}^{\mathsf{r}}\mathbf{M}(\mathscr{D}_X)$). We analyze the relations between both categories in this section. Let us first recall the basic lemmas for generating left or right $\mathscr{D}$-modules. We refer for instance to [**CJ93**, § 1.1] for more details.

Lemma 1.2.1 **(Generating left $\mathscr{D}_X$-modules).** *Let $\mathscr{M}^{\mathrm{left}}$ be an $\mathscr{O}_X$-module and let $\varphi^{\mathrm{left}} : \Theta_X \otimes_{\mathbb{C}_X} \mathscr{M}^{\mathrm{left}} \to \mathscr{M}^{\mathrm{left}}$ be a $\mathbb{C}$-linear morphism such that, for any local sections f of $\mathscr{O}_X$, ξ, η of Θ_X and m of $\mathscr{M}^{\mathrm{left}}$, one has*

(1) $\varphi^{\mathrm{left}}(f\xi \otimes m) = f\varphi^{\mathrm{left}}(\xi \otimes m)$,

(2) $\varphi^{\mathrm{left}}(\xi \otimes fm) = f\varphi^{\mathrm{left}}(\xi \otimes m) + \xi(f)m$,

(3) $\varphi^{\mathrm{left}}([\xi, \eta] \otimes m) = \varphi^{\mathrm{left}}(\xi \otimes \varphi^{\mathrm{left}}(\eta \otimes m)) - \varphi^{\mathrm{left}}(\eta \otimes \varphi^{\mathrm{left}}(\xi \otimes m))$.

Then there exists a unique structure of left $\mathscr{D}_X$-module on $\mathscr{M}^{\mathrm{left}}$ such that $\xi m = \varphi^{\mathrm{left}}(\xi \otimes m)$ for any ξ, m.

***Lemma 1.2.2* (Generating right $\mathscr{D}_X$-modules).** *Let $\mathscr{M}^{\mathrm{right}}$ be an $\mathscr{O}_X$-module and let $\varphi^{\mathrm{right}} : \mathscr{M}^{\mathrm{right}} \otimes_{\mathbb{C}_X} \Theta_X \to \mathscr{M}^{\mathrm{right}}$ be a $\mathbb{C}$-linear morphism such that, for any local sections f of $\mathscr{O}_X$, ξ, η of Θ_X and m of $\mathscr{M}^{\mathrm{right}}$, one has*

(1) $\varphi^{\mathrm{right}}(mf \otimes \xi) = \varphi^{\mathrm{right}}(m \otimes f\xi)$ *(φ^{right} is in fact defined on $\mathscr{M}^{\mathrm{right}} \otimes_{\mathscr{O}_X} \Theta_X$),*

(2) $\varphi^{\mathrm{right}}(m \otimes f\xi) = \varphi^{\mathrm{right}}(m \otimes \xi)f - m\xi(f)$,

(3) $\varphi^{\mathrm{right}}(m \otimes [\xi, \eta]) = \varphi^{\mathrm{right}}(\varphi^{\mathrm{right}}(m \otimes \xi) \otimes \eta) - \varphi^{\mathrm{right}}(\varphi^{\mathrm{right}}(m \otimes \eta) \otimes \xi)$.

Then there exists a unique structure of right $\mathscr{D}_X$-module on $\mathscr{M}^{\mathrm{right}}$ such that $m\xi = \varphi^{\mathrm{right}}(m \otimes \xi)$ for any ξ, m.

***Example 1.2.3* (Most basic examples)**

(1) $\mathscr{D}_X$ is a left and a right $\mathscr{D}_X$-module.

(2) $\mathscr{O}_X$ is a left $\mathscr{D}_X$-module (Exercise 1.2.4).

(3) $\omega_X := \Omega_X^{\dim X}$ is a right $\mathscr{D}_X$-module (Exercise 1.2.5).

***Exercise 1.2.4* ($\mathscr{O}_X$ is a *simple* left $\mathscr{D}_X$-module)**

(1) Use the left action of Θ_X on $\mathscr{O}_X$ to define on $\mathscr{O}_X$ the structure of a left $\mathscr{D}_X$-module.

(2) Let f be a nonzero holomorphic function on $\mathbb{C}^n$. Show that there exists a multi-index $\alpha \in \mathbb{N}^n$ such that $(\partial_\alpha f)(0) \neq 0$.

(3) Conclude that $\mathscr{O}_X$ is a simple left $\mathscr{D}_X$-module, i.e., does not contain any proper non trivial $\mathscr{D}_X$-submodule. Is it simple as a left $\mathscr{O}_X$-module?

***Exercise 1.2.5* (ω_X is a *simple* right $\mathscr{D}_X$-module)**

(1) Use the right action of Θ_X on ω_X to define on ω_X the structure of a right $\mathscr{D}_X$-module.

(2) Show that it is simple as a right $\mathscr{D}_X$-module.

***Exercise 1.2.6* (Tensor products over $\mathscr{O}_X$)**

(1) Let $\mathscr{M}$ and $\mathscr{N}$ be two left $\mathscr{D}_X$-modules.

(a) Show that the $\mathscr{O}_X$-module $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ has the structure of a left $\mathscr{D}_X$-module by setting, by analogy with the Leibniz rule,

$$\xi \cdot (m \otimes n) = \xi m \otimes n + m \otimes \xi n.$$

(b) Notice that, in general, $m \otimes n \mapsto (\xi m) \otimes n$ (or $m \otimes n \mapsto m \otimes (\xi n)$) does not define a left $\mathscr{D}_X$-action on the $\mathscr{O}_X$-module $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$.

(c) Let $\varphi : \mathscr{M} \to \mathscr{M}'$ and $\psi : \mathscr{N} \to \mathscr{N}'$ be $\mathscr{D}_X$-linear morphisms. Show that $\varphi \otimes \psi$ is $\mathscr{D}_X$-linear.

(2) Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module and $\mathscr{N}$ be a right $\mathscr{D}_X$-module. Show that $\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M}$ has the structure of a right $\mathscr{D}_X$-module by setting

$$(n \otimes m) \cdot \xi = n\xi \otimes m - n \otimes \xi m.$$

Remark: one can define a right $\mathscr{D}_X$-module structure on $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ by using the natural involution $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N} \xrightarrow{\sim} \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M}$, so this brings no new structure.

(3) Assume that $\mathscr{M}$ and $\mathscr{N}$ are right $\mathscr{D}_X$-modules. Does there exist a (left or right) $\mathscr{D}_X$-module structure on $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ defined with analogous formulas?

Exercise 1.2.7 **($\mathscr{H}om$ over $\mathscr{O}_X$)**

(1) Let $\mathscr{M}, \mathscr{N}$ be left $\mathscr{D}_X$-modules. Show that $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ has a natural structure of left $\mathscr{D}_X$-module defined by

$$(\xi \cdot \varphi)(m) = \xi \cdot (\varphi(m)) + \varphi(\xi \cdot m),$$

for any local sections ξ of Θ_X, m of $\mathscr{M}$ and φ of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$.

(2) Similarly, if $\mathscr{M}, \mathscr{N}$ are right $\mathscr{D}_X$-modules, then $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ has a natural structure of left $\mathscr{D}_X$-module defined by

$$(\xi \cdot \varphi)(m) = \varphi(m \cdot \xi) - \varphi(m) \cdot \xi.$$

Exercise 1.2.8 **(Tensor product of a left $\mathscr{D}_X$-module with $\mathscr{D}_X$)**

Let $\mathscr{M}^{\mathrm{left}}$ be a left $\mathscr{D}_X$-module. Notice that $\mathscr{M}^{\mathrm{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ has two commuting structures of $\mathscr{O}_X$-module. Similarly $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^{\mathrm{left}}$ has two such structures. The goal of this exercise is to extend them as $\mathscr{D}_X$-structures and examine their relations.

(1) Show that $\mathscr{M}^{\mathrm{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ has the structure of a left and of a right $\mathscr{D}_X$-module *which commute*, given by the formulas:

$$\text{(left)} \qquad (\mathscr{M}^{\mathrm{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{tens}} : \quad \begin{cases} f \cdot (m \otimes P) = (fm) \otimes P = m \otimes (fP), \\ \xi \cdot (m \otimes P) = (\xi m) \otimes P + m \otimes \xi P, \end{cases}$$

$$\text{(right)} \qquad (\mathscr{M}^{\mathrm{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{triv}} : \quad \begin{cases} (m \otimes P) \cdot f = m \otimes (Pf), \\ (m \otimes P) \cdot \xi = m \otimes (P\xi), \end{cases}$$

for any local vector field ξ and any local holomorphic function f. Show that a left $\mathscr{D}_X$-linear morphism $\varphi : \mathscr{M}_1^{\mathrm{left}} \to \mathscr{M}_2^{\mathrm{left}}$ extends as a bi-$\mathscr{D}_X$-linear morphism $\varphi \otimes 1 : \mathscr{M}_1^{\mathrm{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{M}_2^{\mathrm{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X$.

(2) Similarly, show that $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^{\mathrm{left}}$ also has such structures *which commute* and are functorial, given by formulas:

$$\text{(left)} \qquad (\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^{\mathrm{left}})_{\mathrm{triv}} : \quad \begin{cases} f \cdot (P \otimes m) = (fP) \otimes m, \\ \xi \cdot (P \otimes m) = (\xi P) \otimes m, \end{cases}$$

$$\text{(right)} \qquad (\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^{\mathrm{left}})_{\mathrm{tens}} : \quad \begin{cases} (P \otimes m) \cdot f = P \otimes (fm) = (Pf) \otimes m, \\ (P \otimes m) \cdot \xi = P\xi \otimes m - P \otimes \xi m. \end{cases}$$

(3) Show that both morphisms

$$\begin{aligned} \mathscr{M}^{\text{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X &\longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^{\text{left}} & \qquad \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^{\text{left}} &\longrightarrow \mathscr{M}^{\text{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X \\ m \otimes P &\longmapsto (1 \otimes m) \cdot P & P \otimes m &\longmapsto P \cdot (m \otimes 1) \end{aligned}$$

are left and right $\mathscr{D}_X$-linear, induce the identity $\mathscr{M}^{\text{left}} \otimes 1 = 1 \otimes \mathscr{M}^{\text{left}}$, and their composition is the identity of $\mathscr{M}^{\text{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ or $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}^{\text{left}}$, hence both are reciprocal isomorphisms. Show that this correspondence is functorial (i.e., compatible with morphisms).

(4) Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module and let $\mathscr{L}$ be an $\mathscr{O}_X$-module. Justify the following isomorphisms of left $\mathscr{D}_X$-modules and right $\mathscr{O}_X$-modules:

$$\begin{aligned} \mathscr{M} \otimes_{\mathscr{O}_X} (\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{L}) &\simeq (\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \otimes_{\mathscr{O}_X} \mathscr{L} \\ &\simeq (\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}) \otimes_{\mathscr{O}_X} \mathscr{L} \simeq \mathscr{D}_X \otimes_{\mathscr{O}_X} (\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{L}). \end{aligned}$$

Assume moreover that $\mathscr{M}$ and $\mathscr{L}$ are $\mathscr{O}_X$-locally free. Show that $\mathscr{M} \otimes_{\mathscr{O}_X} (\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{L})$ is $\mathscr{D}_X$-locally free.

Exercise 1.2.9 **(Tensor product of a right $\mathscr{D}_X$-module with $\mathscr{D}_X$)**
Let $\mathscr{M}^{\text{right}}$ be a right $\mathscr{D}_X$-module.

(1) Show that $\mathscr{M}^{\text{right}} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ has two structures of right $\mathscr{D}_X$-module *which commute*, denoted $\cdot_{\text{triv}}$ (trivial) and $\cdot_{\text{tens}}$ (tensor; the latter defined by using the left structure on $\mathscr{D}_X$ and Exercise 1.2.6(2)), given by:

$$(\text{right})_{\text{triv}} \qquad (\mathscr{M}^{\text{right}} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\text{triv}} : \quad \begin{cases} (m \otimes P) \cdot_{\text{triv}} f = m \otimes (Pf), \\ (m \otimes P) \cdot_{\text{triv}} \xi = m \otimes (P\xi), \end{cases}$$

$$(\text{right})_{\text{tens}} \qquad (\mathscr{M}^{\text{right}} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\text{tens}} : \quad \begin{cases} (m \otimes P) \cdot_{\text{tens}} f = mf \otimes P = m \otimes fP, \\ (m \otimes P) \cdot_{\text{tens}} \xi = m\xi \otimes P - m \otimes (\xi P). \end{cases}$$

(2) Show that there is a unique involution $\iota : \mathscr{M}^{\text{right}} \otimes_{\mathscr{O}_X} \mathscr{D}_X \xrightarrow{\sim} \mathscr{M}^{\text{right}} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ which exchanges both structures and is the identity on $\mathscr{M}^{\text{right}} \otimes 1$, given by $(m \otimes P)_{\text{triv}} \mapsto (m \otimes 1) \cdot_{\text{tens}} P$. [Hint: show first the properties of ι by using local coordinates, and glue the local constructions by uniqueness of ι.]

(3) For each $p \geqslant 0$, consider the pth term $F_p\mathscr{D}_X$ of the filtration of $\mathscr{D}_X$ by the order (see Exercise 1.1.4) with both structures of $\mathscr{O}_X$-module (one on the left, one on the right) and equip similarly $\mathscr{M}^{\text{right}} \otimes_{\mathscr{O}_X} F_p\mathscr{D}_X$ with two structures (one on the left, one on the right) of $\mathscr{O}_X$-modules. Show that, for each p, ι induces an isomorphism of $\mathscr{O}_X$-modules $(\mathscr{M}^{\text{right}} \otimes_{\mathscr{O}_X} F_p\mathscr{D}_X)_{\text{tens}} \xrightarrow{\sim} (\mathscr{M}^{\text{right}} \otimes_{\mathscr{O}_X} F_p\mathscr{D}_X)_{\text{triv}}$.

Definition 1.2.10 **(Right-left transformation).** Any left $\mathscr{D}_X$-module $\mathscr{M}^{\text{left}}$ gives rise to a right one $\mathscr{M}^{\text{right}}$ by setting (see [**CJ93**] for instance) $\mathscr{M}^{\text{right}} = \omega_X \otimes_{\mathscr{O}_X} \mathscr{M}^{\text{left}}$ and, for any vector field ξ and any function f,

$$(\omega \otimes m) \cdot f = f\omega \otimes m = \omega \otimes fm, \quad (\omega \otimes m) \cdot \xi = \omega\xi \otimes m - \omega \otimes \xi m.$$

Conversely, set $\mathscr{M}^{\text{left}} = \mathscr{H}om_{\mathscr{O}_X}(\omega_X, \mathscr{M}^{\text{right}})$, which also has in a natural way the structure of a left $\mathscr{D}_X$-module (see Exercise 1.2.7(2)).

Exercise 1.2.11 **(Compatibility of right-left transformations)**

Show that the natural morphisms

$$\mathscr{M}^{\text{left}} \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\omega_X, \omega_X \otimes_{\mathscr{O}_X} \mathscr{M}^{\text{left}}), \qquad \omega_X \otimes_{\mathscr{O}_X} \mathscr{H}om_{\mathscr{O}_X}(\omega_X, \mathscr{M}^{\text{right}}) \longrightarrow \mathscr{M}^{\text{right}}$$

are isomorphisms of $\mathscr{D}_X$-modules.

Exercise 1.2.12 **(Compatibility of left-right transformation with tensor product)**

Let $\mathscr{M}^{\text{left}}$ and $\mathscr{N}^{\text{left}}$ be two left $\mathscr{D}_X$-modules and denote by $\mathscr{M}^{\text{right}}, \mathscr{N}^{\text{right}}$ the corresponding right $\mathscr{D}_X$-modules (see Definition 1.2.10). Show that there is a natural isomorphism of right $\mathscr{D}_X$-modules (by using the right structure given in Exercise 1.2.6(2)):

$$\begin{aligned} \mathscr{N}^{\text{right}} \otimes_{\mathscr{O}_X} \mathscr{M}^{\text{left}} &\xrightarrow{\sim} \mathscr{M}^{\text{right}} \otimes_{\mathscr{O}_X} \mathscr{N}^{\text{left}} \\ (\omega \otimes n) \otimes m &\longmapsto (\omega \otimes m) \otimes n \end{aligned}$$

and that this isomorphism is functorial in $\mathscr{M}^{\text{left}}$ and $\mathscr{N}^{\text{left}}$.

Exercise 1.2.13 **(Local expression of the left-right transformation)**

Let U be an open set of $\mathbb{C}^n$.

(1) Show that there exists a unique $\mathbb{C}$-linear involution $P \mapsto {}^tP$ from $\mathscr{D}(U)$ to itself such that

- $\forall\, \varphi \in \mathscr{O}(U)$, ${}^t\varphi = \varphi$,
- $\forall\, i \in \{1, \dots, n\}$, ${}^t\partial_{x_i} = -\partial_{x_i}$,
- $\forall\, P, Q \in \mathscr{D}(U)$, ${}^t(PQ) = {}^tQ \cdot {}^tP$.

(2) Let $\mathscr{M}$ be a left (resp. right) $\mathscr{D}_X$-module and let ${}^t\!\mathscr{M}$ be $\mathscr{M}$ equipped with the right (resp. left) $\mathscr{D}_X$-module structure

$$P \cdot m := {}^tPm.$$

Show that ${}^t\!\mathscr{M} \xrightarrow{\sim} \mathscr{M}^{\text{right}}$ (resp. ${}^t\!\mathscr{M} \xrightarrow{\sim} \mathscr{M}^{\text{left}}$).

Exercise 1.2.14 **(The left-right transformation is an isomorphism of categories)**

To any left $\mathscr{D}_X$-linear morphism $\varphi^{\text{left}} : \mathscr{M}^{\text{left}} \to \mathscr{N}^{\text{left}}$ is associated the $\mathscr{O}_X$-linear morphism $\varphi^{\text{right}} = \mathrm{Id}_{\omega_X} \otimes \varphi^{\text{left}} : \mathscr{M}^{\text{right}} \to \mathscr{N}^{\text{right}}$.

(1) Show that φ^{right} is right $\mathscr{D}_X$-linear.

(2) Define the reverse correspondence $\varphi^{\text{right}} \mapsto \varphi^{\text{left}}$.

(3) Conclude that the left-right correspondence ${}^{\ell}\mathsf{M}(\mathscr{D}_X) \mapsto {}^{\mathsf{r}}\mathsf{M}(\mathscr{D}_X)$ is a functor, which is an isomorphism of categories, having the right-left correspondence ${}^{\mathsf{r}}\mathsf{M}(\mathscr{D}_X) \mapsto {}^{\ell}\mathsf{M}(\mathscr{D}_X)$ as inverse functor.

1.3. Examples of $\mathscr{D}$-modules

We list here some classical examples of $\mathscr{D}$-modules. One may get many other examples by applying various operations on $\mathscr{D}$-modules.

1.3.a. Let $\mathscr{I}$ be a sheaf of left ideals of $\mathscr{D}_X$. We will see in Lecture 2 that, locally on X, $\mathscr{I}$ is generated by a finite set $\{P_1, \dots, P_k\}$ of differential operators (this follows from the noetherianity and coherence properties of $\mathscr{D}_X$). Then the quotient $\mathscr{M} = \mathscr{D}_X/\mathscr{I}$ is a left $\mathscr{D}_X$-module. Locally, $\mathscr{M}$ is the $\mathscr{D}_X$-module associated with $P_1, \dots, P_k$.

Notice that different choices of generators of $\mathscr{I}$ give rise to the same $\mathscr{D}_X$-module $\mathscr{M}$. It may be sometime difficult to guess that two sets of operators generate the same ideal. Therefore, it is useful to develop a systematic procedure to construct from a system of generators a *division basis* of the ideal in order to have a decision algorithm (see Lecture 6 on Gröbner bases).

Exercise 1.3.1. Show that the two sets of differential operators $\{\partial_{x_1}, \dots, \partial_{x_n}\}$ and $\{\partial_{x_1}, x_1\partial_{x_2} + \cdots + x_{n-1}\partial_{x_n}\}$ generate the same ideal of $\mathscr{D}_{\mathbb{C}^n}$.

1.3.b. Let $\mathscr{L}$ be an $\mathscr{O}_X$-module. There is a very simple way to get a right $\mathscr{D}_X$-module from $\mathscr{L}$: consider $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ equipped with the natural right action of $\mathscr{D}_X$. This is called an *induced* $\mathscr{D}_X$-module. Although this construction is very simple, it is also very useful to get cohomological properties of $\mathscr{D}_X$-modules, as we will see in Section 3.2. One can also consider the left $\mathscr{D}_X$-module $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{L}$ (however, this is not the left $\mathscr{D}_X$-module attached to the right one $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ by the left-right transformation of Definition 1.2.10).

1.3.c. One of the main geometrical examples of $\mathscr{D}_X$-modules are the vector bundles on X equipped with an *integrable* connection. Recall that left $\mathscr{D}_X$-modules are $\mathscr{O}_X$-modules with an integrable connection. Among them, the coherent $\mathscr{D}_X$-modules are particularly interesting. We will see (see Exercise 2.4.6), that such modules are $\mathscr{O}_X$-locally free, i.e., correspond to holomorphic vector bundles of finite rank on X.

It may happen that, for some X, such a category does not give any interesting geometric object. Indeed, if for instance X has a trivial fundamental group (e.g. $X = \mathbb{P}^1(\mathbb{C})$), then any vector bundle with integrable connection is isomorphic to the trivial bundle $\mathscr{O}_X^{\text{right}}$ with the connection d. However, on Zariski open sets of X, there may exist interesting vector bundles with connections. This leads to the notion of meromorphic vector bundle with connection.

Let D be a divisor in X and denote by $\mathscr{O}_X(*D)$ the sheaf of meromorphic functions on X with poles along D at most. This is a sheaf of left $\mathscr{D}_X$-modules, being an $\mathscr{O}_X$-module equipped with the natural connection $d : \mathscr{O}_X(*D) \to \Omega^1_X(*D)$.

By definition, a *meromorphic bundle* is a locally free $\mathscr{O}_X(*D)$ module of finite rank. When it is equipped with an integrable connection, it becomes a left $\mathscr{D}_X$-module.

1.3.d. One may *twist* the previous examples. Assume that ω is a *closed* holomorphic form on X. Define $\nabla : \mathscr{O}_X \to \Omega^1_X$ by the formula $\nabla = d + \omega$. As ω is closed, ∇ is an integrable connection on the trivial bundle $\mathscr{O}_X$.

Usually, there only exist meromorphic closed form on X, with poles on some divisor D. Then ∇ is an integrable connection on $\mathscr{O}_X(*D)$.

If ω is exact, $\omega = df$ for some meromorphic function f on X, then ∇ may be written as $e^{-f} \circ d \circ e^f$.

More generally, if $\mathscr{M}$ is any meromorphic bundle with an integrable connection ∇, then, for any such ω, $\nabla + \omega \,\mathrm{Id}$ defines a new $\mathscr{D}_X$-module structure on $\mathscr{M}$.

1.3.e. Denote by $\mathfrak{Db}_X$ the sheaf of distributions on X: given any open set U of X, $\mathfrak{Db}_X(U)$ is the space of distributions on U, which is by definition the week dual of the space of C^∞ forms with compact support on U, of type $(\dim U, \dim U)$. By Exercise 1.2.5, there is a right action of $\mathscr{D}_X$ on such forms. The left action of $\mathscr{D}_X$ on distributions is defined by adjunction: denote by $\langle \varphi, u\rangle$ the natural pairing of a compactly supported C^∞-form φ with a distribution u on U; let P be a holomorphic differential operator on U; define then $P \cdot u$ such that, for any φ, on has

$$\langle \varphi, P \cdot u\rangle = \langle \varphi \cdot P, u\rangle.$$

Given any distribution u on X, the subsheaf $\mathscr{D}_X \cdot u \subset \mathfrak{Db}_X$ is the $\mathscr{D}_X$-module generated by this distribution. Saying that a distribution is a solution of a family $P_1, \dots, P_k$ of differential equation is equivalent to saying that the morphism $\mathscr{D}_X \to \mathscr{D}_X \cdot u$ sending 1 to u induces a surjective morphism $\mathscr{D}_X/(P_1, \dots, P_k) \to \mathscr{D}_X \cdot u$.

Similarly, the sheaf $\mathfrak{C}_X$ of currents of maximal degree on X, dual to $\mathscr{C}^\infty_{c,X}$, is a right $\mathscr{D}_X$-module.

In local coordinates $x_1, \dots, x_n$, a current of maximal degree is nothing but a distribution times the volume form $dx_1 \wedge \cdots \wedge dx_n \wedge d\overline{x}_1 \wedge \cdots \wedge d\overline{x}_n$.

As we are now working with C^∞ forms or with currents, it is natural not to forget the anti-holomorphic part of these objects. Denote by $\mathscr{O}_{\overline{X}}$ the sheaf of anti-holomorphic functions on X and by $\mathscr{D}_{\overline{X}}$ the sheaf of anti-holomorphic differential operators. Then $\mathfrak{Db}_X$ (resp. $\mathfrak{C}_X$) are similarly left (resp. right) $\mathscr{D}_{\overline{X}}$-modules. Of course, the $\mathscr{D}_X$ and $\mathscr{D}_{\overline{X}}$ actions do commute, and they coincide when considering multiplication by constants.

It is therefore natural to introduce the following sheaves of rings:

$$\begin{aligned} \mathscr{O}_{X,\overline{X}} &:= \mathscr{O}_X \otimes_{\mathbb{C}} \mathscr{O}_{\overline{X}}, \\ \mathscr{D}_{X,\overline{X}} &:= \mathscr{D}_X \otimes_{\mathbb{C}} \mathscr{D}_{\overline{X}}, \end{aligned}$$

and consider $\mathfrak{Db}_X$ (resp. $\mathfrak{C}_X$) as left (resp. right) $\mathscr{D}_{X,\overline{X}}$-modules.

1.3.f. One may construct new examples from old ones by using various operations.

- Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. Then $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)$ has a natural structure of right $\mathscr{D}_X$-module. Using a resolution $\mathscr{N}^{\bullet}$ of $\mathscr{M}$ by *left* $\mathscr{D}_X$-modules which are acyclic for $\mathscr{H}om_{\mathscr{D}_X}(\bullet, \mathscr{D}_X)$, one gets a right $\mathscr{D}_X$-module structure on the $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)$.
- Given two left (resp. a left and a right) $\mathscr{D}_X$-modules $\mathscr{M}$ and $\mathscr{N}$, the same argument allows one to put on the various $\mathscr{T}or_{i,\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ a left (resp. a right) $\mathscr{D}_X$-module structure.
- We will see in Lecture 4 the geometric operations "direct image" and "inverse image" of a $\mathscr{D}_X$-module by a holomorphic map.

1.3.g. Solutions. Let $\mathscr{M}, \mathscr{N}$ be two left $\mathscr{D}_X$-modules.

Definition 1.3.2. The sheaf of *solutions* of $\mathscr{M}$ in $\mathscr{N}$ is the sheaf $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$.

Remark 1.3.3

(1) The sheaf $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$ has no structure more than that of a sheaf of $\mathbb{C}$-vector spaces in general, because $\mathscr{D}_X$ is not commutative.

(2) According to Exercise 1.2.7(1), $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ is a left $\mathscr{D}_X$-module, that is, an $\mathscr{O}_X$-module with an integrable connection (Proposition 1.1.11). Then $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$ is the subsheaf of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ consisting of local morphisms $\mathscr{M} \to \mathscr{N}$ which commute with the connections on $\mathscr{M}$ and $\mathscr{N}$, in other words local sections which are annihilated by the connection on $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$.

Example 1.3.4. Let $U \subset X$ be a coordinate chart and let $P \in \mathscr{D}_X(U)$. Let $\mathscr{I} = \mathscr{D}_U \cdot P$ be the left ideal of $\mathscr{D}_U$ generated by P and let $\mathscr{M} = \mathscr{D}_U/\mathscr{I}$. We have a canonical isomorphism $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}) \simeq \operatorname{Ker}[P\cdot : \mathscr{N} \to \mathscr{N}]$, and this explains the terminology "solutions of $\mathscr{M}$ in $\mathscr{N}$".

If $\mathscr{N} = \mathscr{O}_X$, we get the sheaf of holomorphic solutions of P. If $\mathscr{N} = \mathfrak{Db}_X$, we get the sheaf of distributions solutions of P.

If $\mathscr{N} = \mathscr{D}_X$ (with its standard left structure), then $P\cdot : \mathscr{D}_X \to \mathscr{D}_X$ is injective (Exercise 1.1.6), so $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) = 0$. It maybe therefore interesting to consider higher $\mathscr{H}om$, namely, $\mathscr{E}xt$ sheaves. We consider the free resolution of $\mathscr{M}$ defined as

$$0 \longrightarrow \mathscr{D}_U \xrightarrow{\ \cdot P\ } \mathscr{D}_U \longrightarrow \mathscr{M} \longrightarrow 0.$$

The map $\cdot P$ is injective (same argument as for $P\cdot$), so this is indeed a resolution. By definition, $\mathscr{E}xt^1(\mathscr{M}, \mathscr{N})$ is the cokernel of

$$\begin{aligned} \mathscr{H}om_{\mathscr{D}_U}(\mathscr{D}_U, \mathscr{N}) &\longrightarrow \mathscr{H}om_{\mathscr{D}_U}(\mathscr{D}_U, \mathscr{N}) \\ \varphi(\star) &\longmapsto \varphi(\star \cdot P). \end{aligned}$$

If one identifies $\mathscr{H}om_{\mathscr{D}_U}(\mathscr{D}_U, \mathscr{N})$ with $\mathscr{N}$ by $\varphi \mapsto \varphi(1)$, the previous morphism reads

$$\mathscr{N} \xrightarrow{\ P\cdot\ } \mathscr{N},$$

so we recover that $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{N}) = \operatorname{Ker}[P\cdot\ :\mathscr{N}\to\mathscr{N}]$, and we find that $\mathscr{E}xt^1(\mathscr{M},\mathscr{N}) = \operatorname{Coker}[P\cdot\ :\mathscr{N}\to\mathscr{N}]$. In other words, $\mathscr{E}xt^1(\mathscr{M},\mathscr{N})$ measures the obstruction to the solvability of the differential equation $Pm = n$ for $n \in \mathscr{N}$.

Notice that, in this example, since the free resolution of $\mathscr{M}$ has length two, we have $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M},\mathscr{N}) = 0$ for $k \geqslant 2$, for any $\mathscr{N}$.

When $\mathscr{N}$ has a supplementary structure which commutes with its left $\mathscr{D}_X$-structure, then $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{N})$ and the $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M},\mathscr{N})$ inherit this supplementary structure.

Example 1.3.5

(1) Assume $\mathscr{N} = \mathscr{D}_X$. Then the definition of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{N})$ and the $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M},\mathscr{N})$ uses the left $\mathscr{D}_X$-module structure of $\mathscr{D}_X$, which commutes with the right one, so these solution sheaves are right $\mathscr{D}_X$-modules.

(2) Assume that $\mathscr{N} = \mathfrak{Db}_X$. Then the definition of $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{N})$ and the $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M},\mathscr{N})$ uses the left $\mathscr{D}_X$-module structure of $\mathfrak{Db}_X$, which commutes with the left $\mathscr{D}_{\overline{X}}$-structure, so these solution sheaves are left $\mathscr{D}_{\overline{X}}$-modules.

1.4. de Rham and Spencer

Let $\mathscr{M}^{\mathrm{left}}$ be a left $\mathscr{D}_X$-module and let $\mathscr{M}^{\mathrm{right}}$ be a right $\mathscr{D}_X$-module.

***Definition 1.4.1* (de Rham).** The *de Rham complex* $\Omega_X^{n+\bullet}(\mathscr{M}^{\mathrm{left}})$ of $\mathscr{M}^{\mathrm{left}}$ is the complex having as terms the $\mathscr{O}_X$-modules $\Omega_X^{n+\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}^{\mathrm{left}}$ and as differential the $\mathbb{C}$-linear morphism $(-1)^n\nabla$ defined in Exercise 1.1.10.

Notice that the de Rham complex is shifted $n = \dim X$ with respect to the usual convention. The shift produces, by definition, a sign change in the differential, which is then equal to $(-1)^n\nabla$. The previous definition produces a *complex* since $\nabla\circ\nabla = 0$, according to the integrability condition on ∇, as remarked in Exercise 1.1.10.

***Definition 1.4.2* (Spencer).** The *Spencer complex* $(\mathrm{Sp}^{\bullet}_X(\mathscr{M}^{\mathrm{right}}),\delta)$ is the complex having as terms the $\mathscr{O}_X$-modules $\mathscr{M}\otimes_{\mathscr{O}_X}\wedge^{-\bullet}\Theta_X$ (with $\bullet \leqslant 0$) and as differential the $\mathbb{C}$-linear map δ given by

$$\begin{aligned} m\otimes\xi_1\wedge\cdots\wedge\xi_k \xmapsto{\ \delta\ } &\sum_{i=1}^k(-1)^{i-1}m\xi_i\otimes\xi_1\wedge\cdots\wedge\widehat{\xi_i}\wedge\cdots\wedge\xi_k\\ &+\sum_{i<j}(-1)^{i+j}m\otimes[\xi_i,\xi_j]\wedge\xi_1\wedge\cdots\wedge\widehat{\xi_i}\wedge\cdots\wedge\widehat{\xi_j}\wedge\cdots\wedge\xi_k. \end{aligned}$$

Exercise 1.4.3. Check that $(\mathrm{Sp}^{\bullet}_X(\mathscr{M}^{\mathrm{right}}),\delta)$ is indeed a complex, i.e., that $\delta\circ\delta = 0$.

Of special interest will be, of course, the de Rham or Spencer complex of the ring $\mathscr{D}_X$, considered as a left or right $\mathscr{D}_X$-module. Notice that, in $\Omega_X^{n+\bullet}(\mathscr{D}_X)$, the differentials are *right* $\mathscr{D}_X$-linear, and in $\mathrm{Sp}^{\bullet}_X(\mathscr{D}_X)$ they are *left* $\mathscr{D}_X$-linear.

***Exercise 1.4.4* (The Spencer complex is a resolution of $\mathscr{O}_X$ as a left $\mathscr{D}_X$-module)**

Let $F_\bullet\mathscr{D}_X$ be the filtration of $\mathscr{D}_X$ by the order of differential operators. Filter the Spencer complex $\mathrm{Sp}^\bullet_X(\mathscr{D}_X)$ by the subcomplexes $F_p(\mathrm{Sp}^\bullet_X(\mathscr{D}_X))$ defined as

$$\cdots \xrightarrow{\;\delta\;} F_{p-k}\mathscr{D}_X \otimes \wedge^k\Theta_X \xrightarrow{\;\delta\;} F_{p-k+1}\mathscr{D}_X \otimes \wedge^{k-1}\Theta_X \xrightarrow{\;\delta\;} \cdots$$

(1) Show that, locally on X, using coordinates $x_1, \dots, x_n$, the graded complex $\mathrm{gr}^F \mathrm{Sp}^\bullet_X(\mathscr{D}_X) := \oplus_p \mathrm{gr}^F_p \mathrm{Sp}^\bullet_X(\mathscr{D}_X)$ is equal to the Koszul complex of the ring $\mathscr{O}_X[\xi_1, \dots, \xi_n]$ with respect to the regular sequence $\xi_1, \dots, \xi_n$.

(2) Conclude that $\mathrm{gr}^F \mathrm{Sp}^\bullet_X(\mathscr{D}_X)$ is a resolution of $\mathscr{O}_X$.

(3) Check that $F_p \mathrm{Sp}^\bullet_X(\mathscr{D}_X) = 0$ for $p < 0$, $F_0 \mathrm{Sp}^\bullet_X(\mathscr{D}_X) = \mathrm{gr}^F_0 \mathrm{Sp}^\bullet_X(\mathscr{D}_X)$ is isomorphic to $\mathscr{O}_X$ and deduce that the complex

$$\mathrm{gr}^F_p \mathrm{Sp}^\bullet_X(\mathscr{D}_X) := \{\cdots \xrightarrow{[\delta]} \mathrm{gr}^F_{p-k}\mathscr{D}_X \otimes \wedge^k\Theta_X \xrightarrow{[\delta]} \mathrm{gr}^F_{p-k+1}\mathscr{D}_X \otimes \wedge^{k-1}\Theta_X \xrightarrow{[\delta]} \cdots\}$$

is acyclic (i.e., quasi-isomorphic to 0) for $p > 0$.

(4) Show that the inclusion $F_0 \mathrm{Sp}^\bullet_X(\mathscr{D}_X) \hookrightarrow F_p \mathrm{Sp}^\bullet_X(\mathscr{D}_X)$ is a quasi-isomorphism for each $p \geqslant 0$ and deduce, by passing to the inductive limit, that the Spencer complex $\mathrm{Sp}^\bullet_X(\mathscr{D}_X)$ is a resolution of $\mathscr{O}_X$ as a left $\mathscr{D}_X$-module by locally free left $\mathscr{D}_X$-modules.

Exercise 1.4.5. Similarly, show that the complex $\Omega^{n+\bullet}_X(\mathscr{D}_X)$ is a resolution of ω_X as a right $\mathscr{D}_X$-module by locally free right $\mathscr{D}_X$-modules.

Exercise 1.4.6. Let $\mathscr{M}^{\mathrm{right}}$ be a right $\mathscr{D}_X$-module

(1) Show that the natural morphism

$$\mathscr{M}^{\mathrm{right}} \otimes_{\mathscr{D}_X} (\mathscr{D}_X \otimes_{\mathscr{O}_X} \wedge^k\Theta_X) \longrightarrow \mathscr{M}^{\mathrm{right}} \otimes_{\mathscr{O}_X} \wedge^k\Theta_X$$

defined by $m \otimes P \otimes \xi \mapsto mP \otimes \xi$ induces an isomorphism of complexes

$$\mathscr{M}^{\mathrm{right}} \otimes_{\mathscr{D}_X} \mathrm{Sp}^\bullet_X(\mathscr{D}_X) \xrightarrow{\;\sim\;} \mathrm{Sp}^\bullet(\mathscr{M}^{\mathrm{right}}).$$

(2) Similar question for $\Omega^{n+\bullet}_X(\mathscr{D}_X) \otimes_{\mathscr{D}_X} \mathscr{M}^{\mathrm{left}} \to \Omega^{n+\bullet}_X(\mathscr{M}^{\mathrm{left}})$.

Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module and let $\mathscr{M}^{\mathrm{right}}$ the associated right module. We will now compare $\Omega^{n+\bullet}_X(\mathscr{M})$ and $\mathrm{Sp}^\bullet_X(\mathscr{M}^{\mathrm{right}})$. Given any $k \geqslant 0$, the *contraction* is the morphism

$$\begin{aligned} \omega_X \otimes_{\mathscr{O}_X} \wedge^k\Theta_X &\longrightarrow \Omega^{n-k}_X \\ \omega \otimes \xi &\longmapsto \omega(\xi \wedge \bullet). \end{aligned} \tag{1.4.7}$$

Exercise 1.4.8. Show that the isomorphism of right $\mathscr{D}_X$-modules

$$\begin{aligned} \omega_X \otimes_{\mathscr{O}_X} \big(\mathscr{D}_X \otimes_{\mathscr{O}_X} \wedge^k\Theta_X\big) &\xrightarrow[\sim]{\;\iota\;} \Omega^{n-k}_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \\ \big[\omega \otimes (1 \otimes \xi)\big] \cdot P &\longmapsto \omega(\xi \wedge \bullet) \otimes P \end{aligned}$$

(where the right structure of the right-hand term is the natural one and that of the left-hand term is nothing but that induced by the left structure after going from left to right) induces an isomorphism of complexes of right $\mathscr{D}_X$-modules

$$\iota : \omega_X \otimes_{\mathscr{O}_X} \big(\mathrm{Sp}^\bullet_X(\mathscr{D}_X), \delta\big) \xrightarrow{\sim} \big(\Omega^{n+\bullet}_X \otimes_{\mathscr{O}_X} \mathscr{D}_X, (-1)^n\nabla\big).$$

[*Hint*: See [**MHMP**, Lem. 8.4.7].]

Exercise 1.4.9. Similarly, if $\mathscr{M}$ is any left $\mathscr{D}_X$-module and $\mathscr{M}^{\mathrm{right}} = \omega_X \otimes_{\mathscr{O}_X} \mathscr{M}$ is the associated right $\mathscr{D}_X$-module, show that there is an isomorphism

$$\begin{aligned}\mathscr{M}^{\mathrm{right}} \otimes_{\mathscr{D}_X} \big(\mathrm{Sp}^\bullet_X(\mathscr{D}_X), \delta\big) &\simeq \big(\omega_X \otimes_{\mathscr{O}_X} \mathscr{M} \otimes_{\mathscr{O}_X} \wedge^{-\bullet}\Theta_X, \delta\big)\\ &\xrightarrow{\sim} \big(\Omega^{n+\bullet}_X \otimes_{\mathscr{O}_X} \mathscr{M}, (-1)^n\nabla\big) \simeq \big(\Omega^{n+\bullet}_X \otimes_{\mathscr{O}_X} \mathscr{D}_X, (-1)^n\nabla\big) \otimes_{\mathscr{D}_X} \mathscr{M}\end{aligned}$$

given on $\omega_X \otimes_{\mathscr{O}_X} \mathscr{M} \otimes_{\mathscr{O}_X} \wedge^k\Theta_X$ by

$$\omega \otimes m \otimes \xi \longmapsto \omega(\xi \wedge \bullet) \otimes m.$$

[*Hint*: See [**MHMP**, Exer. 8.26(1)].]

Exercise 1.4.10. Consider the function

$$\mathbb{Z} \xrightarrow{\varepsilon} \{\pm 1\}, \quad a \longmapsto \varepsilon(a) = (-1)^{a(a-1)/2},$$

which satisfies in particular

$$\varepsilon(a+1) = \varepsilon(-a) = (-1)^a\varepsilon(a), \quad \varepsilon(a+b) = (-1)^{ab}\varepsilon(a)\varepsilon(b).$$

Using Exercise 1.4.9, show that there is a functorial isomorphism $\mathrm{Sp}^\bullet_X(\mathscr{M}^{\mathrm{right}}) \xrightarrow{\sim} (\Omega^{n+\bullet}_X(\mathscr{M}), (-1)^n\nabla)$ for any left $\mathscr{D}_X$-module $\mathscr{M}$, which is termwise $\mathscr{O}_X$-linear. [*Hint*: See [**MHMP**, Exer. 8.26(3)].]

Remark 1.4.11. We will denote by ${}^{\mathrm{p}}\mathrm{DR}_X(\mathscr{M}^{\mathrm{right}})$ the Spencer complex $\mathrm{Sp}^\bullet_X(\mathscr{M}^{\mathrm{right}})$ and by ${}^{\mathrm{p}}\mathrm{DR}_X(\mathscr{M}^{\mathrm{left}})$ the de Rham complex $\Omega^{n+\bullet}_X(\mathscr{M}^{\mathrm{left}})$. The previous exercise gives an isomorphism ${}^{\mathrm{p}}\mathrm{DR}_X(\mathscr{M}^{\mathrm{right}}) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}_X(\mathscr{M}^{\mathrm{left}})$ and justifies this convention. We will use this notation below. Exercise 1.4.6 clearly shows that ${}^{\mathrm{p}}\mathrm{DR}_X$ is a functor from the category of right (resp. left) $\mathscr{D}_X$-modules to the category of complexes of sheaves of $\mathbb{C}$-vector spaces. It can be extended to a functor between the corresponding derived categories.

However, in § 3.2, we will introduce a functor DR_X to a subcategory, in order to keep the information that the differentials in such a complex are of a special kind, i.e., are differential operators. We will then extend this functor as a functor between suitable localized categories. Then ${}^{\mathrm{p}}\mathrm{DR}_X$ will be the composition of DR_X and the forgetful functor Forget.

Denote by $(\mathscr{E}^{(n+\bullet,0)}_X, (-1)^n d')$ the complex $\mathscr{C}^\infty_X \otimes_{\mathscr{O}_X} \Omega^{n+\bullet}_X$ with the differential induced by $(-1)^n d$ (here, we assume $n + \bullet \geqslant 0$). More generally, let $\mathscr{E}^{(n+p,q)}_X = \mathscr{E}^{(n+p,0)}_X \wedge \mathscr{E}^{(0,q)}_X$ and let d'' be the antiholomorphic differential. For any p, the complex

$(\mathscr{E}_X^{(n+p,\bullet)}, d'')$ is a resolution of Ω_X^{n+p}. We therefore have a complex $(\mathscr{E}_X^{n+\bullet}, (-1)^n d)$, which is the single complex associated to the double complex $(\mathscr{E}_X^{(n+\bullet,\bullet)}, (-1)^n d', d'')$.

In particular, we have a natural quasi-isomorphism of complexes of right $\mathscr{D}_X$-modules:

$$(\Omega_X^{n+\bullet} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \nabla) \xrightarrow{\sim} (\mathscr{E}_X^{n+\bullet} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \nabla)$$

by sending holomorphic k-forms to $(k,0)$-forms. Remark that the terms of these complexes are flat over $\mathscr{O}_X$.

Exercise 1.4.12. Define a sheaf $\mathscr{E}_X^{-k,\ell}$ for $k, \ell \geqslant 0$ and find a Dolbeault resolution of $\mathrm{Sp}^\bullet(\mathscr{D}_X)$ by fine sheaves.

Exercise 1.4.13. Let $\mathscr{L}$ be an $\mathscr{O}_X$-module.

(1) Show that, for any k, we have a (termwise) exact sequence of complexes

$$0 \to \mathscr{L} \otimes_{\mathscr{O}_X} F_{k-1}(\mathrm{Sp}_X^\bullet(\mathscr{D}_X)) \to \mathscr{L} \otimes_{\mathscr{O}_X} F_k(\mathrm{Sp}_X^\bullet(\mathscr{D}_X)) \to \mathscr{L} \otimes_{\mathscr{O}_X} \mathrm{gr}_k^F(\mathrm{Sp}_X^\bullet(\mathscr{D}_X)) \to 0.$$

[Hint: use that the terms of the complexes $F_j(\mathrm{Sp}_X^\bullet(\mathscr{D}_X))$ and $\mathrm{gr}_k^F(\mathrm{Sp}_X^\bullet(\mathscr{D}_X))$ are $\mathscr{O}_X$-locally free.]

(2) Show that $\mathscr{L} \otimes_{\mathscr{O}_X} \mathrm{gr}^F \mathrm{Sp}_X^\bullet(\mathscr{D}_X)$ is a resolution of $\mathscr{L}$ as an $\mathscr{O}_X$-module.

(3) Show that $\mathscr{L} \otimes_{\mathscr{O}_X} \mathrm{Sp}_X^\bullet(\mathscr{D}_X)$ is a resolution of $\mathscr{L}$ as an $\mathscr{O}_X$-module.

***Definition 1.4.14* (Godement resolution)**

(1) The *Godement functor* $\mathcal{C}^0$ (see [**God64**, p. 167]) associates to any sheaf $\mathscr{L}$ the *flabby* sheaf $\mathcal{C}^0(\mathscr{L})$ of its discontinuous sections and to any morphism the corresponding family of germs of morphisms. Then there is a canonical injection $\mathscr{L} \hookrightarrow \mathcal{C}^0(\mathscr{L})$.

(2) Set inductively (see [**God64**, p. 168])

$$\mathcal{Z}^0(\mathscr{L}) = \mathscr{L}, \quad \mathcal{Z}^{k+1}(\mathscr{L}) = \mathcal{C}^k(\mathscr{L})/\mathcal{Z}^k(\mathscr{L}), \quad \mathcal{C}^{k+1}(\mathscr{L}) = \mathcal{C}^0(\mathcal{Z}^{k+1}(\mathscr{L}))$$

and define $\delta : \mathcal{C}^k(\mathscr{L}) \to \mathcal{C}^{k+1}(\mathscr{L})$ as the composition $\mathcal{C}^k(\mathscr{L}) \to \mathcal{Z}^{k+1}(\mathscr{L}) \to \mathcal{C}^0(\mathcal{Z}^{k+1}(\mathscr{L}))$. This defines a complex $(\mathcal{C}^\bullet(\mathscr{L}), \delta)$, that we will denote as $(\mathrm{God}^\bullet \mathscr{L}, \delta)$.

(3) Given any sheaf $\mathscr{L}$, $(\mathrm{God}^\bullet \mathscr{L}, \delta)$ is a resolution of $\mathscr{L}$ by flabby sheaves. For a complex $(\mathscr{L}^\bullet, d)$, we view $\mathrm{God}^\bullet \mathscr{L}^\bullet$ as a double complex ordered as written, i.e., with differential $(\delta_i, (-1)^i d_j)$ on $\mathrm{God}^i \mathscr{L}^j$, and therefore also as the associated simple complex.

The following exercise will be useful when computing direct images of $\mathscr{D}$-modules in Lecture 4

***Exercise 1.4.15* (Compatibility with the Godement functor)**

(1) Show that, if $\mathscr{L}$ and $\mathscr{F}$ are $\mathscr{O}_X$-modules and if $\mathscr{F}$ is locally free, then we have a natural inclusion $\mathcal{C}^0(\mathscr{L})\otimes_{\mathscr{O}_X}\mathscr{F}\hookrightarrow\mathcal{C}^0(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{F})$, which is an equality if $\mathscr{F}$ has finite rank. More generally, show by induction that we have a natural morphism $\mathcal{C}^k(\mathscr{L})\otimes_{\mathscr{O}_X}\mathscr{F}\to\mathcal{C}^k(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{F})$, which is an equality if $\mathscr{F}$ has finite rank.

(2) With the same assumptions, show that both complexes $\mathrm{God}^\bullet(\mathscr{L})\otimes_{\mathscr{O}_X}\mathscr{F}$ and $\mathrm{God}^\bullet(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{F})$ are resolutions of $\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{F}$. Conclude that the natural morphism of complexes $\mathrm{God}^\bullet(\mathscr{L})\otimes_{\mathscr{O}_X}\mathscr{F}\to\mathrm{God}^\bullet(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{F})$ is a quasi-isomorphism.

(3) Let $\mathscr{M}$ be a $\mathscr{D}_X$-module. Show that ${}^{\mathrm{p}}\mathrm{DR}_X\,\mathrm{God}^\bullet\mathscr{M}=\mathrm{God}^\bullet\,{}^{\mathrm{p}}\mathrm{DR}_X\,\mathscr{M}$.

1.5. Filtered objects: the Rees construction

***Definition 1.5.1* (of a filtered $\mathscr{D}_X$-module).** A *filtration* $F_\bullet\mathscr{M}$ of a $\mathscr{D}_X$-module $\mathscr{M}$ will mean an increasing filtration satisfying (for left modules for instance)

$$F_k\mathscr{D}_X\cdot F_\ell\mathscr{M}\subset F_{k+\ell}\mathscr{M}\quad\forall\,k,\ell\in\mathbb{Z}.$$

We usually assume that $F_\ell\mathscr{M}=0$ for $\ell\ll 0$ and that the filtration is exhaustive, i.e., $\bigcup_\ell F_\ell\mathscr{M}=\mathscr{M}$.

***Definition 1.5.2* (of the de Rham complex of a filtered $\mathscr{D}_X$-module)**

Let $F_\bullet\mathscr{M}$ be a filtered $\mathscr{D}_X$-module. The de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ is filtered by sub-complexes $F_p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ defined by

$$F_p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}=\begin{cases}\cdots\xrightarrow{\delta}F_{p-k}\mathscr{M}^{\mathrm{right}}\otimes\wedge^k\Theta_X\xrightarrow{\delta}F_{p-k+1}\mathscr{M}^{\mathrm{right}}\otimes\wedge^{k-1}\Theta_X\xrightarrow{\delta}\cdots\\ \cdots\xrightarrow{\nabla}\Omega_X^{n+k}\otimes F_{p+k}\mathscr{M}^{\mathrm{left}}\xrightarrow{\nabla}\Omega_X^{n+k+1}\otimes F_{p+k+1}\mathscr{M}^{\mathrm{left}}\xrightarrow{\nabla}\cdots\end{cases}$$

and the filtered de Rham complex is denoted by ${}^{\mathrm{p}}\mathrm{DR}\,F_\bullet\mathscr{M}$.

Exercise 1.5.3. Show that the isomorphisms in Exercises 1.2.8 and 1.2.9 are isomorphisms of filtered objects $\mathscr{M}^{\mathrm{left}}\otimes_{\mathscr{O}_X}F_\bullet\mathscr{D}_X$, $F_\bullet\mathscr{D}_X\otimes_{\mathscr{O}_X}\mathscr{M}^{\mathrm{left}}$ and $\mathscr{M}^{\mathrm{right}}\otimes_{\mathscr{O}_X}F_\bullet\mathscr{D}_X$.

It is possible to apply the techniques of the previous sections to filtered objects. A simple way to do that is to introduce the Rees object associated to any filtered object. Introduce a new variable z. We will replace the base field $\mathbb{C}$ with the polynomial ring $\mathbb{C}[z]$.

***Definition 1.5.4* (Rees ring and Rees module).** If $(\mathscr{A},F_\bullet)$ is a filtered ring, we denote by $\widetilde{\mathscr{A}}$ (or $R_F\mathscr{A}$ if we want to insist on the dependence with respect to the filtration) the subring $\oplus_kF_k\mathscr{A}\cdot z^k$ of $\mathscr{A}\otimes_{\mathbb{C}}\mathbb{C}[z,z^{-1}]$. For instance, if $F_k\mathscr{A}=0$ for $k\leqslant -1$ and $F_k\mathscr{A}=\mathscr{A}$ for $k\geqslant 0$, we have $\widetilde{\mathscr{A}}=\mathscr{A}\otimes_{\mathbb{C}}\mathbb{C}[z]$. Any filtered module $(\mathscr{M},F_\bullet)$ on the filtered ring $(\mathscr{A},F_\bullet)$ gives rise similarly to a module $\widetilde{\mathscr{M}}$ on $\widetilde{\mathscr{A}}$.

Notice that $\widetilde{\mathscr{A}}$ is a graded ring with a central element z and that $\widetilde{\mathscr{M}}$ is a graded module on this graded ring. Notice also that, as $\widetilde{\mathscr{M}}$ is contained in $\mathscr{M}\otimes_{\mathbb{C}}\mathbb{C}[z,z^{-1}]$, the multiplication by z is injective on $\mathscr{M}$.

Exercise 1.5.5

(1) Show that the Rees construction gives an equivalence between the category of filtered $(\mathscr{A},F_\bullet)$-modules (the morphisms should preserve the filtrations up to some fixed shift) and the subcategory of the category of graded $\widetilde{\mathscr{A}}$-modules (the morphisms are homogeneous), the object of which have no z-torsion. [Hint: recover $\mathscr{M}$ from $\widetilde{\mathscr{M}}$ by setting $\mathscr{M}=\widetilde{\mathscr{M}}/(z-1)\widetilde{\mathscr{M}}$.]

(2) Recover $\mathrm{gr}^F\mathscr{M}$ as $\widetilde{\mathscr{M}}/z\widetilde{\mathscr{M}}$.

(3) Show that, if one defines the filtration

$$\widetilde{F}_k\widetilde{\mathscr{M}}=\bigoplus_{j\leqslant k}F_j\mathscr{M}z^j\oplus\bigoplus_{j>k}F_k\mathscr{M}z^j,$$

then $\mathrm{gr}^{\widetilde{F}}\widetilde{\mathscr{M}}$ can be identified with $\mathrm{gr}^F\mathscr{M}\otimes_{\mathbb{C}}\mathbb{C}[z]$, where the grading in the previous term is the sum of the grading on $\mathrm{gr}^F\mathscr{M}$ and of the grading of $\mathbb{C}[z]$ by the degree in z.

Applying this construction to the filtered ring $(\mathscr{D}_X,F_\bullet)$ and its (left or right) modules, we obtain the following properties:

- $\widetilde{\mathscr{O}}_X=\mathscr{O}_X[z]$;
- in local coordinates, any local section of $\widetilde{\mathscr{D}}_X$ may be written in a unique way as $\sum_\alpha a_\alpha(x)\eth_x^\alpha=\sum_\alpha\eth_x^\beta b_\alpha(x)$, where we set $\eth_{x_i}:=z\partial_{x_i}$;
- $\widetilde{\Theta}_X$ is the locally free sheaf locally generated by $\eth_{x_1},\dots,\eth_{x_n}$ and we have $[\eth_{x_i},\varphi]=z\partial\varphi/\partial x_i$ for any local section φ of $\widetilde{\mathscr{O}}_X$;
- the sheaf $\widetilde{\Omega}^1_X$ is defined as $z^{-1}\mathbb{C}[z]\otimes_{\mathbb{C}}\Omega^1_X$, and $\widetilde{\Omega}^k_X=\wedge^k\widetilde{\Omega}^1_X$; the differential $\widetilde{d}$ is induced by the differential d on Ω^k_X; the local basis $(\widetilde{dx}_i=z^{-1}dx_i)_i$ is dual to the basis $(\eth_{x_i})_i$ of $\widetilde{\Theta}_X$.

Exercise 1.5.6

(1) Extend the results of §§ 1.1-1.4 to graded $\widetilde{\mathscr{D}}_X$-modules.

(2) Show that the same results hold for unnecessarily graded $\widetilde{\mathscr{D}}_X$-modules and unnecessarily homogeneous morphisms.

***Definition 1.5.7* (z-connection).** Let $\widetilde{\mathscr{M}}$ be a $\widetilde{\mathscr{O}}_X$-module. A z-connection on $\widetilde{\mathscr{M}}$ is a $\mathbb{C}[z]$-linear morphism $\widetilde{\nabla}:\widetilde{\mathscr{M}}\to\widetilde{\Omega}^1_X\otimes\widetilde{\mathscr{M}}$ which satisfies the Leibniz rule

$$\widetilde{\nabla}(\widetilde{f}\widetilde{m})=\widetilde{f}\,\widetilde{\nabla}\widetilde{m}+\widetilde{d}\widetilde{f}\otimes\widetilde{m}.$$

Exercise 1.5.8

(1) Show that $\widetilde{\mathscr{D}}_X$ has a universal z-connection $\widetilde{\nabla}$ for which $\widetilde{\nabla}(1)=\sum_i\widetilde{dx}_i\otimes\eth_{x_i}$.

(2) Show the equivalence between left $\widetilde{\mathscr{D}}_X$-modules and $\widetilde{\mathscr{O}}_X$-modules equipped with an integrable z-connection.

Definition 1.5.9. Let $\widetilde{\mathscr{M}}$ be a left $\widetilde{\mathscr{D}}_X$-module. The de Rham complex ${}^{\mathrm{p}}\widetilde{\mathrm{DR}}\widetilde{\mathscr{M}}$ is the complex having as terms the $\widetilde{\mathscr{O}}_X$-modules $\widetilde{\Omega}_X^{n+\bullet} \otimes \widetilde{\mathscr{M}}$ and as differentials the z-connection $(-1)^n\widetilde{\nabla}$.

Right analogues (in particular, the Spencer complex) are defined similarly as well as the right-left correspondence. All properties of §1.4 extend in this setting.

Exercise 1.5.10. Using Exercise 1.4.4, show that $\widetilde{\mathrm{Sp}}(\widetilde{\mathscr{D}}_X)$ is a resolution of $\widetilde{\mathscr{O}}_X$.

LECTURE 2

COHERENCE

Although it would be natural to develop the theory of coherent $\mathscr{D}_X$-modules in a way similar to that of $\mathscr{O}_X$-modules, some points of the theory are not known to extend to $\mathscr{D}_X$-modules (the lemma on holomorphic matrices). The approach which is therefore classically used consists in using the $\mathscr{O}_X$-theory, and the main tools for that purpose are the good filtrations.

This lecture is much inspired from [**GM93**].

2.0. A reminder on coherence

Let us begin by recalling the definition of coherence. Let $\mathscr{A}$ be a sheaf of rings on a space X.

Definition 2.0.1

(1) A sheaf of $\mathscr{A}$-modules $\mathscr{F}$ is said to be *$\mathscr{A}$-coherent* if it is locally of finite type:

$$\forall\, x \in X,\ \exists\, V_x,\ \exists\, q, \quad \exists\, \mathscr{A}^q_{|V_x} \twoheadrightarrow \mathscr{F}_{|V_x},$$

and if, for any open set U of X and any $\mathscr{A}$-linear morphism $\varphi : \mathscr{A}^r_{|U} \to \mathscr{F}_{|U}$, the kernel of φ is locally of finite type.

(2) The sheaf $\mathscr{A}$ is a coherent sheaf of rings if it is coherent as a (left and right) module over itself.

Lemma 2.0.2. *Assume $\mathscr{A}$ coherent. Let $\mathscr{F}$ be a sheaf of $\mathscr{A}$-module. Then $\mathscr{F}$ is $\mathscr{A}$-coherent if and only if $\mathscr{F}$ is* locally of finite presentation*: $\forall\, x \in X$, $\exists\, V_x$, $\exists\, p, q$ and an exact sequence*

$$\mathscr{A}^p_{|V_x} \longrightarrow \mathscr{A}^q_{|V_x} \longrightarrow \mathscr{F}_{|V_x} \longrightarrow 0.$$

Classical theorems of Cartan and Oka claim the *coherence of $\mathscr{O}_X$*.

2.1. Coherence of $\mathscr{D}_X$

Let K be a compact subset of X. We say K is a *compact polycylinder* if there exist a neighbourhood Ω of K, an analytic chart $\phi : \Omega \to W$ of X, and $(\rho_1, \dots, \rho_n) \in (\mathbb{R}^+)^n$ such that

$$\phi(K) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid \forall\, i \in \{1, \dots, n\},\, |x_i| \leqslant \rho_i\}.$$

In particular a point $x \in X$ is a compact polycylinder. Let $\mathscr{F}$ be a sheaf on X and K a polycylinder. We know by [**God64**], that

$$\varinjlim_{\substack{U \supset K \\ U \text{open}}} \mathscr{F}(U) \simeq \mathscr{F}_{|K}(K)$$

denoted by $\mathscr{F}(K)$. We have $\mathscr{D}_{X|\Omega} \simeq \mathscr{D}_{\mathbb{C}^n|W}$ and this isomorphism is compatible with the filtrations. Thus, to study local properties of $\mathrm{gr}^F \mathscr{D}_X$ or of $\mathscr{D}_X$ in the neighbourhood of a polycylinder K we can assume that $K \subset \mathbb{C}^n$ is a usual polycylinder.

We have $\mathscr{D}_X(K) \subset \mathscr{H}om_{\mathbb{C}}(\mathscr{O}_X, \mathscr{O}_X)(K)$ and any element of $\mathscr{D}_X(K)$ can be written in a unique way as $\sum_{\alpha \in I} c_\alpha \partial^\alpha$, with $c_\alpha \in \mathscr{O}_X(K)$ and $I \subset \mathbb{N}^n$ finite. The relations in Exercise 1.1.6 remain true when we replace U by K. We also have

$$\varinjlim_{\substack{U \supset K \\ U \text{open}}} F_k\mathscr{D}_X(U) = \big\{P \in \mathscr{D}_X(K) \mid P = \textstyle\sum_{|\alpha| \leqslant k} c_\alpha \partial^\alpha \quad \text{with } c_\alpha \in \mathscr{O}_X(K)\big\}.$$

Let $F_k\mathscr{D}_X(K)$ be this $\mathscr{O}(K)$-submodule of $\mathscr{D}(K)$. We get a filtration of $\mathscr{D}_X(K)$ having the same properties as that of $\mathscr{D}_X(U)$. Finally, we deduce from Exercise 1.1.7 the existence of a canonical ring isomorphism

$$\mathrm{gr}^F(\mathscr{D}_X(K)) \xrightarrow{\sim} (\mathrm{gr}^F \mathscr{D}_X)(K).$$

We thus have an isomorphism

$$\mathrm{gr}^F(\mathscr{D}_X(K)) \simeq \mathscr{O}_{\mathbb{C}^n}(K)[\xi_1, \dots, \xi_n]$$

by an inductive limit on $U \supset K$. By a theorem of Frisch [**Fri67**], $\mathscr{O}_{\mathbb{C}^n}(K)$ is a Noetherian ring and, for any $x \in K$, the ring $\mathscr{O}_{\mathbb{C}^n,x}$ is flat over $\mathscr{O}_{\mathbb{C}^n}(K)$. We therefore get:

Proposition 2.1.1. *If K is a compact polycylinder, $\mathrm{gr}^F \mathscr{D}_X(K)$ is a Noetherian ring.* □

Proposition 2.1.2. *The ring $\mathscr{D}_X(K)$ is Noetherian.*

Proof. Let $I \subset \mathscr{D}_X(K)$ be a left ideal. We have to prove that it is finitely generated. Set $F_kI = I \cap F_k\mathscr{D}_X(K)$. Then $\mathrm{gr}^F I = \oplus_{k \in \mathbb{N}} F_kI/F_{k-1}I$ is an ideal in $\mathrm{gr}^F \mathscr{D}_X(K)$, thus is of finite type. Let $e_1, \dots, e_\ell$ be homogeneous generators of $\mathrm{gr}^F I$, of degrees $d_1, \dots, d_\ell$ and $P_1, \dots, P_\ell$ elements of I with $\sigma(P_j) = e_j$. It is easy to prove, by induction on the order of $P \in I$, that $I = \sum_{i=1}^{\ell} \mathscr{D}_X(K) \cdot P_i$ (left to the reader). □

Theorem 2.1.3. *The sheaf of rings $\mathscr{D}_X$ is coherent.*

Proof. If $U \subset X$ is open and

$$\phi : (\mathscr{D}_{X|U})^q \longrightarrow (\mathscr{D}_{X|U})^p$$

is a morphism of left $\mathscr{D}_{X|U}$-modules, we have to prove that $\operatorname{Ker}\phi$ is locally of finite type. We may assume that U is an open chart, thus in fact an open subset of $\mathbb{C}^n$. Let $\varepsilon_1, \dots, \varepsilon_q$ be the canonical base of $\mathscr{D}_X(U)^q$ and $k \in \mathbb{N}$ be such that, for all $i \in \{1, \dots, q\}$, $\phi(\varepsilon_i) \in F_k\mathscr{D}_X(U)^p$. We then have $\phi(F_\ell\mathscr{D}_U^q) \subset F_{k+\ell}\mathscr{D}_U^p$, and $\operatorname{Ker}\phi \cap F_\ell\mathscr{D}_U^q$ is the kernel of a morphism between locally free $\mathscr{O}_U$-modules of finite type

$$F_\ell\mathscr{D}_U^q \longrightarrow F_{k+\ell}\mathscr{D}_U^p.$$

Thus $\operatorname{Ker}\phi \cap F_\ell\mathscr{D}_U^q$ is $\mathscr{O}_U$ coherent, and $\operatorname{Ker}\phi$ is the union of these $\mathscr{O}_U$-modules.

Let $K \subset U$ be a compact polycylinder. By Theorem A of Cartan, for any $x \in K$, the sheaf $[\operatorname{Ker}\phi \cap F_\ell\mathscr{D}_{X|U}^q]_x$ is generated by $\Gamma(K, \operatorname{Ker}\phi \cap F_\ell\mathscr{D}_{X|U}^q)$, which is included in $\Gamma(K, \operatorname{Ker}\phi)$. Thus for any $x \in K$, $(\operatorname{Ker}\phi)_x$ is generated by $\Gamma(K, \operatorname{Ker}\phi)$, i.e., any germ of section of $\operatorname{Ker}\phi$ at x is a linear combination with coefficients in $\mathscr{O}_{X,x}$ of sections of $\operatorname{Ker}\phi$ over K. By left exactness of $\Gamma(K, \bullet)$ we have an exact sequence of left $\mathscr{D}_X(K)$-modules:

$$0 \longrightarrow \Gamma(K, \operatorname{Ker}\phi) \longrightarrow \Gamma(K, \mathscr{D}_U^q) = \mathscr{D}_X(K)^p \xrightarrow{\Gamma(K,\phi)} \mathscr{D}_X(K)^p.$$

Because $\mathscr{D}_X(K)$ is Noetherian, $\Gamma(K, \operatorname{Ker}\phi)$ is then of finite type as a left $\mathscr{D}_X(K)$-module. It is then easy to build a surjective morphism of left $\mathscr{D}_{X|K}$-modules

$$(\mathscr{D}_{X|K})^r \longrightarrow (\operatorname{Ker}\phi)_{|K} \longrightarrow 0$$

using the two properties above. This proves that $\operatorname{Ker}\phi$ is locally of finite type. ☐

Exercise 2.1.4

(1) Prove similarly the coherence of the sheaf of rings $\operatorname{gr}^F\mathscr{D}_X$ and that of the Rees sheaf of rings $R_F\mathscr{D}_X$ (see Definition 1.5.4).

(2) Let $D \subset X$ be a hypersurface and let $\mathscr{O}_X(*D)$ be the sheaf of meromorphic functions on X with poles on D at most (with arbitrary order). Prove similarly that $\mathscr{O}_X(*D)$ is a coherent sheaf of rings.

(3) Prove that $\mathscr{D}_X(*D) := \mathscr{O}_X(*D) \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is a coherent sheaf of rings.

(4) Let $i : Y \hookrightarrow X$ denote the inclusion of a smooth submanifold. Show that $i^*\mathscr{D}_X := \mathscr{O}_Y \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is a coherent sheaf of rings on Y.

2.2. Coherent $\mathscr{D}_X$-modules and good filtrations

Let $\mathscr{M}$ be a $\mathscr{D}_X$-module. From Theorem 2.1.3 and the preliminary reminder on coherence, we know that $\mathscr{M}$ is $\mathscr{D}_X$*-coherent* if it is locally finitely presented, i.e., if for any $x \in X$ there exists an open neighbourhood U of x an an exact sequence $\mathscr{D}_{X|U}^q \to \mathscr{D}_{X|U}^p \to \mathscr{M}_{|U}$.

Exercise 2.2.1

(1) Let $\mathscr{M} \subset \mathscr{N}$ be a $\mathscr{D}_X$-submodule of a coherent $\mathscr{D}_X$-module $\mathscr{N}$. Show that, if $\mathscr{M}$ is locally finitely generated, then it is coherent.

(2) Let $\phi : \mathscr{M} \to \mathscr{N}$ be a morphism between coherent $\mathscr{D}_X$-modules. Show that $\operatorname{Ker} \phi$ and $\operatorname{Coker} \phi$ are coherent.

***Definition 2.2.2* (Good filtrations).** Let $F_\bullet \mathscr{M}$ be a filtration of $\mathscr{M}$ (see § 1.5). We say that the filtration is *good* if the Rees module $R_F\mathscr{M}$ is coherent over the coherent sheaf $R_F\mathscr{D}_X$ (i.e., locally finitely presented).

It is useful to have various criteria for a filtration to be good.

***Exercise 2.2.3* (Characterization of good filtrations).** Show that the following properties are equivalent:

(1) $F_\bullet\mathscr{M}$ is a good filtration;

(2) for any $k \in \mathbb{Z}$, $F_k\mathscr{M}$ is $\mathscr{O}_X$-coherent, and, for any $x \in X$, there exists a neighbourhood U of x and $k_0 \in \mathbb{Z}$ such that, for any $k \geqslant 0$, $F_k\mathscr{D}_{X|U} \cdot F_{k_0}\mathscr{M}_{|U} = F_{k+k_0}\mathscr{M}_{|U}$;

(3) the graded module $\operatorname{gr}^F\mathscr{M}$ is $\operatorname{gr}^F\mathscr{D}_X$-coherent.

Conclude that, if $F_\bullet\mathscr{M}, G_\bullet\mathscr{M}$ are two good filtrations of $\mathscr{M}$, then, locally on X, there exists k_0 such that, for any k, we have

$$F_{k-k_0}\mathscr{M} \subset G_k\mathscr{M} \subset F_{k+k_0}\mathscr{M}.$$

***Proposition 2.2.4* (Local existence of good filtrations).** *If $\mathscr{M}$ is $\mathscr{D}_X$-coherent, then it admits locally on X a good filtration.*

Proof. Exercise 2.2.5. ☐

***Exercise 2.2.5* (Local existence of good filtrations)**

(1) Show that, if $\mathscr{M}$ has a good filtration, then it is $\mathscr{D}_X$-coherent and $\operatorname{gr}^F\mathscr{M}$ is $\operatorname{gr}^F\mathscr{D}_X$-coherent. In particular, a good $\mathscr{D}_X$-module is coherent. [Hint: use that the tensor product $\mathbb{C}[z]/(z-1) \otimes_{\mathbb{C}[z]} \bullet$ is right exact.]

(2) Conversely, show that any coherent $\mathscr{D}_X$-module admits locally a good filtration. [Hint: choose a local presentation $\mathscr{D}^q_{X|U} \xrightarrow{\varphi} \mathscr{D}^p_{X|U} \to \mathscr{M}_{|U} \to 0$, and show that the filtration induced on $\mathscr{M}_{|U}$ by $F_\bullet\mathscr{D}^p_{X|U}$ is good by using Exercise 2.2.3: Set $\mathscr{K} = \operatorname{Im}\varphi$ and reduce the assertion to showing that $F_j\mathscr{D}_X \cap \mathscr{K}$ is $\mathscr{O}_X$-coherent; prove that, up to shrinking U, there exists $k_o \in \mathbb{N}$ such that $\varphi(F_k\mathscr{D}^q_{X|U}) \subset F_{k+k_o}\mathscr{D}^p_{X|U}$ for each k; deduce that $\varphi(F_k\mathscr{D}^q_{X|U})$, being locally of finite type and contained in a coherent $\mathscr{O}_X$-module, is $\mathscr{O}_X$-coherent for each k; conclude by using the fact that an increasing sequence of coherent $\mathscr{O}_X$-modules in a coherent $\mathscr{O}_X$-module is locally stationary.]

(3) Show that, locally, any coherent $\mathscr{D}_X$-module is generated over $\mathscr{D}_X$ by a coherent $\mathscr{O}_X$-submodule.

(4) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module and let $\mathscr{F}$ be an $\mathscr{O}_X$-submodule which is locally finitely generated. Show that $\mathscr{F}$ is $\mathscr{O}_X$-coherent. [Hint: choose a good filtration $F_\bullet\mathscr{M}$ and show that, locally, $\mathscr{F} \subset F_k\mathscr{M}$ for some k; apply then the analogue of Exercise 2.2.1(1) for $\mathscr{O}_X$-modules.]

Good filtrations are the main tool to get results on coherent $\mathscr{D}_X$-modules from theorems on coherent $\mathscr{O}_X$-modules. This justifies the following definition:

***Definition 2.2.6* (Good $\mathscr{D}$-modules, see [SS94]).** A $\mathscr{D}_X$-module is *good* if, for any compact set $K \subset X$, there exists, on some neighbourhood U of K, a finite filtration of $\mathscr{M}_{|U}$ by $\mathscr{D}_U$-submodules such that each successive quotient has a good filtration.

Remark 2.2.7. It is not known whether any coherent $\mathscr{D}_X$-module has *globally* a good filtration, or even whether it is good. Nevertheless, it is known that any *holonomic* $\mathscr{D}_X$-module (see Definition 5.2.1) has a good filtration (see [**Mal94a, Mal94b, Mal96**]); in fact, if such is the case, there even exists a coherent $\mathscr{O}_X$-submodule $\mathscr{F}$ of $\mathscr{M}$ which generates $\mathscr{M}$, i.e., such that the natural morphism $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F} \to \mathscr{M}$ is onto (this is a little stronger than the existence of a good filtration, if the manifold X is not compact).

The main results concerning coherent $\mathscr{D}_X$-modules are obtained from the theorems of Cartan and Oka for $\mathscr{O}_X$-modules.

***Theorem 2.2.8* (Theorems of Cartan-Oka for $\mathscr{D}_X$-modules).** *Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module and let K be a compact polycylinder contained in an open subset U of X, such that $\mathscr{M}$ has a good filtration on U. Then,*

(1) *$\Gamma(K,\mathscr{M})$ generates $\mathscr{M}_{|K}$ as an $\mathscr{O}_K$-module,*

(2) *For any $i \geqslant 1$, $H^i(K,\mathscr{M}) = 0$.*

Proof. This is easily obtained from the theorems A and B for $\mathscr{O}_X$-modules, by using inductive limits (for A it is obvious and, for B, see [**God64**, Th. 4.12.1]). ☐

***Theorem 2.2.9* (Characterization of coherence for $\mathscr{D}_X$-modules)**

(1) *Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. Then, for any small enough compact polycylinder K, we have the following properties:*

(a) *$\mathscr{M}(K)$ is a finite type $\mathscr{D}(K)$-module,*

(b) *For any $x \in K$, $\mathscr{O}_x \otimes_{\mathscr{O}(K)} \mathscr{M}(K) \to \mathscr{M}_x$ is an isomorphism.*

(2) *Conversely, if there exists a covering $\{K_\alpha\}$ by polycylinders K_α such that $X = \cup \overset{\circ}{K}_\alpha$ and that on any K_α the properties* (1a) *and* (1b) *are fulfilled, then $\mathscr{M}$ is $\mathscr{D}_X$-coherent.*

Proof. Let $U \subset X$ be an open subset small enough for $\mathscr{M}$ to have a presentation

$$0 \longrightarrow \mathscr{N} \longrightarrow \mathscr{D}_U^p \longrightarrow \mathscr{M}_{|U} \longrightarrow 0.$$

The $\mathscr{D}_U$-module $\mathscr{N}$ is coherent, therefore we have $H^1(K, \mathscr{N}) = 0$ for any small enough compact polycylinder $K \subset U$, and $\mathscr{D}^p(K) \to \mathscr{M}(K)$ is surjective. This proves (1a).

The $\mathscr{O}_X$-module $F_k\mathscr{M}_{|U} := \operatorname{image} F_k\mathscr{D}_U^p$ being coherent we also have for any k an isomorphism $\mathscr{O}_x \otimes F_k\mathscr{M}(K) \to F_k\mathscr{M}_x$, by Theorem A of Cartan-Oka. From this we get (1b) by using an inductive limit

$$\mathscr{O}_x \otimes_{\mathscr{O}(K)} \mathscr{M}(K) \simeq \varinjlim_k \mathscr{O}_x \otimes_{\mathscr{O}(K)} F_k\mathscr{M}(K).$$

Conversely, if Condition (1a) is fulfilled we have, since $\mathscr{D}(K)$ is Noetherian, a finite presentation

$$\mathscr{D}^q(K) \xrightarrow{\ \phi\ } \mathscr{D}^p(K) \xrightarrow{\ \pi\ } \mathscr{M}(K) \longrightarrow 0,$$

which gives sheaf morphisms which we denote again by ϕ, π:

$$\mathscr{D}^q_{X|K} \xrightarrow{\ \phi\ } \mathscr{D}^p_{X|K} \xrightarrow{\ \pi\ } \mathscr{M}_{|K} \longrightarrow 0,$$

and, by the exactness of the functor $\mathscr{D}_x \otimes_{\mathscr{D}(K)}$, an exact sequence

$$\mathscr{D}_x^q \xrightarrow{\ \phi_x\ } \mathscr{D}_x^p \xrightarrow{\ p_x\ } \mathscr{D}_x \otimes_{\mathscr{D}(K)} \mathscr{M}(K) \longrightarrow 0.$$

By Condition (1b) and the lemma below, the morphism

$$c_x : \mathscr{D}_x \otimes_{\mathscr{D}(K)} \mathscr{M}(K) \longrightarrow \mathscr{M}_x$$

is an isomorphism. We deduce from this, and from the equality $\pi_x = c_x \circ p_x$ that $\mathscr{M}_{|K} = \operatorname{Coker}(\phi)$ is finitely presented on K. ☐

Lemma 2.2.10. *For any left $\mathscr{D}(K)$-module N, the canonical homomorphism*

$$\mathscr{O}_x \otimes_{\mathscr{O}(K)} N \longrightarrow \mathscr{D}_x \otimes_{\mathscr{D}(K)} N$$

is an isomorphism.

Proof. This is clear when $N = \mathscr{D}(K)$, hence for any free module and finally, in the general case, by the right exactness of the functors $\mathscr{O}_x \otimes_{\mathscr{O}(K)} \bullet$ and $\mathscr{D}_x \otimes_{\mathscr{D}(K)} \bullet$. ☐

Exercise 2.2.11

(1) Show similar statements for $R_F\mathscr{D}_X$-modules, $\operatorname{gr}^F\mathscr{D}_X$-modules, $\mathscr{O}_X(*D)$-modules, $\mathscr{D}_X(*D)$-modules and $i^*\mathscr{D}_X$-modules (see Exercise 2.1.4).

(2) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Show that $\mathscr{D}_X(*D) \otimes_{\mathscr{D}_X} \mathscr{M}$ is $\mathscr{D}_X(*D)$-coherent and that $i^*\mathscr{M}$ is $i^*\mathscr{D}_X$-coherent.

***Exercise 2.2.12* (External product)**

(1) Let A, B be two Noetherian $\mathbb{C}$-algebras. Show that $A \otimes_{\mathbb{C}} B$ is Noetherian.

(2) Let X_1, X_2 be two complex manifolds and let p_1, p_2 be the projections from $X_1 \times X_2$ to X and Y respectively. For any pair of sheaves of $\mathbb{C}$-vector spaces $\mathscr{F}_1, \mathscr{F}_2$ on X and Y respectively, set $\mathscr{F}_1 \boxtimes_{\mathbb{C}} \mathscr{F}_2 := p_1^{-1}\mathscr{F}_1 \otimes_{\mathbb{C}} p_2^{-1}\mathscr{F}_2$. Show that $\mathscr{O}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{O}_{X_2}$ is a coherent sheaf of rings on $X_1 \times X_2$. [Hint: Use an analogue of Theorem 2.2.9(2).]

(3) Prove similar properties for $\mathscr{D}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{D}_{X_2}$.

(4) Show that $\mathscr{O}_{X_1 \times X_2}$ is faithfully flat over $\mathscr{O}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{O}_{X_2}$. [Hint:.] A compléter

(5) Show that

$$\mathscr{D}_{X_1 \times X_2} = \mathscr{O}_{X_1 \times X_2} \otimes_{(\mathscr{O}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{O}_{X_2})} (\mathscr{D}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{D}_{X_2}) = (\mathscr{D}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{D}_{X_2}) \otimes_{(\mathscr{O}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{O}_{X_2})} \mathscr{O}_{X_1 \times X_2}.$$

(6) For an $\mathscr{O}_{X_1}$-module $\mathscr{L}_1$ (resp. a $\mathscr{D}_{X_1}$-module $\mathscr{M}_1$) and an $\mathscr{O}_{X_2}$-module $\mathscr{L}_2$ (resp. a $\mathscr{D}_{X_2}$-module $\mathscr{M}_2$), set

$$\begin{aligned} \mathscr{L}_1 \boxtimes_{\mathscr{O}} \mathscr{L}_2 &= (\mathscr{L}_1 \boxtimes_{\mathbb{C}} \mathscr{L}_2) \otimes_{\mathscr{O}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{O}_{X_2}} \mathscr{O}_{X_1 \times X_2} \\ \text{resp.} \qquad \mathscr{M}_1 \boxtimes_{\mathscr{D}} \mathscr{M}_2 &= (\mathscr{M}_1 \boxtimes_{\mathbb{C}} \mathscr{M}_2) \otimes_{\mathscr{O}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{O}_{X_2}} \mathscr{O}_{X_1 \times X_2} \\ &= (\mathscr{M}_1 \boxtimes_{\mathbb{C}} \mathscr{M}_2) \otimes_{\mathscr{D}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{D}_{X_2}} \mathscr{D}_{X_1 \times X_2} \\ &\simeq \mathscr{M}_1 \boxtimes_{\mathscr{O}} \mathscr{M}_2. \end{aligned}$$

Show that if $\mathscr{L}_1, \mathscr{L}_2$ are $\mathscr{O}$-coherent (resp. $\mathscr{M}_1, \mathscr{M}_2$ are $\mathscr{D}$-coherent), then $\mathscr{L}_1 \boxtimes_{\mathscr{O}} \mathscr{L}_2$ is $\mathscr{O}_{X_1 \times X_2}$-coherent (resp. $\mathscr{M}_1 \boxtimes_{\mathscr{D}} \mathscr{M}_2$ is $\mathscr{D}_{X_1 \times X_2}$-coherent).

(7) Show that, if $F_\bullet\mathscr{M}_1, F_\bullet\mathscr{M}_2$ are good filtrations, then $F_j(\mathscr{M}_1 \boxtimes_{\mathscr{D}} \mathscr{M}_2) := \sum_{k+\ell=j} F_k\mathscr{M}_1 \boxtimes_{\mathscr{O}} F_\ell\mathscr{M}_2$ is a good filtration of $\mathscr{M}_1 \boxtimes_{\mathscr{D}} \mathscr{M}_2$ for which

$$\operatorname{gr}^F(\mathscr{M}_1 \boxtimes_{\mathscr{D}} \mathscr{M}_2) = \operatorname{gr}^F\mathscr{M}_1 \boxtimes_{\operatorname{gr}^F\mathscr{D}} \operatorname{gr}^F\mathscr{M}_2.$$

[Hint: See [**Kas03**, §4.3].]

A first application of Theorem 2.2.9 is a variant of the classical Artin-Rees Lemma:

Corollary 2.2.13. *Let $\mathscr{M}$ be a $\mathscr{D}_X$-module with a good filtration $F_\bullet\mathscr{M}$ and let $\mathscr{N}$ be a coherent $\mathscr{D}_X$-submodule of $\mathscr{M}$. Then the filtration $F_\bullet\mathscr{N} := \mathscr{N} \cap F_\bullet\mathscr{M}$ is good.*

Proof. Let K be a small compact polycylinder for $R_F\mathscr{M}$. Then $\Gamma(K, R_F\mathscr{M})$ is finitely generated, hence so is $\Gamma(K, R_F\mathscr{N})$, as $\Gamma(K, R_F\mathscr{D}_X)$ is Noetherian. It remains to be proved that, for any $x \in K$ and any k, the natural morphism

$$\mathscr{O}_x \otimes_{\mathscr{O}(K)} (F_k\mathscr{M}(K) \cap \mathscr{N}(K)) \longrightarrow F_k\mathscr{M}_x \cap \mathscr{N}_x$$

is an isomorphism. This follows from the flatness of $\mathscr{O}_x$ over $\mathscr{O}(K)$ (see [**Fri67**]). □

Exercise 2.2.14

(1) Similarly, prove that if $\varphi : \mathscr{M} \to \mathscr{N}$ is a surjective morphism of coherent $\mathscr{D}_X$-modules and if $F_\bullet\mathscr{M}$ is good, then $F_\bullet\mathscr{N} := \varphi(F_\bullet\mathscr{M})$ is good as well.

(2) Let $\mathscr{M}$ be a good $\mathscr{D}_X$-module (see Definition 2.2.6). Show that any coherent $\mathscr{D}_X$-submodule and any coherent quotient $\mathscr{D}_X$-module of $\mathscr{M}$ is good.

2.3. Support

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Being a sheaf on X, $\mathscr{M}$ has a support $\operatorname{Supp}\mathscr{M}$, which is the closed subset complement to the set of $x \in X$ in the neighbourhood of which $\mathscr{M}$ is zero. Recall that the support of a coherent $\mathscr{O}_X$-module is a closed analytic subset of X. Such a property extends to coherent $\mathscr{D}_X$-modules:

Proposition 2.3.1. *The support* $\operatorname{Supp}\mathscr{M}$ *of a coherent* $\mathscr{D}_X$*-module* $\mathscr{M}$ *is a closed analytic subset of* X*.*

Proof. The property of being an analytic subset being local, we may assume that $\mathscr{M}$ is generated over $\mathscr{D}_X$ by a coherent $\mathscr{O}_X$-submodule $\mathscr{F}$ (see Exercise 2.2.5(3)). Then the support of $\mathscr{M}$ is equal to the support of $\mathscr{F}$. □

Let $Y \subset X$ be a complex submanifold. The following is known as “Kashiwara's equivalence”.

Proposition 2.3.2. *There is a natural equivalence between coherent* $\mathscr{D}_X$*-modules supported on* Y *and coherent* $\mathscr{D}_Y$*-modules.*

Proof. We will prove this in the special case where X is an open set in $\mathbb{C}^n$ with coordinates $x_1, \dots, x_n$ and Y is defined by $x_n = 0$. Given a coherent $\mathscr{D}_X$-module $\mathscr{M}$ supported on Y, we set $\mathscr{N} := \operatorname{Ker}[x_n \cdot : \mathscr{M} \to \mathscr{M}]$: this is a $\mathscr{D}_Y$-module. We also set $\mathscr{N}[\partial_{x_n}] := \mathscr{N} \otimes_{\mathbb{C}} \mathbb{C}[\partial_{x_n}]$: this is a $\mathscr{D}_X$-module by the following rule; let $f(x_1, \dots, x_n) = \sum_k f_k(x_1, \dots, x_{n-1}) x_n^k$ be a holomorphic function and n be a section of $\mathscr{N}$; then we set

$$f \cdot (n \otimes \partial_j) = \sum_{k \leqslant j} \frac{(-1)^k j!}{(j-k)!} \, (f_k \cdot n) \otimes \partial_{x_n}^{j-k}.$$

We first claim that $\mathscr{M} = \mathscr{N}[\partial_{x_n}]$. Indeed, let m be a local section of $\mathscr{M}$. We will prove that it decomposes uniquely as $\sum_j \otimes \partial_{x_n}^j n_j$ for some local sections n_j of $\mathscr{N}$. The section m generates an $\mathscr{O}_X$-submodule of $\mathscr{M}$, which is finitely generated, hence coherent since $\mathscr{M}$ is coherent, and is supported on Y. Therefore, locally, there exists ℓ such that $x_n^{\ell+1} m = 0$. Then

$$0 = \partial_{x_n}(x_n^{\ell+1} m) = (\ell+1) x_n^\ell m + x_n^{\ell+1} \partial_{x_n} m = x_n^\ell \big((\ell+1) + x_n \partial_{x_n}\big) m = x_n^\ell (\ell + \partial_{x_n} x_n) m,$$

so $m = m_1 + \partial_{x_n} m_2$, $m_1 = (\ell + \partial_{x_n} x_n) m/\ell$, $m_2 = -x_n m/\ell$, with $x_n^\ell m_1 = x_n^\ell m_2 = 0$. This gives the existence by decreasing induction on ℓ. Uniqueness is obtained similarly. It remains to be proved that the natural $\mathscr{D}_Y$-linear morphism $\mathscr{N}[\partial_{x_n}] \to \mathscr{M}$ defined by $\sum_j n_j \otimes \partial_{x_n}^j \mapsto \sum_j \partial_{x_n}^j n_j$ is a $\mathscr{D}_X$-linear isomorphism, which is straightforward.

Lastly, the proof that $\mathscr{N}$ is $\mathscr{D}_Y$-coherent is obtained by using the coherence criterion given by Theorem 2.2.9. □

2.4. Characteristic variety

The support is usually not the right geometric object attached to a $\mathscr{D}_X$-module $\mathscr{M}$, as it does not provide enough information on $\mathscr{M}$. A finer object is the *characteristic variety* that we introduce below. The following lemma will justify its definition.

Lemma 2.4.1. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then there exists a coherent sheaf $\mathscr{I}(\mathscr{M})$ of ideals of $\operatorname{gr}^F\mathscr{D}_X$ such that, for any open set U of X and any good filtration $F_\bullet\mathscr{M}_{|U}$, we have $\mathscr{I}(\mathscr{M})_{|U} = \operatorname{Rad}(\operatorname{ann}_{\operatorname{gr}^F\mathscr{D}_U}\operatorname{gr}^F\mathscr{M}_{|U})$.*

We denote by $\operatorname{Rad}(I)$ the radical of the ideal I and by ann the annihilator ideal of the corresponding module. Hence, for any $x \in U$, we have

$$\operatorname{Rad}(\operatorname{ann}_{\operatorname{gr}^F\mathscr{D}_{X,x}}\operatorname{gr}^F\mathscr{M}_x) = \{\varphi \in \operatorname{gr}^F\mathscr{D}_{X,x} \mid \exists\,\ell,\ \varphi^\ell\operatorname{gr}^F\mathscr{M}_x = 0\}.$$

Proof. It is a matter of showing that, if $F_\bullet\mathscr{M}_{|U}$ and $G_\bullet\mathscr{M}_{|U}$ are two good filtrations, then the corresponding ideals coincide. Notice first that these ideals are homogeneous, i.e., if φ belongs to the ideal, then so does any homogeneous component of φ. Let φ be a homogeneous element of degree j in the ideal corresponding to $F_\bullet\mathscr{M}$ and let $\widetilde{\varphi}$ be a lifting of φ in $F_j\mathscr{D}_X$. Then, locally, there exists ℓ such that, for any k, we have $\widetilde{\varphi}^\ell F_k\mathscr{M} \subset F_{k+j\ell-1}\mathscr{M}$ and thus, for any $p \geqslant 0$,

$$\widetilde{\varphi}^{(p+1)\ell}F_k\mathscr{M} \subset F_{k+j(p+1)\ell-p-1}\mathscr{M}.$$

Taking k_0 as in Exercise 2.2.3, associated to $F_\bullet\mathscr{M}, G_\bullet\mathscr{M}$, we have

$$\begin{aligned}\widetilde{\varphi}^{(2k_0+1)\ell}G_k\mathscr{M} \subset \widetilde{\varphi}^{(2k_0+1)\ell}F_{k+k_0}\mathscr{M} &\subset F_{k+k_0+j(2k_0+1)\ell-2k_0-1}\mathscr{M}\\ &\subset G_{k+2k_0+j(2k_0+1)\ell-2k_0-1}\mathscr{M}\\ &= G_{k+j(2k_0+1)\ell-1}\mathscr{M}.\end{aligned}$$

This shows that, by setting $\ell' = (2k_0+1)\ell$, $\widetilde{\varphi}^{\ell'}G_k\mathscr{M} \subset G_{k+j\ell'-1}\mathscr{M}$, and thus φ is in the ideal corresponding to $G_\bullet\mathscr{M}$. By a symmetric argument, we find that both ideals are identical. ☐

Notice that we consider the radicals of the annihilator ideals, and not these annihilator ideals themselves, because of the shift k_0. In fact, the annihilator ideals may not be equal, as shown by the following example.

Exercise 2.4.2. Let t be a coordinate on $\mathbb{C}$ and set $\mathscr{M} = \mathscr{O}_\mathbb{C}(*0)/\mathscr{O}_\mathbb{C}$. Consider the two elements $m_1 = [1/t]$ and $m_2 = [1/t^2]$, where $[\bullet]$ denotes the class modulo $\mathscr{O}_\mathbb{C}$. Show that the good filtrations generated respectively by m_1 and m_2 do not give rise to the same annihilator ideals.

***Definition 2.4.3* (Characteristic variety).** The *characteristic variety* $\operatorname{Char}\mathscr{M}$ is the subset of the cotangent space T^*X defined by the ideal $\mathscr{I}(\mathscr{M})$.

Locally, given any good filtration of $\mathscr{M}$, the characteristic variety is defined as the set of common zeros of the elements of $\operatorname{ann}_{\operatorname{gr}^F\mathscr{D}_X}\operatorname{gr}^F\mathscr{M}$.

Assume that $\mathscr{M}$ is the quotient of $\mathscr{D}_X$ by the left ideal $\mathscr{J}$. Then one may choose for $F_\bullet\mathscr{M}$ the filtration induced by $F_\bullet\mathscr{D}_X$, so that $\operatorname{Char}\mathscr{M}$ is the locus of common zeros of the elements of $\operatorname{gr}^F\mathscr{J}$. In general, finding generators of $\operatorname{gr}^F\mathscr{J}$ from generators of $\mathscr{J}$ needs the use of Gröbner bases.

In local coordinates $x_1,\dots,x_n$, denote by $\xi_1,\dots,\xi_n$ the complementary symplectic coordinates in the cotangent space. Then $\operatorname{gr}^F\mathscr{J}$ is generated by a finite set of homogeneous elements $a_\alpha(x)\xi^\alpha$, where α belongs to a finite set of multi-indices. Hence the homogeneity of the ideal $\mathscr{I}(\mathscr{M})$ implies that

$$\operatorname{Supp}\mathscr{M}=\pi(\operatorname{Char}\mathscr{M})=\operatorname{Char}\mathscr{M}\cap T^*_XX, \tag{2.4.4}$$

where $\pi: T^*X\to X$ denotes the bundle projection and T^*_XX denotes the zero section of the cotangent bundle.

Exercise 2.4.5. Let $0\to\mathscr{M}'\to\mathscr{M}\to\mathscr{M}''\to 0$ be an exact sequence of $\mathscr{D}_X$-modules. Show that $\operatorname{Char}\mathscr{M}=\operatorname{Char}\mathscr{M}'\cup\operatorname{Char}\mathscr{M}''$. [Hint: take a good filtration on $\mathscr{M}$ and induce it on $\mathscr{M}'$ and $\mathscr{M}''$.]

Exercise 2.4.6 **(Coherent $\mathscr{D}_X$-modules with characteristic variety T^*_XX)**

Recall that a local section m of a left $\mathscr{D}_X$-module $\mathscr{M}$ is said to be horizontal if $\nabla m=0$, i.e., in local coordinates, forall $i\in\mathbb{N}$, $(\partial/\partial x_i)m=0$. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module such that $\operatorname{Char}\mathscr{M}=T^*_XX$. Show that

(1) for every $x\in X$, $\mathscr{M}_x$ is an $\mathscr{O}_{X,x}$-module of finite type;

(2) $\mathscr{M}_x$ is therefore free over $\mathscr{O}_{X,x}$;

(3) $\mathscr{M}_x$ has an $\mathscr{O}_{X,x}$-basis made of horizontal sections;

(4) $\mathscr{M}$ is locally isomorphic, *as a $\mathscr{D}_X$-module*, to $\mathscr{O}_X^d$ for some d.

Exercise 2.4.7 **(Coherent $\mathscr{D}_X$-modules with characteristic variety contained in T^*_YX)**

Let $i: Y\hookrightarrow X$ be the inclusion of a smooth codimension p closed submanifold. Define the p-th algebraic local cohomology with support in Y by $R^p\Gamma_{[Y]}\mathscr{O}_X=\varinjlim_k\mathscr{E}xt^p(\mathscr{O}_X/\mathscr{J}_Y^k,\mathscr{O}_X)$, where $\mathscr{J}_Y$ is the ideal defining Y. $R^p\Gamma_{[Y]}\mathscr{O}_X$ has a natutal structure of $\mathscr{D}_X$-module. In local coordinates $(x_1,\dots,x_n)$ where Y is defined by $x_1=\cdots=x_p=0$, we have

$$R^p\Gamma_{[Y]}\mathscr{O}_X\simeq\frac{\mathscr{O}_{\mathbb{C}^n}[1/x_1\cdots x_n]}{\sum_{i=1}^p\mathscr{O}_{\mathbb{C}^n}(x_i/x_1\cdots x_n)}.$$

Denote this $\mathscr{D}_X$-module by $\mathscr{B}_YX$.

(1) Show that $\mathscr{B}_YX$ has support contained in Y and characteristic variety equal to T^*_YX.

(2) Identify $\mathscr{B}_YX$ with $i_+\mathscr{O}_Y$.

(3) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module with characteristic variety equal to $T^*_Y X$. Show that $\mathscr{M}$ is locally isomorphic to $(\mathscr{B}_Y X)^d$ for some d.

2.5. Involutiveness of the characteristic variety

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, $\operatorname{Char}\mathscr{M} \subset T^*X$ its characteristic variety and $\operatorname{Supp}\mathscr{M}$ its support. For $(x,0) \in T^*X$, we denote by $\dim_{(x,0)} \operatorname{Char}\mathscr{M}$ the dimension at $(x,0)$ of the analytic space $\operatorname{Char}\mathscr{M}$.

Proposition 2.5.1. *Let $\mathscr{M}$ be a nonzero coherent $\mathscr{D}_X$-module. Then, for any $x \in X$, $\dim_{(x,0)} \operatorname{Char}\mathscr{M} \geqslant \dim X$.*

This inequality is called *Bernstein's inequality.*

Proof. We can assume that $X = \mathbb{C}^n$. The proposition is proved by induction on $\dim X$. If $\operatorname{Supp}\mathscr{M}$ is n-dimensional, the inequality is obvious. Then, it is enough to prove the proposition for every x in the smooth part of $\operatorname{Supp}\mathscr{M}$. Therefore, we have to consider the case where $\operatorname{Supp}\mathscr{M}$ is contained in the hypersurface $x_n = 0$. The proposition follows from Kashiwara's equivalence of categories between coherent $\mathscr{D}_{\mathbb{C}^n}$-modules supported on $x_n = 0$ and coherent $\mathscr{D}_{\mathbb{C}^{n-1}}$-modules (Proposition 2.3.2) (for the details, see [**GM93**, p. 129]). ☐

But there exists a more precise result. In order to state it, consider on T^*X the fundamental 2-form ω. In local coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$, it is written $\omega = \sum_{1=1}^n d\xi_i \wedge dx_i$. For any $(x,\xi) \in T^*X$, ω defines on $T_{(x,\xi)}(T^*X)$ a nondegenerate bilinear form. We denote by $E^\perp$ the orthogonal space in the sense of ω of the vector subspace E of $T_{(x,\xi)}(T^*X)$. Recall that if V is a reduced analytic subspace of T^*X, with smooth part V_0,

- V is said to be *isotropic* if, for any $a \in V_0$, we have $T_aV \subset (T_aV)^\perp$,
- V is said to be *involutive* (or *co-isotropic*) if, for any $a \in V_0$, we have $(T_aV)^\perp \subset T_aV$,
- V is said to be *Lagrangean* if, for any $a \in V_0$, we have $(T_aV)^\perp = T_aV$.

We observe that if V is involutive, the dimension of any irreducible components of V is bigger than $\dim X$.

Exercise 2.5.2. Let V be a reduced analytic subspace of T^*X of pure dimension $\dim V$.

(1) Show that if $\dim V = 1$, then V is isotropic.

(2) Show that if $\operatorname{codim} V = 1$, then V is involutive. (Hint: for $E \subset T_{(x,\xi)}(T^*X)$ of codimension 1, show that $E^\perp$ has dimension 1, and thus $E^\perp \subset E^{\perp\perp} = E$.)

Theorem 2.5.3. *Let $\mathscr{M}$ be a nonzero coherent $\mathscr{D}_X$-module. Then* $\operatorname{Char}\mathscr{M}$ *is an involutive set in T^*X.* ☐

The first proof has been given by Sato, Kawai, Kashiwara [**SKK73**]. Next, Malgrange gave a very simple proof in a seminar Bourbaki talk ([**Mal78**], see also [**GM93**, p. 165]). And finally, Gabber gave the proof of a general algebraic version of this theorem (see [**Gab81**], see also [**Bjö93**, p. 473]).

A consequence is that any irreducible component of the characteristic variety of a coherent $\mathscr{D}_X$-module has a dimension $\geqslant \dim X$. On the other hand, we can get homological consequences of this result by using the homological theory of dimension.

Let $\mathscr{M}$ be a $\mathscr{D}_X$-module coherent and $x \in X$ and let $F_\bullet\mathscr{M}$ a local good filtration of $\mathscr{M}$. The dimension of the characteristic variety $\operatorname{Char}\mathscr{M}$ at $x \in X$ can be determined with $\operatorname{gr}^F\mathscr{M}$. Let $(x,\xi) \in \operatorname{Char}\mathscr{M}$ and let $\mathfrak{m}_{(x,\xi)}$ be the maximal ideal defining (x,ξ). For d sufficiently large, $\dim \operatorname{gr}^F\mathscr{M}/\mathfrak{m}^d_{(x,\xi)}$ is a polynomial in d. Let $d(x,\xi)$ be its degree. We have

$$\dim_x \operatorname{Char}\mathscr{M} = \sup\{d(x,\xi) \mid (x,\xi) \in T^*X\}.$$

Then, the following results can be proved using algebraic properties of $\operatorname{gr}^F\mathscr{D}_X$ (see e.g. [**Bjö79, GM93**]).

Proposition 2.5.4. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. We have*

$$\mathscr{E}xt^i_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}_X) = 0 \quad \textit{for } i \geqslant n+1. \qquad \square$$

Theorem 2.5.5. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module and $x \in \operatorname{Supp}\mathscr{M}$. Then*

$$2n - \dim_x \operatorname{Char}\mathscr{M} = \inf\{i \in \mathbb{N} \mid \mathscr{E}xt^i_{\mathscr{D}_{X,x}}(\mathscr{M}_x,\mathscr{D}_{X,x}) = 0\}. \qquad \square$$

Another useful consequence of the homology theory of the dimension is the following proposition:

Proposition 2.5.6. *Let $\mathscr{M}$ a coherent $\mathscr{D}_X$-module. Then, the $\mathscr{D}_X$-submodule of $\mathscr{M}$ consisting of local sections m such that $\dim \mathscr{D}_X m \leqslant k$ is coherent.* ☐

2.6. Non-characteristic restrictions

Let $i : Y \hookrightarrow X$ denote the inclusion of a closed submanifold with ideal $\mathscr{I}_Y$ (in local coordinates $(x_1,\dots,x_n)$, $\mathscr{I}_Y$ is generated by $x_1,\dots,x_p$, where $p = \operatorname{codim} Y$). A local section ξ of $i^{-1}\Theta_X$ (vector field on X, considered at points of Y only; we denote by i^{-1} the sheaf-theoretic pull-back) is said to be tangent to Y if, for any local section f of $\mathscr{I}_Y$, $\xi(f) \in \mathscr{I}_Y$. This defines a subsheaf $\Theta_{X|Y}$ of $i^{-1}\Theta_X$. Then $\Theta_Y = \mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_X} \Theta_{X|Y} = i^*\Theta_{X|Y}$ is a subsheaf of $i^*\Theta_X$.

Given a left $\mathscr{D}_X$-module, the action of $i^{-1}\Theta_X$ on $i^{-1}\mathscr{M}$ restricts to an action of Θ_Y on $i^*\mathscr{M} = \mathscr{O}_Y \otimes_{i^{-1}\mathscr{O}_X} i^{-1}\mathscr{M}$. The criterion of Lemma 1.2.1 is fulfilled since it is fulfilled for Θ_X and $\mathscr{M}$, defining therefore a left $\mathscr{D}_Y$-module structure on $i^*\mathscr{M}$. We denote this left $\mathscr{D}_Y$-module by $i^+\mathscr{M}$.

Without any other assumption, coherence is not preserved by i^+. For example, $i^+\mathscr{D}_X$ is not $\mathscr{D}_Y$-coherent if $\operatorname{codim} Y \geqslant 1$. A criterion for coherence of the pull-back is given below.

The cotangent map to the inclusion defines a natural bundle morphism

$$\varpi : T^*X_{|Y} \longrightarrow T^*Y,$$

the kernel of which is by definition the conormal bundle $T^*_Y X$ of Y in X.

Lemma-Definition 2.6.1 **(Non-characteristic property).** We say that Y is *non-characteristic* with respect to the coherent $\mathscr{D}_X$-module $\mathscr{M}$ if one of the following equivalent conditions is satisfied:

- $T^*_Y X \cap \operatorname{Char} \mathscr{M} \subset T^*_X X$,
- $\varpi : \operatorname{Char} \mathscr{M}_{|Y} \to T^*Y$ is finite, i.e., proper with finite fibres.

Exercise 2.6.2. Show that both conditions in Definition 2.6.1 are indeed equivalent. (Hint: use the homogeneity property of $\operatorname{Char} \mathscr{M}$.)

Theorem 2.6.3 **(Coherence of non-characteristic restrictions)**

Assume that $\mathscr{M}$ is $\mathscr{D}_X$-coherent and that Y is non-characteristic with respect to $\mathscr{M}$. Then $i^+\mathscr{M}$ is $\mathscr{D}_Y$-coherent and $\operatorname{Char} i^+\mathscr{M} \subset \varpi(\operatorname{Char} \mathscr{M}_{|Y})$.

Sketch of proof. The question is local near a point $x \in Y$. We may therefore assume that $\mathscr{M}$ has a good filtration $F_\bullet \mathscr{M}$.

(1) Set $F_k i^+\mathscr{M} = \operatorname{image}[i^* F_k \mathscr{M} \to i^*\mathscr{M}]$. Then, using Exercise 2.2.11(2), one shows that $F_\bullet i^+\mathscr{M}$ is a good filtration with respect to $F_\bullet i^+\mathscr{D}_X$.

(2) The module $\operatorname{gr}^F i^+\mathscr{M}$ is a quotient of $i^*\operatorname{gr}^F\mathscr{M}$, hence its support is contained in $\operatorname{Char} \mathscr{M}_{|Y}$. By Remmert's Theorem, it is a coherent $\operatorname{gr}^F\mathscr{D}_Y$-module.

(3) The filtration $F_\bullet i^+\mathscr{M}$ is thus a good filtration of the $\mathscr{D}_Y$-module $i^+\mathscr{M}$. By Exercise 2.2.5(1), $i^+\mathscr{M}$ is $\mathscr{D}_Y$-coherent. Using the good filtration above, it is clear that $\operatorname{Char} i^+\mathscr{M} \subset \varpi(\operatorname{Char} \mathscr{M}_{|Y})$. $\square$

LECTURE 3

DIFFERENTIAL COMPLEXES AND LOCAL DUALITY

3.1. Introduction

This lecture relies on [**Sai89a**]. The de Rham functor is a functor between two very different derived categories, that of $\mathscr{D}_X$-modules and that of sheaves of $\mathbb{C}$-vector spaces. This makes complicated checking compatibility of various functors with the de Rham functor. The idea of differential complexes is to replace the derived category of $\mathscr{D}_X$-modules with a category that looks like that of sheaves of $\mathbb{C}$-vector spaces: this is the category of differential complexes, that is, the category complexes whose terms are $\mathscr{O}_X$-modules and differentials are differential operators. Passing from a $\mathscr{D}_X$-module to a differential complex is best seen if one starts from an *induced $\mathscr{D}_X$-module*, that is, a right $\mathscr{D}_X$-module of the form $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ for some $\mathscr{O}_X$-module $\mathscr{L}$. The associated differential complex is simply $\mathscr{L}$ in degree zero, with differential equal to zero. The *differential de Rham functor* ${}^{\mathrm{diff}}\mathrm{DR}$ induces an equivalence between bounded derived categories of $\mathscr{D}_X$-modules and differential complexes. It has an explicit quasi-inverse functor ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$, which takes values in complexes of induced $\mathscr{D}_X$-modules. In that way, the composed functor ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}$ replaces a bounded complex of $\mathscr{D}_X$-modules with an isomorphic complex whose terms are induced $\mathscr{D}_X$-modules.

Before starting the main course, let us illustrate the previous considerations on the case of vector bundles with flat connection, as an appetizer.

Let $(\mathscr{V}, \nabla)$ be a vector bundle of finite rank on X with a flat connection, that we also regard as a left $\mathscr{D}_X$-module, then simply denoted by $\mathscr{V}$. We also denote by ∇ the differentials $\Omega_X^k \otimes \mathscr{V} \to \Omega_X^{k+1} \otimes \mathscr{V}$. On the other hand, let ∇ also denote the connection on $\mathscr{D}_X$ regarded as a left $\mathscr{D}_X$-module.

Proposition 3.1.1. *The complex*

$$(3.1.1\,*)\qquad \cdots \longrightarrow \Omega_X^k \otimes \mathscr{V} \otimes \mathscr{D}_X \xrightarrow{\nabla \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla} \Omega_X^{k+1} \otimes \mathscr{V} \otimes \mathscr{D}_X \longrightarrow \cdots,$$

where we put the term $\Omega_X^n \otimes \mathscr{V} \otimes \mathscr{D}_X = \omega_X \otimes \mathscr{V} \otimes \mathscr{D}_X$ in degree zero, is a complex of right $\mathscr{D}_X$-modules when $\mathscr{D}_X$ is equipped with its right $\mathscr{D}_X$-module structure, and is a resolution of the right $\mathscr{D}_X$-module $\mathscr{V}^{\mathrm{right}} = \omega_X \otimes \mathscr{V}$ by locally free right $\mathscr{D}_X$-modules.

This proposition is instrumental when comparing operations on vector bundles with flat connection with the similar operations on $\mathscr{D}_X$-modules.

Proof. Let us make precise the augmentation morphism $\omega_X \otimes \mathscr{V} \otimes \mathscr{D}_X \to \omega_X \otimes \mathscr{V}$: it is defined, for local sections ω, v, P of each sheaf, by $\omega \otimes v \otimes P \mapsto (\omega \otimes v) \cdot P$.

The question is local, so that we will work with germs, and with a local coordinate system $(x_1, \dots, x_n)$. For $J \subset \{1, \dots, n\}$, let $\mathscr{D}^{(J)}$ denote the subring of $\mathscr{D}$ consisting of germs of differential operators not containing ∂_{x_j} for $j \in J$. We represent the complex (3.1.1 $*$) as the simple complex associated to the n-cube with vertices $\mathscr{V} \otimes \mathscr{D}$ and all arrows in the i-th direction equal to $\partial_{x_i} \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_{x_i}$. By a straightforward induction, it is enough to prove that for each $J \subset \{1, \dots, n\}$ and each $i \notin j$, the morphism

$$\mathscr{V} \otimes \mathscr{D}^{(J)} \xrightarrow{\partial_{x_i} \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_{x_i}} \mathscr{V} \otimes \mathscr{D}^{(J)} \tag{3.1.2}$$

is injective and has cokernel isomorphic to $\mathscr{V}' := \mathscr{V} \otimes \mathscr{D}^{(J \cup \{i\})}$ with right action of ∂_{x_i} induced by the left action of $-\partial_{x_i}$ on $\mathscr{V}'$. We write $\mathscr{V} \otimes \mathscr{D}^{(J)} = \bigoplus_{k \geqslant 0} \mathscr{V}' \otimes \partial_{x_i}^k$ and

$$(\partial_{x_i} \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_{x_i}) \sum_k v'_k \otimes \partial_{x_i}^k = \sum_{k \geqslant 0} (\partial_{x_i} v'_k + v'_{k-1}) \otimes \partial_{x_i}^k.$$

Injectivity is then clear. The morphism $\bigoplus_{k \geqslant 0} \mathscr{V}' \otimes \partial_{x_i}^k \to \mathscr{V}'$ defined by

$$\sum_k v'_k \otimes \partial_{x_i}^k \longmapsto \sum_k (-\partial_{x_i})^k v'_k$$

identifies the cokernel of (3.1.2) with $\mathscr{V}$, and the formula above shows that the right action of ∂_{x_i} induces the left action of $-\partial_{x_i}$ on $\mathscr{V}'$. ☐

3.2. Induced $\mathscr{D}$-modules and differential morphisms

3.2.a. Right induced $\mathscr{D}$-modules.

Let $\mathscr{L}$ be an $\mathscr{O}_X$-module. It induces a *right* $\mathscr{D}_X$-module $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ (the left $\mathscr{O}_X$-module structure of $\mathscr{D}_X$ is used for the tensor product).

Remark 3.2.1. We note that $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ has two structures of $\mathscr{O}_X$-module, one on the left and one on the right, and they do not coincide. We will mainly use the right one. The "left" $\mathscr{O}_X$-module structure on $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ will only be used when noticing that some naturally defined sheaves of $\mathbb{C}$-vector spaces are in fact sheaves of $\mathscr{O}_X$-modules.

When we need to distinguish both structures, we denote them by $(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X)^{\mathrm{left}}$ and $(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X)^{\mathrm{right}}$, and in general $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ will mean $(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X)^{\mathrm{right}}$.

The category $\mathsf{M}_{\mathrm{i}}(\mathscr{D}_X)$ of right induced differential modules is the full subcategory of $\mathsf{M}(\mathscr{D}_X)$ (right $\mathscr{D}_X$-modules) consisting of induced $\mathscr{D}_X$-modules (i.e., we consider as morphisms all $\mathscr{D}_X$-linear morphisms).

There is a natural $\mathscr{O}_X$-linear morphism (with the right structure on the right-hand term)

$$\mathscr{L} \longrightarrow \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \quad \ell \longmapsto \ell \otimes 1.$$

There is also a (only) $\mathbb{C}$-linear morphism

$$\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \mathscr{L} \tag{3.2.2}$$

defined at the level of local sections by $\ell \otimes P \mapsto P(1)\ell$, where $P(1)$ is the result of the action of the differential operator P on 1, which is equal to the degree 0 coefficient of P if P is locally written as $\sum_\alpha a_\alpha(x)\partial_x^\alpha$. In an intrinsic way, consider the natural augmentation morphism $\mathscr{D}_X \to \mathscr{O}_X$, which is left $\mathscr{D}_X$-linear, hence left $\mathscr{O}_X$-linear; then apply $\mathscr{L} \otimes_{\mathscr{O}_X} \bullet$ to it.

Remark 3.2.3. Notice however that (3.2.2) is $\mathscr{O}_X$-linear on $(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X)^{\mathrm{left}}$.

For the right $\mathscr{D}_X$-module $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$, the de Rham complex ${}^{\mathrm{p}}\mathrm{DR}(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ is by definition the Spencer complex $\mathrm{Sp}_X^\bullet(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X)$. By using the left structure on $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$, one checks that this is a complex in the category of $\mathscr{O}_X$-modules (i.e., the terms are $\mathscr{O}_X$-modules and the differential is $\mathscr{O}_X$-linear). The proof of the following lemma is straightforward.

Lemma 3.2.4. *The Spencer complex* $\mathrm{Sp}_X^\bullet(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ *is a resolution of* $\mathscr{L}$ *as an* $\mathscr{O}_X$*-module and the morphism* (3.2.2) *is the augmentation morphism*

$$\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X = \mathrm{Sp}_X^0(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \longrightarrow \mathscr{L}. \qquad \square$$

The category $\mathsf{M}_{\mathrm{i}}(\mathscr{D}_X)$ is an additive category. One can associate to it the category $\mathsf{C}_{\mathrm{i}}^\star(\mathscr{D}_X)$ of complexes which are $\star$-bounded (i.e., for $\star = \varnothing, +, -, \mathrm{b}$, no condition, bounded from below, bounded from above, bounded), and the category $\mathsf{K}_{\mathrm{i}}^\star(\mathscr{D}_X)$ of complexes up to homotopy. The category $\mathsf{C}_{\mathrm{i}}^\star(\mathscr{D}_X)$ is a full subcategory of $\mathsf{C}^\star(\mathscr{D}_X)$. Note also that two morphisms in $\mathsf{C}_{\mathrm{i}}^\star(\mathscr{D}_X)$ are homotopic when regarded as morphisms in $\mathsf{C}^\star(\mathscr{D}_X)$ if and only if they are homotopic in $\mathsf{C}_{\mathrm{i}}^\star(\mathscr{D}_X)$. Therefore, $\mathsf{K}_{\mathrm{i}}^\star(\mathscr{D}_X)$ is also a full subcategory of $\mathsf{K}^\star(\mathscr{D}_X)$. Moreover, since the mapping cone of a morphism in $\mathsf{C}_{\mathrm{i}}^\star(\mathscr{D}_X)$ is equal to the mapping cone of this morphism considered in $\mathsf{C}^\star(\mathscr{D}_X)$, a triangle in $\mathsf{K}_{\mathrm{i}}^\star(\mathscr{D}_X)$ is distinguished if and only if it is distinguished when regarded as a triangle in $\mathsf{K}^\star(\mathscr{D}_X)$.

Since $\mathsf{M}(\mathscr{D}_X)$ is an abelian category, we have the usual definition of a *null system* $\mathscr{N}$ in $\mathsf{K}^\star(\mathscr{D}_X)$ (see [**KS90**, Def. 1.6.6] and Lemma 3.3.3 below for the definition of a null system in a triangulated category, and [**KS90**, (1.7.1)]): an object $K^\bullet$ belongs to $\mathscr{N}$ if $H^jK^\bullet = 0$ for all j. Inverting the associated multiplicative system $S(\mathscr{N})$ gives rise to the derived category $\mathsf{D}^\star(\mathscr{D}_X)$ (see [**KS90**, §1.6]).

Let us now define $\mathscr{N}_{\mathrm{i}}$ as the family of objects of $\mathsf{K}_{\mathrm{i}}^\star(\mathscr{D}_X)$ which belong to $\mathscr{N}$ when regarded as objects of $\mathsf{K}^\star(\mathscr{D}_X)$, that is, which are quasi-isomorphic to 0 in $\mathsf{K}^\star(\mathscr{D}_X)$. This clearly defines a null system in the triangulated category $\mathsf{K}_{\mathrm{i}}^\star(\mathscr{D}_X)$. Inverting the associated multiplicative system gives rise to the category that we denote by $\mathbf{D}^\star_{\mathrm{i}}(\mathscr{D}_X)$. By definition, there is a natural functor $\mathbf{D}^\star_{\mathrm{i}}(\mathscr{D}_X) \mapsto \mathbf{D}^\star(\mathscr{D}_X)$.

3.2.b. Differential morphisms. Let $\mathscr{L}, \mathscr{L}'$ be two $\mathscr{O}_X$-modules. A (right) $\mathscr{D}_X$-linear morphism

$$v : \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X \tag{3.2.5}$$

is uniquely determined by the $\mathscr{O}_X$-linear morphism

$$w : \mathscr{L} \longrightarrow \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X \tag{3.2.6}$$

that it induces (where the right $\mathscr{O}_X$-module structure is chosen on $\mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X$) by the formula, for any differential operator P and any local section ℓ of $\mathscr{L}$:

$$v(\ell \otimes P) = w(\ell) \cdot P.$$

In other words, the natural morphism

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X) \longrightarrow \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$$

is an isomorphism. We also have, at the sheaf level,

$$\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X) \xrightarrow{\sim} \mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X). \tag{3.2.7}$$

Notice that $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ is naturally equipped with an $\mathscr{O}_X$-module structure by using the left $\mathscr{O}_X$-module structure on $\mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X$ (see Remark 3.2.1), and similarly $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ is a $\Gamma(X, \mathscr{O}_X)$-module.

Now, w induces a $\mathbb{C}$-linear morphism

$$u : \mathscr{L} \longrightarrow \mathscr{L}', \tag{3.2.8}$$

by composition with (3.2.2): $\mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{L}'$. By Lemma 3.2.4, u is nothing but the morphism

$$\mathscr{H}^0({}^{\mathrm{p}}\mathrm{DR}(v)) : \mathscr{H}^0\big({}^{\mathrm{p}}\mathrm{DR}(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X)\big) \longrightarrow \mathscr{H}^0\big({}^{\mathrm{p}}\mathrm{DR}(\mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)\big).$$

To any such morphism u corresponds at most one v:

Lemma 3.2.9. *The morphism*

$$\begin{aligned} \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X) &\longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathscr{L}, \mathscr{L}') \\ v &\longmapsto u = \mathscr{H}^0({}^{\mathrm{p}}\mathrm{DR}(v)) \end{aligned}$$

is injective.

Proof. Recall that, for any multi-index β, we have $\partial_x^\alpha(x^\beta) = 0$ if $\beta_i < \alpha_i$ for some i, and $\partial_x^\alpha(x^\alpha) = \alpha!$. Assume that $u \equiv 0$. Let ℓ be a local section of $\mathscr{L}$ and, using local coordinates $(x_1, \dots, x_n)$, write in a unique way $w(\ell) = \sum_\alpha w(\ell)_\alpha \otimes \partial_x^\alpha$, where the sum is taken on multi-indices α, and w is as in (3.2.6). If $w(\ell) \neq 0$, let β be minimal (with respect to the usual partial ordering on $\mathbb{N}^n$) among the multi-indices α such that $w(\ell)_\alpha \neq 0$. Then $w(x^\beta\ell) = \sum_\alpha w(\ell)_\alpha \otimes \partial_x^\alpha x^\beta$ and the coefficient of order zero with respect to ∂_x is by definition $u(x^\beta\ell)$, so that

$$0 = u(x^\beta\ell) = \sum_\alpha \partial_x^\alpha(x^\beta)w(\ell)_\alpha = \beta! w(\ell)_\beta,$$

a contradiction. ☐

Definition 3.2.10 **(Differential operators between two $\mathscr{O}_X$-modules)**
The $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathrm{Diff}}(\mathscr{L},\mathscr{L}')$ of *differential operators from $\mathscr{L}$ to $\mathscr{L}'$* is the image of the *injective* morphism

$$\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X, \mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X) \xrightarrow{\mathscr{H}^0({}^{\mathrm{p}}\mathrm{DR}(\bullet))} \mathrm{Hom}_{\mathbb{C}}(\mathscr{L},\mathscr{L}').$$

Similarly we define the sheaf of $\mathbb{C}$-vector spaces $\mathscr{H}\!om_{\mathrm{Diff}}(\mathscr{L},\mathscr{L}')$ as the image of the *injective* morphism

$$\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X, \mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X) \xrightarrow{\mathscr{H}^0({}^{\mathrm{p}}\mathrm{DR}(\bullet))} \mathscr{H}\!om_{\mathbb{C}}(\mathscr{L},\mathscr{L}').$$

By using the left $\mathscr{O}_X$-module structure, $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)$ has a natural structure of $\mathscr{O}_X$-module, which can be transferred to its image $\mathscr{H}\!om_{\mathrm{Diff}}(\mathscr{L},\mathscr{L}')$ in $\mathscr{H}\!om_{\mathbb{C}}(\mathscr{L},\mathscr{L}')$. We note that $\mathscr{H}\!om_{\mathrm{Diff}}(\mathscr{L},\mathscr{L}')$ contains $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}')$: indeed, any $\mathscr{O}_X$-linear morphism $u:\mathscr{L}\to\mathscr{L}'$ is a differential operator from $\mathscr{L}$ to $\mathscr{L}'$ with corresponding v being $u\otimes 1$. However, $\mathscr{H}\!om_{\mathrm{Diff}}(\mathscr{L},\mathscr{L}')$ in general bigger than $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}')$.

Caveat 3.2.11. Both $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}')\otimes_{\mathscr{O}_X}\mathscr{D}_X$ and $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},(\mathscr{L}'\otimes\mathscr{D}_X)^{\mathrm{left}})$ have a right $\mathscr{D}_X$-module structure and a left $\mathscr{O}_X$-module structure, and the natural morphism

$$\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}')\otimes_{\mathscr{O}_X}\mathscr{D}_X \longrightarrow \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},(\mathscr{L}'\otimes\mathscr{D}_X)^{\mathrm{left}}),$$

is linear for both structures. However, we have used above $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}'\otimes\mathscr{D}_X) = \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},(\mathscr{L}'\otimes\mathscr{D}_X)^{\mathrm{right}})$ and, in general, there does not exist a morphism from $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}')\otimes_{\mathscr{O}_X}\mathscr{D}_X$ to $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}'\otimes\mathscr{D}_X)$. As a consequence, the subsheaf $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}')$ of $\mathscr{H}\!om_{\mathrm{Diff}}(\mathscr{L},\mathscr{L}')$ is *not* obtained as the image of

$$\mathscr{H}^0\,{}^{\mathrm{p}}\mathrm{DR}(\bullet) : \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}')\otimes_{\mathscr{O}_X}\mathscr{D}_X \longrightarrow \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}').$$

(See however Proposition 3.2.24 for the case where $\mathscr{L}'$ is a right $\mathscr{D}_X$-module.)

Examples 3.2.12

(1) Let us checks that $\mathscr{H}\!om_{\mathrm{Diff}}(\mathscr{O}_X,\mathscr{O}_X) = \mathscr{D}_X$. Using the notation u,v,w as above, for any holomorphic function φ we have $w(f) = w(1)\cdot f$ and $w(1)$ is a differential operator P. Then $u(f) = (P\cdot f)(1) = P(f)$, so that u is identified with the differential operator P.

(2) Let $\mathscr{L}$ be an $\mathscr{O}_X$-module and let $\nabla : \mathscr{L} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{L}$ be an integrable connection on $\mathscr{L}$. Then ∇ is a differential morphism, i.e., belongs to $\mathrm{Hom}_{\mathsf{Diff}}(\mathscr{L}, \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{L})$: this is seen by considering the right $\mathscr{D}_X$-linear morphism

$$v(\ell \otimes P) := \nabla(\ell) \otimes P + \ell \otimes \nabla(P),$$

for any local section ℓ of $\mathscr{L}$ and P of $\mathscr{D}_X$, and where ∇P is defined in Exercise 1.1.9. This extends to the connections ${}^{(k)}\nabla$.

Similarly, let $\mathscr{L}', \mathscr{L}''$ be $\mathscr{O}_X$-submodules of $\mathscr{L}$ such that ${}^{(k)}\nabla$ induces a $\mathbb{C}$-linear morphism ${}^{(k)}\nabla' : \Omega^k_X \otimes_{\mathscr{O}_X} \mathscr{L}' \to \Omega^{k+1}_X \otimes_{\mathscr{O}_X} \mathscr{L}''$. Then ${}^{(k)}\nabla'$ is a differential morphism.

Lemma 3.2.13. *The composition of differential morphisms is a differential morphism.*

Proof. We have to check that, under the composition

$$\mathrm{Hom}_{\mathbb{C}}(\mathscr{L}, \mathscr{L}') \times \mathrm{Hom}_{\mathbb{C}}(\mathscr{L}', \mathscr{L}'') \xrightarrow{\circ} \mathrm{Hom}_{\mathbb{C}}(\mathscr{L}, \mathscr{L}''),$$

the subspace $\mathrm{Hom}_{\mathsf{Diff}}(\mathscr{L}, \mathscr{L}') \times \mathrm{Hom}_{\mathsf{Diff}}(\mathscr{L}', \mathscr{L}'')$ is sent to $\mathrm{Hom}_{\mathsf{Diff}}(\mathscr{L}, \mathscr{L}'')$. This amounts to checking, with the notation above, that

$$\mathscr{H}^0\, {}^{\mathrm{p}}\mathrm{DR}(v' \circ v) = \mathscr{H}^0\, {}^{\mathrm{p}}\mathrm{DR}(v') \circ \mathscr{H}^0\, {}^{\mathrm{p}}\mathrm{DR}(v).$$

By using the notation as in Lemma 3.2.9, we have

$$v'v(\ell \otimes 1) = v'\Big(\sum_\alpha w(\ell)_\alpha \partial_x^\alpha\Big) = \sum_{\alpha,\beta} w'(w(\ell)_\alpha)_\beta \partial_x^{\alpha+\beta},$$

so that $\mathscr{H}^0\, {}^{\mathrm{p}}\mathrm{DR}(v' \circ v)(\ell \otimes 1) = w'(w(\ell)_0)_0$. On the other hand, $u(\ell) = w(\ell)_0$ and $u'u(\ell) = w'(w(\ell)_0)_0$. ☐

***Definition 3.2.14* (The category $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X)$).** We denote by $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X)$ the category whose objects are $\mathscr{O}_X$-modules and morphisms are $\mathrm{Hom}_{\mathsf{Diff}}(\mathscr{L}, \mathscr{L}')$ (this is justified by Lemma 3.2.13).

We note that $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X)$ is an additive category, i.e.,

- $\mathrm{Hom}_{\mathsf{Diff}}(\mathscr{L}, \mathscr{L}')$ is a $\mathbb{C}$-vector space and the composition is $\mathbb{C}$-bilinear,
- the $\mathscr{O}_X$-module 0 satisfies $\mathrm{Hom}_{\mathsf{Diff}}(0,0) = 0$,
- $\mathrm{Hom}_{\mathsf{Diff}}(\mathscr{L}_1 \oplus \mathscr{L}_2, \mathscr{L}') = \mathrm{Hom}_{\mathsf{Diff}}(\mathscr{L}_1, \mathscr{L}') \oplus \mathrm{Hom}_{\mathsf{Diff}}(\mathscr{L}_2, \mathscr{L}')$ and similarly with $\mathscr{L}'_1, \mathscr{L}'_2$.

Also, $\mathsf{M}(\mathscr{O}_X)$ is a subcategory of $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X)$, since any $\mathscr{O}_X$-linear morphism is a differential operator (of degree zero). It has the same objects but less morphisms. For the restriction of this inclusion functor to $\mathsf{M}(\mathscr{D}_X)$, it will be convenient to have a notation

$$\mathrm{can} : \mathsf{M}(\mathscr{D}_X) \longmapsto \mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X). \tag{3.2.15}$$

Note that any $\mathscr{D}_X$-linear morphism (on the left category) is regarded as a differential operator of degree zero (on the right category).

Exercise 3.2.16 **(A relative version of differential morphisms)**

Let $f : X \to Y$ be a holomorphic map between to complex manifolds. Let $\mathscr{L}, \mathscr{L}'$ be $\mathscr{O}_X$-modules.

(1) Define a $\mathbb{C}$-linear morphism $\mathscr{L} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y \to \mathscr{L}$ and a morphism

$$\mathrm{Hom}_{f^{-1}\mathscr{D}_Y}(\mathscr{L} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y, \mathscr{L}' \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathscr{L}, \mathscr{L}').$$

Define the space $\mathrm{Hom}_{\mathrm{Diff}_Y}(\mathscr{L}, \mathscr{L}')$ as the image of this morphism

(2) Show that the category $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_Y)$ is additive. Introduce the category $\mathsf{M}(f^{-1}\mathscr{O}_Y, \mathrm{Diff}_Y)$ and show that $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_Y)$ is a full subcategory of $\mathsf{M}(f^{-1}\mathscr{O}_Y, \mathrm{Diff}_Y)$.

(3) Using the projection formula (see [**KS90**, Prop. 2.5.13]), define a functor $f_!$ (direct image with proper support) from $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_Y)$ (or $\mathsf{M}(f^{-1}\mathscr{O}_Y, \mathrm{Diff}_Y)$) to $\mathsf{M}(\mathscr{O}_Y, \mathrm{Diff}_Y)$, sending $\mathscr{L}$ to $f_!\mathscr{L}$ (in the usual sheaf-theoretical sense).

3.2.c. The inverse de Rham functor. We will now show that the correspondence $\mathscr{L} \mapsto \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ induces a functor $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{M}_{\mathrm{i}}(\mathscr{D}_X)$. We will then compose with the inclusion $\mathsf{M}_{\mathrm{i}}(\mathscr{D}_X) \mapsto \mathsf{M}(\mathscr{D}_X)$. According to Lemma 3.2.9, the following definition is meaningful.

Definition 3.2.17 **(The inverse de Rham functor).** The functor

$$^{\mathrm{diff}}\mathrm{DR}^{\text{-}1} : \mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X) \longrightarrow \mathsf{M}_{\mathrm{i}}(\mathscr{D}_X)$$

is defined by $^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}(\mathscr{L}) = \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ and $^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}(u) = v$.

Definition 3.2.10 then reads

$$\begin{aligned} (3.2.18)\qquad & \mathscr{H}\!om_{\mathrm{Diff}}(\mathscr{L}, \mathscr{L}') \\ & \qquad = \mathrm{image}[\mathscr{H}\!om_{\mathscr{D}_X}(^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}\mathscr{L}, {}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}\mathscr{L}') \to \mathscr{H}\!om_{\mathbb{C}}(\mathscr{L}, \mathscr{L}')].\end{aligned}$$

Let $\mathscr{L}$ be an $\mathscr{O}_X$-module. Recall that $\mathscr{H}^k\big({}^{\mathrm{p}}\mathrm{DR}(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X)\big) = 0$ for $k \neq 0$ and $\mathscr{H}^0\big({}^{\mathrm{p}}\mathrm{DR}(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X)\big) = \mathscr{L}$ (Lemma 3.2.4). By definition, $\mathscr{H}^0\,{}^{\mathrm{p}}\mathrm{DR}$ is a functor $\mathsf{M}_{\mathrm{i}}(\mathscr{D}_X) \mapsto \mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X)$, that will be denoted by $^{\mathrm{diff}}\mathrm{DR}$.

Lemma 3.2.19. *The functor* $^{\mathrm{diff}}\mathrm{DR}^{\text{-}1} : \mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{M}_{\mathrm{i}}(\mathscr{D}_X)$ *is an equivalence of categories, a quasi-inverse functor being* $^{\mathrm{diff}}\mathrm{DR} : \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X \mapsto \mathscr{L}$, $^{\mathrm{diff}}\mathrm{DR}(v) = u$.

Proof. This follows from Lemma 3.2.9. ☐

Furthermore, the composed functor $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{M}_{\mathrm{i}}(\mathscr{D}_X) \mapsto \mathsf{M}(\mathscr{D}_X)$, still denoted by $^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}$, is *fully faithful*, i.e., it induces a bijective morphism

$$\mathrm{Hom}_{\mathrm{Diff}}(\mathscr{L}, \mathscr{L}') \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}_X}(^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}\mathscr{L}, {}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}\mathscr{L}').$$

(One may think that we "embed" the additive category $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X)$, which is non-abelian, in the abelian category $\mathsf{M}(\mathscr{D}_X)$; we will use this "embedding" to define below acyclic objects).

Remark 3.2.20. When considered as taking values in $\mathsf{M}(\mathscr{D}_X)$, the functor ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ is not, however, an equivalence of categories, i.e., is not essentially surjective. The reason is that, first, not all $\mathscr{D}_X$-modules are isomorphic to some $\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X$ and, next, its natural quasi-inverse is the de Rham functor ${}^{\mathrm{diff}}\mathrm{DR}$ which takes values in a category of complexes. Nevertheless, if one extends suitably these functors to categories of complexes, they become equivalences (see below Theorem 3.3.7).

3.2.d. Induced $\mathscr{D}_X$-modules from $\mathscr{D}_X$-modules. We give more details on the functor can of (3.2.15). Recall that, for a right $\mathscr{D}_X$-module $\mathscr{M}$, there are two natural $\mathscr{D}_X$-module structures on $\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_X$, denoted $(\mathrm{right})_{\mathrm{triv}}$ and $(\mathrm{right})_{\mathrm{tens}}$ that are isomorphic via the involution ι (see Exercise 1.2.9). We can write

$${}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathrm{can}(\mathscr{M})=(\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_X)_{\mathrm{triv}}.$$

On the other hand, the $\mathscr{O}_X$-module structure underlying $(\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_X)_{\mathrm{tens}}$ has been denoted $(\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_X)^{\mathrm{left}}$ when regarding $\mathscr{M}$ only as an $\mathscr{O}_X$-module. Remark 3.2.3 is more precise in the present setting.

Lemma 3.2.21. *The augmentation morphism* (3.2.2), $(\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_X)_{\mathrm{triv}}\to\mathscr{M}$*, becomes* $\mathscr{D}_X$*-linear when composed by* ι*, in other words, it is* $\mathscr{D}_X$*-linear when we equip* $\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_X$ *with its (right)*$_{\mathrm{tens}}$ *structure.*

Proof. In a local coordinate system, let us check for example that of

$$\big[(3.2.2)(m\otimes P)\big]\cdot\partial_{x_i}=(3.2.2)\big(m\partial_{x_i}\otimes P-m\otimes\partial_{x_i}P\big)=:(3.2.2)\big[(m\otimes P)\cdot_{\mathrm{tens}}\partial_{x_i}\big].$$

The left-hand side is equal to $(m\cdot P(1))\cdot\partial_{x_i}=(m\partial_{x_i})\cdot P(1)-m(\partial_{x_i}(P(1)))$ and the right-hand side is equal to $(m\partial_{x_i})\cdot P(1)-m(\partial_{x_i}P)(1))$. The conclusion follows from the equality $(\partial_{x_i}P)(1)=[\partial_{x_i},P](1)=\partial_{x_i}(P(1))$. ☐

The difficulty emphasized in Caveat 3.2.11 can be overcome when $\mathscr{L}'$ is a right $\mathscr{D}_X$-module, and not only an $\mathscr{O}_X$-module. Let us explain this. Let $\mathscr{L},\mathscr{L}'$ be $\mathscr{O}_X$-modules. There is a natural morphism $\mathscr{L}'\hookrightarrow\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X$ which is $\mathscr{O}_X$-linear for both $\mathscr{O}_X$-structures $(\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)^{\mathrm{right}}$ and $(\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)^{\mathrm{left}}$ (see Remark 3.2.1) and which induces $\mathscr{O}_X$-linear morphisms

$$\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}')\longrightarrow\begin{cases}\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},(\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)^{\mathrm{left}})\\ \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},(\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)^{\mathrm{right}})\end{cases}\tag{3.2.22}$$

On the other hand, $\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},(\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)^{\mathrm{left}})$ is naturally equipped with a right $\mathscr{D}_X$-module structure induced by that on $(\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)^{\mathrm{right}}$, and that we denote by $\big[\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},(\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)^{\mathrm{left}})\big]_{\mathrm{triv}}$. Therefore, there is a natural $\mathscr{D}_X$-linear morphism

$$\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}')\otimes_{\mathscr{O}_X}\mathscr{D}_X\longrightarrow\big[\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{L},(\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)^{\mathrm{left}})\big]_{\mathrm{triv}}.\tag{3.2.23}$$

As indicated in Caveat 3.2.11, the identification (3.2.7) uses $(\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)^{\mathrm{right}}$. However, when $\mathscr{L}'$ is a right $\mathscr{D}_X$-module, the involution ι considered in Exercise 1.2.9 can be used to circumvent this difficulty.

For any $\mathscr{O}_X$-module $\mathscr{L}$, the sheaf $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}, (\mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)^{\mathrm{right}})$ is equipped with the right $\mathscr{D}_X$-module structure coming from $(\mathrm{right})_{\mathrm{tens}}$ on $\mathscr{L}' \otimes \mathscr{D}_X$. Then

$$\mathscr{L} \longmapsto \mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}, (\mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{triv}})_{\mathrm{tens}}$$

defines a contravariant functor $\mathbf{M}_{\mathrm{i}}(\mathscr{D}_X) \mapsto \mathbf{M}(\mathscr{D}_X)$. This functor can be extended as a functor $\mathbf{C}^\star_{\mathrm{i}}(\mathscr{D}_X) \mapsto \mathbf{C}^\star(\mathscr{D}_X)$. The right-hand side also reads

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X, (\mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{triv}})_{\mathrm{tens}}.$$

Proposition 3.2.24. *There exists a natural bi-functorial morphism of right $\mathscr{D}_X$-modules, for $\mathscr{L} \in \mathbf{M}(\mathscr{O}_X)$ and $\mathscr{L}' \in \mathbf{M}(\mathscr{D}_X)$:*

$$\begin{aligned}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}') \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow {} & \big[\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, (\mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{triv}})\big]_{\mathrm{tens}} \\ & = \big[\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X, (\mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{triv}})\big]_{\mathrm{tens}},\end{aligned}$$

and the image of the composed morphism

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}') \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, (\mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{triv}}) \longrightarrow \mathscr{H}om_{\mathbb{C}}(\mathscr{L}, \mathscr{L}')$$

is equal to $\mathscr{H}^0\,{}^{\mathrm{p}}\mathrm{DR}[\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}') \otimes_{\mathscr{O}_X} \mathscr{D}_X]$, that is, $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}')$.

Proof. The natural morphism (3.2.23) is a $\mathscr{D}_X$-linear morphism

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}') \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \big[\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, (\mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{tens}})\big]_{\mathrm{triv}},$$

and the involution ι changes the right-hand side to the desired one.

We check the second assertion locally. Let $v = \sum_\alpha v_\alpha \otimes \partial_x^\alpha$, where v_α is a local section of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}')$. Its image in $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, (\mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{tens}})_{\mathrm{triv}}$ is $\widetilde{v} : \ell \mapsto \sum_\alpha (v_\alpha(\ell) \otimes 1) \cdot_{\mathrm{tens}} \partial_x^\alpha$, and thus $\iota(\widetilde{v})$ sends ℓ to $\sum_\alpha (v_\alpha(\ell) \otimes 1) \cdot_{\mathrm{triv}} \partial_x^\alpha$. The corresponding u is the morphism $\ell \mapsto v_0(\ell)$. In other words, the image of the section v is v_0. This is nothing but the image of v by the augmentation morphism $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}') \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}')$ of the Spencer complex. ☐

3.3. Differential complexes

Since $\mathbf{M}(\mathscr{O}_X, \mathrm{Diff}_X)$ is an additive category, one can consider the category $\mathbf{C}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$ of $\star$-bounded complexes of objects of $\mathbf{M}(\mathscr{O}_X, \mathrm{Diff}_X)$ (with $\star = \varnothing, +, -, \mathrm{b}$), and the category $\mathbf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$ of $\star$-bounded complexes up to homotopy (see [**KS90**, Def. 1.3.4]). These are called $\star$-bounded *differential complexes.* Furthermore, we can extend can of (3.2.15) as a functor $\mathrm{can} : \mathbf{C}^\star(\mathscr{D}_X) \mapsto \mathbf{C}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$.

Since connections are differential operators, the de Rham complex of a left $\mathscr{D}_X$-module $\mathscr{M}$ is a complex in $\mathbf{C}^{\mathrm{b}}(\mathscr{O}_X, \mathrm{Diff}_X)$, that we denote ${}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}$ as such. By using the isomorphism ${}^{\mathrm{p}}\mathrm{DR}(\mathscr{M}) \simeq {}^{\mathrm{p}}\mathrm{DR}(\mathscr{M}^{\mathrm{left}})$ of Exercise 1.4.10 for a right $\mathscr{D}_X$-module $\mathscr{M}$, one makes ${}^{\mathrm{p}}\mathrm{DR}(\mathscr{M})$ a complex in $\mathbf{C}^{\mathrm{b}}(\mathscr{O}_X, \mathrm{Diff}_X)$, that is also denoted by ${}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}$. We can then extend ${}^{\mathrm{diff}}\mathrm{DR}$ as a functor ${}^{\mathrm{diff}}\mathrm{DR} : \mathbf{C}^\star(\mathscr{D}_X) \mapsto \mathbf{C}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$, and then as a functor of triangulated categories $\mathbf{K}^\star(\mathscr{D}_X) \to \mathbf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$.

On the other hand, there is a natural forget functor Forget from $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X)$ to $\mathsf{M}(\mathbb{C}_X)$, and by extension a functor Forget at the level of $\mathsf{C}^\star$ and $\mathsf{K}^\star$. The previous considerations show that we can decompose the ${}^{\mathrm{p}}\mathrm{DR}$ functor as

$$\begin{tikzcd}
\mathsf{M}(\mathscr{D}_X) \arrow[r, "{}^{\mathrm{diff}}\mathrm{DR}"'] \arrow[rr, bend left, "{}^{\mathrm{p}}\mathrm{DR}"] & \mathsf{C}^{\mathrm{b}}(\mathscr{O}_X, \mathrm{Diff}_X) \arrow[r, "\mathsf{Forget}"'] & \mathsf{C}^{\mathrm{b}}(\mathbb{C}_X)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{K}^\star(\mathscr{D}_X) \arrow[r, "{}^{\mathrm{diff}}\mathrm{DR}"'] \arrow[rr, bend left, "{}^{\mathrm{p}}\mathrm{DR}"] & \mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X) \arrow[r, "\mathsf{Forget}"'] & \mathsf{K}^\star(\mathbb{C}_X)
\end{tikzcd}$$

***Exercise 3.3.1* (Stability by the Godement functor).** Let $\mathscr{M}$ be a $\mathscr{D}_X$-module. Use Exercise 1.4.15 to show that $\mathrm{God}^\bullet\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}$ is a differential complex. [Hint: Identify this complex with ${}^{\mathrm{diff}}\mathrm{DR}\,\mathrm{God}^\bullet\,\mathscr{M}$.]

In order to define the "derived category" of the additive category $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X)$, one needs to define the notion of null system in $\mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$ and localize the category with respect to the associated multiplicative system. A possible choice would be to say that an object belongs to the null system if it belongs to the null system of $\mathsf{C}^\star(\mathbb{C}_X)$ when forgetting the Diff structure, i.e., which has zero cohomology when considered as a complex of sheaves of $\mathbb{C}$-vector spaces. This is not the choice made below. One says that a differential morphism $u : \mathscr{L} \to \mathscr{L}'$ as in (3.2.8) is a Diff-quasi-isomorphism if the corresponding v as in (3.2.5) is a quasi-isomorphism of right $\mathscr{D}_X$-modules.

The functor ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ of Definition 3.2.17 extends as a functor $\mathsf{C}^\star(\mathscr{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{C}^\star_{\mathrm{i}}(\mathscr{D}_X)$ and $\mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{K}^\star_{\mathrm{i}}(\mathscr{D}_X)$ in a natural way, and is a functor of triangulated categories on K. Moreover, according to Lemma 3.2.19, it is an equivalence of triangulated categories.

We wish now to define *acyclic objects* in the triangulated category $\mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$, and show that they form a *null system* in the sense of [**KS90**, Def. 1.6.6].

Definition 3.3.2. We say that a object $\mathscr{L}^\bullet$ of $\mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$ is Diff-*acyclic* if ${}^{\mathrm{diff}}\mathrm{DR}^{-1}(\mathscr{L}^\bullet)$ is acyclic in $\mathsf{K}^\star_{\mathrm{i}}(\mathscr{D}_X)$ (equivalently, in $\mathsf{K}^\star(\mathscr{D}_X)$).

Lemma 3.3.3. *The family $\mathscr{N}$ of* Diff*-acyclic objects forms a* null system *in the category $\mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$, i.e.,*

- *the object 0 belongs to $\mathscr{N}$,*
- *an object $\mathscr{L}^\bullet$ belongs to $\mathscr{N}$ iff $\mathscr{L}^\bullet[1]$ does so,*
- *if $\mathscr{L}^\bullet \to \mathscr{L}'^\bullet \to \mathscr{L}''^\bullet \to \mathscr{L}^\bullet[1]$ is a distinguished triangle of $\mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$, and if $\mathscr{L}^\bullet, \mathscr{L}'^\bullet$ are objects in $\mathscr{N}$, then so is $\mathscr{L}''^\bullet$.*

Proof. It follows from the property that the extension of ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ to the categories $\mathsf{K}^\star$ is a functor of triangulated categories. ☐

Define, as in [**KS90**, (1.6.4)], the family $S(\mathscr{N})$ as the family of morphisms which can be embedded in a distinguished triangle of $\mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$, with the third term being an object of $\mathscr{N}$. We call such morphisms Diff-*quasi-isomorphisms*. Clearly, they correspond exactly via ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ to quasi-isomorphisms in $\mathsf{K}^\star(\mathscr{D}_X)$.

We now may localize the category $\mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$ with respect to the null system $\mathscr{N}$ and get a category denoted by $\mathsf{D}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$. By construction, we still get a functor

$$ {}^{\mathrm{diff}}\mathrm{DR}^{-1} : \mathsf{D}^\star(\mathscr{O}_X, \mathrm{Diff}_X) \xrightarrow{\sim} \mathsf{D}^\star_{\mathrm{i}}(\mathscr{D}_X) \longrightarrow \mathsf{D}^\star(\mathscr{D}_X), \tag{3.3.4}$$

where the first equivalence is by definition of the null system (since we have an equivalence at the level of the categories $\mathsf{K}^\star$).

On the other hand, the functor can defined by (3.2.15) extends as a functor between the triangulated categories $\mathsf{K}^\star(\mathscr{D}_X)$ and $\mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$, factoring through $\mathsf{K}^\star(\mathscr{O}_X)$. Since $\mathscr{D}_X$ is $\mathscr{O}_X$-flat, if $\mathscr{L}^\bullet$ is acyclic in $\mathsf{K}^\star(\mathscr{O}_X)$, then $\mathscr{L}^\bullet \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is acyclic in $\mathsf{K}^\star(\mathscr{D}_X)$. Then the previous functor extends as a functor

$$\mathsf{D}^\star(\mathscr{O}_X) \longmapsto \mathsf{D}^\star(\mathscr{O}_X, \mathrm{Diff}_X).$$

By composing with the forgetful functor $\mathsf{D}^\star(\mathscr{D}_X) \mapsto \mathsf{D}^\star(\mathscr{O}_X)$, we extend can as a functor

$$\mathrm{can} : \mathsf{D}^\star(\mathscr{D}_X) \longmapsto \mathsf{D}^\star(\mathscr{O}_X, \mathrm{Diff}_X).$$

Caveat 3.3.5. Do not confuse can and ${}^{\mathrm{p}}\mathrm{DR}$.

Remark 3.3.6. The category $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}_X)$ is also naturally a subcategory of the category $\mathsf{M}(\mathbb{C}_X)$ of sheaves of $\mathbb{C}$-vector spaces because $\mathrm{Hom}_{\mathsf{Diff}}(\mathscr{L}, \mathscr{L}')$ is a subset of $\mathrm{Hom}_{\mathbb{C}}(\mathscr{L}, \mathscr{L}')$. We therefore have a natural functor $\mathsf{Forget} : \mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X) \to \mathsf{K}^\star(\mathbb{C}_X)$, forgetting that the differentials of a complex are differential operators, and forgetting also that the homotopies should be differential operators too. As a consequence of Theorem 3.3.7, we will see in Proposition 3.3.12 that any object in the null system $\mathscr{N}$ defined above is sent to an object in the usual null system of $\mathsf{K}^\star(\mathbb{C}_X)$, i.e., objects with zero cohomology. In other words, a Diff-quasi-isomorphism is sent into a usual quasi-isomorphism. But there may exist morphisms in $\mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$ which are quasi-isomorphisms when viewed in $\mathsf{K}^\star(\mathbb{C}_X)$, but are not Diff-quasi-isomorphisms.

Theorem 3.3.7. *The functors* ${}^{\mathrm{diff}}\mathrm{DR}$ *and* ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ *induce quasi-inverse and induce equivalences of categories*

$$\begin{tikzcd} \mathsf{D}^\star(\mathscr{D}_X) \arrow[rr, bend left, "{}^{\mathrm{diff}}\mathrm{DR}"] & & \mathsf{D}^\star(\mathscr{O}_X, \mathrm{Diff}_X). \arrow[ll, bend left, "{}^{\mathrm{diff}}\mathrm{DR}^{-1}"] \end{tikzcd}$$

Lemma 3.3.8. *There is an isomorphism of functors* ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\circ{}^{\mathrm{diff}}\mathrm{DR}\xrightarrow{\sim}\mathrm{Id}$ *from* $\mathbf{D}^{\star}(\mathscr{D}_X)$ *(right* $\mathscr{D}_X$*-modules) to itself.*

This lemma enables one to attach to each object of $\mathbf{D}^{\star}(\mathscr{D}_X)$ a canonical resolution by induced $\mathscr{D}_X$-modules since ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ takes values in $\mathbf{D}^{\star}_{\mathrm{i}}(\mathscr{D}_X)$.

Proof. Let us recall that there exists an explicit side-changing isomorphism of complexes ${}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}^{\mathrm{left}}\simeq{}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}^{\mathrm{right}}$ which is given by termwise $\mathscr{O}_X$-linear morphisms. If we regard these complexes as objects of $\mathbf{C}^{\mathrm{b}}(\mathscr{O}_X,\mathrm{Diff})$, we deduce that the side-changing isomorphism is an isomorphism in this category. In other words, we have ${}^{\mathrm{diff}}\mathrm{DR}(\mathscr{M}^{\mathrm{left}})\simeq{}^{\mathrm{diff}}\mathrm{DR}(\mathscr{M}^{\mathrm{right}})$.

For the proof of the lemma, start with a left $\mathscr{D}_X$-module $\mathscr{M}^{\mathrm{left}}$. By definition, ${}^{\mathrm{diff}}\mathrm{DR}^{-1}{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^{\mathrm{left}}=(\Omega_X^{n+\bullet}\otimes\mathscr{M}^{\mathrm{left}})\otimes\mathscr{D}_X$ with differential ${}^{\mathrm{diff}}\mathrm{DR}^{-1}(\nabla)$. This is nothing but the complex $\Omega_X^{n+\bullet}\otimes(\mathscr{M}^{\mathrm{left}}\otimes\mathscr{D}_X)$ where the differential is the connection on the left $\mathscr{D}_X$-module $(\mathscr{M}^{\mathrm{left}}\otimes\mathscr{D}_X)_{\mathrm{tens}}$. Furthermore, this identification is right $\mathscr{D}_X$-linear with respect to the $(\mathrm{right})_{\mathrm{triv}}$ structure on both terms.

We note that $\big[(\mathscr{M}^{\mathrm{left}}\otimes_{\mathscr{O}_X}\mathscr{D}_X)^{\mathrm{right}}\big]_{\mathrm{tens}}\simeq(\mathscr{M}^{\mathrm{right}}\otimes_{\mathscr{O}_X}\mathscr{D}_X)_{\mathrm{tens}}$, i.e., both with the tensor structure, respectively left and right, and this isomorphism is compatible with the right $\mathscr{D}_X$-structure $(\mathrm{right})_{\mathrm{triv}}$ on both terms. By side-changing we find

$$\big[{}^{\mathrm{p}}\mathrm{DR}(\mathscr{M}^{\mathrm{left}}\otimes_{\mathscr{O}_X}\mathscr{D}_X)_{\mathrm{tens}}\big]_{\mathrm{triv}}\simeq\big[{}^{\mathrm{p}}\mathrm{DR}(\mathscr{M}^{\mathrm{right}}\otimes_{\mathscr{O}_X}\mathscr{D}_X)_{\mathrm{tens}}\big]_{\mathrm{triv}},$$

and by using the involution of Exercise 1.2.9,

$$\big[{}^{\mathrm{p}}\mathrm{DR}(\mathscr{M}^{\mathrm{right}}\otimes_{\mathscr{O}_X}\mathscr{D}_X)_{\mathrm{tens}}\big]_{\mathrm{triv}}\simeq\big[{}^{\mathrm{p}}\mathrm{DR}(\mathscr{M}^{\mathrm{right}}\otimes_{\mathscr{O}_X}\mathscr{D}_X)_{\mathrm{triv}}\big]_{\mathrm{tens}}.$$

Lastly, we have

$${}^{\mathrm{p}}\mathrm{DR}(\mathscr{M}^{\mathrm{right}}\otimes_{\mathscr{O}_X}\mathscr{D}_X)_{\mathrm{triv}}=\mathscr{M}^{\mathrm{right}}\otimes_{\mathscr{O}_X}\mathrm{Sp}^{\bullet}(\mathscr{D}_X)\simeq\mathscr{M}^{\mathrm{right}}\otimes_{\mathscr{O}_X}\mathscr{O}_X=\mathscr{M}^{\mathrm{right}},$$

and the remaining right $\mathscr{D}_X$-structure is deduced from the tens one, which is the natural right structure on $\mathscr{M}^{\mathrm{right}}$. We conclude that, functorially, ${}^{\mathrm{diff}}\mathrm{DR}^{-1}{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^{\mathrm{left}}\simeq\mathscr{M}^{\mathrm{right}}$. Since ${}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^{\mathrm{left}}\simeq{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^{\mathrm{right}}$, the lemma follows. ☐

Proof of Theorem 3.3.7. From the previous lemma, it is now enough to show that, if $\mathscr{L}^{\bullet}$ is a complex in $\mathbf{C}^{\star}(\mathscr{O}_X,\mathrm{Diff}_X)$, there exists a a Diff-quasi-isomorphism ${}^{\mathrm{diff}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathscr{L}^{\bullet}\to\mathscr{L}^{\bullet}$, and, by definition, this is equivalent to showing the existence of a quasi-isomorphism ${}^{\mathrm{diff}}\mathrm{DR}^{-1}{}^{\mathrm{diff}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathscr{L}^{\bullet}\to{}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathscr{L}^{\bullet}$, that we know from the previous result applied to $\mathscr{M}={}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathscr{L}^{\bullet}$. ☐

Remark 3.3.9. Lemma 3.2.4 only shows the existence of a quasi-isomorphism

$${}^{\mathrm{p}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathscr{L}^{\bullet}\xrightarrow{\sim}\mathsf{Forget}\,\mathscr{L}^{\bullet}$$

in the category of sheaves of $\mathbb{C}$-vector spaces.

Exercise 3.3.10. Show that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{D}^{\star}(\mathscr{D}_X) \arrow[r, "{}^{\mathrm{diff}}\mathrm{DR}"'] \arrow[rr, bend left, "{}^{\mathrm{p}}\mathrm{DR}"] & \mathsf{D}^{\star}(\mathscr{O}_X, \mathrm{Diff}_X) \arrow[r, "\mathsf{Forget}"'] & \mathsf{D}^{\star}(\mathbb{C}_X)
\end{tikzcd}$$

Corollary 3.3.11. *The natural functor $\mathsf{D}^{\star}_{\mathrm{i}}(\mathscr{D}_X) \mapsto \mathsf{D}^{\star}(\mathscr{D}_X)$ is an equivalence of categories.*

Proof. This follows from the isomorphism of functors ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR} \xrightarrow{\sim} \mathrm{Id}$ from $\mathsf{D}^{\star}(\mathscr{D}_X)$ to itself, and (3.3.4). □

Proposition 3.3.12. *The functor Forget induces a functor $\mathsf{D}^{\star}(\mathscr{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{D}^{\star}(\mathbb{C}_X)$, and we have an isomorphism of functors*

$$ {}^{\mathrm{p}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1} \xrightarrow{\sim} \mathit{Forget} : \mathsf{D}^{\star}(\mathscr{O}_X, \mathrm{Diff}_X) \longmapsto \mathsf{D}^{\star}(\mathbb{C}_X). $$

Proof. If $\mathscr{L}^{\bullet}$ is Diff-acyclic, $\mathsf{Forget}\,\mathscr{L}^{\bullet}$ is acyclic indeed, by definition, ${}^{\mathrm{diff}}\mathrm{DR}^{-1}(\mathscr{L}^{\bullet})$ is acyclic; then ${}^{\mathrm{p}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}(\mathscr{L}^{\bullet})$ is also acyclic and quasi-isomorphic to : $\mathsf{Forget}\,\mathscr{L}^{\bullet}$. This shows the first part of the statement. The second part follows from Theorem 3.3.7 and of the commutativity of the diagram of Exercise 3.3.10. □

Remark 3.3.13 **(The Godement resolution of a differential complex)**

Let $\mathscr{L}^{\bullet}$ be an object of $\mathsf{C}^{+}(\mathscr{O}_X, \mathrm{Diff}_X)$. Then $\mathrm{God}^{\bullet}\mathscr{L}^{\bullet}$ is maybe not a differential complex (see Exercise 1.4.15(1)). However, $\mathrm{God}^{\bullet}\,{}^{\mathrm{diff}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathscr{L}^{\bullet}$ is a differential complex, being equal to ${}^{\mathrm{diff}}\mathrm{DR}\,\mathrm{God}^{\bullet}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathscr{L}^{\bullet}$. Therefore, the composite functor $\mathrm{God}^{\bullet}\,{}^{\mathrm{diff}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}$ plays the role of Godement resolutions in the category of differential complexes.

Exercise 3.3.14 **(Differential complexes in the relative situation)**

Keep notation of Exercise 3.2.16.

(1) Show the analogue of Lemma 3.2.9 and deduce the existence of a functor and ${}^{\mathrm{diff}}\mathrm{DR}^{-1}{}_{Y} : \mathsf{M}(f^{-1}\mathscr{O}_Y, \mathrm{Diff}_Y) \to \mathsf{M}(f^{-1}\mathscr{D}_Y)$.

(2) Construct the derived categories $\mathsf{D}^{\star}(f^{-1}\mathscr{O}_Y, \mathrm{Diff}_Y)$ and show that ${}^{\mathrm{diff}}\mathrm{DR}$ and ${}^{\mathrm{diff}}\mathrm{DR}^{-1}{}_{Y}$ are quasi-inverse functors.

(3) Using Godement resolutions, define a functor

$$ \boldsymbol{R}f_{!} : \mathsf{D}^{+}(f^{-1}\mathscr{O}_Y, \mathrm{Diff}_Y) \longrightarrow \mathsf{D}^{+}(\mathscr{O}_Y, \mathrm{Diff}_Y). $$

[Hint: Given a complex $\mathscr{L}^{\bullet}$ in $\mathsf{C}^{+}(f^{-1}\mathscr{O}_Y, \mathrm{Diff}_Y)$, replace it with the complex ${}^{\mathrm{diff}}\mathrm{DR}\,\mathrm{God}^{\bullet}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}{}_{Y}\,\mathscr{L}^{\bullet}$.]

(4) Show that the following diagrams commute:

$$\begin{tikzcd}
\mathsf{D}^+(f^{-1}\mathscr{D}_Y) \arrow[r, "{}^{\mathrm{diff}}\mathrm{DR}"] \arrow[d, "\boldsymbol{R}f_!"'] & \mathsf{D}^+(f^{-1}\mathscr{O}_Y, \mathrm{Diff}_Y) \arrow[d, "\boldsymbol{R}f_!"] \\
\mathsf{D}^+(\mathscr{D}_Y) \arrow[r, "{}^{\mathrm{diff}}\mathrm{DR}"] & \mathsf{D}^+(\mathscr{O}_Y, \mathrm{Diff}_Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{D}^+(f^{-1}\mathscr{D}_Y) \arrow[d, "\boldsymbol{R}f_!"'] & \mathsf{D}^+(f^{-1}\mathscr{O}_Y, \mathrm{Diff}_Y) \arrow[l, "{}^{\mathrm{diff}}\mathrm{DR}^{-1}_Y"'] \arrow[d, "\boldsymbol{R}f_!"] \\
\mathsf{D}^+(\mathscr{D}_Y) & \mathsf{D}^+(\mathscr{O}_Y, \mathrm{Diff}_Y) \arrow[l, "{}^{\mathrm{diff}}\mathrm{DR}^{-1}_Y"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{D}^+(f^{-1}\mathscr{O}_Y, \mathrm{Diff}_Y) \arrow[r, "\mathsf{Forget}"] \arrow[d, "\boldsymbol{R}f_!"'] & \mathsf{D}^+(\mathbb{C}_X) \arrow[d, "\boldsymbol{R}f_!"] \\
\mathsf{D}^+(\mathscr{O}_Y, \mathrm{Diff}_Y) \arrow[r, "\mathsf{Forget}"] & \mathsf{D}^+(\mathbb{C}_Y)
\end{tikzcd}$$

Of special interest for us will be the composed functor can defined by (3.2.15),

$$ {}^{\mathrm{diff}}\mathrm{DR}^{-1} \circ \mathrm{can} : \mathsf{D}^+(\mathscr{D}_X) \longrightarrow \mathsf{D}^+(\mathscr{D}_X). $$

Lemma 3.3.15. *We have a functorial isomorphism of functors*

$$ {}^{\mathrm{diff}}\mathrm{DR} \circ ({}^{\mathrm{diff}}\mathrm{DR}^{-1} \circ \mathrm{can}) \simeq \mathrm{can} : \mathsf{D}^+(\mathscr{D}_X) \longrightarrow \mathsf{D}^+(\mathscr{O}_X, \mathrm{Diff}_X). $$

In other words, ${}^{\mathrm{diff}}\mathrm{DR} \circ ({}^{\mathrm{diff}}\mathrm{DR}^{-1} \circ \mathrm{can})$, which a priori takes values in $\mathsf{D}^+(\mathscr{O}_X, \mathrm{Diff}_X)$, takes in fact values in $\mathsf{D}^+(\mathscr{D}_X)$.

Proof. If $\mathscr{M}$ is a right $\mathscr{D}_X$-module, then ${}^{\mathrm{diff}}\mathrm{DR}^{-1} \circ \mathrm{can}(\mathscr{M}) = (\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{triv}}$. It follows that

$$\begin{aligned} {}^{\mathrm{diff}}\mathrm{DR}({}^{\mathrm{diff}}\mathrm{DR}^{-1} \circ \mathrm{can})(\mathscr{M}) &= {}^{\mathrm{diff}}\mathrm{DR}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{triv}} \\ &= \mathscr{M} \otimes_{\mathscr{O}_X} \mathrm{Sp}^\bullet(\mathscr{D}_X) \simeq \mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{O}_X = \mathscr{M}. \end{aligned}$$

A priori, these equalities hold as $\mathscr{O}_X$-modules. However, putting the $(\mathrm{right})_{\mathrm{tens}}$ $\mathscr{D}_X$-module structure on $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ makes ${}^{\mathrm{diff}}\mathrm{DR}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{triv}}$ a complex of right $\mathscr{D}_X$-modules and the above equalities are compatible with this structure (see Lemma 3.2.21). These isomorphisms are functorial with respect to $\mathscr{M} \in \mathsf{M}(\mathscr{D}_X)$, so that this identification also holds for morphisms in $\mathsf{D}^+(\mathscr{D}_X)$. □

3.4. Differential complexes of finite order

In order to deal with local duality, we would need that, when $\mathscr{L}'^\bullet$ is a complex of right $\mathscr{D}_X$-module, $\mathscr{H}om_{\mathrm{Diff}}(\mathscr{L}, \mathscr{L}'^\bullet)$ is a differential complex. This would be the case if the morphism $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}') \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ of Proposition 3.2.24 were an isomorphism, according to the second part of this proposition. This is not the case in general. For that reason, we introduce the notion of differential morphism of finite order.

Recall (see (3.2.7)) that we have a natural identification

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)\xrightarrow{\sim}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X).$$

and that $\mathscr{D}_X$ is filtered by $F_\bullet\mathscr{D}_X$ (see Definition 1.1.4), that we consider with its $\mathscr{O}_X$-module structures by multiplication on the left and on right. This defines a filtration

$$F_p\,\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X):=\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}'\otimes_{\mathscr{O}_X}F_p\mathscr{D}_X),$$

(where the left $\mathscr{O}_X$-module structure of $F_p\mathscr{D}_X$ is used for the tensor product and the right one for $\mathscr{H}om_{\mathscr{O}_X}$) that we also write as $F_p\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)$, according to the previous identification. This filtration is not exhaustive in general, and we set

$$\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)=\bigcup_p F_p\,\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X),$$

and the space of $\mathscr{D}_X$-linear morphisms locally of finite order from $\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X$ to $\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X$ is defined as

$$\mathrm{Hom}^{\mathrm{i,f}}_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)=\Gamma(X,\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)).$$

(The exponent i recalls that this is only defined for induced $\mathscr{D}_X$-modules.) In other words, a $\mathscr{D}_X$-linear morphism $\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X\to\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X$ is locally of finite order if and only if, in the neighbourhood of each point of X there exists p_o such that the morphism sends $\mathscr{L}\otimes_{\mathscr{O}_X}F_p\mathscr{D}_X$ to $\mathscr{L}'\otimes_{\mathscr{O}_X}F_{p+p_o}\mathscr{D}_X$ for each p.

The following is easily checked:

Lemma 3.4.1

(1) *The composition of morphisms of finite order has finite order.*

(2) *Any $\mathscr{O}_X$-linear morphism $\mathscr{L}\to\mathscr{L}'$ has order zero, and that any integrable connection $\nabla:\mathscr{L}\to\Omega^1_X\otimes_{\mathscr{O}_X}\mathscr{L}$ has order $\leqslant 1$, as well as any of its extensions $\nabla^{(k)}$ (see Example* 3.2.12(2)*).*

Exercise 3.4.2

(1) Let $\mathscr{L}=\mathscr{D}_X$ (with its right structure of $\mathscr{O}_X$-module) and $\mathscr{L}'=\mathscr{O}_X$. Let $w=\mathrm{Id}\in\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{D}_X,\mathscr{O}_X\otimes_{\mathscr{O}_X}\mathscr{D}_X)=\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{D}_X,\mathscr{D}_X)$. Show that w does not belong to any $F_p\,\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{D}_X,\mathscr{D}_X)$.

(2) Assume that $\mathscr{L}$ is $\mathscr{O}_X$-coherent. Show that

$$\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)=\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X).$$

Definition 3.4.3. The category $\mathsf{M}_{\mathrm{i,f}}(\mathscr{D}_X)$ is the additive category of induced $\mathscr{D}$-modules with $\mathscr{D}_X$-linear morphisms locally of finite order. It gives rise to the categories of complexes $\mathsf{C}^\star_{\mathrm{i,f}}(\mathscr{D}_X)$ and that of complexes up to homotopy $\mathsf{K}^\star_{\mathrm{i,f}}(\mathscr{D}_X)$.

Note that $\mathsf{M}_{\mathrm{i,f}}(\mathscr{D}_X)$ is not a full subcategory of $\mathsf{M}(\mathscr{D}_X)$. However, one can construct the corresponding category of complexes $\mathsf{C}^\star_{\mathrm{i,f}}(\mathscr{D}_X)$, that of complexes up to homotopy $\mathsf{K}^\star_{\mathrm{i,f}}(\mathscr{D}_X)$ and there is a natural functor $\mathsf{K}^\star_{\mathrm{i,f}}(\mathscr{D}_X) \mapsto \mathsf{K}^\star_{\mathrm{i}}(\mathscr{D}_X)$ since a homotopy of finite order is a homotopy. Defining then the null system in $\mathsf{K}^\star_{\mathrm{i,f}}(\mathscr{D}_X)$ as consisting of complexes which are acyclic in $\mathsf{K}^\star_{\mathrm{i}}(\mathscr{D}_X)$ (equivalently, in $\mathsf{K}^\star(\mathscr{D}_X)$), we define the derived category $\mathsf{D}^\star_{\mathrm{i,f}}(\mathscr{D}_X)$, which is equipped with a natural functor to $\mathsf{D}^\star_{\mathrm{i}}(\mathscr{D}_X)$. We can then enhance Proposition 3.2.24 with the assumption of finite order.

***Definition 3.4.4* (Differential operators of finite order).** The space $\mathrm{Hom}^{\mathrm{f}}_{\mathrm{Diff}}(\mathscr{L}, \mathscr{L}')$ of differential operator $\mathscr{L} \to \mathscr{L}'$ which have (locally) *finite order* is the image of

$$\Gamma\big(X, \mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)\big)$$

in $\mathrm{Hom}_{\mathrm{Diff}}(\mathscr{L}, \mathscr{L}')$. The filtration on $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ defines a filtration $F_\bullet \mathscr{H}om_{\mathrm{Diff}}(\mathscr{L}, \mathscr{L}')$. We also set $\mathscr{H}om_{\mathrm{Diff}^{\mathrm{f}}}(\mathscr{L}, \mathscr{L}') = \bigcup_p F_p \mathscr{H}om_{\mathrm{Diff}}(\mathscr{L}, \mathscr{L}') \subset \mathscr{H}om_{\mathrm{Diff}}(\mathscr{L}, \mathscr{L}')$

Examples 3.4.5

(1) We have $F_\bullet \mathscr{H}om_{\mathrm{Diff}}(\mathscr{O}_X, \mathscr{O}_X) = F_\bullet \mathscr{D}_X$ (see Example 3.2.12(1)).

(2) If $\mathscr{L}$ is $\mathscr{O}_X$-coherent, the filtration $F_\bullet \mathscr{H}om_{\mathrm{Diff}}(\mathscr{L}, \mathscr{L}')$ is exhaustive, i.e., $\mathscr{H}om_{\mathrm{Diff}^{\mathrm{f}}}(\mathscr{L}, \mathscr{L}') = \mathscr{H}om_{\mathrm{Diff}}(\mathscr{L}, \mathscr{L}')$.

Due to the composition property seen above, one can define the category $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}^{\mathrm{f}}_X)$, whose objects are $\mathscr{O}_X$-modules and morphisms are differential operators of finite order. This category is additive. By the same proof as in Lemma 3.2.19, one checks that the functor ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ sends $\mathsf{M}(\mathscr{O}_X, \mathrm{Diff}^{\mathrm{f}}_X)$ to $\mathsf{M}_{\mathrm{i,f}}(\mathscr{D}_X)$ and that it is an equivalence of categories, a quasi-inverse functor being ${}^{\mathrm{diff}}\mathrm{DR} : \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X \mapsto \mathscr{L}$, ${}^{\mathrm{diff}}\mathrm{DR}(v) = u$.

One then constructs the categories denoted by $\mathsf{C}^\star(\mathscr{O}_X, \mathrm{Diff}^{\mathrm{f}}_X)$ and $\mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}^{\mathrm{f}}_X)$, respectively equivalent to $\mathsf{C}^\star_{\mathrm{i,f}}(\mathscr{D}_X)$ and $\mathsf{K}^\star_{\mathrm{i,f}}(\mathscr{D}_X)$. By defining acyclic objects in $\mathsf{K}^\star_{\mathrm{i,f}}(\mathscr{D}_X)$ as objects whose image by the functor $\mathsf{K}^\star_{\mathrm{i,f}}(\mathscr{D}_X) \mapsto \mathsf{K}^\star(\mathscr{D}_X)$ is acyclic, and correspondingly acyclic objects in $\mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}^{\mathrm{f}}_X)$, one checks that they form a null system in $\mathsf{K}^\star(\mathscr{O}_X, \mathrm{Diff}^{\mathrm{f}}_X)$. This leads to the derived category $\mathsf{D}^\star(\mathscr{O}_X, \mathrm{Diff}^{\mathrm{f}}_X)$.

Since any $\mathscr{O}_X$-linear morphism has order zero, the natural functor

$$\mathsf{C}^\star(\mathscr{O}_X) \longmapsto \mathsf{C}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$$

takes values in $\mathsf{C}^\star(\mathscr{O}_X, \mathrm{Diff}^{\mathrm{f}}_X)$ and

$$\mathsf{D}^\star(\mathscr{O}_X) \longmapsto \mathsf{D}^\star(\mathscr{O}_X, \mathrm{Diff}_X)$$

takes values in $\mathsf{D}^\star(\mathscr{O}_X, \mathrm{Diff}^{\mathrm{f}}_X)$. As a consequence, the functor can of (3.2.15) can be regarded as a functor

$$\mathrm{can} : \mathsf{D}^\star(\mathscr{D}_X) \longmapsto \mathsf{D}^\star(\mathscr{O}_X, \mathrm{Diff}^{\mathrm{f}}_X). \tag{3.4.6}$$

Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. Then ${}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}$ is an object of $\mathsf{C}^{\mathrm{b}}(\mathscr{O}_X,\mathrm{Diff}^{\mathrm{f}}_X)$, since the connection has order one. By the side-changing isomorphism, which has termwise finite order (order zero), we conclude that the same property holds for a right $\mathscr{D}_X$-module $\mathscr{M}$.

Let $\mathscr{M}^\bullet$ be an object of $\mathsf{C}^\star(\mathscr{D}_X)$. Then that ${}^{\mathrm{diff}}\mathrm{DR}(\mathscr{M}^\bullet)$ is an object of $\mathsf{C}^\star(\mathscr{O}_X,\mathrm{Diff}^{\mathrm{f}}_X)$, because the differentials of the complex $\mathscr{M}^\bullet$ induce differential morphisms of order zero.

It follows that ${}^{\mathrm{diff}}\mathrm{DR}$ defines a functor $\mathsf{D}^\star(\mathscr{D}_X)\mapsto\mathsf{D}^\star(\mathscr{O}_X,\mathrm{Diff}^{\mathrm{f}}_X)$. As a consequence, the functor ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}$ is regarded as a functor $\mathsf{D}^\star(\mathscr{D}_X)\mapsto\mathsf{D}^\star_{\mathrm{i,f}}(\mathscr{D}_X)$.

In a way analogous to Theorem 3.3.7, and since Lemma 3.3.8 only makes use of $\mathsf{C}^{\mathrm{b}}(\mathscr{O}_X,\mathrm{Diff}^{\mathrm{f}})$ according to the above remarks, one obtains:

Theorem 3.4.7. *The functors* ${}^{\mathrm{diff}}\mathrm{DR}$ *and* ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ *induce quasi-inverse and induce equivalences of categories*

$$\begin{tikzcd}\mathsf{D}^\star(\mathscr{D}_X) \arrow[r, bend left, "{}^{\mathrm{diff}}\mathrm{DR}"] & \mathsf{D}^\star(\mathscr{O}_X,\mathrm{Diff}^{\mathrm{f}}_X), \arrow[l, bend left, "{}^{\mathrm{diff}}\mathrm{DR}^{-1}"]\end{tikzcd}$$

the functor ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ *takes values in* $\mathsf{D}^\star_{\mathrm{i,f}}(\mathscr{D}_X)$*, and the natural functor* $\mathsf{D}^\star_{\mathrm{i,f}}(\mathscr{D}_X)\mapsto\mathsf{D}^\star(\mathscr{D}_X)$ *is an equivalence of categories.* ☐

3.5. A prelude to local duality

3.5.a. A refinement of Proposition 3.2.24

Proposition 3.5.1. *Assume that* $\mathscr{L}'$ *is a right* $\mathscr{D}_X$*-module. Then the sheaf of* $\mathbb{C}$*-vector spaces* $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)$ *has a natural structure of right* $\mathscr{D}_X$*-module, denoted* $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)_{\mathrm{tens}}$*, making it isomorphic to the induced* $\mathscr{D}_X$*-module* $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}')\otimes_{\mathscr{O}_X}\mathscr{D}_X$*.*

More precisely, for each $p\geqslant 0$*, the morphism of Proposition* 3.2.24 *induces an* $\mathscr{O}_X$*-linear isomorphism*

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}')\otimes_{\mathscr{O}_X}F_p\mathscr{D}_X\xrightarrow{\sim}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L},(\mathscr{L}'\otimes_{\mathscr{O}_X}F_p\mathscr{D}_X)_{\mathrm{triv}}),$$

where the $\mathscr{O}_X$*-structure on the left-hand term is induced by the right one on* $F_p\mathscr{D}_X$*, and thus a* $\mathscr{D}_X$*-linear isomorphism which is bi-functorial with respect to* $\mathscr{L}\in\mathsf{M}(\mathscr{O}_X)$ *and* $\mathscr{L}'\in\mathsf{M}(\mathscr{D}_X)$*:*

$$(3.5.1\,*)\qquad \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}')\otimes_{\mathscr{O}_X}\mathscr{D}_X \xrightarrow{\sim}\big[\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,(\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)_{\mathrm{triv}})\big]_{\mathrm{tens}}.$$

Proof. This follows from the strictness of ι with respect to the filtration $F_\bullet$ (see Exercise 1.2.9(3)) and from the $\mathscr{O}_X$-coherence of $F_p\mathscr{D}_X$. ☐

It follows from Exercise 3.4.2(2) that, if $\mathscr{L}$ is $\mathscr{O}_X$-coherent, we have an isomorphism of right $\mathscr{D}_X$-modules

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L},\mathscr{L}')\otimes_{\mathscr{O}_X}\mathscr{D}_X\xrightarrow{\sim}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X).$$

Similarly, if $\mathscr{L}^\bullet$ is a bounded complex of $\mathscr{O}_X$-modules (and $\mathscr{O}_X$-linear differentials) with coherent cohomology, then the natural right $\mathscr{D}_X$-linear morphism is a quasi-isomorphism:

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}^\bullet,\mathscr{L}')\otimes_{\mathscr{O}_X}\mathscr{D}_X\xrightarrow{\sim}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{L}^\bullet\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X) \tag{3.5.2}$$

The isomorphism (3.5.1 $*$) shows that, if $\mathscr{L}'$ is a right $\mathscr{D}_X$-module, the contravariant functor $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}(\bullet,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)$ from $\mathsf{M}_{\mathrm{i,f}}(\mathscr{D}_X)$ to $\mathsf{M}(\mathscr{D}_X)$ takes values in $\mathsf{M}_{\mathrm{i}}(\mathscr{D}_X)$.

Proposition 3.5.3 **(Refinement of Proposition 3.5.1)**

> (1) *The functor* $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}(\bullet,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)$ *from* $\mathsf{M}_{\mathrm{i,f}}(\mathscr{D}_X)$ *to* $\mathsf{M}(\mathscr{D}_X)$ *takes values in* $\mathsf{M}_{\mathrm{i,f}}(\mathscr{D}_X)$.
>
> (2) *The functor* $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}(\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\bullet\otimes_{\mathscr{O}_X}\mathscr{D}_X)$ *is a covariant functor from* $\mathsf{M}(\mathscr{D}_X)$ *to* $\mathsf{M}_{\mathrm{i}}(\mathscr{D}_X)$ *which takes values in* $\mathsf{M}_{\mathrm{i,f}}(\mathscr{D}_X)$.

Proof. For (1), it amounts to proving that a morphism $\mathscr{L}_1\otimes_{\mathscr{O}_X}\mathscr{D}_X\to\mathscr{L}_2\otimes_{\mathscr{O}_X}\mathscr{D}_X$ of finite order p induces a morphism of finite order

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}_2,\mathscr{L}')\otimes_{\mathscr{O}_X}\mathscr{D}_X\longrightarrow\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}_1,\mathscr{L}')\otimes_{\mathscr{O}_X}\mathscr{D}_X.$$

The result follows from considering the composition

$$\begin{aligned}F_q\,\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}(\mathscr{L}_2\otimes_{\mathscr{O}_X}\mathscr{D}_X,(\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)_{\mathrm{triv}})&\simeq\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}_2,(\mathscr{L}'\otimes_{\mathscr{O}_X}F_q\mathscr{D}_X)_{\mathrm{triv}})\\&\hookrightarrow\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}_2\otimes_{\mathscr{O}_X}F_p\mathscr{D}_X,(\mathscr{L}'\otimes_{\mathscr{O}_X}F_{p+q}\mathscr{D}_X)_{\mathrm{triv}})\\&\longrightarrow\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}_1,(\mathscr{L}'\otimes_{\mathscr{O}_X}F_{p+q}\mathscr{D}_X)_{\mathrm{triv}})\\&\simeq F_{p+q}\,\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}(\mathscr{L}_1\otimes_{\mathscr{O}_X}\mathscr{D}_X,(\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)_{\mathrm{triv}}).\end{aligned}$$

The argument for (2) is similar. □

Corollary 3.5.4. *Let* $\mathscr{L}^\bullet$ *be a complex in* $\mathsf{D}^{\mathrm{b}}(\mathscr{O}_X,\mathrm{Diff}^{\mathrm{f}}_X)$ *and let* $\mathscr{L}'^\bullet$ *be in* $\mathsf{D}^+(\mathscr{D}_X)$. *Then the simple complex associated to* $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathscr{L}^\bullet,({}^{\mathrm{diff}}\mathrm{DR}^{-1}\circ\mathrm{can})\mathscr{L}'^\bullet)$ *is a complex in* $\mathsf{D}^+_{\mathrm{i,f}}(\mathscr{D}_X)$, *and there is an isomorphism in* $\mathsf{C}^+(\mathscr{O}_X,\mathrm{Diff}^{\mathrm{f}}_X)$*:*

$$\mathscr{H}om_{\mathrm{Diff}^{\mathrm{f}}}(\mathscr{L}^\bullet,\mathrm{can}\,\mathscr{L}'^\bullet)\simeq{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathscr{L}^\bullet,({}^{\mathrm{diff}}\mathrm{DR}^{-1}\circ\mathrm{can})\mathscr{L}'^\bullet). \tag{3.5.4 $*$}$$

Proof. The assertion for $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}$ follows from Proposition 3.5.3. The isomorphism (3.5.4 $*$) then follows from the isomorphism (3.5.1 $*$) together with the last assertion in Proposition 3.2.24. □

3.5.b. Properties of the functor $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}$ applied to $\mathscr{D}_X$-modules. Let $\mathscr{N}^\bullet$ be a complex in $\mathsf{C}^{\mathrm{b}}_{\mathrm{i,f}}(\mathscr{D}_X)$, and let $\mathscr{L}'$ be a right $\mathscr{D}_X$-module. It follows from Proposition 3.5.3(1) that $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}(\mathscr{N}^\bullet, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ is a complex in $\mathsf{C}^{\mathrm{b}}_{\mathrm{i,f}}(\mathscr{D}_X)$ (i.e., the differentials are morphisms of finite order). From Theorem 3.4.7 we deduce that, for any bounded complex $\mathscr{M}^\bullet$ in $\mathsf{C}^{\mathrm{b}}(\mathscr{D}_X)$, the complex $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^\bullet, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ is a complex in $\mathsf{C}^{\mathrm{b}}_{\mathrm{i,f}}(\mathscr{D}_X)$, and we have a natural morphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$:

$$\begin{aligned}(3.5.5)\quad &\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^\bullet, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)\\ &\qquad\longrightarrow \mathscr{H}om_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^\bullet, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)\\ &\qquad\qquad\longrightarrow \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^\bullet, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X).\end{aligned}$$

Let us also notice that, according to Lemma 3.3.8,

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^\bullet, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X) \simeq \big[\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}^\bullet, (\mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{triv}})\big]_{\mathrm{tens}}.$$

The morphism (3.5.5) is functorial with respect to $\mathscr{L}'$. Indeed, a $\mathscr{D}_X$-linear morphism $u : \mathscr{L}'_1 \to \mathscr{L}'_2$ gives rise to a $\mathscr{D}_X$-linear morphism $v = {}^{\mathrm{diff}}\mathrm{DR}^{-1}(u) : (\mathscr{L}'_1 \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{triv}} \to (\mathscr{L}'_2 \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{triv}}$. The morphism u being $\mathscr{O}_X$-linear, we have $v = u \otimes 1$, and it is also $\mathscr{D}_X$-linear with respect to the $(\mathrm{right})_{\mathrm{tens}}$ structure. It induces therefore a morphism between the corresponding last two terms of (3.5.5) (for $\mathscr{L}'_1$ and $\mathscr{L}'_2$). Since $u \otimes 1$ sends $\mathscr{L}'_1 \otimes F_p\mathscr{D}_X$ to $\mathscr{L}'_2 \otimes F_p\mathscr{D}_X$ for each p, it also induces a morphism between the corresponding first terms of (3.5.5). Finally, the diagram $(3.5.5)(\mathscr{L}'_1) \to (3.5.5)(\mathscr{L}'_2)$ commutes.

The main reason for using differential complexes of finite order instead of differential complexes is given by the following property.

Proposition 3.5.6. *Let $\mathscr{L}'$ be a right $\mathscr{D}_X$-module which is $\mathscr{O}_X$-injective and let $\mathscr{M}^\bullet$ be a bounded complex of right $\mathscr{D}_X$-modules which is acyclic. Then the complex of right $\mathscr{D}_X$-modules $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^\bullet, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ is also acyclic.*

Proof. Recall (see the proof of Theorem 3.3.7) that, if δ denotes the differential of $\mathscr{M}^\bullet$ and ∇ is the connection on each $\mathscr{M}^{\mathrm{left},k}$, ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^\bullet$ is the simple complex associated to the bi-complex

$$(\Omega^{n+\bullet}_X \otimes_{\mathscr{O}_X} \mathscr{M}^{\mathrm{left},\bullet} \otimes_{\mathscr{O}_X} \mathscr{D}_X, {}^{\mathrm{diff}}\mathrm{DR}^{-1}(\nabla), \widetilde{\delta}),$$

where ${}^{\mathrm{diff}}\mathrm{DR}^{-1}(\nabla)$ is the connection on $(\mathscr{M}^{\mathrm{left},\bullet} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\mathrm{tens}}$ and $\widetilde{\delta}$ is induced by $\delta \otimes 1$ (see Exercise 1.2.8(1)). Recall also that ${}^{\mathrm{diff}}\mathrm{DR}^{-1}(\nabla)$ is of finite order (in fact, order $\leqslant 1$), as well as $\widetilde{\delta}$ (order zero). It is then enough to prove the acyclicity for each k of the complex

$$\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}\big((\Omega^{n+k}_X \otimes_{\mathscr{O}_X} \mathscr{M}^{\mathrm{left},\bullet} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \widetilde{\delta}), \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X\big).$$

By definition of $\widetilde{\delta}$ and of $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}$ and according to functoriality in Proposition 3.5.3, this complex is nothing but the complex $\mathscr{H}om_{\mathscr{O}_X}((\Omega^{n+k}_X \otimes_{\mathscr{O}_X} \mathscr{M}^{\mathrm{left},\bullet}, \delta), \mathscr{L}') \otimes_{\mathscr{O}_X} \mathscr{D}_X$, where we still denote by δ the $\mathscr{O}_X$-linear morphism associated to δ. Since $\mathscr{L}'$ is $\mathscr{O}_X$-injective and $(\mathscr{M}^{\mathrm{left},\bullet},\delta)$ is acyclic, the complex $\mathscr{H}om_{\mathscr{O}_X}((\Omega_X^{n+k}\otimes_{\mathscr{O}_X}\mathscr{M}^{\mathrm{left},\bullet},\delta),\mathscr{L}')$ is acyclic, hence so is the complex $\mathscr{H}om_{\mathscr{O}_X}((\Omega_X^{n+k}\otimes_{\mathscr{O}_X}\mathscr{M}^{\mathrm{left},\bullet},\delta),\mathscr{L}')\otimes_{\mathscr{O}_X}\mathscr{D}_X$ since $\mathscr{D}_X$ is $\mathscr{O}_X$-flat. ☐

According to Proposition 3.5.3, $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1\,\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^\bullet,\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)$ is an object of $\mathsf{C}^{\mathrm{b}}_{\mathrm{i,f}}(\mathscr{D}_X)$, therefore of $\mathsf{K}^{\mathrm{b}}_{\mathrm{i,f}}(\mathscr{D}_X)$, and acyclicity in $\mathsf{K}^{\mathrm{b}}(\mathscr{D}_X)$ is by definition equivalent to acyclicity in $\mathsf{K}^{\mathrm{b}}_{\mathrm{i,f}}(\mathscr{D}_X)$.

One can choose for $\mathscr{L}'$ an injective $\mathscr{D}_X$-module.[1] Note that $\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X$ is not $\mathscr{D}_X$-injective in general, and we cannot assert the property of Proposition 3.5.6 for $\mathscr{H}om_{\mathscr{D}_X}$. Since we are mainly interested in the latter, we will complement this result with the following extension of Exercise 3.4.2(2) and of (3.5.2).

Proposition 3.5.7. *Let $\mathscr{M}^\bullet$ be an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ and let $\mathscr{L}'$ be a right $\mathscr{D}_X$-module which is $\mathscr{O}_X$-injective. Then* (3.5.5) *is an isomorphism.*

For the proof of Proposition 3.5.7, we cannot directly apply (3.5.2), since ${}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1\,\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^\bullet$ is in general not of the form $\mathscr{L}^\bullet\otimes_{\mathscr{O}_X}\mathscr{D}_X$ with $\mathscr{L}^\bullet$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$. We will use the following.

***Lemma 3.5.8* (see [Har75, Prop. I.4.4]).** *Let $(\mathscr{F}_p^\bullet)_{p\in\mathbb{N}}$ be a projective system of complexes of sheaves of $\mathbb{C}$-vector spaces. Assume that*

(1) *for each $k\in\mathbb{Z}$, the morphisms $\mathscr{F}^k_{p+1}\to\mathscr{F}^k_p$ are onto (in particular the Mittag-Leffler condition is satisfied for each projective system $(\mathscr{F}^k_p)_{p\in\mathbb{N}}$),*

(2) *there exists p_o such that, for each $p\geqslant p_o$, the morphism $\mathscr{F}^\bullet_{p+1}\to\mathscr{F}^\bullet_p$ is a quasi-isomorphism (in particular the Mittag-Leffler condition is satisfied for each projective system $(\mathscr{H}^k\mathscr{F}^\bullet_p)_{p\in\mathbb{N}}$).*

Then, for each $k\in\mathbb{Z}$, the natural morphisms

$$\mathscr{H}^k\Big(\varprojlim_p\mathscr{F}^\bullet_p\Big)\longrightarrow\varprojlim_p(\mathscr{H}^k\mathscr{F}^\bullet_p)\longrightarrow\mathscr{H}^k\mathscr{F}^\bullet_{p_o}$$

---

[1] Let $\mathscr{A},\mathscr{B}$ be sheaves of rings with unit, $\mathscr{A}$ being commutative, and assume that $\mathscr{B}$ is a left and right $\mathscr{A}$-module, which is left $\mathscr{A}$-flat. Let $\mathscr{K}$ be an injective right $\mathscr{B}$-module. Then $\mathscr{K}$ is injective as a right $\mathscr{A}$-module. Indeed, let $\mathscr{F}\hookrightarrow\mathscr{G}$ be a monomorphism of sheaves of right $\mathscr{A}$-modules and let $\varphi:\mathscr{F}\to\mathscr{K}$ be $\mathscr{A}$-linear. Since $\mathscr{B}$ is left $\mathscr{A}$-flat, $\mathscr{F}\otimes_{\mathscr{A}}\mathscr{B}\to\mathscr{G}\otimes_{\mathscr{A}}\mathscr{B}$ is a monomorphism and the composed $\mathscr{B}$-linear morphism

$$\varphi_{\mathscr{B}}:\mathscr{F}\otimes_{\mathscr{A}}\mathscr{B}\longrightarrow\mathscr{K}\otimes_{\mathscr{A}}\mathscr{B}\longrightarrow\mathscr{K}$$

extends as a morphism $\psi_{\mathscr{B}}:\mathscr{G}\otimes_{\mathscr{A}}\mathscr{B}\to\mathscr{K}$ since $\mathscr{K}$ is $\mathscr{B}$-injective. On the other hand, the composition of $\varphi_{\mathscr{B}}$ with

$$\mathscr{F}\longrightarrow\mathscr{F}\otimes 1\lhook\joinrel\longrightarrow\mathscr{F}\otimes_{\mathscr{A}}\mathscr{B}$$

is equal to φ, hence the composition ψ of $\psi_{\mathscr{B}}$ with

$$\mathscr{G}\longrightarrow\mathscr{G}\otimes 1\lhook\joinrel\longrightarrow\mathscr{G}\otimes_{\mathscr{A}}\mathscr{B}$$

extends φ.

are isomorphisms. □

Exercise 3.5.9 **($\mathscr{H}om$, $\varinjlim$ and $\varprojlim$)**

(1) Let A be a ring and let F, G be A-modules. Let $(F_p)_{p\in\mathbb{N}}$ be an exhaustive increasing filtration of F. Show that

$$\mathrm{Hom}_A(F,G)=\varprojlim_p \mathrm{Hom}_A(F_p,G),$$

i.e., giving a morphism $\phi: F\to G$ is equivalent to giving morphisms $\phi_p: F_p\to G$ subject to the condition that $\phi_{p+1|F_p}=\phi_p$.

(2) Let $\mathscr{A}$ be a sheaf of rings on a topological space X and let $\mathscr{F},\mathscr{G}$ be two sheaves of $\mathscr{A}$-modules. Let $(\mathscr{F}_p)_{p\in\mathbb{N}}$ be an exhaustive increasing filtration of $\mathscr{F}$.

(a) Show that, for each open set $U\subset X$, the natural morphism

$$\Gamma(U,\mathscr{H}om_{\mathscr{A}}(\mathscr{F},\mathscr{G}))\longrightarrow \varprojlim_p \Gamma(U,\mathscr{H}om_{\mathscr{A}}(\mathscr{F}_p,\mathscr{G}))=\Gamma(U,\varprojlim_p \mathscr{H}om_{\mathscr{A}}(\mathscr{F}_p,\mathscr{G}))$$

is an isomorphism. [Hint: use (1) and Exercise 1.1.2.]

(b) Conclude that the natural morphism

$$\mathscr{H}om_{\mathscr{A}}(\mathscr{F},\mathscr{G})\longrightarrow \varprojlim_p \mathscr{H}om_{\mathscr{A}}(\mathscr{F}_p,\mathscr{G})$$

is an isomorphism.

Proof of Proposition 3.5.7. We first assume that $\mathscr{M}^\bullet=\mathscr{M}$ consists of a coherent $\mathscr{D}_X$-module. The question is local on X, so we can assume that $\mathscr{M}$ has a good filtration $(F_p\mathscr{M})_{p\in\mathbb{Z}}$ with $F_p\mathscr{M}=0$ for $p\ll 0$. The de Rham complex ${}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}$ is naturally filtered, by setting $F_p(\mathscr{M}\otimes_{\mathscr{O}_X}\wedge^k\Theta_X)=(F_{p-k}\mathscr{M})\otimes_{\mathscr{O}_X}\wedge^k\Theta_X$, and $F_p\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}$ is an object of $\mathsf{C}^{\mathrm{b}}(\mathscr{O}_X,\mathrm{Diff}_X^{\mathrm{f}})$, according to Lemma 3.4.1(2).

Lemma 3.5.10. *For each p, $\mathrm{gr}_p^F\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}$ is a complex in $\mathsf{C}^{\mathrm{b}}(\mathscr{O}_X)$.*

Proof. Let us prove the lemma for left $\mathscr{D}_X$-modules. It is a matter of showing that, if ∇ denotes the connection on $\mathscr{M}$, that we regard as a connection $F_p\mathscr{M}\to\Omega_X^1\otimes F_{p+1}\mathscr{M}$, then the induced map $\mathrm{gr}_1\nabla:\mathrm{gr}_p^F\mathscr{M}\to\Omega_X^1\otimes\mathrm{gr}_{p+1}^F\mathscr{M}$ is $\mathscr{O}_X$-linear. This follows from the Leibniz rule. □

Lemma 3.5.11. *Locally, there exists p_o such that ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathrm{gr}_p^F\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}$ is acyclic for $p\geqslant p_o$.*

Proof. According to the previous lemma,

$${}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathrm{gr}_p^F\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}=(\mathrm{gr}_p^F\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M})\otimes_{\mathscr{O}_X}\mathscr{D}_X.$$

It suffices therefore to prove the acyclicity of $\mathrm{gr}_p^F\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathscr{M}$ in $\mathsf{C}^{\mathrm{b}}(\mathscr{O}_X)$. One can find, locally, a resolution the filtered $\mathscr{D}_X$-module $(\mathscr{M},F_\bullet\mathscr{M})$ (more precisely, a resolution of the associated Rees module, see §1.5) by coherent filtered $\mathscr{D}_X$-modules of the form $\bigoplus_k\mathscr{L}_k\otimes_{\mathscr{O}_X}F_\bullet[n_k]\mathscr{D}_X$, where $\mathscr{L}_k$ is $\mathscr{O}_X$-free of finite rank, $F_\bullet\mathscr{D}_X$ is the standard filtration of $\mathscr{D}_X$, $F_\bullet[n_k]$ is the same filtration shifted by some integer n_k, and the direct sum is finite.

It is therefore enough to prove the assertion for such summands, by a classical argument of homological algebra, and then for $(\mathscr{D}_X, F_\bullet\mathscr{D}_X)$, for which the assertion was proved in Exercise 1.4.4(3) for $p \geqslant 1$. $\square$

We now consider the commutative diagram

$$\begin{tikzcd}
\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathrm{DR}\,\mathscr{M}, \mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X) \arrow[r, "a"] \arrow[d, "b"'] & \mathscr{H}om_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathrm{DR}\,\mathscr{M}, \mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X) \arrow[d, "c"'] \\
\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}F_p\,\mathrm{DR}\,\mathscr{M}, \mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X) \arrow[r, "a_p"] & \mathscr{H}om_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}F_p\,\mathrm{DR}\,\mathscr{M}, \mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)
\end{tikzcd}$$

and we will prove that a_p, c, b, hence a, are quasi-isomorphisms locally for $p \gg 0$.

(1) For a_p, we identify

$$ {}^{\mathrm{diff}}\mathrm{DR}^{-1}F_p\,\mathrm{DR}\,\mathscr{M} \quad \text{to} \quad (\Omega_X^{n+\bullet}\otimes_{\mathscr{O}_X} F_{p+\bullet}\mathscr{M}^{\mathrm{left}}\otimes_{\mathscr{O}_X}\mathscr{D}_X, {}^{\mathrm{diff}}\mathrm{DR}^{-1}(\nabla)),$$

as in the proof of Proposition 3.5.6. Then a_p is termwise an isomorphism for each p, according to Exercise 3.4.2(2).

(2) According to Exercise 3.5.9(2), we identify

$$\mathscr{H}om_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathrm{DR}\,\mathscr{M}, \mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)$$

to

$$\varprojlim_p \mathscr{H}om_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}F_p\,\mathrm{DR}\,\mathscr{M}, \mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)$$

and thus to

$$\varprojlim_p \mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}F_p\,\mathrm{DR}\,\mathscr{M}, \mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X).$$

That Condition 3.5.8(1) is fulfilled follows from the $\mathscr{O}_X$-injectivity of $\mathscr{L}'$ and (3.5.2). Condition 3.5.8(2) follows from Lemma 3.5.11. Therefore, c is a quasi-isomorphism for $p \gg 0$, according to Lemma 3.5.8.

(3) In order to prove that b is a quasi-isomorphism, we will first analyze the differential of the corresponding complexes. We cannot apply directly Lemma 3.5.8 to these complexes, since the terms are of the form

$$\mathscr{H}om_{\mathscr{O}_X}(\Omega_X^k\otimes\mathscr{M}^{\mathrm{left}}, \mathscr{L}')\otimes_{\mathscr{O}_X}\mathscr{D}_X$$

and we cannot assert that $\varprojlim_p(\bullet_p)\otimes_{\mathscr{O}_X}\mathscr{D}_X = \varprojlim_p(\bullet_p\otimes_{\mathscr{O}_X}\mathscr{D}_X)$ for a projective system $\bullet_p$ of $\mathscr{O}_X$-modules, since $\mathscr{D}_X$ is not of finite rank over $\mathscr{O}_X$. We will thus consider the filtration $F_\bullet\mathscr{D}_X$ in order to use $\varprojlim_p$.

The differential ∂ of the complex $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathrm{DR}\,\mathscr{M}, \mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)$ sends a local section φ of $\mathscr{H}om_{\mathscr{O}_X}(\Omega_X^{n+k+1}\otimes\mathscr{M}^{\mathrm{left}}, \mathscr{L}')$ to the section

$$\varphi\circ\nabla + \varphi(\bullet\otimes\nabla(1))$$

of $\mathscr{H}om_{\mathscr{O}_X}(\Omega_X^{n+k} \otimes \mathscr{M}^{\mathrm{left}}, \mathscr{L}' \otimes F_1\mathscr{D}_X)$, where $\nabla(1)$ is the same as in Exercise 1.1.9(1).

We can therefore filter the complex $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}\mathrm{DR}\,\mathscr{M}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ by setting

$$G_p\,\mathscr{H}om_{\mathscr{O}_X}(\Omega_X^{n-k} \otimes \mathscr{M}^{\mathrm{left}}, \mathscr{L}' \otimes \mathscr{D}_X) = \mathscr{H}om_{\mathscr{O}_X}(\Omega_X^{n-k} \otimes \mathscr{M}^{\mathrm{left}}, \mathscr{L}' \otimes F_{p+k}\mathscr{D}_X).$$

The differential $\mathrm{gr}_1\partial$ of the graded complex satisfies $\mathrm{gr}_1\partial(\varphi) = \varphi(\bullet \otimes \nabla(1))$. The graded complex can then be identified, up to a shift by n in the grading, to the complex $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{L}' \otimes \mathrm{gr}^F \mathrm{Sp}^\bullet \mathscr{D}_X)$. According to Exercise 1.1.9, we find that the inclusion of complexes

$$G_0\,\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}\mathrm{DR}\,\mathscr{M}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X) \hookrightarrow \mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}\mathrm{DR}\,\mathscr{M}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$$

is a quasi-isomorphism. A similar argument applies for each p to the complex $\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}F_p\,\mathrm{DR}\,\mathscr{M}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$. We can now apply Exercise 3.5.9(2) to terms of the complexes $G_0(\bullet)$ in order to obtain

$$\begin{aligned} &G_0\,\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}\mathrm{DR}\,\mathscr{M}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X) \\ &\qquad = \varprojlim_p G_0\,\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}F_p\,\mathrm{DR}\,\mathscr{M}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X). \end{aligned}$$

Since

$$\begin{aligned} &\big[G_0\,\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}F_p\,\mathrm{DR}\,\mathscr{M}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)\big]^k \\ &\qquad = \mathscr{H}om_{\mathscr{O}_X}(\Omega_X^{n-k} \otimes F_{p-k}\mathscr{M}^{\mathrm{left}}, \mathscr{L}' \otimes F_{p+k}\mathscr{D}_X) =: \mathscr{F}_p^k, \end{aligned}$$

Condition 3.5.8(1) holds by the $\mathscr{O}_X$-injectivity of $\mathscr{L}' \otimes F_{p+k}\mathscr{D}_X$ and Condition 3.5.8(2) follows moreover from the local acyclicity of ${}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1}\mathrm{gr}_p^F\,\mathrm{DR}\,\mathscr{M}$ for $p \gg 0$.

End of the proof of Proposition 3.5.7. We continue to assume that $\mathscr{M}^\bullet = \mathscr{M}$, and we will prove that the composition of the arrows in (3.5.5) is an isomorphism. This is a local question. Let $\mathscr{M}^\bullet$ be a resolution of $\mathscr{M}$ by locally free $\mathscr{D}_X$-modules of finite rank. Then ${}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1\,\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^\bullet$ is a resolution of ${}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1\,\mathrm{diff}}\mathrm{DR}\,\mathscr{M}$ by locally free right $\mathscr{D}_X$-modules, so by definition

$$\mathscr{H}om_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1\,\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^\bullet, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X) = \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1\,\mathrm{diff}}\mathrm{DR}\,\mathscr{M}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X).$$

On the other hand, Proposition 3.5.6 applied to the acyclic complex $\cdots \to \mathscr{M}^1 \to \operatorname{Ker} d_0 \to 0$, with $d_0 : \mathscr{M}^0 \to \mathscr{M}$, shows that

$$\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1\,\mathrm{diff}}\mathrm{DR}\,\mathscr{M}, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X) \longrightarrow \mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1\,\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^\bullet, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X).$$

is a quasi-isomorphism. Since

$$\mathscr{H}om^{\mathrm{i,f}}_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1\,\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^k, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X) \longrightarrow \mathscr{H}om_{\mathscr{D}_X}({}^{\mathrm{diff}}\mathrm{DR}^{\text{-}1\,\mathrm{diff}}\mathrm{DR}\,\mathscr{M}^k, \mathscr{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$$

is an isomorphism for each k, we obtain that (3.5.5) is an isomorphism when $\mathscr{M}$ is $\mathscr{D}_X$-coherent. By using similar arguments, we get that (3.5.5) is an isomorphism for any bounded complex $\mathscr{M}^\bullet$ with $\mathscr{D}_X$-coherent cohomology. ☐

3.6. Behaviour with respect to external tensor product

3.6.a. The $\mathbb{C}$-external tensor product. It is straightforward to adapt the results of §§3.2–3.5 to the sheaves $\mathscr{O}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{O}_{X_2}$ and $\mathscr{D}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{D}_{X_2}$. Furthermore, we have a natural identification

$$(3.6.1)\quad (\mathscr{L}_1 \otimes_{\mathscr{O}_{X_1}} \mathscr{D}_{X_1}) \boxtimes_{\mathbb{C}} (\mathscr{L}_2 \otimes_{\mathscr{O}_{X_2}} \mathscr{D}_{X_2}) \simeq (\mathscr{L}_1 \boxtimes_{\mathbb{C}} \mathscr{L}_2) \otimes_{\mathscr{O}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{O}_{X_2}} (\mathscr{D}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{D}_{X_2}).$$

In particular, the $\mathbb{C}$-external product $\boxtimes_{\mathbb{C}}$ defines functors

$$\mathsf{M}(\mathscr{O}_{X_1}, \mathrm{Diff}^{\mathrm{f}}_{X_1}) \times \mathsf{M}(\mathscr{O}_{X_2}, \mathrm{Diff}^{\mathrm{f}}_{X_2}) \longmapsto \mathsf{M}(\mathscr{O}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{O}_{X_2}, \mathrm{Diff}^{\mathrm{f}}_{X_1} \boxtimes_{\mathbb{C}} \mathrm{Diff}^{\mathrm{f}}_{X_2}),$$

where $\mathrm{Diff}^{\mathrm{f}}_{X_1} \boxtimes_{\mathbb{C}} \mathrm{Diff}^{\mathrm{f}}_{X_2}$ refers to differential complexes over $X_1 \times X_2$ with respect to the sheaves of rings $\mathscr{O}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{O}_{X_2}$ and $\mathscr{D}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{D}_{X_2}$. Such functors extend to the derived categories D^b and D^+. Then one checks that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{M}(\mathscr{O}_{X_1}, \mathrm{Diff}^{\mathrm{f}}_{X_1}) \times \mathsf{M}(\mathscr{O}_{X_2}, \mathrm{Diff}^{\mathrm{f}}_{X_2}) \arrow[r, mapsto] \arrow[d, mapsto, "\mathsf{Forget}"'] \arrow[d, mapsto, "\mathsf{Forget}"] & \mathsf{M}(\mathscr{O}_{X_1} \boxtimes_{\mathbb{C}} \mathscr{O}_{X_2}, \mathrm{Diff}^{\mathrm{f}}_{X_1} \boxtimes_{\mathbb{C}} \mathrm{Diff}^{\mathrm{f}}_{X_2}) \arrow[d, mapsto, "\mathsf{Forget}"] \\
\mathsf{M}(\mathbb{C}_{X_1}) \times \mathsf{M}(\mathbb{C}_{X_2}) \arrow[r, mapsto] & \mathsf{M}(\mathbb{C}_{X_1 \times X_2})
\end{tikzcd}$$

and similarly for D^{b} and D^+.

3.6.b. The Diff-external tensor product. We keep the notation of Exercise 2.2.12. Recall that we have defined bifunctors

$$\boxtimes_{\mathscr{O}} : \mathsf{M}(\mathscr{O}_{X_1}) \times \mathsf{M}(\mathscr{O}_{X_2}) \longmapsto \mathsf{M}(\mathscr{O}_{X_1 \times X_2}),$$
$$\boxtimes_{\mathscr{D}} : \mathsf{M}(\mathscr{D}_{X_1}) \times \mathsf{M}(\mathscr{D}_{X_2}) \longmapsto \mathsf{M}(\mathscr{D}_{X_1 \times X_2}).$$

From (3.6.1) it immediately follows that

$$(3.6.2)\qquad (\mathscr{L}_1 \otimes_{\mathscr{O}_{X_1}} \mathscr{D}_{X_1}) \boxtimes_{\mathscr{D}} (\mathscr{L}_2 \otimes_{\mathscr{O}_{X_2}} \mathscr{D}_{X_2}) \simeq (\mathscr{L}_1 \boxtimes_{\mathscr{O}} \mathscr{L}_2) \otimes_{\mathscr{O}_{X_1 \times X_2}} \mathscr{D}_{X_1 \times X_2}.$$

Given $\mathscr{D}_{X_i}$-linear morphisms $v_i : \mathscr{L}_i \otimes_{\mathscr{O}_{X_i}} \mathscr{D}_{X_i} \to \mathscr{L}'_i \otimes_{\mathscr{O}_{X_i}} \mathscr{D}_{X_1}$ $(i = 1, 2)$, $v_1 \boxtimes_{\mathbb{C}} v_2$ induces a morphism

$$v_1 \boxtimes_{\mathscr{D}} v_2 : (\mathscr{L}_1 \otimes_{\mathscr{O}_{X_1}} \mathscr{D}_{X_1}) \boxtimes_{\mathscr{D}} (\mathscr{L}_2 \otimes_{\mathscr{O}_{X_2}} \mathscr{D}_{X_2}) \longrightarrow (\mathscr{L}'_1 \otimes_{\mathscr{O}_{X_1}} \mathscr{D}_{X_1}) \boxtimes_{\mathscr{D}} (\mathscr{L}'_2 \otimes_{\mathscr{O}_{X_2}} \mathscr{D}_{X_2}),$$

that is,

$$v_1 \boxtimes_{\mathscr{D}} v_2 : (\mathscr{L}_1 \boxtimes_{\mathscr{O}} \mathscr{L}_2) \otimes_{\mathscr{O}_{X_1 \times X_2}} \mathscr{D}_{X_1 \times X_2} \longrightarrow (\mathscr{L}'_1 \boxtimes_{\mathscr{O}} \mathscr{L}'_2) \otimes_{\mathscr{O}_{X_1 \times X_2}} \mathscr{D}_{X_1 \times X_2},$$

so that the external product $\boxtimes_{\mathscr{D}}$ is a bifunctor $\mathsf{M}_{\mathrm{i}}(\mathscr{D}_{X_1}) \times \mathsf{M}_{\mathrm{i}}(\mathscr{D}_{X_2}) \mapsto \mathsf{M}_{\mathrm{i}}(\mathscr{D}_{X_1 \times X_2})$. Similarly, it extends as a bifunctor $\mathsf{M}_{\mathrm{i,f}}(\mathscr{D}_{X_1}) \times \mathsf{M}_{\mathrm{i,f}}(\mathscr{D}_{X_2}) \mapsto \mathsf{M}_{\mathrm{i,f}}(\mathscr{D}_{X_1 \times X_2})$. Furthermore, this construction can be enhanced to the categories of complexes and the derived categories (the $\mathscr{D}$-external tensor product of complexes, one of which is quasi-isomorphic to zero, is also quasi-isomorphic to zero).

From (3.6.2) we deduce that

$$^{\mathrm{diff}}\mathrm{DR}\left[(\mathscr{L}_1 \otimes_{\mathscr{O}_{X_1}} \mathscr{D}_{X_1}) \boxtimes_{\mathscr{D}} (\mathscr{L}_2 \otimes_{\mathscr{O}_{X_2}} \mathscr{D}_{X_2})\right] = \mathscr{L}_1 \boxtimes_{\mathscr{O}} \mathscr{L}_2.$$

Lemma 3.6.3. *The bifunctor* $\boxtimes_{\mathscr{O}} : \mathbf{M}(\mathscr{O}_{X_1}) \times \mathbf{M}(\mathscr{O}_{X_2}) \mapsto \mathbf{M}(\mathscr{O}_{X_1\times X_2})$ *extends as a bifunctor*

$$\boxtimes_{\mathrm{Diff}} : \mathbf{M}(\mathscr{O}_{X_1}, \mathrm{Diff}^{\mathrm{f}}_{X_1}) \times \mathbf{M}(\mathscr{O}_{X_2}, \mathrm{Diff}^{\mathrm{f}}_{X_2}) \longmapsto \mathbf{M}(\mathscr{O}_{X_1\times X_2}, \mathrm{Diff}^{\mathrm{f}}_{X_1\times X_2}),$$

by setting

$$\mathscr{L}_1 \boxtimes_{\mathrm{Diff}} \mathscr{L}_2 = \mathscr{L}_1 \boxtimes_{\mathscr{O}} \mathscr{L}_2 \quad \textit{and} \quad u_1 \boxtimes_{\mathrm{Diff}} u_2 = \mathscr{H}^0\, {}^{\mathrm{p}}\mathrm{DR}(v_1 \boxtimes_{\mathscr{D}} v_2),$$

if $u_i = \mathscr{H}^0\, {}^{\mathrm{p}}\mathrm{DR}(v_i)$ *(*$i = 1, 2$*).* □

These constructions extend to the derived categories $\mathbf{D}^{\mathrm{b}}$ and $\mathbf{D}^{+}$, and by definition they are compatible with the functors ${}^{\mathrm{diff}}\mathrm{DR}$ and ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$, i.e., the following diagrams commute:

$$\begin{tikzcd}
\mathbf{D}^{\mathrm{b}}(\mathscr{D}_{X_1}) \arrow[d, mapsto, "{}^{\mathrm{diff}}\mathrm{DR}"'] & \times & \mathbf{D}^{\mathrm{b}}(\mathscr{D}_{X_2}) \arrow[r, mapsto, "\boxtimes_{\mathscr{D}}"] \arrow[d, mapsto, "{}^{\mathrm{diff}}\mathrm{DR}"] & \mathbf{D}^{\mathrm{b}}(\mathscr{D}_{X_1\times X_2}) \arrow[d, mapsto, "{}^{\mathrm{diff}}\mathrm{DR}"] \\
\mathbf{D}^{\mathrm{b}}(\mathscr{O}_{X_1}, \mathrm{Diff}^{\mathrm{f}}_{X_1}) & \times & \mathbf{D}^{\mathrm{b}}(\mathscr{O}_{X_2}, \mathrm{Diff}^{\mathrm{f}}_{X_2}) \arrow[r, mapsto, "\boxtimes_{\mathrm{Diff}}"] & \mathbf{D}^{\mathrm{b}}(\mathscr{O}_{X_1\times X_2}, \mathrm{Diff}^{\mathrm{f}}_{X_1\times X_2})
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{D}^{\mathrm{b}}(\mathscr{D}_{X_1}) & \times & \mathbf{D}^{\mathrm{b}}(\mathscr{D}_{X_2}) \arrow[r, mapsto, "\boxtimes_{\mathscr{D}}"] & \mathbf{D}^{\mathrm{b}}(\mathscr{D}_{X_1\times X_2}) \\
\mathbf{D}^{\mathrm{b}}(\mathscr{O}_{X_1}, \mathrm{Diff}^{\mathrm{f}}_{X_1}) \arrow[u, mapsto, "{}^{\mathrm{diff}}\mathrm{DR}^{-1}"] & \times & \mathbf{D}^{\mathrm{b}}(\mathscr{O}_{X_2}, \mathrm{Diff}^{\mathrm{f}}_{X_2}) \arrow[u, mapsto, "{}^{\mathrm{diff}}\mathrm{DR}^{-1}"'] \arrow[r, mapsto, "\boxtimes_{\mathrm{Diff}}"] & \mathbf{D}^{\mathrm{b}}(\mathscr{O}_{X_1\times X_2}, \mathrm{Diff}^{\mathrm{f}}_{X_1\times X_2}) \arrow[u, mapsto, "{}^{\mathrm{diff}}\mathrm{DR}^{-1}"']
\end{tikzcd} \tag{3.6.4}$$

Let us denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X_1}, \mathrm{Diff}^{\mathrm{f}}_{X_1})$ (resp. $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{O}_{X_1}, \mathrm{Diff}^{\mathrm{f}}_{X_1})$) the full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathscr{O}_{X_1}, \mathrm{Diff}^{\mathrm{f}}_{X_1})$ whose objects $\mathscr{L}_1^{\bullet}$ are such that ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\mathscr{L}_1^{\bullet}$ has $\mathscr{D}_{X_1}$-coherent (resp. $\mathscr{D}_{X_1}$-holonomic) cohomology. Then, according to Proposition 5.5.2 and Exercise 3.3.10, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{O}_{X_1}, \mathrm{Diff}^{\mathrm{f}}_{X_1}) \arrow[d, mapsto, "\mathsf{Forget}"'] & \times & \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X_2}, \mathrm{Diff}^{\mathrm{f}}_{X_2}) \arrow[r, mapsto, "\boxtimes_{\mathrm{Diff}}"] \arrow[d, mapsto, "\mathsf{Forget}"] & \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X_1\times X_2}, \mathrm{Diff}^{\mathrm{f}}_{X_1\times X_2}) \arrow[d, mapsto, "\mathsf{Forget}"] \\
\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X_1}) & \times & \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X_2}) \arrow[r, mapsto, "\boxtimes_{\mathbb{C}}"] & \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X_1\times X_2})
\end{tikzcd}$$

We deduce the commutativity of

$$\begin{tikzcd}
\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X_1}) \times \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X_2}) \arrow[r, mapsto, "\boxtimes_{\mathscr{D}}"] \arrow[d, mapsto, "{}^{\mathrm{p}}\mathrm{DR}"'] & \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X_1\times X_2}) \arrow[d, mapsto, "{}^{\mathrm{p}}\mathrm{DR}"] \\
\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_{X_1}) \times \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_{X_2}) \arrow[r, mapsto, "\boxtimes_{\mathbb{C}}"] & \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_{X_1\times X_2})
\end{tikzcd} \tag{3.6.5}$$

LECTURE 4

DIRECT IMAGES OF $\mathscr{D}_X$-MODULES

The notion of direct image of a $\mathscr{D}$-module answers the following problem: given a C^∞ differential form η of maximal degree on a complex manifold X, which satisfies a linear system of holomorphic differential equations (recall that $\mathscr{D}_X$ acts on the right on the sheaf $\mathscr{E}_X^{n,n}$ of forms of maximal degree), what can be said of the form (or more generally the current) obtained by integrating η along the fibres of a holomorphic map $f : X \to Y$? Does it satisfy a finite (i.e., coherent) system of holomorphic differential equations on Y? How can one define intrinsically this system?

Such a question arises in many domains of algebraic geometry. The system of differential equation is often called the "Picard-Fuchs system", or the Gauss-Manin system. A way of "solving" a linear system of holomorphic or algebraic differential equations on a space Y consists in recognizing in this system the Gauss-Manin system attached to some holomorphic or algebraic function $f : X \to Y$. The geometric properties of f induce interesting properties of the system. Practically, this reduces to expressing solutions of the system as integrals over the fibers of f of some differential forms.

The definition of the direct image of a $\mathscr{D}$-module cannot be as simple as that of the direct image of a sheaf. One is faced to a problem which arises in differential geometry: the cotangent map of a holomorphic map $f : X \to Y$ is not a map from the cotangent space T^*Y of Y to that of X, but is a bundle map from the pull-back bundle f^*T^*Y to T^*X. In other words, a vector field on X does not act as a derivation on functions on Y. The transfer module $\mathscr{D}_{X\to Y}$ will give a reasonable solution to this problem.

We have seen that the notion of a left $\mathscr{D}_X$-module is equivalent to that of an $\mathscr{O}_X$-module equipped with a flat connection. Correspondingly, there are two notions of direct images.

- The direct image of an $\mathscr{O}_X$-module with a flat connection is known as the *Gauss-Manin* connection attached to te original one. This notion is only cohomological. Although many examples were given some centuries ago (related to the differential equations satisfied by the periods of a family of elliptic curves), the systematic construction was only achieved in [**KO68**]. The construction with a filtration is due to Griffiths [**Gri70a, Gri70b**] (the main result is called *Griffiths' transversality theorem*). There is a strong constraint however: the map should be smooth (i.e., without critical points).

- The direct image of left $\mathscr{D}$-modules was constructed in [**SKK73**]. This construction has the advantage of being very functorial, and defined at the level of derived categories, not only at the cohomology level as is the first one. It is very flexible. The filtered analogue is straightforward. It appears as a basic tool in various questions in algebraic geometry.

4.1. Example of computation of a Gauss-Manin differential equation

Let $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial in n variables. Consider the following integral depending on a parameter t:

$$I(t) = \int_{\mathbf{T}^n} \frac{\omega}{f-t}, \qquad \omega = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n},$$

where $\mathbf{T}^n$ is the real torus $\{|x_i| = 1 \,\forall\, i\}$.

Proposition 4.1.1. *There exists a non-zero differential operator $q(t, \partial_t)$ with polynomial coefficients such that $q(t, \partial_t)I(t) = 0$.*

We will show how to compute algebraically this differential operator. Denote by $\Omega^k = \Omega^k((\mathbb{C}^*)^n)$ the space of differential forms of degree k with Laurent polynomials as coefficients and by $\Omega^k[\tau]$ the space of polynomials in the new variable τ with coefficients in Ω^k. The differential $d : \Omega^k \to \Omega^{k+1}$ gives rise to a twisted differential

$$d - \tau df \wedge : \Omega^k[\tau] \longrightarrow \Omega^{k+1}[\tau].$$

Lemma 4.1.2. *We have $(d - \tau df\wedge)^2 = 0$, hence $(\Omega^\bullet[\tau], d - \tau df\wedge)$ is a complex.*

Definition 4.1.3. The k-th Gauss-Manin system $\mathrm{GM}^k(f)$ is defined as the k-th cohomology $H^k\big(\Omega^\bullet[\tau], d - \tau df \wedge\big)$.

Lemma 4.1.4. *The following action:*

$$\begin{aligned} \partial_t \cdot (\textstyle\sum \eta_i \tau^i) &= \textstyle\sum \eta_i \tau^{i+1} \\ t \cdot (\textstyle\sum \eta_i \tau^i) &= \textstyle\sum (f\eta_i - (i+1)\eta_{i+1})\tau^i \end{aligned}$$

defines an action of the Weyl algebra $\mathbb{C}[t]\langle\partial_t\rangle := \mathscr{D}(\mathbf{A}_t^1)$ on $\mathrm{GM}^k(f)$ for each k. □

The Gauss-Manin systems $\mathrm{GM}^\bullet(f)$ are an algebraic version of the direct image $f_+\mathscr{O}_{(\mathbb{C}^*)^n}$ that we will consider later.

Theorem 4.1.5 **(Bernstein).** *Each non-zero element of $\mathrm{GM}^k(f)$ is annihilated by a non-zero element of $\mathbb{C}[t]\langle\partial_t\rangle$.* □

Proposition 4.1.6. *Let $p \in \mathbb{C}[t]\langle\partial_t\rangle$ be such that $p \cdot [\omega] = 0$ in $\mathrm{GM}^n(f)$. Then there exists $N \geqslant 0$ such that*

$$\partial_t^N \cdot p(t, \partial_t) \cdot I(t) = 0.$$

Proof. We write p as $p = \sum_{i=0}^d \partial_t^i a_i(t)$. The relation $p \cdot [\omega] = 0$ shows that there exists $k \geqslant 0$ and $\eta_0, \dots, \eta_{d+k} \in \Omega^{n-1}$ such that

$$\begin{aligned} 0 &= \qquad\quad - df \wedge \eta_{d+k} \\ 0 &= d\eta_{d+k} - df \wedge \eta_{d+k-1} \\ &\;\;\vdots \\ 0 &= d\eta_{d+1} - df \wedge \eta_d \\ a_d \circ f \cdot \omega &= d\eta_d - df \wedge \eta_{d-1} \\ &\;\;\vdots \\ a_0 \circ f \cdot \omega &= d\eta_0. \end{aligned} \tag{4.1.7}$$

Claim. *We have for all j, ℓ:*

$$\int_{\mathbf{T}^n} \frac{d\eta_j}{(f-t)^\ell} = \ell \int_{\mathbf{T}^n} \frac{df \wedge \eta_j}{(f-t)^{\ell+1}}.$$

This follows from Stokes formula: $\displaystyle\int_{\mathbf{T}^n} d\big(\eta_j/(f-t)^\ell\big) = 0$. From (4.1.7) we get

$$\int_{\mathbf{T}^n} \frac{d\eta_d}{(f-t)} = \int_{\mathbf{T}^n} \frac{df \wedge \eta_d}{(f-t)^2} = \int_{\mathbf{T}^n} \frac{d\eta_{d+1}}{(f-t)^2} = \cdots = 0.$$

On the other hand, let us work modulo $\mathbb{C}[t]$, and use the sign $\equiv$ instead of $=$. For any polynomial $a(t)$ we thus have

$$a(t) \int_{\mathbf{T}^n} \frac{\omega}{(f-t)} \equiv \int_{\mathbf{T}^n} \frac{a \circ f \cdot \omega}{(f-t)} \quad \mathrm{mod}\ \mathbb{C}[t],$$

since $[a(t) - a \circ f]/(f-t) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}][t]$. Now,

$$\frac{d}{dt} a_d(t) \int_{\mathbf{T}^n} \frac{\omega}{f-t} \equiv \frac{d}{dt} \int_{\mathbf{T}^n} \frac{a_d \circ f \cdot \omega}{f-t} \equiv \frac{d}{dt} \int_{\mathbf{T}^n} -\frac{df \wedge \eta_d}{f-t} = \int_{\mathbf{T}^n} -\frac{df \wedge \eta_d}{(f-t)^2},$$

and by using Stokes formula,

$$\frac{d}{dt} a_d(t) \int_{\mathbf{T}^n} \frac{\omega}{f-t} \equiv \int_{\mathbf{T}^n} -\frac{d\eta_{d-1}}{f-t} \equiv -a_{d-1}(t) \int_{\mathbf{T}^n} \frac{\omega}{f-t} - \int_{\mathbf{T}^n} \frac{df \wedge \eta_{d-2}}{f-t}.$$

Iterating this reasoning (by applying d/dt once more, etc.) gives:

$$p(t, \partial_t) \int_{\mathbf{T}^n} \frac{\omega}{f-t} \equiv 0, \quad \text{i.e.,}\ \in \mathbb{C}[t].$$

Applying now a sufficiently high power of d/dt to kill the polynomial, we get

$$\partial_t^N p(t, \partial_t) \int_{\mathbf{T}^n} \frac{\omega}{f-t} = 0. \qquad \square$$

4.2. Inverse images of left $\mathscr{D}$-modules

Let us begin with some relative complements to § 1.2. Let $f : X \to Y$ be a holomorphic map between analytic manifolds. For any section ξ of the sheaf Θ_X of vector fields on X, $Tf(\xi)$ is a local section of $\mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\Theta_Y$. We hence have an $\mathscr{O}_X$-linear map

$$Tf : \Theta_X \longrightarrow \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\Theta_Y,$$

and dually

$$T^*f : \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} \Omega^1_Y \longrightarrow \Omega^1_X.$$

Therefore, if $\mathscr{N}$ is any left $\mathscr{D}_Y$-module, the connection ∇^Y on $\mathscr{N}$ can be lifted as a connection

$$\nabla^X : \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{N} \longrightarrow \Omega^1_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{N}$$

by setting

$$\nabla^X = d \otimes \mathrm{Id} + (T^*f \otimes \mathrm{Id}_{\mathscr{N}}) \circ (1 \otimes \nabla^Y). \tag{4.2.1}$$

Exercise 4.2.2 **(Definition of the inverse image of a left $\mathscr{D}_X$-module)**

(1) Show that the connection ∇^X on $f^*\mathscr{N} := \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{N}$ is integrable and defines the structure of a left $\mathscr{D}_X$-module on $f^*\mathscr{N}$. The corresponding $\mathscr{D}_X$-module is denoted by $f^+\mathscr{N}$.

(2) Show that, if $\mathscr{N}$ also has a right $\mathscr{D}_Y$-module structure commuting with the left one, then ∇^X is right $f^{-1}\mathscr{D}_Y$-linear, and $f^+\mathscr{N}$ is a right $f^{-1}\mathscr{D}_Y$-module.

Exercise 4.2.3

(1) Express the previous connection in local coordinates on X and Y.

(2) Show that, if $\mathscr{M}^{\text{left}}$ is any left $\mathscr{D}_X$-module and $\mathscr{N}$ any left $f^{-1}\mathscr{D}_Y$-module, then $\mathscr{M}^{\text{left}} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{N}$ may be equipped with a left $\mathscr{D}_X$-module structure: if ξ is a local vector field on X, set

$$\xi \cdot (m \otimes n) = (\xi m) \otimes n + Tf(\xi)(m \otimes n).$$

[Hint: identify $\mathscr{M}^{\text{left}} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{N}$ with $\mathscr{M}^{\text{left}} \otimes_{\mathscr{O}_X} f^+\mathscr{N}$ and use Exercise 4.2.2.]

Definition 4.2.4 **(Transfer modules, see, e.g. [CJ93] for details)**

(1) The sheaf $\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y = f^+\mathscr{D}_Y$ is a left-right $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule when using the natural right $f^{-1}\mathscr{D}_Y$-module structure and the left $\mathscr{D}_X$-module introduced above.

(2) The sheaf $\mathscr{D}_{Y\leftarrow X}$ is obtained from $\mathscr{D}_{X\to Y}$ by using the usual left-right transformation on both sides:

$$\mathscr{D}_{Y\leftarrow X} = \mathscr{H}om_{f^{-1}\mathscr{O}_Y}\big(\omega_Y, \omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y}\big).$$

Notice that, $\mathscr{D}_Y$ being a locally free $\mathscr{O}_Y$-module, $f^+\mathscr{D}_Y$ is a locally free $\mathscr{O}_X$-module.

Choose local coordinates $x_1, \dots, x_n$ on X and $y_1, \dots, y_m$ on Y. Then $\mathscr{D}_{X\to Y} = \mathscr{O}_X[\partial_{y_1}, \dots, \partial_{y_m}]$. The left $\mathscr{D}_X$-structure is given by

$$\partial_{x_i} \cdot \sum_\alpha a_\alpha(x)\partial_y^\alpha = \sum_\alpha \Big(\frac{\partial a_\alpha}{\partial x_i} + \sum_{j=1}^m a_\alpha(x)\frac{\partial f_j}{\partial x_i}\partial_{y_j}\Big)\partial_y^\alpha.$$

Exercise 4.2.5 **($\mathscr{D}_{X\to Y}$ for a closed embedding).** Assume that X is a complex submanifold of Y of codimension d, defined by $g_1 = \cdots = g_d = 0$, where the g_i are holomorphic functions on Y. Show that

$$\mathscr{D}_{X\to Y} = \mathscr{D}_Y \big/ \textstyle\sum_{i=1}^d g_i\mathscr{D}_Y$$

with its natural right $\mathscr{D}_Y$ structure. In local coordinates $(x_1, \dots, x_n, y_1, \dots, y_d)$ such that $g_i = y_i$, show that $\mathscr{D}_{X\to Y} = \mathscr{D}_X[\partial_{y_1}, \dots, \partial_{y_d}]$.

Conclude that, if f is an embedding, the sheaves $\mathscr{D}_{X\to Y}$ and $\mathscr{D}_{Y\leftarrow X}$ are locally free over $\mathscr{D}_X$.

Exercise 4.2.6 **(Filtration of $\mathscr{D}_{X\to Y}$).** Put $F_k\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}F_k\mathscr{D}_X$. Show that this defines a filtration (see Definition 1.5.1) of $\mathscr{D}_{X\to Y}$ as a left $\mathscr{D}_X$-module and as a right $f^{-1}\mathscr{D}_Y$-module, and that $\operatorname{gr}^F\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\operatorname{gr}^F\mathscr{D}_Y$.

Exercise 4.2.7 **(The chain rule).** Consider holomorphic maps $f : X \to Y$ and $g : Y \to Z$.

(1) Give an canonical isomorphism $\mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{D}_{Y\to Z} \xrightarrow{\sim} \mathscr{D}_{X\to Z}$ as right $(g\circ f)^{-1}\mathscr{D}_Z$-modules.

(2) Use the chain rule to show that this isomorphism is left $\mathscr{D}_X$-linear.

(3) Same question with filtrations $F_\bullet$.

We can now give a better definition of the inverse image of a left $\mathscr{D}_Y$-module $\mathscr{N}$, better in the sense that it is defined inside of the category of $\mathscr{D}$-modules. It also allows one to give a definition of a derived inverse image.

Definition 4.2.8 **(of the inverse image of a left $\mathscr{D}_Y$-module).** Let $\mathscr{N}$ be a left $\mathscr{D}_Y$-module. The inverse image $f^+\mathscr{N}$ is the left $\mathscr{D}_X$-module $\mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathscr{D}_Y} \mathscr{N}$.

Exercise 4.2.9

(1) Show that the provious definition coincides with that of Exercise 4.2.2(1).

(2) Let $f : X \to Y$, $g : Y \to Z$ be holomorphic maps and let $\mathscr{N}$ be a left $\mathscr{D}_Z$-module. Show that $(g\circ f)^+\mathscr{N} \simeq f^+(g^+\mathscr{N})$.

The *derived inverse image* $\boldsymbol{L}f^+\mathscr{N}$ is now defined by the usual method, i.e., by taking a flat resolution of $\mathscr{N}$ as a left $\mathscr{D}_Y$-module, or by taking a right $f^{-1}\mathscr{D}_Y$-flat resolution of $\mathscr{D}_{X\to Y}$ by $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodules. The cohomology modules $\boldsymbol{L}^j f^+\mathscr{N} := \mathscr{T}\!or_j^{f^{-1}\mathscr{D}_Y}(\mathscr{D}_{X\to Y}, f^{-1}\mathscr{N})$ are left $\mathscr{D}_X$-modules.

4.3. Direct images of right $\mathscr{D}$-modules

Recall that the Spencer complex $\mathrm{Sp}^\bullet_X(\mathscr{D}_X)$, which was defined in 1.4.2, is a complex of left $\mathscr{D}_X$-modules. Denote by $\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ the complex $\mathrm{Sp}^\bullet_X(\mathscr{D}_X)\otimes_{\mathscr{O}_X}\mathscr{D}_{X\to Y}$ (the left $\mathscr{O}_X$-structure on each factor is used for the tensor product). It is a complex of $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodules: the right $f^{-1}\mathscr{D}_Y$ structure is the trivial one; the left $\mathscr{D}_X$-structure is that defined by Exercise 1.2.6(1).

***Exercise 4.3.1* (The (filtered) relative Spencer complex)**

(1) Show that $\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ is a resolution of $\mathscr{D}_{X\to Y}$ as a bimodule.

(2) Show that the terms of the complex $\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ are locally free left $\mathscr{D}_X$-modules. [Hint: use Exercise 1.2.8(4).]

(3) Define the filtration of $\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ by the formula

$$F_\ell\,\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)=\sum_{j+k=\ell}F_j\,\mathrm{Sp}^\bullet_X(\mathscr{D}_X)\otimes_{f^{-1}\mathscr{O}_Y}F_k\mathscr{D}_{X\to Y},$$

where the filtration on the Spencer complex is defined in Exercise 1.4.4. Show that, for any ℓ, $F_\ell\,\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ is a resolution of $F_\ell\mathscr{D}_{X\to Y}$.

Examples 4.3.2

(1) For $f=\mathrm{Id}:X\to X$, the complex $\mathrm{Sp}^\bullet_{X\to X}(\mathscr{D}_X)=\mathscr{D}_X\otimes_{\mathscr{O}_X}\mathrm{Sp}^\bullet_X(\mathscr{D}_X)$ is a resolution of $\mathscr{D}_{X\to X}=\mathscr{D}_X$ as a left and right $\mathscr{D}_X$-module (notice that the left structure of $\mathscr{D}_X$ is used for the tensor product).

(2) For $f:X\to\mathrm{pt}$, the complex $\mathrm{Sp}^\bullet_{X\to\mathrm{pt}}(\mathscr{D}_X)=\mathrm{Sp}^\bullet_X(\mathscr{D}_X)$ is a resolution of $\mathscr{D}_{X\to\mathrm{pt}}=\mathscr{O}_X$.

(3) If $X=Y\times Z$ and f is the projection, denote by $\Theta_{X/Y}$ the sheaf of *relative* tangent vector fields, i.e., which do not contain ∂_{y_j} in their local expression in coordinates adapted to the product $Y\times Z$. The complex $\mathscr{D}_X\otimes_{\mathscr{O}_X}\wedge^{-\bullet}\Theta_{X/Y}$ is also a resolution of $\mathscr{D}_{X\to Y}$ as a bimodule by locally free left $\mathscr{D}_X$-modules (Exercise: describe the right $f^{-1}\mathscr{D}_Y$-module structure). We moreover have a canonical quasi-isomorphism as bimodules

$$\begin{aligned}\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)&=\big(\mathscr{D}_X\otimes_{\mathscr{O}_X}\wedge^{-\bullet}\Theta_{X/Y}\big)\underset{f^{-1}\mathscr{O}_Y}{\otimes}f^{-1}\big(\wedge^{-\bullet}\Theta_Y\otimes_{\mathscr{O}_Y}\mathscr{D}_Y\big)\\&=\big(\mathscr{D}_X\otimes_{\mathscr{O}_X}\wedge^{-\bullet}\Theta_{X/Y}\big)\underset{f^{-1}\mathscr{D}_Y}{\otimes}f^{-1}\big(\mathrm{Sp}^\bullet_Y(\mathscr{D}_Y)\otimes_{\mathscr{O}_Y}\mathscr{D}_Y\big)\\&\xrightarrow{\sim}\big(\mathscr{D}_X\otimes_{\mathscr{O}_X}\wedge^{-\bullet}\Theta_{X/Y}\big)\underset{f^{-1}\mathscr{D}_Y}{\otimes}f^{-1}\mathscr{D}_{Y\to Y}\\&=\mathscr{D}_X\otimes_{\mathscr{O}_X}\wedge^{-\bullet}\Theta_{X/Y}.\end{aligned}$$

Recall that the flabby sheaves are injective with respect to the functor f_* (direct image) in the category of sheaves (of modules over a ring) and, being c-soft, are injective with respect to the functor $f_!$ (direct image with proper support).

***Definition 4.3.3* (Direct images of $\mathscr{D}$-modules)**

(1) The *direct image* f_+ is the functor from $\mathsf{M}(\mathscr{D}_X)$ to $\mathsf{C}^+(\mathscr{D}_Y)$ defined by (we take the single complex associated to the double complex)

$$f_+\mathscr{M} = f_* \operatorname{God}^\bullet\Big(\mathscr{M} \otimes_{\mathscr{D}_X} \operatorname{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)\Big).$$

It is a realization of $\boldsymbol{R}f_*\big(\mathscr{M} \otimes^{\boldsymbol{L}}_{\mathscr{D}_X} \mathscr{D}_{X\to Y}\big)$.

(2) The *direct image with proper support* $f_\dagger$ is the functor from $\mathsf{M}(\mathscr{D}_X)$ to $\mathsf{C}^+(\mathscr{D}_Y)$ defined by (we take the single complex associated to the double complex)

$$f_\dagger\mathscr{M} = f_! \operatorname{God}^\bullet\Big(\mathscr{M} \otimes_{\mathscr{D}_X} \operatorname{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)\Big).$$

It is a realization of $\boldsymbol{R}f_!\big(\mathscr{M} \otimes^{\boldsymbol{L}}_{\mathscr{D}_X} \mathscr{D}_{X\to Y}\big)$.

Remarks 4.3.4

(1) If $\mathscr{M}$ is a left $\mathscr{D}_X$-module, one *defines* $f_\dagger\mathscr{M}$ as $(f_\dagger\mathscr{M}^{\text{right}})^{\text{left}}$.

(2) If f is proper, or proper on the support of $\mathscr{M}$, we have an isomorphism in the category $\mathsf{D}^+(\mathscr{D}_Y)$:

$$f_\dagger\mathscr{M} \xrightarrow{\sim} f_+\mathscr{M}.$$

(3) One may replace $\mathscr{M}$ with a complex of right $\mathscr{D}_X$-modules which is bounded from below. Then $\mathscr{M} \otimes_{\mathscr{D}_X} \operatorname{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$ is first replaced with the associated single complex. Up to this modification, one defines similarly $f_+, f_\dagger$. These can be extended as functors from $\mathsf{D}^+(\mathscr{D}_X)$ to $\mathsf{D}^+(\mathscr{D}_Y)$.

(4) If $\mathscr{F}$ is any sheaf on X, we have $R^jf_*\mathscr{F} = 0$ and $R^jf_!\mathscr{F} = 0$ for $j \notin [0, 2\dim X]$. Therefore, taking into account the length $\dim X$ of the relative Spencer complex, we find that $\mathscr{H}^jf_+\mathscr{M}$ and $\mathscr{H}^jf_\dagger\mathscr{M}$ are zero for $j \notin [-\dim X, 2\dim X]$: we say that $f_+\mathscr{M}, f_\dagger\mathscr{M}$ have *bounded amplitude*. Similarly, if $\mathscr{M}^\bullet$ has bounded amplitude, then so has $f_\dagger\mathscr{M}^\bullet$.

Theorem 4.3.5

(1) *Let $f : X \to Y$ and $g : Y \to Z$ be two maps. There is a functorial canonical isomorphism of functors $(g\circ f)_\dagger = g_\dagger f_\dagger$. If f is proper, we also have $(g\circ f)_+ = g_+f_+$.*

(2) *If f is an embedding, then $f_\dagger\mathscr{M} = f_+\mathscr{M} = f_*(\mathscr{M} \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y})$.*

(3) *If $f : X = Y\times Z \to Y$ is the projection, we have*

$$f_+\mathscr{M} = f_* \operatorname{God}^\bullet\big(\mathscr{M} \otimes_{\mathscr{O}_X} \wedge^{-\bullet}\Theta_{X/Y}\big) \quad \textit{and} \quad f_\dagger\mathscr{M} = f_! \operatorname{God}^\bullet\big(\mathscr{M} \otimes_{\mathscr{O}_X} \wedge^{-\bullet}\Theta_{X/Y}\big).$$

Proof. Let us begin with (1). We have a natural morphism of complexes

$$\operatorname{Sp}^\bullet_{X\to Y}(\mathscr{D}_X) \underset{f^{-1}\mathscr{D}_Y}{\otimes} f^{-1}\operatorname{Sp}^\bullet_{Y\to Z}(\mathscr{D}_Y) \longrightarrow \operatorname{Sp}^\bullet_{X\to Y}(\mathscr{D}_X) \underset{f^{-1}\mathscr{D}_Y}{\otimes} f^{-1}\mathscr{D}_{Y\to Z},$$

lifting the identity morphism of $\mathscr{D}_{X\to Y}\otimes_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{D}_{Y\to Z}$, obtained by using the augmentation morphism $\operatorname{Sp}^\bullet_{Y\to Z}(\mathscr{D}_Y)\to\mathscr{D}_{Y\to Z}$.

On the one hand, the left-hand term is a resolution (in the category of $(\mathscr{D}_X,(g\circ f)^{-1}\mathscr{D}_Z)$-bimodules) of $\mathscr{D}_{X\to Z}$ by locally free $\mathscr{D}_X$-modules. Indeed, remark that, as $\operatorname{Sp}^\bullet_{Y\to Z}(\mathscr{D}_Y)$ is $\mathscr{D}_Y$ locally free, one has

$$\begin{aligned}\operatorname{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)\underset{f^{-1}\mathscr{D}_Y}{\otimes} f^{-1}\operatorname{Sp}^\bullet_{Y\to Z}(\mathscr{D}_Y)&\xrightarrow{\sim}\mathscr{D}_{X\to Y}\underset{f^{-1}\mathscr{D}_Y}{\otimes} f^{-1}\operatorname{Sp}^\bullet_{Y\to Z}(\mathscr{D}_Y)\\ &=\mathscr{O}_X\underset{f^{-1}\mathscr{O}_Y}{\otimes} f^{-1}\operatorname{Sp}^\bullet_{Y\to Z}(\mathscr{D}_Y)\\ &=\mathscr{O}_X\underset{f^{-1}\mathscr{O}_Y}{\overset{\boldsymbol{L}}{\otimes}} f^{-1}\mathscr{D}_{Y\to Z}\\ &=\mathscr{O}_X\underset{g^{-1}f^{-1}\mathscr{O}_Z}{\otimes} g^{-1}f^{-1}\mathscr{D}_Z\quad(\mathscr{D}_{Y\to Z}\text{ is }\mathscr{O}_Y\text{ locally free})\\ &=\mathscr{D}_{X\to Z}.\end{aligned}$$

On the other hand, there is a natural morphism

$$\operatorname{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)\underset{f^{-1}\mathscr{D}_Y}{\otimes} f^{-1}\mathscr{D}_{Y\to Z}\longrightarrow\operatorname{Sp}^\bullet_{X\to Z}(\mathscr{D}_X).$$

Indeed, we have a natural morphism

$$\Big[\operatorname{Sp}^\bullet_X(\mathscr{D}_X)\underset{\mathscr{O}_X}{\otimes}\mathscr{D}_{X\to Y}\Big]\underset{f^{-1}\mathscr{D}_Y}{\otimes} f^{-1}\mathscr{D}_{Y\to Z}\xrightarrow{\sim}\operatorname{Sp}^\bullet_X(\mathscr{D}_X)\underset{\mathscr{O}_X}{\otimes}\mathscr{D}_{X\to Z},$$

which, according to the chain rule (Exercise 4.2.7), is an isomorphism of $(\mathscr{D}_X,(g\circ f)^{-1}\mathscr{D}_Z)$-bimodules.

We have found a morphism, lifting the identity,

$$\operatorname{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)\underset{f^{-1}\mathscr{D}_Y}{\otimes} f^{-1}\operatorname{Sp}^\bullet_{Y\to Z}(\mathscr{D}_Y)\longrightarrow\operatorname{Sp}^\bullet_{X\to Z}(\mathscr{D}_X),$$

between two resolutions (in the category of $(\mathscr{D}_X,(g\circ f)^{-1}\mathscr{D}_Z)$-bimodules) of $\mathscr{D}_{X\to Z}$ by locally free $\mathscr{D}_X$-modules. This morphism is therefore a quasi-isomorphism.

We now have, for an object $\mathscr{M}$ of $\mathbf{M}(\mathscr{D}_X)$ of $\mathbf{D}^+(\mathscr{D}_X)$:

$$\begin{aligned}(g\circ f)_\dagger\mathscr{M}&=(g\circ f)_!\operatorname{God}^\bullet\big(\mathscr{M}\underset{\mathscr{D}_X}{\otimes}\operatorname{Sp}^\bullet_{X\to Z}(\mathscr{D}_X)\big)\\ &\simeq(g\circ f)_!\operatorname{God}^\bullet\big(\mathscr{M}\underset{\mathscr{D}_X}{\otimes}\operatorname{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)\underset{f^{-1}\mathscr{D}_Y}{\otimes} f^{-1}\operatorname{Sp}^\bullet_{Y\to Z}(\mathscr{D}_Y)\big)\\ &\simeq g_!\operatorname{God}^\bullet f_!\operatorname{God}^\bullet\big(\mathscr{M}\underset{\mathscr{D}_X}{\otimes}\operatorname{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)\underset{f^{-1}\mathscr{D}_Y}{\otimes} f^{-1}\operatorname{Sp}^\bullet_{Y\to Z}(\mathscr{D}_Y)\big)\qquad(4.3.6)\\ &\simeq g_!\operatorname{God}^\bullet f_!\Big[\operatorname{God}^\bullet\big(\mathscr{M}\underset{\mathscr{D}_X}{\otimes}\operatorname{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)\big)\underset{f^{-1}\mathscr{D}_Y}{\otimes} f^{-1}\operatorname{Sp}^\bullet_{Y\to Z}(\mathscr{D}_Y)\Big]\qquad(4.3.7)\\ &\simeq g_!\operatorname{God}^\bullet\Big[f_!\operatorname{God}^\bullet\big(\mathscr{M}\underset{\mathscr{D}_X}{\otimes}\operatorname{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)\big)\underset{\mathscr{D}_Y}{\otimes}\operatorname{Sp}^\bullet_{Y\to Z}(\mathscr{D}_Y)\Big]\qquad(4.3.8)\\ &=g_\dagger f_\dagger\mathscr{M}.\end{aligned}$$

Indeed, (4.3.6), as $f_! \operatorname{God}^\bullet$ is c-soft, it is acyclic for $g_!$, hence the natural morphism $g_! f_! \operatorname{God}^\bullet \to g_! \operatorname{God}^\bullet f_! \operatorname{God}^\bullet$ is an isomorphism. Next, (4.3.7) follows from Exercise 1.4.15, as the terms of $\operatorname{Sp}^\bullet_{Y\to Z}(\mathscr{D}_Y)$ are $\mathscr{D}_Y$-locally free. Last, (4.3.8) follows from the projection formula for $f_!$ (see, e.g. [**KS90**, Prop. 2.5.13]).

If f is proper, then $f_! = f_*$ and $f_! \operatorname{God}^\bullet$ is flabby, so (4.3.6) still holds with g_*, and the same reasoning gives $(g\circ f)_+ = g_+ f_+$.

Remark 4.3.9. If f is not proper, we cannot assert in general that $(g\circ f)_+ = g_+ f_+$. However, such an identity still holds when applied to suitable subcategories of $\mathbf{D}^+(\mathscr{D}_X)$, the main examples being:

- the restriction of f to the support of $\mathscr{M}$ is proper,
- $\mathscr{M}$ has $\mathscr{D}_X$-coherent cohomology.

In such cases, the natural morphism coming in the projection formula for f_* is a quasi-isomorphism (see [**MN93**, § II.5.4] for the coherent case).

The point (2) is easy, as $\mathscr{D}_{X\to Y}$ is then $\mathscr{D}_X$ locally free. For (3), use Example 4.3.2(3). Remark that the analogous result holds with f_+ if f is proper on the support of $\mathscr{M}$. ☐

This theorem reduces the computation of the direct image by any morphism $f : X \to Y$ by decomposing it as $f = p \circ i_f$, where $i_f : X \hookrightarrow X\times Y$ denotes the graph inclusion $x \mapsto (x, f(x))$. As i_f is an embedding, it is proper, so we have $f_+ = p_+ i_{f+}$.

***Exercise 4.3.10* (Direct image of left $\mathscr{D}_X$-modules).** Let $f : X \to Y$ be a holomorphic map ($\dim X = n$, $\dim Y = m$) and let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. As $\mathscr{D}_{X\to Y}$ is a left $\mathscr{D}_X$-module, $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y}$ has a natural structure of left $\mathscr{D}_X$-module (see Exercise 1.2.6(2)) and of course a compatible structure of right $f^{-1}\mathscr{D}_Y$-module.

(1) Show that the de Rham complex $\Omega_X^{n+\bullet}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y})$ is isomorphic, as a complex of right $f^{-1}\mathscr{D}_Y$-modules, to $\mathscr{M}^{\text{right}} \otimes_{\mathscr{D}_X} \operatorname{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)$.

(2) Conclude that $f_+\mathscr{M}$ is the complex of left $\mathscr{D}_Y$-modules associated to the double complex $f_* \operatorname{God}^\bullet \Omega_X^{n+\bullet}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y})$.

Choose local coordinates $y_1, \dots, y_m$ on Y and write $f_j = y_j \circ f$.

(3) Show that $i_{f+}\mathscr{M} = \mathscr{M}[\partial_{y_1}, \dots, \partial_{y_m}]$ with left $\mathscr{D}_{X\times Y}$ structure given locally by

$$\begin{aligned}\partial_{y_j} \cdot m\partial_y^\alpha &= m\partial_y^{\alpha+\mathbf{1}_j},\\ \partial_{x_i} \cdot m\partial_y^\alpha &= (\partial_{x_i} m)\partial_y^\alpha - \sum_{j=1}^m \frac{\partial f_j}{\partial x_i} m\partial_y^{\alpha+\mathbf{1}_j}.\end{aligned}$$

(4) Show that $\mathscr{D}_{Y\leftarrow X} \otimes_{\mathscr{D}_X} \mathscr{M}$ is isomorphic to the complex $\Omega_X^{n+m+\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}[\partial_y]$ with differential $(-1)^{n+m}\triangledown$ defined by

$$\triangledown(\omega\otimes m\partial_y^\alpha) = \nabla(\omega\otimes m)\partial_y^\alpha - \sum_j df_j \wedge \omega \otimes m\partial_y^{\alpha+\mathbf{1}_j},$$

where $\nabla : \Omega^k_X \otimes \mathscr{M} \to \Omega^{k+1}_X \otimes \mathscr{M}$ is the differential of the de Rham complex (see Exercise 1.1.10), and with left $f^{-1}\mathscr{D}_Y$ structure given by

$$\begin{aligned} y_j(\omega \otimes m\partial_y^\alpha) &= \omega \otimes f_j m\partial_y^\alpha - \omega \otimes m[\partial_y^\alpha, f_j], \\ \partial_{y_j}(\omega \otimes m\partial_y^\alpha) &= \omega \otimes m\partial_y^{\alpha+\mathbf{1}_j}. \end{aligned}$$

Exercise 4.3.11 **(Direct image of induced $\mathscr{D}$-modules, see [Sai89a, Lemma 3.2])**

Let $\mathscr{L}$ be an $\mathscr{O}_X$-module and let $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ be the associated induced right $\mathscr{D}_X$-module. Let $f : X \to Y$ be a holomorphic map. Show that $f_\dagger(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ is quasi-isomorphic to $\boldsymbol{R}f_!\mathscr{L} \otimes_{\mathscr{O}_Y} \mathscr{D}_Y$. [Hint: use that $\mathscr{L} \otimes_{\mathscr{O}_X} \mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X) \to \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y}$ is a quasi-isomorphism as $\mathscr{D}_X$ is $\mathscr{O}_X$-locally free, and use the projection formula.]

Exercise 4.3.12 **(Direct image of $\mathscr{D}$-modules and direct image of $\mathscr{O}$-modules)**

Let $f : X \to Y$ be a holomorphic map and let $\mathscr{M}$ be a right $\mathscr{D}_X$-module. It is also an $\mathscr{O}_X$-module. The goal of this exercise is to exhibit natural $\mathscr{O}_Y$-linear morphisms

$$R^i f_* \mathscr{M} \longrightarrow \mathscr{H}^i f_+ \mathscr{M} \quad \text{and} \quad R^i f_! \mathscr{M} \longrightarrow \mathscr{H}^i f_\dagger \mathscr{M}.$$

(1) Show that $\mathscr{D}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$ has a natural global section $\mathbf{1}$.

(2) Show that there is a natural $f^{-1}\mathscr{O}_Y$-linear morphism of complexes

$$\mathscr{M} \longrightarrow \mathscr{M} \otimes_{\mathscr{D}_X} \mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X), \quad m \longmapsto m \otimes \mathbf{1},$$

where $\mathscr{M}$ is considered as a complex with $\mathscr{M}$ in degree 0 and all other terms equal to 0, so the differential are all equal to 0. [Hint: use Exercise 1.2.8(3) to identify $\mathrm{Sp}^0_{X\to Y}(\mathscr{D}_X) = f^+\mathscr{D}_Y \otimes_{\mathscr{O}_X} \mathscr{D}_X$ with $\mathscr{D}_X \otimes_{\mathscr{O}_X} f^+\mathscr{D}_Y$ equipped with its trivial left $\mathscr{D}_X$ structure, and then identify $\mathscr{M} \otimes_{\mathscr{D}_X} (\mathscr{D}_X \otimes_{\mathscr{O}_X} f^+\mathscr{D}_Y)$ with $\mathscr{M} \otimes_{\mathscr{O}_X} f^+\mathscr{D}_Y = \mathscr{M} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$.]

(3) Conclude with the existence of the desired morphisms.

4.4. Direct images of differential complexes

We wish to define, in this section, the direct images as functors

$$f_{\mathrm{Diff}\,\dagger}, f_{\mathrm{Diff}\,+} : \mathbf{D}^+(\mathscr{O}_X, \mathrm{Diff}_X) \longrightarrow \mathbf{D}^+(\mathscr{O}_Y, \mathrm{Diff}_Y).$$

These functors should make the following natural diagrams to commute up to functorial isomorphism:

$$\begin{tikzcd}[column sep=large] \mathbf{D}^+(\mathscr{D}_X) \arrow[r, shift left, "\mathrm{DR}_X"] \arrow[d, "f_\dagger"'] & \mathbf{D}^+(\mathscr{O}_X, \mathrm{Diff}_X) \arrow[l, shift left, "{}^{\mathrm{diff}}\mathrm{DR}^{-1}_X"] \arrow[r, "\mathsf{Forget}"] \arrow[d, "f_{\mathrm{Diff}\,\dagger}"] & \mathbf{D}^+(\mathbb{C}_X) \arrow[d, "\boldsymbol{R}f_! = f_!\,\mathrm{God}^\bullet"] \\ \mathbf{D}^+(\mathscr{D}_Y) \arrow[r, shift left, "\mathrm{DR}_Y"] & \mathbf{D}^+(\mathscr{O}_Y, \mathrm{Diff}_Y) \arrow[l, shift left, "{}^{\mathrm{diff}}\mathrm{DR}^{-1}_Y"] \arrow[r, "\mathsf{Forget}"] & \mathbf{D}^+(\mathbb{C}_Y) \end{tikzcd} \tag{4.4.1}$$

and a similar diagram for $f_{\mathrm{Diff}\,+}$.

Let $\mathscr{M}=\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X$ be an induced $\mathscr{D}_X$-module. Recall that, after Exercise 4.3.11, we have

$$\mathscr{M}\otimes_{\mathscr{D}_X}\mathrm{Sp}^{\bullet}_{X\to Y}(\mathscr{D}_X)=\mathscr{M}\otimes_{\mathscr{D}_X}\mathscr{D}_{X\to Y}=\mathscr{L}\otimes_{f^{-1}\mathscr{O}_Y}f^{-1}\mathscr{D}_Y.$$

Let $\mathscr{L},\mathscr{L}'$ be two $\mathscr{O}_X$-modules and let $u:\mathscr{L}\to\mathscr{L}'$ be a differential morphism. Let $v:\mathscr{M}=\mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X\to\mathscr{M}'=\mathscr{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X$ be the corresponding $\mathscr{D}_X$-linear morphism (see Lemma 3.2.9). It defines a $f^{-1}\mathscr{D}_Y$-linear morphism $v\otimes\mathbf{1}:\mathscr{M}\otimes_{\mathscr{D}_X}\mathscr{D}_{X\to Y}\to\mathscr{M}'\otimes_{\mathscr{D}_X}\mathscr{D}_{X\to Y}$, where $\mathbf{1}$ is the section introduced in Exercise 4.3.12(1). This is therefore a morphism $\widetilde{v}:\mathscr{L}\otimes_{f^{-1}\mathscr{O}_Y}f^{-1}\mathscr{D}_Y\to\mathscr{L}'\otimes_{f^{-1}\mathscr{O}_Y}f^{-1}\mathscr{D}_Y$.

Proposition 4.4.2 **([Sai89a]).** *Under these conditions, we have $\mathscr{H}^0\,\mathrm{DR}_Y(\widetilde{v})=u$.*

In other words, $\mathbf{M}(\mathscr{O}_X,\mathrm{Diff}_X)$ is a subcategory of $\mathbf{M}(\mathscr{O}_X,\mathrm{Diff}_Y)$.

Proof. We keep notation of the beginning of §3.2. The problem is local, so that we can use coordinates on X and Y and write $f=(f_1,\dots,f_m)$. Let ℓ be a local section of $\mathscr{L}$, and let $\mathbf{1}_X$ be the unit of $\mathscr{D}_X$. Then, $v(\ell\otimes\mathbf{1}_X)=w(\ell)=\sum_\alpha w(\ell)_\alpha\otimes\partial_x^\alpha$ and $\widetilde{v}(\ell\otimes\mathbf{1}_X)=v(\ell\otimes\mathbf{1}_X)\otimes\mathbf{1}_{X\to Y}$. If $\alpha_i\neq 0$, we have

$$\partial_{x_i}^{\alpha_i}\otimes\mathbf{1}_{X\to Y}=\partial_{x_i}^{\alpha_i-1}\sum_j\frac{\partial f_j}{\partial x_i}\otimes\partial_{y_j}.$$

The image of $\widetilde{v}(\ell\otimes\mathbf{1}_X)$ by the map $\mathscr{L}\otimes_{f^{-1}\mathscr{O}_Y}f^{-1}\mathscr{D}_Y\to\mathscr{L}$ is therefore equal to the image of $w(\ell)_0$, which is nothing but $u(\ell)$ by definition of u. ☐

Let $\mathscr{L}^\bullet$ be a differential complex on X, i.e., an object of $\mathbf{C}^+(\mathscr{O}_X,\mathrm{Diff}_X)$. It is therefore also an object of $\mathbf{C}^+(\mathscr{O}_X,\mathrm{Diff}_Y)$ and we have (see Exercise 3.3.14) a quasi-isomorphism in this category:

$$\mathrm{DR}_Y\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}{}_Y\,\mathscr{L}^\bullet\xrightarrow{\sim}\mathscr{L}^\bullet.$$

We *define* the functor $f_{\mathrm{Diff}\,\dagger}$ from $\mathbf{C}^+(\mathscr{O}_X,\mathrm{Diff}_X)$ to $\mathbf{C}^+(\mathscr{O}_Y,\mathrm{Diff}_Y)$ as the composite $f_!\,\mathrm{DR}_Y\,\mathrm{God}^\bullet\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}{}_Y=f_!\,\mathrm{God}^\bullet\,\mathrm{DR}_Y\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}{}_Y$.

Corollary 4.4.3. *If $\mathscr{L}^\bullet$ is Diff_X-acyclic, i.e., if ${}^{\mathrm{diff}}\mathrm{DR}^{-1}{}_X\,\mathscr{L}^\bullet$ is acyclic, then $f_{\mathrm{Diff}\,\dagger}\mathscr{L}^\bullet$ is Diff_Y-acyclic. The functor $f_{\mathrm{Diff}\,\dagger}$ can be extended as a functor from the derived category $\mathbf{D}^+(\mathscr{O}_X,\mathrm{Diff}_X)$ to $\mathbf{D}^+(\mathscr{O}_Y,\mathrm{Diff}_Y)$ and the following diagrams commutes up to functorial isomorphisms:*

$$\begin{tikzcd}
\mathbf{D}^+(\mathscr{D}_X) \arrow[d, "f_\dagger"'] & \mathbf{D}^+(\mathscr{O}_X,\mathrm{Diff}_X) \arrow[l, "{}^{\mathrm{diff}}\mathrm{DR}^{-1}{}_X"'] \arrow[r, "\mathsf{Forget}"] \arrow[d, "f_{\mathrm{Diff}\,\dagger}"] & \mathbf{D}^+(\mathbb{C}_X) \arrow[d, "\boldsymbol{R}f_!=f_!\,\mathrm{God}^\bullet"] \\
\mathbf{D}^+(\mathscr{D}_Y) & \mathbf{D}^+(\mathscr{O}_Y,\mathrm{Diff}_Y) \arrow[l, "{}^{\mathrm{diff}}\mathrm{DR}^{-1}{}_Y"'] \arrow[r, "\mathsf{Forget}"] & \mathbf{D}^+(\mathbb{C}_Y).
\end{tikzcd}$$

Proof. By Proposition 4.4.2 and Exercise 4.3.11, we have by definition

$$^{\mathrm{diff}}\mathrm{DR}^{-1}_Y\,\mathscr{L}^\bullet = (^{\mathrm{diff}}\mathrm{DR}^{-1}_X\,\mathscr{L}^\bullet)\otimes_{\mathscr{D}_X}\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X).$$

Hence, $f_\dagger\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}_X\,\mathscr{L}^\bullet = f_!\,\mathrm{God}^\bullet\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}_Y\,\mathscr{L}^\bullet$ and therefore

$$f_{\mathrm{Diff}\,\dagger}\mathscr{L}^\bullet = \mathrm{DR}_Y\,f_\dagger\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}_X\,\mathscr{L}^\bullet.$$

If $\mathscr{L}^\bullet$ is Diff_X-acyclic, then $f_\dagger\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}_X\,\mathscr{L}^\bullet$ is acyclic, so $f_{\mathrm{Diff}\,\dagger}\mathscr{L}^\bullet$ is Diff_Y-acyclic.

The other points are then left to the reader. □

Using the isomorphisms ${}^{\mathrm{diff}}\mathrm{DR}^{-1}_X\,\mathrm{DR}_X\to\mathrm{Id}$ and ${}^{\mathrm{diff}}\mathrm{DR}^{-1}_Y\,\mathrm{DR}_Y\to\mathrm{Id}$, we get the essential commutativity of the diagram

$$\begin{tikzcd}
\mathbf{D}^+(\mathscr{D}_X) \arrow[r, "\mathrm{DR}_X"] \arrow[d, "f_\dagger"'] & \mathbf{D}^+(\mathscr{O}_X, \mathrm{Diff}_X) \arrow[d, "f_{\mathrm{Diff}\,\dagger}"] \\
\mathbf{D}^+(\mathscr{D}_Y) \arrow[r, "\mathrm{DR}_Y"] & \mathbf{D}^+(\mathscr{O}_Y, \mathrm{Diff}_Y).
\end{tikzcd}$$

We also get:

***Corollary 4.4.4* (Compatibility of direct images with the de Rham functor)**

Let $\mathscr{M}^{\mathrm{right}}$ be a right $\mathscr{D}_X$-module or, more generally, an object of $\mathbf{D}^+(\mathscr{D}_X)$. Let $f: X\to Y$ be a holomorphic map. Then there is a functorial canonical isomorphism

$$^{\mathrm{p}}\mathrm{DR}_Y\,f_\dagger\mathscr{M}^{\mathrm{right}} \xrightarrow{\sim} \boldsymbol{R}f_!\,{}^{\mathrm{p}}\mathrm{DR}_X\,\mathscr{M}^{\mathrm{right}}.$$

If f is proper on $\mathrm{Supp}\,\mathscr{M}^{\mathrm{right}}$ *or if $\mathscr{M}^{\mathrm{right}}$ has $\mathscr{D}_X$-coherent cohomology, there is a similar isomorphism with f_+ and $\boldsymbol{R}f_*$.* □

Notice that this corollary can also be obtained directly, by applying the proof of Theorem 4.3.5(1) with $Z=\mathrm{pt}$ and forgetting $g_!$ in the proof.

4.5. Direct image of currents

Let φ be a C^∞ form of maximal degree on X. If $f: X\to Y$ is a proper holomorphic map which is *smooth*, then the integral of φ in the fibres of f is a C^∞ form of maximal degree on Y, that one denotes by $\int_f\varphi$.

If f is not smooth, then $\int_f\varphi$ is only defined as a current of maximal degree on Y, and the definition extends to the case where φ is itself a current of maximal degree on X (see § 1.3.e for the notion of current).

Exercise 4.5.1. Extend the notion and properties of direct image of a right (resp. left) $\mathscr{D}_{X,\overline{X}}$-module, by introducing the transfer module $\mathscr{D}_{X\to Y,\overline{X}\to\overline{Y}}=\mathscr{D}_{X\to Y}\otimes_{\mathbb{C}}\mathscr{D}_{\overline{X}\to\overline{Y}}$. One denotes these direct images by f_{++} or $f_{\dagger\dagger}$.

Definition 4.5.2 **(Integration of currents of maximal degree)**
Let $f: X \to Y$ be a proper holomorphic map and let T be a current of maximal degree on X. The current $\int_f T$ of maximal degree on Y is defined by

$$\Big\langle \int_f T, \varphi \Big\rangle = \langle T, \varphi \circ f \rangle.$$

We continue to assume that f is proper. We will now show how the integration of currents is used to defined a natural $\mathscr{D}_{Y,\overline{Y}}$ morphism $\mathscr{H}^0 f_{++}\,\mathfrak{C}_X \to \mathfrak{C}_Y$. Let us first treat as an exercise the case of a closed embedding.

Exercise 4.5.3. Assume that X is a closed submanifold of Y and denote by $f: X \hookrightarrow Y$ the embedding (which is a proper map). Denote by $\mathbf{1}$ the canonical section of $\mathscr{D}_{X\to Y,\overline{X}\to\overline{Y}}$. Show that the natural map

$$\mathscr{H}^0 f_{++}\,\mathfrak{C}_X = f_*\big(\,\mathfrak{C}_X \otimes_{\mathscr{D}_{X,\overline{X}}} \mathscr{D}_{X\to Y,\overline{X}\to\overline{Y}}\big) \longrightarrow \mathfrak{C}_Y, \quad T\otimes\mathbf{1} \longmapsto \int_f T$$

induces an isomorphism of the right $\mathscr{D}_{Y,\overline{Y}}$-module $\mathscr{H}^0 f_{++}\,\mathfrak{C}_X$ with the submodule of $\mathfrak{C}_Y$ consisting of currents supported on X. [Hint: use a local computation.]

By going from right to left, identify $\mathscr{H}^0 f_{++}\,\mathfrak{Db}_X$ with the sheaf of distributions on Y supported on X.

Denote by $\mathfrak{Db}_X^{n-p,n-q}$ or $\mathfrak{Db}_{X,p,q}$ the sheaf of currents of degree p,q, which are linear forms on C^∞ differential forms of degree p,q. The integration of currents is a morphism

$$\int_f : \mathfrak{Db}_X^{n-p,n-q} \longrightarrow \mathfrak{Db}_Y^{m-p,m-q},$$

if $m=\dim Y$ and $n=\dim X$, which is compatible with the d' or d'' differential of currents on X and Y.

Exercise 4.5.4

(1) Show that the complex $f_{++}\,\mathfrak{C}_X$ is quasi-isomorphic to the single complex associated to the double complex $f_*(\mathfrak{Db}_X^{n-\bullet,n-\bullet} \otimes_{\mathscr{O}_{X,\overline{X}}} \mathscr{D}_{X\to Y,\overline{X}\to\overline{Y}})$. [Hint: use Exercise 4.3.10(1).]

(2) Show that the integration of currents $\int_f$ induces a $\mathscr{D}_{Y,\overline{Y}}$-linear morphism of complexes

$$\int_f : f_{++}\,\mathfrak{C}_X \longrightarrow \mathfrak{C}_Y \otimes_{\mathscr{D}_{Y,\overline{Y}}} \mathrm{Sp}^{\bullet,\bullet}_{Y\to Y,\overline{Y}\to\overline{Y}} \simeq \mathfrak{C}_Y\,.$$

4.6. The Gauss-Manin connection

Let $\mathscr{M}$ be a *left* $\mathscr{D}_X$-module and let $f: X \to Y$ be a holomorphic mapping. On the one hand, one may define the direct images $f_+\mathscr{M}$ or $f_\dagger\mathscr{M}$ of $\mathscr{M}$ viewed as a $\mathscr{D}_X$-modules. These are objects in $\mathbf{D}^+(\mathscr{D}_Y)^{\mathrm{left}}$. On the other hand, it is possible, *when f is a smooth holomorphic mapping*, to define a flat connection, called the *Gauss-Manin connection* on the relative de Rham cohomology of $\mathscr{M}$. We will compare both constructions, when f is smooth. Such a comparison has yet been done *when f is the projection of a product $X = Y \times Z \to Z$* (see Example 4.3.2(3) and Remark 4.3.4(3)).

Let us begin with the Gauss-Manin connection. We assume in this section that $f : X \to Y$ is a *smooth* holomorphic map. We set $n = \dim X$, $m = \dim Y$ and $d = n - m$ (we assume that X and Y are pure dimensional, otherwise one works on each connected component of X and Y).

Consider the *Kozsul filtration* $\mathrm{L}^\bullet$ on the complex $(\Omega_X^{n+\bullet}, (-1)^n d)$, defined by

$$\mathrm{L}^p\,\Omega_X^{n+i} = \mathrm{Im}(f^*\Omega_Y^{m+p} \otimes_{\mathscr{O}_X} \Omega_X^{d+i-p} \longrightarrow \Omega_X^{n+i}).$$

Exercise 4.6.1

(1) Show that the Kozsul filtration is a decreasing finite filtration and that it is compatible with the differential.

(2) Show that, locally, being in L^p means having at least $m + p$ factors dy_i in any summand.

Then, as f is smooth, we have (by computing with local coordinates adapted to f),

$$\mathrm{gr}^p_{\mathrm{L}}\Omega_X^{n+i} = f^*\Omega_Y^{m+p} \otimes_{\mathscr{O}_X} \Omega_{X/Y}^{d+i-p},$$

where $\Omega^k_{X/Y}$ is the sheaf of relative differential forms: $\Omega^k_{X/Y} = \wedge^k\Omega^1_{X/Y}$ and $\Omega^1_{X/Y} = \Omega^1_X \big/ f^*\Omega^1_Y$. Notice that $\Omega^k_{X/Y}$ is $\mathscr{O}_X$-locally free.

Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module or an object of $\mathbf{D}^+(\mathscr{D}_X)^{\mathrm{left}}$. As f is smooth, the sheaf $\mathscr{D}_{X/Y}$ of relative differential operators is well defined and, composing the flat connection $\nabla : \mathscr{M} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{M}$ with the projection $\Omega^1_X \to \Omega^1_{X/Y}$, we get a relative flat connection $\nabla_{X/Y}$ on $\mathscr{M}$, and thus the structure of a left $\mathscr{D}_{X/Y}$-module on $\mathscr{M}$. In particular, the relative de Rham complex is defined as

$${}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathscr{M} = (\Omega^{d+\bullet}_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{M}, \nabla_{X/Y}).$$

We have ${}^{\mathrm{p}}\mathrm{DR}_X\,\mathscr{M} = (\Omega_X^{n+\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}, \nabla)$ (see Definition 1.4.1) and the Kozsul filtration $\mathrm{L}^p\,\Omega_X^{n+\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}$ is preserved by the differential ∇ (recall that being in L^p means having at least $m+p$ factors dy_i in any summand). We may therefore induce the filtration $\mathrm{L}^\bullet$ on the complex ${}^{\mathrm{p}}\mathrm{DR}_X\,\mathscr{M}$. We then have an equality of complexes

$$\mathrm{gr}^p_{\mathrm{L}}\,{}^{\mathrm{p}}\mathrm{DR}_X\,\mathscr{M} = f^*\Omega_Y^{m+p} \otimes_{\mathscr{O}_X} {}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathscr{M}[-p].$$

Notice that the differential of these complexes are $f^{-1}\mathscr{O}_Y$-linear.

The complex $f_*\,\mathrm{God}^\bullet\,{}^{\mathrm{p}}\mathrm{DR}_X\,\mathscr{M}$ (resp. the complex $f_!\,\mathrm{God}^\bullet\,{}^{\mathrm{p}}\mathrm{DR}_X\,\mathscr{M}$) is filtered by subcomplexes $f_*\,\mathrm{God}^\bullet\,\mathrm{L}^p\,{}^{\mathrm{p}}\mathrm{DR}_X\,\mathscr{M}$ (resp. $f_!\,\mathrm{God}^\bullet\,\mathrm{L}^p\,{}^{\mathrm{p}}\mathrm{DR}_X\,\mathscr{M}$). We therefore get a spectral sequence (the Leray spectral sequence in the category of sheaves of $\mathbb{C}$-vector spaces, see, e.g. [**God64**]). Using the projection formula for $f_!$ and the fact that Ω_Y^{m+p}

is $\mathscr{O}_Y$-locally free, one obtains that the E_1 term for the complex $f_!\operatorname{God}^\bullet {}^{\mathrm{p}}\mathrm{DR}_X\mathscr{M}$ is given by

$$E_{1,!}^{p,q}=\Omega_Y^{m+p}\otimes_{\mathscr{O}_Y}R^qf_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\mathscr{M},$$

and the spectral sequence converges to (a suitable graded object associated with) $R^{p+q}f_!\,{}^{\mathrm{p}}\mathrm{DR}_X\mathscr{M}$. If f is proper on $\operatorname{Supp}\mathscr{M}$ or if $\mathscr{M}$ has $\mathscr{D}_X$-coherent cohomology, one may also apply the projection formula to f_* (see [**MN93**, § II.5.4]).

By definition of the spectral sequence, the differential $d_1:E_1^{p,q}\to E_1^{p+1,q}$ is the connecting morphism (see Exercise 4.6.2 below) in the long exact sequence associated to the short exact sequence of complexes

$$0\longrightarrow \mathrm{gr}_{\mathrm{L}}^{p+1}\,{}^{\mathrm{p}}\mathrm{DR}_X\mathscr{M}\longrightarrow \mathrm{L}^p\,{}^{\mathrm{p}}\mathrm{DR}_X\mathscr{M}\big/\mathrm{L}^{p+2}\,{}^{\mathrm{p}}\mathrm{DR}_X\mathscr{M}\longrightarrow \mathrm{gr}_{\mathrm{L}}^{p}\,{}^{\mathrm{p}}\mathrm{DR}_X\mathscr{M}\longrightarrow 0$$

after applying $f_!\operatorname{God}^\bullet$ (or $f_*\operatorname{God}^\bullet$ if one of the previous properties is satisfied).

***Exercise 4.6.2* (The connecting morphism).** Let $0\to C_1^\bullet\to C_2^\bullet\to C_3^\bullet\to 0$ be an exact sequence of complexes. Let $[\mu]\in H^kC_3^\bullet$ and choose a representative in C_3^k with $d\mu=0$. Lift μ as $\widetilde{\mu}\in C_2^k$.

(1) Show that $d\widetilde{\mu}\in C_1^{k+1}$ and that its differential is zero, so that the class $[d\widetilde{\mu}]\in H^{k+1}C_1^\bullet$ is well defined.

(2) Show that $\delta:[\mu]\mapsto[d\widetilde{\mu}]$ is a well defined morphism $H^kC_3^\bullet\to H^{k+1}C_1^\bullet$.

(3) Deduce the existence of the cohomology long exact sequence, having δ as its connecting morphism.

***Lemma 4.6.3* (The Gauss-Manin connection).** *The morphism*

$$\nabla^{\mathrm{GM}}:=d_1:R^qf_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\mathscr{M}=E_1^{-m,q}\longrightarrow E_1^{-m+1,q}=\Omega_Y^1\otimes_{\mathscr{O}_Y}R^qf_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\mathscr{M}$$

is a flat connection on $R^qf_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\mathscr{M}$, called the Gauss-Manin connection *and the complex $(E_1^{\bullet,q},d_1)$ is equal to the de Rham complex ${}^{\mathrm{p}}\mathrm{DR}_Y(R^qf_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\mathscr{M},\nabla^{\mathrm{GM}})$.*

Sketch of proof of Lemma 4.6.3. Instead of using the Godement resolution, one may use C^∞ differential forms $\mathscr{E}_X^\bullet$. One considers the complex $\mathscr{E}_X^{n+\bullet}\otimes_{\mathscr{O}_X}\mathscr{M}$, with the differential D defined by

$$D(\varphi\otimes m)=(-1)^nd\varphi\otimes m+(-1)^k\varphi\wedge\nabla m,$$

if φ is a local section of $\mathscr{E}_X^{n+k}$ $(k\leqslant 0)$. This C^∞ de Rham complex is quasi-isomorphic to the holomorphic one, and is equipped with the Kozsul filtration. The quasi-isomorphism is strict with respect to $\mathrm{L}^\bullet$. One may therefore compute with the C^∞ de Rham complex.

Choose a partition of unity (χ_α) such that f is locally a product on a neighbourhood of $\operatorname{Supp}\chi_\alpha$ for any α.

Let $\eta\wedge(\varphi\otimes m)$ be a section of $\mathscr{E}_Y^{m+p}\otimes f_!(\mathscr{E}_{X/Y}^{d+q}\otimes\mathscr{M})$. In the neighbourhood of $\operatorname{Supp}\chi_\alpha$, we can choose a decomposition $D=D_Y^{(\alpha)}+D_{X/Y}^{(\alpha)}$. As $\sum_\alpha\chi_\alpha\equiv 1$, we have

$$\begin{aligned}d_1[\eta\wedge(\varphi\otimes m)]&=\sum_\alpha\chi_\alpha d_1[\eta\wedge(\varphi\otimes m)]=\sum_\alpha\chi_\alpha D_Y^{(\alpha)}[\eta\wedge(\varphi\otimes m)]\\&=\sum_\alpha\chi_\alpha\Big[(-1)^m d\eta\wedge(\varphi\otimes m)+(-1)^r\eta\wedge(\varphi\nabla_Y^{(\alpha)}m)\Big],\end{aligned}$$

for a suitable r. One gets the desired result by a local computation. □

This lemma shows in particular that the E_1 complex is a complex in $\mathsf{C}^+(\mathscr{O}_Y,\mathrm{Diff}_X)$, and ${}^{\mathrm{diff}}\mathrm{DR}^{-1}{}_Y(E_1^{\bullet,q},d_1)\simeq(R^q f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\mathscr{M},\nabla^{\mathrm{GM}})^{\mathrm{right}}$.

Theorem 4.6.4. *Let $f:X\to Y$ be a smooth holomorphic map and let $\mathscr{M}$ be left $\mathscr{D}_X$-module—or more generally an object of $\mathsf{D}^+(\mathscr{D}_X)^{\mathrm{left}}$. Then there is a functorial isomorphism of left $\mathscr{D}_Y$-modules*

$$R^k f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\mathscr{M}\longrightarrow\mathscr{H}^k f_\dagger\mathscr{M}$$

when one endows the left-hand term with the Gauss-Manin connection ∇^{GM}. The same result holds for f_+ instead of $f_\dagger$ if f is proper on $\operatorname{Supp}\mathscr{M}$ *or $\mathscr{M}$ is $\mathscr{D}_X$-coherent (or has coherent cohomology).*

Proof. Recall that, for a left $\mathscr{D}_X$-module $\mathscr{M}$, we have

$$\mathscr{M}^{\mathrm{right}}\otimes_{\mathscr{D}_X}\mathrm{Sp}^\bullet_{X\to Y}(\mathscr{D}_X)\simeq\Omega_X^{n+\bullet}(\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_{X\to Y}),$$

so that the direct image of $\mathscr{M}$ is

$$f_\dagger\mathscr{M}=f_!\,\mathrm{God}^\bullet\mathrm{DR}_X(\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_{X\to Y})=f_!\,\mathrm{DR}_X((\mathrm{God}^\bullet\mathscr{M})\otimes_{\mathscr{O}_X}\mathscr{D}_{X\to Y}),$$

by using Exercise 1.4.15(1). There is a Kozsul filtration $\mathrm{L}^\bullet\mathrm{DR}_X(\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_{X\to Y})$. Notice that $\mathrm{gr}^p_{\mathrm{L}}\mathrm{DR}_X(\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_{X\to Y})$ is equal to the complex

$$f^*\Omega_Y^{m+p}\otimes_{\mathscr{O}_X}{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\mathscr{M}\otimes_{f^{-1}\mathscr{O}_Y}f^{-1}\mathscr{D}_Y[-p],$$

with differential induced by $\nabla_{X/Y}$ on $\mathscr{M}$ (remark that the part of the differential involving T^*f is killed by taking $\mathrm{gr}^p_{\mathrm{L}}$). The differential is now $f^{-1}\mathscr{O}_Y$-linear.

The filtered complex $f_!\,\mathrm{L}^\bullet\mathrm{DR}_X(\mathscr{M}\otimes_{\mathscr{O}_X}\mathscr{D}_{X\to Y})$ gives rise to a spectral sequence in the category of right $\mathscr{D}_Y$-modules. By the previous computation, the $E_1^{p,q}$ term of this spectral sequence is an induced $\mathscr{D}_Y$-module, equal to ${}^{\mathrm{diff}}\mathrm{DR}^{-1}{}_Y$ of the corresponding Gauss-Manin term. We will show below that the differential d_1 becomes the Gauss-Manin d_1 after applying $\mathscr{H}^0\mathrm{DR}_Y$. This will prove that the Gauss-Manin E_1 complex is equal to DR_Y of the E_1 complex of $\mathscr{D}_Y$-modules, i.e., this complex is isomorphic to ${}^{\mathrm{diff}}\mathrm{DR}^{-1}{}_Y$ of the Gauss-Manin E_1 complex. By Lemma 4.6.3 and the remark after its proof, the E_1 complex of the $\mathscr{D}_Y$-Leray spectral sequence has therefore cohomology in degree 0 only, hence this spectral sequence degenerates at E_2, this cohomology being equal to $(R^q f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\mathscr{M},\nabla^{\mathrm{GM}})^{\mathrm{right}}$. But the spectral sequences converges (the Kozsul filtration is finite) and its limit is $\oplus_p\mathrm{gr}^p\mathscr{H}^q f_\dagger\mathscr{M}$ for some suitable filtration on $\mathscr{H}^q f_\dagger \mathscr{M}$. We conclude that this implicit filtration is trivial and that $\mathscr{H}^q f_\dagger \mathscr{M} = (R^q f_! \,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\, \mathscr{M}, \nabla^{\mathrm{GM}})^{\mathrm{right}}$.

Let us now compare the d_1 of both spectral sequences. As the construction is clearly functorial with respect to $\mathscr{M}$, we can assume that $\mathscr{M}$ is flabby, by replacing $\mathscr{M}$ by $\mathrm{God}^\ell \mathscr{M}$ for any ℓ. It is also enough to make the computation locally on Y, so that we can write $f = (f_1, \dots, f_m)$, using local coordinates $(y_1, \dots, y_m)$. If μ is a section of $\Omega_X^{n+k} \otimes \mathscr{M}$ and $\mathbf{1}_Y$ is the unit of $\mathscr{D}_Y$, then (4.2.1) can be written as

$$\nabla^X(\mu \otimes \mathbf{1}_Y) = (\nabla\mu) \otimes \mathbf{1}_Y + \sum_{j=1}^m \mu \wedge df_j \otimes \partial_{y_j}.$$

Using the definition of d_1 given by Exercise 4.6.2 and an argument similar to that of Proposition 4.4.2, one gets the desired assertion. ☐

4.7. Coherence of direct images

Let $f : X \to Y$ be a holomorphic map and $\mathscr{M}$ be a $\mathscr{D}_X$-module. We say that $\mathscr{M}$ is *f-good* if there exists a covering of Y by open sets V_j such that $\mathscr{M}$ is good on each $f^{-1}(V_j)$. As we indicated in Remark 2.2.7, any holonomic $\mathscr{D}_X$-module is good with respect to any holomorphic map.

Theorem 4.7.1. *Let $\mathscr{M}$ be a f-good $\mathscr{D}_X$-module. Assume that f is proper on the support of $\mathscr{M}$. Then $f_\dagger \mathscr{M} = f_+ \mathscr{M}$ has $\mathscr{D}_Y$-coherent cohomology.*

This theorem is an application of Grauert's coherence theorem for $\mathscr{O}_X$-modules, and this is why we restrict to f-good $\mathscr{D}_X$-modules. In general, it is not known whether the theorem holds for any coherent $\mathscr{D}_X$-module or not. Notice, however, that one may relax the geometric condition on $f_{|\operatorname{Supp}\mathscr{M}}$ (properness) by using more specific properties of $\mathscr{D}$-modules: as we have seen, the characteristic variety is a finer geometrical object attached to the $\mathscr{D}$-module, and one should expect that the right condition on f has to be related with the characteristic variety. The most general statement in this direction is the coherence theorem for *elliptic pairs*, due to P. Schapira and J.-P. Schneiders [**SS94**]. For instance, if X is an open set of X' and f is the restriction of $f' : X' \to Y$, and if the boundary of X is f-noncharacteristic with respect to $\mathscr{M}$ (a relative variant of Definition 2.6.1) then the direct image of $\mathscr{M}$ has $\mathscr{D}_Y$-coherent cohomology.

Proof of Theorem 4.7.1. As the coherence property is a local property on Y, the statement one proves is, more precisely, that the direct image of a good $\mathscr{D}_X$-module $\mathscr{M}$ is a good $\mathscr{D}_Y$-module when f is proper on $\operatorname{Supp}\mathscr{M}$. By an extension argument, it is even enough to assume that $\mathscr{M}$ has a good filtration and show that, locally on Y, the cohomology modules of $f_\dagger \mathscr{M}$ have a good filtration.

First step: induced $\mathscr{D}$-modules. Assume that $\mathscr{M} = \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ and $\mathscr{L}$ is $\mathscr{O}_X$-coherent. By Exercise 4.3.10, it is enough to prove that the cohomology of $\boldsymbol{R}f_!\mathscr{L}$ is $\mathscr{O}_Y$-coherent when f is proper on $\operatorname{Supp} \mathscr{L}$: this is Grauert's Theorem.

Second step: finite complexes of induced $\mathscr{D}$-modules. Let $\mathscr{L}_\bullet \otimes_{\mathscr{O}_X} \mathscr{D}_X$ be a *finite* complex of induced $\mathscr{D}_X$-modules. Recall that its direct image complex was defined in Remark 4.3.4(3). Assume that f restricted to the support of each term is proper. Using Exercise 2.2.1(2) and Artin-Rees (Corollary 2.2.13), one shows by induction on the length of the complex that the cohomology modules of $f_\dagger(\mathscr{L}_\bullet \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ have a good filtration.

Third step: general case. Fix a compact set K of Y. We will show that the cohomology modules of $f_\dagger \mathscr{M}$ have a good filtration in a neighbourhood of K. Fix a good filtration $F_\bullet \mathscr{M}$ of $\mathscr{M}$. As $f^{-1}(K) \cap \operatorname{Supp} \mathscr{M}$ is compact, there exists k such that $\mathscr{L}^0 := F_k \mathscr{M}$ generates $\mathscr{M}$ as a $\mathscr{D}_X$-module in some neighbourhood of $f^{-1}(K)$. Hence $\mathscr{L}^0$ is a coherent $\mathscr{O}_X$-module with support contained in $\operatorname{Supp} \mathscr{M}$ and we have a surjective morphism $\mathscr{L}^0 \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{M}$ in some neighbourhood of $f^{-1}(K)$ that we still call X. The kernel of this morphism is therefore $\mathscr{D}_X$-coherent, has support contained in $\operatorname{Supp} \mathscr{M}$ and, by Artin-Rees (Corollary 2.2.13), has a good filtration.

The process may therefore be continued and leads to the existence, in some neighbourhood of K, of a (maybe infinite) resolution $\mathscr{L}^{-\bullet} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ by coherent induced $\mathscr{D}_X$-modules with support contained in $\operatorname{Supp} \mathscr{M}$.

Fix some ℓ and stop the resolution at the ℓ-th step. Denote by $\mathscr{N}^{-\bullet}$ this bounded complex and by $\mathscr{M}'$ the kernel of $\mathscr{N}^{-\ell} \to \mathscr{N}^{-\ell+1}$. We have an exact sequence of complexes

$$0 \longrightarrow \mathscr{M}'[\ell] \longrightarrow \mathscr{N}^{-\bullet} \longrightarrow \mathscr{M} \longrightarrow 0,$$

where $\mathscr{M}$ is considered as a complex with only one term in degree 0, and $\mathscr{M}'[\ell]$ a complex with only one term in degree $-\ell$. This sequence induces a long exact sequence

$$\cdots \longrightarrow \mathscr{H}^{j+\ell}(f_\dagger \mathscr{M}') \longrightarrow \mathscr{H}^j(f_\dagger \mathscr{N}^{-\bullet}) \longrightarrow \mathscr{H}^j(f_\dagger \mathscr{M}) \longrightarrow \mathscr{H}^{j+\ell+1}(f_\dagger \mathscr{M}') \longrightarrow \cdots$$

Recall (see Remark 4.3.4(4)) that $\mathscr{H}^j(f_\dagger \mathscr{M}) = 0$ for $j \notin [-\dim X, 2\dim X]$. Choose then ℓ big enough so that, for any $j \in [-\dim X, 2\dim X]$, both numbers $j+\ell$ and $j+\ell+1$ do not belong to $[-\dim X, 2\dim X]$. With such a choice, we have $\mathscr{H}^j(f_\dagger \mathscr{M}) \simeq \mathscr{H}^j(f_\dagger \mathscr{N}^{-\bullet})$ for $j \in [-\dim X, 2\dim X]$ and $\mathscr{H}^j(f_\dagger \mathscr{M}) = 0$ otherwise. By the second step, $\mathscr{H}^j(f_\dagger \mathscr{M})$ has a good filtration in some neighbourhood of K. $\square$

4.8. Kashiwara's estimate for the behaviour of the characteristic variety

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module with characteristic variety $\operatorname{Char} \mathscr{M}$. Let $f : X \to Y$ be a holomorphic map and assume that the cohomology modules $\mathscr{H}^j(f_\dagger \mathscr{M})$ are $\mathscr{D}_Y$-coherent (for instance, assume that all conditions in Theorem 4.7.1 are fulfilled). Is it possible to give an upper bound of the characteristic variety of each $\mathscr{H}^j(f_\dagger \mathscr{M})$

in terms of that of $\mathscr{M}$? There is such an estimate which is known as *Kashiwara's estimate.*

The most natural approach to this question is to introduce the sheaf of microdifferential operators and to show that the characteristic variety is nothing but the support of the microlocalized module associated with $\mathscr{M}$. The behaviour of the support of a microdifferential module with respect to direct images is then easy to understand (see, e.g. [**Bjö79, Mal93, Bjö93**] for such a proof, see [**SS94**] for a very general result and [**Lau85**] for an algebraic approach).

Nevertheless, we will not introduce here microdifferential operators (see however [**Sch85**] for a good introduction to the subject). Therefore, we will give a direct proof of Kashiwara's estimate.

This estimate may be understood as a weak version of a general Riemann-Roch theorem for $\mathscr{D}_X$-modules (see, e.g. [**Sab97**] and the references given therein).

Let $f : X \to Y$ be a holomorphic map. We will consider the following associated cotangent diagram:

$$\begin{tikzcd} T^*X & f^*T^*Y \arrow[l, "T^*f"'] \arrow[r, "f"] & T^*Y. \end{tikzcd}$$

Theorem 4.8.1 **(Kashiwara's estimate for the characteristic variety)**
Let $\mathscr{M}$ be a f-good $\mathscr{D}_X$-module such that f is proper on $\operatorname{Supp}\mathscr{M}$. *Then, for any $j \in \mathbb{Z}$, we have*

$$\operatorname{Char} \mathscr{H}^j(f_\dagger \mathscr{M}) \subset f\big((T^*f)^{-1}(\operatorname{Char}\mathscr{M})\big).$$

Exercise 4.8.2. Explain more precisely this estimate when f is the inclusion of a closed submanifold.

Sketch of proof. As in the proof of Theorem 4.7.1, we first reduce to the case where $\mathscr{M}$ has a good filtration $F_\bullet\mathscr{M}$.

Notice first that it is possible to define a functor $f_\dagger$ for $\operatorname{gr}^F\mathscr{D}_X$-modules, by the formula $f_\dagger(\bullet) = \boldsymbol{R}f_!(\boldsymbol{L}(T^*f)^*(\bullet))$. Moreover, the inverse image $(T^*f)^*$ is nothing but the tensor product $\otimes_{f^{-1}\mathscr{O}_Y}\operatorname{gr}^F\mathscr{D}_Y$. We therefore clearly have the inclusion

$$\operatorname{Supp}\mathscr{H}^j f_\dagger \operatorname{gr}^F\mathscr{M} \subset f\big((T^*f)^{-1}(\operatorname{Supp}\operatorname{gr}^F\mathscr{M})\big) = f\big((T^*f)^{-1}(\operatorname{Char}\mathscr{M})\big).$$

The problem consists now in understanding the difference between $f_\dagger \operatorname{gr}^F$ and $\operatorname{gr}^F f_\dagger$. In order to analyse this difference, we will put $\mathscr{M}$ and $\operatorname{gr}^F\mathscr{M}$ in a one parameter family, i.e., we will consider the associated Rees module.

One then defines direct images of $R_F\mathscr{D}_X$-modules, still denoted by $f_\dagger$, and shows the R_F analogue of Theorem 4.7.1. Therefore, $f_\dagger R_F\mathscr{M}$ has $R_F\mathscr{D}_Y$-coherent cohomology. One has to be careful that the cohomology of $f_\dagger R_F\mathscr{M}$ may have z-torsion, hence does not take the form R_F of something. Nevertheless, as $\mathscr{H}^j(f_\dagger R_F\mathscr{M})$ is $R_F\mathscr{D}_Y$-coherent,

- the kernel sequence $\operatorname{Ker}\big[z^\ell : \mathscr{H}^j(f_\dagger R_F\mathscr{M}) \to \mathscr{H}^j(f_\dagger R_F\mathscr{M})\big]$ is locally stationary,

- the quotient of $\mathscr{H}^j(f_\dagger R_F\mathscr{M})$ by its z-torsion (i.e., locally by $\operatorname{Ker} z^\ell$ for ℓ big enough) is $R_F\mathscr{D}_Y$-coherent, hence is the Rees module associated with some good filtration $F_\bullet$ on $\mathscr{H}^j(f_\dagger\mathscr{M})$.

Consider the exact sequence

$$\cdots \longrightarrow \mathscr{H}^j(f_\dagger R_F\mathscr{M}) \xrightarrow{z^\ell} \mathscr{H}^j(f_\dagger R_F\mathscr{M}) \longrightarrow \mathscr{H}^j\big(f_\dagger(R_F\mathscr{M}/z^\ell R_F\mathscr{M})\big) \longrightarrow \cdots$$

Then,

- $\mathscr{H}^j(f_\dagger R_F\mathscr{M})/z^\ell\mathscr{H}^j(f_\dagger R_F\mathscr{M})$ is a submodule of $\mathscr{H}^j\big(f_\dagger(R_F\mathscr{M}/z^\ell R_F\mathscr{M})\big)$
- and, on the other hand, if ℓ is big enough, $R_F\mathscr{H}^j(f_\dagger\mathscr{M})/z^\ell\mathscr{H}^j(f_\dagger\mathscr{M})$ is a quotient of $\mathscr{H}^j(f_\dagger R_F\mathscr{M})/z^\ell\mathscr{H}^j(f_\dagger R_F\mathscr{M})$.

For $\ell \geqslant 1$, let us denote by $\operatorname{gr}^F_{[\ell]}$ the grading with step ℓ, namely $\bigoplus_k F_k/F_{k-\ell}$, and let us define f_+ for $\operatorname{gr}^F_{[\ell]}\mathscr{D}_X$-modules in a way similar to what is done for $\mathscr{D}_X$-modules. The conclusion is that $\operatorname{gr}^F_{[\ell]}\mathscr{H}^j(f_\dagger\mathscr{M})$ is a $\operatorname{gr}^F_{[\ell]}\mathscr{D}_Y$-submodule of $\mathscr{H}^j(f_\dagger\operatorname{gr}^F_{[\ell]}\mathscr{M})$.

The sheaf of rings $\operatorname{gr}^F_{[\ell]}\mathscr{D}_X$ is filtered by the finite filtration $G_j\operatorname{gr}^F_{[\ell]}\mathscr{D}_X = \bigoplus_k F_{k+j-\ell}\mathscr{D}_X/F_{k-\ell}\mathscr{D}_X$, and there is the notion of a G-filtration of a $\operatorname{gr}^F_{[\ell]}\mathscr{D}_X$-module (these filtrations should be finite). Moreover, $\operatorname{gr}^G\operatorname{gr}^F_{[\ell]}\mathscr{D}_X \simeq \operatorname{gr}^F\mathscr{D}_X[u]/u^\ell$ by suitably defining the grading on the left-hand term. Given a coherent $\operatorname{gr}^F_{[\ell]}\mathscr{D}_X$-module, the graded module with respect to any G-filtration is $\operatorname{gr}^F\mathscr{D}_X[u]/u^\ell$-coherent, hence $\operatorname{gr}^F\mathscr{D}_X$-coherent, and its support as such does not depend on the choice of such a filtration (same proof as that for the characteristic variety, in a simpler way).

Since the filtration $G_\bullet$ is finite, there is a finite spectral sequence having E_2 term equal to $\mathscr{H}^j(f_\dagger\operatorname{gr}^G\operatorname{gr}^F_{[\ell]}\mathscr{M}) = \mathscr{H}^j(f_\dagger\operatorname{gr}^F\mathscr{M}[u]/u^\ell) \simeq \mathscr{H}^j(f_\dagger\operatorname{gr}^F\mathscr{M})^\ell$ abutting to $\operatorname{gr}^G\mathscr{H}^j(f_\dagger\operatorname{gr}^F_{[\ell]}\mathscr{M})$ for a suitable G-filtration on $\mathscr{H}^j(f_\dagger\operatorname{gr}^F_{[\ell]}\mathscr{M})$. It follows that the support of $\operatorname{gr}^G\mathscr{H}^j(f_\dagger\operatorname{gr}^F_{[\ell]}\mathscr{M})$ is contained in $\widetilde{f}\big((T^*f)^{-1}(\operatorname{Char}\mathscr{M})\big)$.

The filtration $G_\bullet\mathscr{H}^j(f_\dagger\operatorname{gr}^F_{[\ell]}\mathscr{M})$ induces in a natural way a G-filtration on any submodule and any quotient of it, and therefore on $\operatorname{gr}^F_{[\ell]}\mathscr{H}^j(f_\dagger\mathscr{M})$. The support of $\operatorname{gr}^G\operatorname{gr}^F_{[\ell]}\mathscr{H}^j(f_\dagger\mathscr{M})$ as a $\operatorname{gr}^F\mathscr{D}_X$-module is therefore included in that of $\operatorname{gr}^G\mathscr{H}^j(f_\dagger\operatorname{gr}^F_{[\ell]}\mathscr{M})$, hence in $\widetilde{f}\big((T^*f)^{-1}(\operatorname{Char}\mathscr{M})\big)$. Now, as already remarked, as $\operatorname{gr}^F\mathscr{D}_X$-modules we have $\operatorname{gr}^G\operatorname{gr}^F_{[\ell]}\mathscr{H}^j(f_\dagger\mathscr{M}) \simeq \big(\operatorname{gr}^F\mathscr{H}^j(f_\dagger\mathscr{M})\big)^\ell$, which has the same support as $\operatorname{gr}^F\mathscr{H}^j(f_\dagger\mathscr{M})$, that is, $\operatorname{Char}\mathscr{H}^j(f_\dagger\mathscr{M})$. $\square$

LECTURE 5

HOLONOMIC $\mathscr{D}_X$-MODULES

5.1. Motivation: division of distributions

Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a non-zero polynomial. In general the function $1/f$ is not locally integrable, hence does not define a distribution on $\mathbb{C}^n$.

Question. *Does there exist a distribution T on $\mathbb{C}^n$ (or, better, a temperate distribution) such that $f \cdot T = 1$? (More generally, given any distribution (resp. temperate distribution) S on $\mathbb{C}^n$, does there exists a distribution (resp. temperate distribution) T such that $fT = S$.)*

The solution given by J. Bernstein [**Ber72**] proceeds along the following steps.

(1) For $s \in \mathbb{C}$ such that $\operatorname{Re} s \geqslant 0$, the function $|f|^{2s}$ is continuous, hence defines a distribution T_s on $\mathbb{C}^n$: for each test (n,n)-form $\varphi \in C_c^\infty(\mathbb{C}^n) d\boldsymbol{x} \wedge d\overline{\boldsymbol{x}}$, set

$$T_s(\varphi) = \int |f|^{2s} \varphi.$$

One reduces the question to prove that, for each φ, the holomorphic function $s \mapsto T_s(\varphi)$ on the half plane $\operatorname{Re} s > 0$ extends as a meromorphic function on $\mathbb{C}$. One also shows that, denoting by $S(\varphi)$ the constant term in the Laurent expansion of $T_s(\varphi)$ at $s = -1$, the correspondence $\varphi \mapsto S(\varphi)$ defines a distribution (i.e., a continuous linear form on test (n,n)-forms). Lastly, $|f|^2 S(\varphi) = S(|f|^2 \varphi)$ is seen to be equal to the constant term of the Laurent expansion of $T_s(\varphi)$ at $s = 0$. This is nothing but $\int \varphi$. In other words, $|f|^2 S = 1$, hence $T := \overline{f} S$ is a solution to $fT = 1$.

(2) In order to obtain the meromorphic extension of $s \mapsto T_s(\varphi)$, one looks for a pair of differential operators $P \in \mathbb{C}[s][\boldsymbol{x}]\langle \partial_{\boldsymbol{x}} \rangle$ and $Q \in \mathbb{C}[s][\overline{\boldsymbol{x}}]\langle \partial_{\overline{\boldsymbol{x}}} \rangle$ and polynomials $b'(s), b''(s) \in \mathbb{C}[s]$ such that

$$b'(s)|f|^{2s} = P \cdot |f|^{2s} f,$$
$$b''(s)|f|^{2s} = Q \cdot |f|^{2s} \overline{f}.$$

Assume P, Q, b', b'' are found. Then, for $\operatorname{Re} s > 0$,

$$b'(s)\int |f|^{2s}\varphi = \int (P|f|^{2s}f)\varphi = \int |f|^{2s} f P^*\varphi,$$

where P^* denotes the adjoint differential operator (the previous formula corresponds to an iteration of integrations by parts). Therefore,

$$b''(s)b'(s)\int |f|^{2s}\varphi = b''(s)\int |f|^{2s} f P^*\varphi = \int (Q|f|^{2s}\overline{f}) f P^*\varphi = \int |f|^{2(s+1)}\psi,$$

where $\psi = Q^*P^*\varphi$. The right-hand term is holomorphic on $\operatorname{Re} s > -1$, and thus the expression

$$\frac{1}{b'(s)b''(s)}\int |f|^{2(s+1)}\psi$$

is a meromorphic function on $\operatorname{Re} s > -1$ which coincides with $\int |f|^{2s}\varphi$ on $\operatorname{Re} s > 0$. Iterating this process gives the desired meromorphic extension.

(3) It remains to find P, Q, b', b''. Let us try to find P and b'. Then Q and b'' are obtained similarly, by working with $\overline{\boldsymbol{x}}$ and $\partial_{\overline{\boldsymbol{x}}}$. Consider the ring of differential operators $\mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$. We wish to find $\widetilde{P} \in \mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$ such that $\widetilde{P}|f|^{2s}f = |f|^{2s}$ (we then get P, b' by eliminating denominators in $\widetilde{P}$). Note that $\mathbb{C}(s)[\boldsymbol{x}, 1/f] \cdot |f|^{2s}$ is naturally a left $\mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$-module.

The main observation of Bernstein is that this $\mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$-module *has finite length*. This means that any decreasing sequence of submodules is stationary.

Consider the decreasing sequence consisting of $\mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$-submodules M_j of $\mathbb{C}(s)[\boldsymbol{x}, 1/f] \cdot |f|^{2s}$ generated by $f^j|f|^{2s}$ ($j \geqslant 0$). There exists therefore $k \geqslant 1$ such that $f^k|f|^{2s} \in M_{k+1}$, hence there exists $\widetilde{P}_k \in \mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$ such that $f^k|f|^{2s} = \widetilde{P}_k f^{k+1}|f|^{2s}$. Multiplying by $\overline{f}^k$ and using that $\widetilde{P}_k$ is holomorphic, we get $|f|^{2(s+k)} = \widetilde{P}_k f|f|^{2(s+k)}$. We can change the variable s to $s-k$ to get the desired relation.

The property that $\mathbb{C}(s)[\boldsymbol{x}, 1/f] \cdot |f|^{2s}$ has finite length as a $\mathbb{C}(s)[\boldsymbol{x}]\langle\partial_{\boldsymbol{x}}\rangle$-module is the main property used, which follows from a finer property called *holonomy*, concerning dimension. In the next sections, we make explicit this notion in the analytic framework. We come back to the algebraic framework at the end of §5.4.

Exercise 5.1.1. Let $f(x_1, \dots, x_n) = x_1^{m_1} \cdots x_n^{m_n}$, $m_j \in \mathbb{N}$. Show that

$$\Big[\prod_{i=1}^{n}\prod_{k=1}^{m_i}(m_i s + k)\Big] \cdot |f|^{2s} = \Big[\partial_{x_1}^{m_1} \cdots \partial_{x_n}^{m_n}\Big](f|f|^{2s}).$$

Exercise 5.1.2. Consider the quadratic form

$$f(x_1, \dots, x_n) = a_1x_1^2 + \cdots + a_nx_n^2$$

with $a_i \neq 0$ for each i. By using

$$\partial_{x_i}^2(f|f|^{2s}) = 2a_i(s+1)\big(|f|^{2s} + 2s(a_ix_i^2)|f|^{2(s-1)}\overline{f}\big)$$

show that

$$4(s+1)(s+n/2)|f|^{2s} = \Big(\sum_i \frac{\partial_{x_i}^2}{a_i}\Big)(f|f|^{2s}).$$

Exercise 5.1.3. Consider the semi-cubic parabola

$$f(x_1,x_2) = x_1^2 + x_2^3.$$

Show the following relations

$$\begin{aligned}\partial_{x_1}^2(f|f|^{2s}) &= 2(s+1)\big(|f|^{2s} + 2sx_1^2|f|^{2(s-1)}\overline{f}\big)\\ \partial_{x_2}^2(f|f|^{2s}) &= 3(s+1)x_2\big(2|f|^{2s} + 3sx_2^3|f|^{2(s-1)}\overline{f}\big).\end{aligned}$$

Deduce that

$$(9x_2\partial_{x_1}^2 + 4\partial_{x_2}^2)(f|f|^{2s}) = 6(s+1)(6s+7)x_2|f|^{2s}$$

and then

$$\big(\partial_{x_2}(9x_2\partial_{x_1}^2 + 4\partial_{x_2}^2)\big)(f|f|^{2s}) = 6(s+1)(6s+7)\big(|f|^{2s} + 3sx_2^3|f|^{2(s-1)}\overline{f}\big).$$

As in the previous exercise, show that

$$(6s+5)(6s+6)(6s+7)|f|^{2s} = \big(9(6s+7)\partial_{x_1}^2 + 2\partial_{x_2}(9x_2\partial_{x_1}^2 + 4\partial_{x_2}^2)\big)(f|f|^{2s}).$$

Notice that the operator $P(s,\boldsymbol{x},\partial_{\boldsymbol{x}})$ depends on s.

5.2. First properties of holonomic $\mathscr{D}_X$-modules

We consider a complex analytic manifold X of pure dimension n and we introduce in this general setting the notion of holonomic $\mathscr{D}_X$-module.

Definition 5.2.1. A coherent $\mathscr{D}_X$-module is said to be *holonomic* if its characteristic variety $\operatorname{Char}\mathscr{M}$ has dimension $\dim X$.

It follows from the involutiveness theorem 2.5.3 that, if $\mathscr{M}$ is holonomic, $\operatorname{Char}\mathscr{M}$ is a Lagrangean conical subspace of T^*X.

Let us recall a result on conical Lagrangean subspaces of T^*X. Denote by $\pi : T^*X \to X$ the canonical projection. Let Y be a closed analytic subset of X and Y_o the smooth part of Y. The conormal bundle $T^*_{Y_o}X \subset T^*X$ is the following vector subbundle of T^*X:

$$T^*_{Y_o}X = \{v \in T^*X \mid p = \pi(v) \in Y_o \text{ and } v \text{ annihilates } T_pY\}.$$

The *conormal space of* Y, denoted by T^*_YX, is by definition the closure in T^*X of $T^*_{Y_o}X$. We say that a subspace V of T^*X is *conical* if $(x,\xi) \in V \Rightarrow (x,\lambda\xi) \in V$, for any $\lambda \in \mathbb{C}$.

Lemma 5.2.2. *If $V \subset T^*X$ is an analytic conical and Lagrangean subset of T^*X, then there exists a locally finite set $(Y_\alpha)_{\alpha\in A}$ of closed irreducible analytic subsets of Y such that $V = \bigcup_\alpha T^*_{Y_\alpha}X$. Moreover, the $Y_\alpha \subset X$ are the projections of the irreducible components of V.*

As a consequence, if $\mathscr{M}$ is holonomic, there exists a locally finite family (Y_α) of irreducible closed analytic subset of X such that $\operatorname{Char}\mathscr{M} = \bigcup_\alpha T^*_{Y_\alpha}X$, and if we know $\operatorname{Char}\mathscr{M}$, we can recover the sets Y_α as the projections of the irreducible components of $\operatorname{Char}\mathscr{M}$.

Examples 5.2.3

(1) $\mathscr{O}_X$ is a holonomic $\mathscr{D}_X$-module and $\operatorname{Char}\mathscr{O}_X = T^*_XX$.

(2) For any smooth hypersurface H of X, $\mathscr{O}_X(*H)$ is holonomic, $\operatorname{Char}\mathscr{O}_X(*H) = T^*_XX \cup T^*_YX$.

(3) If $n = 1$, a $\mathscr{D}_X$-module is holonomic if and only if each local section of m is annihilated by a non-zero differential operator.

(4) For $n \geqslant 2$, if P is a section of $\mathscr{D}_X$, the quotient $\mathscr{D}_X/\mathscr{D}_XP$ is never holonomic. Its characteristic variety is a hypersurface of T^*X.

From Exercise 2.4.5 we get:

Corollary 5.2.4. *In an exact sequence $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ of coherent $\mathscr{D}_X$-modules, $\mathscr{M}$ is holonomic if and only if $\mathscr{M}'$ and $\mathscr{M}''$ are so.* □

Remark 5.2.5. It is possible to make this result more precise. One can attach to each irreducible component of $\operatorname{Char}\mathscr{M}$ a multiplicity, which is a strictly positive number. This produces a cycle in T^*X, that is, a linear combination of irreducible analytic subsets of T^*X with multiplicities. Then one can prove that the characteristic cycle behaves in an *additive* way in exact sequences.

Corollary 5.2.6. *A decreasing sequence $\mathscr{M}_1 \supset \mathscr{M}_2 \supset \cdots$ of holonomic $\mathscr{D}_X$-modules is locally stationary.*

Proof. By considering the exact sequences $0 \to \mathscr{M}_{j+1} \to \mathscr{M}_j \to \mathscr{M}_j/\mathscr{M}_{j+1} \to 0$, one checks that the family of characteristic cycles is decreasing. In the neighbourhood of a given compact set, we have a decreasing family of cycles with a finite number of components and integral coefficients. It is therefore stationary. Now for $j \gg 0$, the characteristic cycle of $\mathscr{M}_j/\mathscr{M}_{j+1}$ is zero, hence this module is zero, so the sequence $\mathscr{M}_j$ is stationary. □

Corollary 5.2.7. *Each holonomic $\mathscr{D}_X$-module has a Jordan-Hölder sequence which is locally finite.* □

***Corollary 5.2.8* (of Th. 2.5.5).** *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then $\mathscr{M}$ is holonomic if and only if $\mathscr{E}xt^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) = 0$ for $i \neq n$. In this case the natural right $\mathscr{D}_X$-module $\mathscr{E}xt^n_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)$ is a holonomic $\mathscr{D}_X$-module. The associated left $\mathscr{D}_X$-module is called the dual of $\mathscr{M}$.*

5.3. Vector bundles with integrable connections

Let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of finite rank r, equipped with an *integrable* connection $\nabla : \mathscr{E} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{E}$.

***Lemma 5.3.1* (Cauchy-Kowalevski's theorem).** *In the neighbourhood of each point of X there exist a local frame of $\mathscr{E}$ consisting of horizontal sections, i.e., annihilated by ∇.*

This classical theorem is equivalent to the property that, as a vector bundle with connection, $(\mathscr{E}, \nabla)$ is locally isomorphic to $(\mathscr{O}_X, d)^r$. As an immediate consequence, the corresponding $\mathscr{D}_X$-module (see Proposition 1.1.11) has characteristic variety equal to $T^*_X X$ (see Example 5.2.3(1)), and is therefore holonomic. In fact the converse is true (see Exercise 2.4.6).

What happens now if the connection has a pole along a hypersurface $D \subset X$? In such a case, $\mathscr{E}(*D) := \mathscr{O}_X(*D) \otimes_{\mathscr{O}_X} \mathscr{E}$ has an integrable connection, hence is a left $\mathscr{D}_X$-module. Is it coherent, or holonomic, as such?

***Theorem 5.3.2* (Kashiwara [Kas78]).** *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Assume that $\mathscr{M}_{|X \smallsetminus D}$ is holonomic. Then $\mathscr{M}(*D)$ is a holonomic (hence coherent) $\mathscr{D}_X$-module.*

Note that the coherence property of $\mathscr{M}(*D)$ is already not obvious. This theorem extends the algebraic result of Bernstein used in §5.1 to the analytic setting.

Corollary 5.3.3. *Let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module equipped with an integrable meromorphic connection, having poles along a hypersurface $D \subset X$. Then $(\mathscr{E}(*D), \nabla)$ defines a holonomic (hence coherent) $\mathscr{D}_X$-module.*

Proof. We have $\mathscr{E} \subset \mathscr{E}(*D)$. Let us consider the $\mathscr{D}_X$-submodule $\mathscr{M} = \mathscr{D}_X \cdot \mathscr{E} \subset \mathscr{E}(*D)$ generated by $\mathscr{E}$. Consider the filtration $F_k\mathscr{M} = F_k\mathscr{D}_X \cdot \mathscr{E}$. The criterion of Exercise 2.2.3(2) shows that it is a good filtration, hence $\mathscr{M}$ is $\mathscr{D}_X$-coherent. Moreover, $\mathscr{M}_{|X \smallsetminus D}$ is the left $\mathscr{D}_X$-module associated with the holomorphic bundle $\mathscr{E}_{|X \smallsetminus D}$ with holomorphic connection $\nabla_{|X \smallsetminus D}$. From Kashiwara's theorem 5.3.2 we conclude that $\mathscr{M}(*D)$ is $\mathscr{D}_X$-holonomic. But from the inclusions $\mathscr{E} \subset \mathscr{M} \subset \mathscr{E}(*D)$ we deduce that $\mathscr{M}(*D) = \mathscr{E}(*D)$. □

The following converse holds.

Theorem 5.3.4 **(Malgrange [Mal94a, Mal94b, Mal96]).** *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module.*

(1) *If $\mathscr{M}_{|X\smallsetminus D}$ is a vector bundle with integrable connection, there exists a coherent $\mathscr{O}_X$-module $\mathscr{E}$ equipped with an integrable meromorphic connection ∇ such that $\mathscr{M}(*D) = (\mathscr{E}(*D), \nabla)$.*

(2) *In general, $\mathscr{M}$ has a globally defined good filtration.*

5.4. Direct images of holonomic $\mathscr{D}_X$-modules

Theorem 5.4.1 **(Kashiwara [Kas76]).** *Let $f : X \to Y$ be a holomorphic map between complex manifolds and let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Assume that $f_{|\operatorname{Supp}\mathscr{M}}$ is proper. Then the cohomology sheaves $\mathscr{H}^j(f_+\mathscr{M})$ are holonomic.*

Proof. Since $\mathscr{M}$ has a globally defined good filtration (Theorem 5.3.4), it is f-good, and $\mathscr{H}^j(f_+\mathscr{M})$ are coherent $\mathscr{D}_Y$-module whose characteristic variety is controlled by the estimate of Kashiwara (Theorem 4.8.1). The result is then a consequence of the following geometric lemma.

Lemma 5.4.2. *Let $\Lambda = T^*_Z X$ be a Lagrangean closed analytic subvariety in T^*X (i.e., Z is a closed analytic set in X). Then each irreducible component of $\widetilde{f}\big((T^*f)^{-1}\Lambda\big)$ is isotropic in T^*Y.*

Assume that the lemma is proved. It follows that each irreducible component of $\widetilde{f}\big((T^*f)^{-1}\operatorname{Char}\mathscr{M}\big)$ is isotropic. As a consequence, according to Kashiwara's estimate, each irreducible component of $\operatorname{Char}\mathscr{H}^j(f_+\mathscr{M})$ is isotropic. Since such a component is also involutive (Theorem 2.5.3), it is therefore Lagrangean, so $\mathscr{H}^j(f_+\mathscr{M})$ is holonomic. □

Proof of Lemma 5.4.2. It is convenient to decompose f in the following way:

$$\begin{tikzcd}[row sep=tiny] f : X \arrow[r, hook, "i_f"] & X \times Y \arrow[r, "p_2"] & Y \\ x & (x, f(x)) & f(x) \end{tikzcd}$$

and we are reduced to showing the lemma when f is an inclusion (like i_f) and when f is a submersion (like p_2). The first case is easy, and we will only consider the second one. We will use the following property, which follows from a theorem due to H. Whitney: let $\Lambda' \subset \Lambda$ be to closed analytic subsets of T^*X; if Λ is isotropic (i.e., ω_X vanishes when restricted to pairs of vectors tangent to the smooth part Λ^o), then Λ' is also isotropic.

Recall the basic diagram:

$$\begin{tikzcd}[row sep=tiny] T^*X & f^*T^*Y \arrow[l, hook', "T^*f =: \rho"'] \arrow[r, "\widetilde{f}"] & T^*Y \\ \omega_X & \rho^*\omega_X = \widetilde{f}^*\omega_Y & \omega_Y \end{tikzcd}$$

Then ω_X vanishes on any $\Lambda'^o \subset (\Lambda \cap f^*T^*Y)$ (i.e., on any pair of vectors tangent to Λ'^o), hence so does $\rho^*\omega_X$, that is, $\widetilde{f}^*\omega_Y$. Argue now by contradiction: assume there is an irreducible component Λ_Y of $\widetilde{f}((T^*f)^{-1}\Lambda)$ such that $\omega_Y \not\equiv 0$ on Λ_Y^o; let Λ' be an irreducible component of $\Lambda \cap f^*TY$ whose image by $\widetilde{f}$ is Λ_Y; then $\widetilde{f} : \Lambda' \to \Lambda_Y$ is generically a submersion; therefore, $\widetilde{f}^*\omega_Y$ cannot vanish identically on Λ'^o, a contradiction. □

Application to the proof of Theorem 4.1.5*.* The algebraic analogues of the previous results are due to J. Bernstein (note however that the algebraic analogue of Malgrange's theorem is much easier than the analytic one). We explain now how they can be combined to obtain a proof of Theorem 4.1.5.

Firstly, $\mathrm{GM}^k(f)$ is identified with the algebraic direct image $\mathscr{H}^{k-n}(f_+\mathscr{O}_{(\mathbb{C}^*)^n})$. Since f is not proper, one cannot apply the coherence and holonomy theorem. However, choose a smooth quasi-projective variety X and a projective morphism $g : X \to \mathbb{C}$ such that $(\mathbb{C}^*)^n$ is a dense Zariski open set of X whose complement is a hypersurface D, and $g_{|(\mathbb{C}^*)^n} = f$. Let $i : (\mathbb{C}^*)^n \hookrightarrow X$ denote the inclusion. Since $g \circ i = f$, we have

$$\mathscr{H}^{k-n}(f_+\mathscr{O}_{(\mathbb{C}^*)^n}) = \mathscr{H}^{k-n}\big(g_+(\mathscr{O}_X(*D))\big).$$

Since $\mathscr{O}_X(*D)$ is $\mathscr{D}_X$-holonomic, these $\mathscr{D}_{\mathbb{C}}$-modules are holonomic, that is, $\mathrm{GM}^k(f)$ are holonomic. In particular (Exercise 5.2.3(3)), any section of $\mathrm{GM}^k(f)$ is annihilated by a non-zero differential operator. □

5.5. The de Rham complex of a holonomic $\mathscr{D}_X$-module

Theorem 5.5.1. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then ${}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ has constructible cohomology. More precisely, ${}^{\mathrm{p}}\mathrm{DR}\,\mathscr{M}$ is a perverse sheaf on X.*

***Proposition 5.5.2* (Behaviour with respect to external product, see [Meb89, Prop. 10.19])** *Let X_1, X_2 be complex manifolds and let $\mathscr{M}_i$ (($i = 1, 2$) be a $\mathscr{D}_{X_i}$-module (or an object of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_{X_i})$. There is a natural and functorial morphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_1 \times X_2})$:*

$${}^{\mathrm{p}}\mathrm{DR}_{X_1}(\mathscr{M}_1) \boxtimes_{\mathbb{C}} {}^{\mathrm{p}}\mathrm{DR}_{X_2}(\mathscr{M}_2) = {}^{\mathrm{p}}\mathrm{DR}_{X_1\times X_2}(\mathscr{M}_1 \boxtimes_{\mathbb{C}} \mathscr{M}_2) \longrightarrow {}^{\mathrm{p}}\mathrm{DR}_{X_1\times X_2}(\mathscr{M}_1 \boxtimes_{\mathscr{D}} \mathscr{M}_2),$$

which is an isomorphism if $\mathscr{M}_1$ has holonomic cohomology and $\mathscr{M}_2$ has $\mathscr{D}_{X_2}$-coherent cohomology.

5.6. Recent advances

5.6.a. Local normal form of a meromorphic connection near a pole. Classical asymptotic analysis in one complex variable produces a normal form for a meromorphic connection in one variable near one of its pole. It is now standard to present this result in three steps:

(1) existence of a local formal normal form for the matrix of the connection,

(2) asymptotic – or multisummable – liftings of this normal form in sectors around the pole,

(3) comparison between the various liftings, which gives rise to the Stokes phenomenon.

In higher dimension, such results have only been obtained recently. More precisely, work of Majima [**Maj84**] and C.S. [**Sab93, Sab00**] answer the second point, if the first one is assumed to be solved.

A precise conjecture for a statement analogous to the first point is given in [**Sab00**]. It has been solved recently in two different ways:

- T. Mochizuki has solved the conjecture as stated in [**Sab00**], that is, in dimension two, by using a reduction to characteristic p and the notion of p-curvature [**Moc09**]. He then solved the analogue of the conjecture in arbitrary dimension by using techniques of differential geometry (Higgs bundles and harmonic metrics) [**Moc11a**].
- K. Kedlaya used techniques inspired from p-adic differential equations (in particular, a systematic use of Berkovich spaces) to solve the conjecture in any dimension [**Ked10, Ked11**].

The third point has been generalized by T. Mochizuki [**Moc11a, Moc11b**] and C.S. [**Sab13**] by developing the notion of Stokes filtration, following a previous approach in dimension one by P. Deligne [**Del07**] and B. Malgrange [**Mal91**]. This leads to a Riemann-Hilbert correspondence in arbitrary dimension, and in the setting of possibly irregular singularities.

5.6.b. A conjecture of Kashiwara. One of the consequences of the previous results is the following theorem, proved by M. Kashiwara [**Kas86**] under the assumption of regular singularity, and in general (a conjecture of Kashiwara) by T. Mochizuki T. Mochizuki [**Moc11b**] and C.S. [**Sab00, Sab13**].

Theorem 5.6.1. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathfrak{Db}_X)$, equipped with its left $\mathscr{D}_{\overline{X}}$-module structure coming from that on $\mathfrak{Db}_X$, see* §1.3.e, *is a holonomic (hence coherent) $\mathscr{D}_{\overline{X}}$-module. Moreover, for each $i > 0$, $\mathscr{E}xt^i_{\mathscr{D}_X}(\mathscr{M}, \mathfrak{Db}_X) = 0$.*

This statement has many consequences, already noted by M. Kashiwara [**Kas86**] (see also [**Bjö93**]). One of them says that any holonomic $\mathscr{D}_X$-module is locally a $\mathscr{D}_X$-submodule of $\mathfrak{Db}_X$.

5.6.c. Wild Hodge theory. The theory of Hodge $\mathscr{D}$-modules developed by M. Saito [**Sai88, Sai90**] allows one to consider Hodge theory for singular spaces. The basic objects are holonomic $\mathscr{D}_X$-modules equipped with a specific good filtration (the Hodge filtration). Following Deligne, Beilinson and Bernstein [**BBDG82**], this makes an analogy with the theory of pure perverse ℓ-adic sheaves with tame ramification on an algebraic variety in characteristic p.

In order to extend such an analogy in the case of wild ramification, a generalization of Hodge theory (and thus of Hodge $\mathscr{D}$-modules) is needed. This has been developed by T. Mochizuki and C.S. in the tame case first [**Moc07, Sab05**] and then in the wild case [**Moc11a**], see also [**Sab09**].

LECTURE 6

COMPUTATIONAL ASPECTS IN $\mathscr{D}$-MODULE THEORY

6.1. Review on the Gröbner basis of an ideal in a polynomial ring

Let $\boldsymbol{k}$ a commutative field. We choose on $\mathbb{N}^{\mathrm{right}}$ a well-ordering. By definition, for such an ordering, all nonempty subset in $\mathbb{N}^{\mathrm{right}}$ has a smallest element. We assume that this relation is compatible with addition in the semi group $\mathbb{N}^{\mathrm{right}}$:

$$\forall\, \alpha, \beta, \gamma \in \mathbb{N}^{\mathrm{right}}, \quad \alpha \geqslant \beta \Longrightarrow \alpha + \beta \geqslant \beta + \gamma.$$

Exercise 6.1.1. Show that an ordering on $\mathbb{N}^{\mathrm{right}}$ is a well-ordering if and only if every strictly decreasing sequence in $\mathbb{N}^{\mathrm{right}}$ is finite.

The first example is the *lexicographic ordering*, which is defined by induction:

$$\alpha = (\alpha_1, \dots, \alpha_r) > \beta = (\beta_1, \dots, \beta_r)$$

when $\alpha_1 > \beta_1$ or when $\alpha_1 = \beta_1$ and $(\alpha_2, \dots, \alpha_r) > (\beta_2, \dots, \beta_r)$ (the natural ordering of $\mathbb{N}$ for $r = 1$).

The second example is the *graded lexicographic ordering*, defined by induction:

$$\alpha = (\alpha_1, \dots, \alpha_r) > \beta = (\beta_1, \dots, \beta_r)$$

if $|\alpha| > |\beta|$, or if $|\alpha| = |\beta|$ and $\alpha > \beta$ for the lexicographic ordering define before ($|\alpha| = \sum_{i=1}^{\mathrm{right}} \alpha_i$ denotes the length of the multi-index $\alpha \in \mathbb{N}^{\mathrm{right}}$).

Exercise 6.1.2. Show that the lexicographic ordering (resp. the graded lexicographic ordering) is a well-ordering compatible with addition in the semi group $\mathbb{N}^{\mathrm{right}}$.

Given a well-ordering on $\mathbb{N}^{\mathrm{right}}$, we will use the following terminology:

Definition 6.1.3. Let $P = \sum_\alpha a_\alpha X^\alpha$ be a nonzero polynomial in $\boldsymbol{k}[X_1, \dots, X_r]$.

- The multidegree of P is $d(P) = \max\{\alpha \in \mathbb{N}^{\mathrm{right}} \mid a_\alpha \neq 0\}$.
- The leading coefficient of P is $\mathrm{LC}(P) = a_{d(P)} \in k$.
- The leading monomial of P is $\mathrm{LM}(P) = X^{d(P)} \in \boldsymbol{k}[X_1, \dots, X_r]$.

Proposition 6.1.4. *Let $P, Q \in \boldsymbol{k}[X_1, \dots, X_n]$ be two nonzero polynomials. We have:*

- *if $d(P) > d(Q)$, then $d(P + Q) = d(P)$ and $\mathrm{LT}(P + Q) = \mathrm{LT}(P)$,*
- *if $d(Q) > d(P)$, then $d(P + Q) = d(Q)$ and $\mathrm{LT}(P + Q) = \mathrm{LT}(Q)$,*

- *if* $d(P) = d(Q)$ *and* $\mathrm{LC}(P) + \mathrm{LC}(Q) \neq 0$*, then*
$$\begin{cases} d(P+Q) = d(P) = d(Q), \\ \mathrm{LT}(P+Q) = \mathrm{LT}(P) + \mathrm{LT}(Q), \end{cases}$$
- *if* $d(P) = d(Q)$ *and* $\mathrm{LC}(P) + \mathrm{LC}(Q) = 0$*, then* $d(P+Q) < d(P)$*.*

Proof. Write the polynomials P and Q as
$$P = \mathrm{LC}(P)X^{d(P)} + \sum_{d(P)>\alpha} a_\alpha X^\alpha, \quad Q = \mathrm{LC}(Q)X^{d(Q)} + \sum_{d(Q)>\beta} b_\beta X^\beta$$
If $d(P) > d(Q)$, we obtain:
$$P + Q = \mathrm{LC}(P)X^{d(P)} + \sum_{d(P)>\gamma} (a_\gamma + b_\gamma)X^\gamma$$
If $d(P) = d(Q)$, we obtain:
$$P + Q = (\mathrm{LC}(P) + \mathrm{LC}(Q))X^{d(P)} + \sum_{d(P)>\gamma} (a_\gamma + b_\gamma)X^\gamma$$
The proposition follows from these equalities. □

We can now give a division algorithm for polynomials in $\boldsymbol{k}[X_1, \ldots, X_r]$ that extends the Euclidean algorithm for one variable polynomials. The goal is to divide $F \in \boldsymbol{k}[X_1, \ldots, X_r]$ by nonzero polynomials $F_1, \ldots, F_s \in \boldsymbol{k}[X_1, \ldots, X_r]$.

***Proposition 6.1.5* (Division of a polynomial by a family).** *Let* $F_1, \ldots, F_s \in \boldsymbol{k}[X_1, \ldots, X_r]$ *be nonzero polynomials and let* $F \in \boldsymbol{k}[X_1, \ldots, X_r]$*. Associate to* F *a sequence* $(A_1^{(l)}, \ldots, A_s^{(l)}, P^{(l)}, R^{(l)})_{l\in\mathbb{N}}$ *in* $\boldsymbol{k}[X_1, \ldots, X_r]^{s+2}$ *defined inductively by:*

(1) $(A_1^{(0)}, \ldots, A_s^{(0)}, P^{(0)}, R^{(0)}) = (0, \ldots, 0, F, 0)$*,*

(2) (2.1) *if* $P^{(k)} = 0$*, then for* $\ell \geqslant k$*,*
$$(A_1^{(\ell)}, \ldots, A_s^{(\ell)}, P^{(\ell)}, R^{(\ell)}) = (A_1^{(k)}, \ldots, A_s^{(k)}, P^{(k)}, R^{(k)}),$$

(2.2) *if for every* i*,* $1 \leqslant i \leqslant s$*,* $\mathrm{LM}(F_i)$ *does not divide* $\mathrm{LM}(P^{(k)})$*, then*
$$\begin{aligned} A_i^{(k+1)} &= A_i^{(k)} \quad 1 \leqslant i \leqslant s \\ P^{(k+1)} &= P^{(k)} - \mathrm{LT}(P^{(k)}) \\ R^{(k+1)} &= R^{(k)} + \mathrm{LT}(P^{(k)}), \end{aligned}$$

(2.3) *or else, let* $j = \inf\{i \mid 1 \leqslant i \leqslant s,\ \mathrm{LM}(F_i)$ *divides* $\mathrm{LM}(P^{(k)})\}$*; then,*
$$\begin{aligned} A_i^{(k+1)} &= A_i^{(k)} \text{ for } i \neq j \\ A_j^{(k+1)} &= A_j^{(k)} + (\mathrm{LC}(P^{(k)})/\mathrm{LC}(F_j))\, X^{d(P^{(k)})-d(F_j)} \\ P^{(k+1)} &= P^{(k)} - (\mathrm{LC}(P^{(k)})/\mathrm{LC}(F_j))\, X^{d(P^{(k)})-d(F_j)}\, F_j \\ R^{(k+1)} &= R^{(k)}. \end{aligned}$$

Then, for every integer k, we have:

$$F = A_1^{(k)}F_1 + \cdots + A_s^{(k)}F_s + P^{(k)} + R^{(k)}.$$

Moreover, $d(P^{(k)}) \geqslant d(P^{(k+1)})$, $d(A_i^{(k)}) \leqslant d(F) - d(F_i)$ and $P^{(k)}$ are zero for k large enough. We set $k_0 = \inf\{k \mid P^{(k)} = 0\}$ and call the ordered family

$$(A_1^{(k_0)}, \ldots, A_s^{(k_0)}, R^{(k_0)})$$

the result of the division of F by $F_1, \ldots, F_s$ *and* $R^{(k_0)}$ *the* remainder of F in the division by $F_1, \ldots, F_s$. *We obtain:*

$$F = A_1^{(k_0)}F_1 + \cdots A_s^{(k_0)}F_s + R^{(k_0)}$$

with $d(A_i^{k_0}) + d(F_i) \leqslant d(F)$ for $1 \leqslant i \leqslant s$ and none of the terms of $R^{(k_0)}$ is divisible by any of the $\mathrm{LM}(F_i)$ *for $1 \leqslant i \leqslant s$.*

Proof. By induction, it is easy to check that for every integer k, we have:

$$F = A_1^{(k)}F_1 + \cdots A_s^{(k)}F_s + P^{(k)} + R^{(k)}.$$

By construction, if $P^{(k)} \neq 0$, the polynomial $P^{(k+1)}$ is zero or has a strictly smaller multidegree. This proves the existence of the integer k_0. By induction, it is obvious that no term of $R^{(k)}$ is divisible by the leading monomial of any F_i. Moreover, for every k, we have $d(F) = d(P^{(0)}) \geqslant d(P^{(k)})$. By induction, we deduce that, for $1 \leqslant i \leqslant s$, $d(A_i^k) + d(F_i) \leqslant d(F)$. We take $k = k_0$ to finish of the proof. $\square$

Example 6.1.6. Let us consider the lexicographic ordering on $\mathbb{N}^2$. The result of the division of $F = X^3 - XY^2 + X^2 - Y^2$ by $(F_1 = X^2 + Y, F_2 = XY + X)$ is:

$$F = (X+1)F_1 - YF_2 - Y^2 - Y.$$

We remark that the remainder is not zero. But, as $Y^2 + Y = (1+Y)F_1 - XF_2$, we have:

$$F = (X - Y)F_1 + (X - Y)F_2$$

and F is nevertheless in the ideal generated by F_1, F_2.

Exercise 6.1.7. Give a example where the remainder of a polynomial on division by a family does depend on the order of this family.

Definition 6.1.8. An ideal of $\boldsymbol{k}[X_1, \ldots, X_r]$ is a monomial ideal if it is generated by a family of monomials

Proposition 6.1.9. *Let I be a monomial ideal generated and $(X^\beta)_{\beta \in A}$ a generating set. Let $P = \in \sum_\alpha a_\alpha X^\alpha$ be a polynomial. Then, P belongs to I if and only if every term of P is multiple of some X^β for $\beta \in A$.*

Definition 6.1.10. Let I be an ideal of $\boldsymbol{k}[X_1, \ldots, X_r]$. We denote by $\mathrm{in}(I)$ the monomial ideal of $\boldsymbol{k}[X_1, \ldots, X_n]$ generated by the leading monomials of the nonzero elements of I.

Definition 6.1.11. Let I be an ideal of $\boldsymbol{k}[X_1, \ldots, X_r]$. We call *Gröbner basis of I* a finite set $F_1 \ldots, F_s$ of I such that $\mathrm{LM}(F_1), \ldots, \mathrm{LM}(F_s)$ generate $\mathrm{in}(I)$.

If we lift a finite generating set of $\mathrm{in}(I)$, we obtain a Gröbner basis of I. Therefore, Gröbner bases exist.

Proposition 6.1.12. *Let I be a nonzero ideal of $\boldsymbol{k}[X_1, \ldots, X_r]$ and let $G_1, \ldots, G_s$ be a Gröbner basis of I. Then, for every $F \in \boldsymbol{k}[X_1, \ldots, X_r]$, there exists $A_1, \ldots, A_s \in \boldsymbol{k}[X_1, \ldots, X_r]$ and a unique $R \in \boldsymbol{k}[X_1, \ldots, X_r]$ such that:*

$$F = \sum_{i=1}^{s} A_i G_i + R,$$

where $R = 0$ or none of the terms of R is in $\mathrm{in}(I)$.

The polynomial R is called the *remainder of F* in the division by the Gröbner basis $G_1, \ldots, G_s$.

Proof. We consider the division of F by the family $G_1, \ldots, G_s$. None of the terms of the remainder is divisible by $\mathrm{LM}(G_1), \ldots, \mathrm{LM}(G_s)$. If follows from Proposition 6.1.9 that none of these terms is in $\mathrm{in}(I)$. Then, the result of the division of F by the family $G_1, \ldots, G_s$ gives a decomposition satisfying conditions of the proposition. If we assume that there are two distinct remainders R and R', we obtain $R - R' \in I$; it follows that $\mathrm{LM}(R - R') \in \mathrm{in}(I)$; this is impossible because, after the condition on the remainder, none of the monomials of $R - R'$ is in $\mathrm{in}(I)$. ☐

Proposition 6.1.13. *Let I be a nonzero ideal of $\boldsymbol{k}[X_1, \ldots, X_s]$. Then, every Gröbner basis $G_1, \ldots, G_s$ of I is a generating set of I. And for every $F \in \boldsymbol{k}[X_1, \ldots, X_n]$, we have $F \in I$ if and only if the remainder of F in the division by $G_1, \ldots, G_s$ is zero.*

Proof. The key is the last assertion. Let us show it. Let R be the remainder of F in division by the Gröbner basis $G_1, \ldots, G_s$. As $G_1, \ldots, G_s \in I$, the polynomial R is in I. So, $R = 0$ or $\mathrm{LM}(R) \in \mathrm{in}(I)$. From the assumption on R, this implies $R = 0$. ☐

Definition 6.1.14. Let $F, G \in \boldsymbol{k}[X_1, \ldots, X_s]$ be two nonzero polynomials, $X^\alpha = \mathrm{LM}(F)$ and $X^\beta = \mathrm{LM}(G)$. Let γ be the least common multiple of X^α and X^β. The *S-polynomial of F and G* is the combination:

$$S(F, G) := \frac{X^{\gamma-\alpha}}{\mathrm{LC}(F)} F - \frac{X^{\gamma-\beta}}{\mathrm{LC}(G)} G = \frac{X^\gamma}{\mathrm{LT}(F)} F - \frac{X^\gamma}{\mathrm{LT}(G)} G.$$

Remark 6.1.15. We have $d(S(F, G)) < \gamma$.

Proposition 6.1.16. *Let I be a nonzero ideal of $\boldsymbol{k}[X_1, \ldots, X_s]$ and $G_1, \ldots, G_s$ a generating set of I. Then, $G_1, \ldots, G_s$ is a Gröbner basis of I if and only if the remainders of $S(G_i, G_j)$, $1 \leqslant i < j \leqslant s$ in the division by $G_1, \ldots, G_s$ are zero.*

Proof. Let $G_1, \ldots, G_s$ be a Gröbner basis of I. As $S(G_i, G_j) \in I$ it follows from Proposition 6.1.13 that the remainder of $S(G_i, G_j)$, $1 \leqslant i < j \leqslant s$ in the division by $G_1, \ldots, G_s$ is zero.

Conversely, assume that these remainders are zero. We have to show that, if $P \in I$ is not zero, we have $\mathrm{LM}(P) \in (\mathrm{LM}(G_1), \ldots, \mathrm{LM}(G_s))$. As $G_1, \ldots, G_s$ is a generating set of I, there exists $P_1, \ldots, P_s \in \boldsymbol{k}[X_1, \ldots, X_s]$ such that

$$(*) \qquad P = \sum_{i=1}^{s} P_i G_i$$

Let $d = \max\{d(P_i) + d(G_i)\}$. Choose $(*)$ with d minimum. Then, set

$$\Gamma = \{i \mid d = d(P_i) + d(G_i)\}.$$

If $d(P) = d$, we have

$$\mathrm{LT}(P) = \sum_{i \in I} \mathrm{LT}(P_i)\mathrm{LT}(G_i) \quad \text{and} \quad \mathrm{LM}(P) \in (\mathrm{LM}(G_1), \ldots, \mathrm{LM}(G_s))$$

If $d(P) < d$, we have

$$\sum_{i \in \Gamma} \mathrm{LT}(P_i)\mathrm{LT}(G_i) = X^d \sum_{i \in \Gamma} \mathrm{LC}(P_i)\mathrm{LC}(G_i) = 0.$$

Then, choose $i_0 \in \Gamma$ and let $\gamma(i, i_0) \in \mathbb{N}^n$ be such that

$$X^{\gamma(i,i_0)} = \mathrm{lcm}(\mathrm{LM}(G_i), \mathrm{LM}(G_{i_0}))$$

For every $i \in \Gamma$, X^d is a multiple of $X^{\gamma(i,i_0)}$. Therefore, we obtain

$$\mathrm{LT}(P_i)G_i = \mathrm{LM}(P_i)\mathrm{LC}(P_i)G_i = X^{d-\gamma(i,i_0)}\mathrm{LC}(P_i)\mathrm{LC}(G_i)\Big(\frac{X^{\gamma(i,i_0)}}{\mathrm{LT}(G_i)}G_i\Big).$$

But the remainder of $S(G_i, G_{i_0})$ in the division by $G_1, \ldots, G_s$ is zero. Hence, we have

$$S(G_i, G_{i_0}) = \frac{X^{\gamma(i,i_0)}}{\mathrm{LT}(G_i)}G_i - \frac{X^{\gamma(i,i_0)}}{\mathrm{LT}(G_{i_0})}G_{i_0} = \sum_{j=1}^{s} U_{i,i_0,j}G_j,$$

with $d(U_{i,i_0,j}) + d(G_j) < \gamma(i, i_0)$. It follows that

$$\begin{aligned}\sum_{i \in \Gamma} \mathrm{LT}(P_i)G_i &= \sum_{i \in \Gamma} X^{d-\gamma(i,i_0)}\mathrm{LT}(P_i)\mathrm{LC}(G_i)\Big(\frac{X^{\gamma(i,i_0)}}{\mathrm{LC}(G_{i_0})}G_{i_0} + \sum_{j=1}^{s} U_{i,i_0,j}G_j\Big) \\ &= \sum_{i \in \Gamma} X^{d-\gamma(i,i_0)}\mathrm{LT}(P_i)\mathrm{LC}(G_i)\Big(\sum_{j=1}^{s} U_{i,i_0,j}G_j\Big)\end{aligned}$$

As $P = \sum_{i \in \Gamma} \mathrm{LT}(P_i)G_i + \sum_{i \in \Gamma}(P_i - \mathrm{LT}(P_i))G_i + \sum_{i \notin \Gamma} P_i G_i$, we deduce a new way of writing P. This shows that d is not minimum. □

We now explain an algorithm that computes a Gröbner basis of an ideal I of $\boldsymbol{k}[X_1, \ldots, X_s]$ from a given finite generating set.

***Proposition 6.1.17* (Algorithm for a Gröbner basis).** *Let I be an ideal of $\boldsymbol{k}[X_1,\dots,X_s]$ generated by a finite set $F_1,\dots,F_s$. Define by induction a sequence $(F_1^{(k)},\dots,F_{n_k}^{(k)})$ of polynomials*

$$n_0 = s, \quad F_1^{(0)} = F_1, \dots, F_{n_0}^{(0)} = F_s.$$

If $R_1^{(k)},\dots R_{l_k}^{(k)}$ are the nonzero remainders of $S(F_i^{(k)},F_j^{(k)})$, $1 \leqslant i < j \leqslant n_k$ in the division by $F_1^{(k)},\dots,F_{n_k}^{(k)}$, set

$$n_{k+1} = n_k + l_k, \quad F_i^{(k+1)} = \begin{cases} F_i^{(k)} & \text{if } i \leqslant n_k, \\ R_{i-n_k}^{(k)} & \text{if } i > n_k. \end{cases}$$

Then, for k_0 large enough, $(F_1^{(k_0)},\dots,F_{n_{k_0}}^{(k_0)})$ is a Gröbner basis of I.

Proof. At each step, $F_1^{(k)},\dots,F_{n_k}^{(k)}$ is a generating set of I. If a remainder R of $S(F_i^{(k)},F_j^{(k)})$ in the division by $(F_1^{(k)},\dots,F_{n_k}^{(k)})$ is not zero, $\mathrm{LM}(R)$ is not in the monomial ideal $J_k = \big(\mathrm{LM}(F_1^{(k)}),\dots,\mathrm{LM}(F_{n_k}^{(k)})\big)$ and $F_1^{(k)},\dots,F_{n_k}^{(k)}$ is not Gröbner basis of I. The sequence J_k is an ascending chain of ideals in $\boldsymbol{k}[X_1,\dots,X_s]$. As $\boldsymbol{k}[X_1,\dots,X_s]$ is a Noetherian ring, this chain stabilizes. Therefore, there exists k_0 such that every remainder of $S(F_i^{(k_0)},F_j^{(k_0)})$ in the division by $F_1^{(k_0)},\dots,F_{n_{k_0}}^{(k_0)}$ is zero. By Proposition 6.1.16, we deduce that $F_1^{(k_0)},\dots,F_{n_{k_0}}^{(k_0)}$ is Gröbner basis of I. □

We end this section with an algorithm computing the intersection $I \cap \boldsymbol{k}[X_\ell,\dots,X_s]$ of an ideal I in $\boldsymbol{k}[X_1,\dots,X_s]$ given by a finite generating set with the subring $\boldsymbol{k}[X_\ell,\dots,X_s]$ for $1 \leqslant \ell \leqslant s$. We have seen there exists an algorithm computing a Gröbner basis of I, hence an algorithm computing the ideal $I \cap \boldsymbol{k}[X_\ell,\dots,X_s]$ in $\boldsymbol{k}[X_\ell,\dots,X_s]$ is a consequence from the following proposition:

Proposition 6.1.18. *Let I an ideal in $\boldsymbol{k}[X_1,\dots,X_s]$ and let $\ell \in \mathbb{N}$, with $1 \leqslant \ell \leqslant s$. Choose a well-ordering compatible with the addition such that, for $1 \leqslant i,j \leqslant \ell-1$, the multidegree of X_i is larger than the multidegree of any monomial which is independent of $X_1,\dots,X_{\ell-1}$ (for instance, a lexicographic ordering). Let G be a Gröbner basis of I, then the set $G \cap \boldsymbol{k}[X_\ell,\dots,X_s]$ is a Gröbner basis of $I \cap \boldsymbol{k}[X_\ell,\dots,X_s]$*

Proof. It is enough to use the criterion of Proposition 6.1.16 to get the result. □

6.2. Gröbner basis of a noncommutative algebra

We choose on $\mathbb{N}^{\mathrm{right}}$ a well-ordering compatible with the addition. In this subsection, we consider a commutative unitary ring A and we assume that A is an integral domain. We denote by K the quotient field of A. And we consider a commutative, unitary, associative A-algebra $\mathscr{B}$. We will make some assumptions on $\mathscr{B}$ in order to develop a theory of Gröbner basis similar to that of the previous section.

Assumption 1. *There exist elements $z_1, \dots, z_r \in \mathscr{B}$ so that the set of monomials $(z_1^{\alpha_1} \dots z_r^{\alpha_r})_{\alpha \in \mathbb{N}^{\text{right}}}$ is a basis of $\mathscr{B}$ as a A-module.*

We will set $z^\alpha = z_1^{\alpha_1} \dots z_r^{\alpha_r}$ (be careful however: in this way of writing, the ordering of the z_i is fixed).

Thus, we can define the multidegree of a nonzero element $P = \sum a_\alpha z^\alpha$ of $\mathscr{B}$ as an element in $\mathbb{N}^{\text{right}}$:

$$d(P) = \max\{\alpha \mid a_\alpha \neq 0\}$$

Then $c(P)$, the leading coefficient of P, is defined as the coefficient of the the highest degree term in P.

By convention, $d(0) = -\infty$. We then have

$$\forall\, P, Q \in \mathscr{B}, \quad d(P+Q) \leqslant \sup(d(P), d(Q))$$

with equality when $d(P) \neq d(Q)$ or when $d(P) = d(Q)$ and $c(P) + c(Q)$ is nonzero.

For every $\alpha \in \mathbb{N}^{\text{right}}$, we set

$$\mathscr{B}_\alpha = \{P \in \mathscr{B} \mid d(P) \leqslant \alpha\}$$
$$\mathscr{B}_{<\alpha} = \{P \in \mathscr{B} \mid d(P) < \alpha\}$$
$$\mathrm{gr}\mathscr{B} = \bigoplus_{\alpha \in \mathbb{N}^{\text{right}}} \mathscr{B}_\alpha / \mathscr{B}_{<\alpha}$$

Assumption 2. *For all $\alpha, \beta \in \mathbb{N}^{\text{right}}$, $\mathscr{B}_\alpha \mathscr{B}_\beta \subset \mathscr{B}_{\alpha+\beta}$.*

The graded ring $\mathrm{gr}\mathscr{B}$ is then naturally a A-algebra.

Assumption 3. *The A-algebra $\mathrm{gr}\mathscr{B}$ is commutative.*

We can now define a natural map $\mathrm{gr}\mathscr{B} \to A[Z_1, \dots, Z_r]$ that associates to $[P] \in \mathscr{B}_\alpha / \mathscr{B}_{<\alpha}$ the monomial $\mathrm{in}(P) = c(P) Z^{d(P)}$. This monomial is called the *initial part of* P.

Assumptions 2 and 3 on $\mathscr{B}$ are in fact equivalent to the condition that, for all $i > j \in \{1, \dots, r\}$, one has $d(z_i z_j - z_j z_i) < d(z_j z_i)$.

They also imply that, for all $P, Q \in \mathscr{B}$, one has $d(PQ) = d(P) + d(Q)$ and $\mathrm{in}(PQ) = c(P)c(Q)Z^{d(P)+d(Q)}$.

We can now define a division of $P \in \mathscr{B}$ by a family $(P_1, \dots, P_m)$ of elements in $\mathscr{B}$. Let a be the product of the $c(P_i)$. If $d(P) \geqslant \max\{d(P_i) \mid 1 \leqslant i \leqslant r\}$, let $j = \inf\{i \mid d(P) = d(P_i)\}$. We consider

$$P(1) = aP - \frac{a\, c(P)}{c(P_j)} P_j.$$

This element $P(1)$ of $\mathscr{B}$ satisfies $d(P(1)) < d(P)$. By iterating the process, we obtain an algorithm that provides, for every $P \in \mathscr{B}$, an integer ℓ and a family $Q_1, \dots, Q_m, R$ of elements in $\mathscr{B}$ such that

$$a^\ell P = \sum_{i=1}^{m} Q_i P_i + R,$$

and, for every $i \in \{1, 2, \ldots, r\}$, $d(Q_i) + d(P_i) \leqslant d(P^)$, $d(R) < d(P_i)$. As in the commutative case, the family $(Q_1, \ldots, Q_m, R)$ depends on the ordering in the family $(P_1, \ldots, P_m)$. We call R the *remainder* of P in the division by $P_1, \ldots, P_m$.

Remark that, if $\mathscr{B}$ satisfies Assumptions 1, 2, 3 and if B is a A-algebra and an integral domain, the B-algebra $B \otimes_A \mathscr{B}$ also satisfies Assumptions 1, 2, 3. In particular, we will consider the case where $B = K$ is the quotient field of A, or $B = A_{\mathfrak{p}}$ is the localization of A along the multiplicative system $A - \mathfrak{p}$, or also $B = \kappa(\mathfrak{p})$ is the residual field of $A_{\mathfrak{p}}$.

Definition 6.2.1. Let I a left ideal of $\mathscr{B}$, Denote by $\mathrm{in}_K(I)$ the ideal of $\boldsymbol{k}[Z_1, \ldots, Z_r]$ generated by the initial part of the elements of I. A finite family $G_1, \ldots, G_s$ of I is called a Gröbner basis of I if $(\mathrm{in}(G_1), \ldots, \mathrm{in}(G_s))$ is a generating set of $\mathrm{in}_K(I)$.

The existence of a Gröbner basis of I follows from a choice, for every $\alpha \in F$, of one element P_α in I such that $\mathrm{in}(P_\alpha) = c(P)Z^\alpha$. The family $(P_\alpha)_{\alpha \in F}$ is then a Gröbner basis of I.

As $\mathrm{in}_K(I) = \mathrm{in}_K(K \otimes_A I)$, a Gröbner basis of I is also a Gröbner basis of the left ideal $K \otimes_A I$ of $K \otimes_A \mathscr{B}$. By means of the division algorithm, we obtain:

Proposition 6.2.2. *The remainder (by the previous division algorithm) of $P \in \mathscr{B}$ in the division by a Gröbner basis of I belongs to $\sum_{z^\alpha \notin \mathrm{in}_K(I)} Az^\alpha$. It does not depends of the Gröbner basis up to a multiplicative constant.*

Proposition 6.2.3. *Let $P_1, \ldots, P_m \in \mathscr{B}$ be a Gröbner basis of a left ideal I in $\mathscr{B}$, $\boldsymbol{a} = \prod_{k=1}^m c(P_k)$ and $P \in \mathscr{B}$. The following properties are equivalent:*

- *there exists an integer ℓ such that $\boldsymbol{a}^\ell P \in I$,*
- *the remainder of P in the division by $P_1, \ldots, P_m$ is zero,*
- *$P \in K \otimes_A I$.*

It follows from this proposition that a Gröbner of I is a generating set of $K \otimes_A I$.

Definition 6.2.4. Let $P, Q \in \mathscr{B} \smallsetminus \{0\}$, $\alpha = d(P)$, $\beta = d(Q)$. For $i \in \{1, \ldots, r\}$, set $\gamma_i = \max(\alpha_i, \beta_i)$ and $\gamma = (\gamma_1, \ldots, \gamma_r)$. We call

$$S(P, Q) = c(Q)z^{\gamma - \alpha}P - c(P)z^{\gamma - \beta}Q$$

the *S-polynomial* of P and Q.

Proposition 6.2.5. *Let I be a left ideal in $\mathscr{B}$ and let $P_1, \ldots, P_m \in I$ be a generating set of $K \otimes_A I$. Then, $P_1, \ldots, P_m$ is a Gröbner basis of I if and only if, for $1 \leqslant i, j \leqslant m$, the remainders of $S(P_i, P_j)$ in the division by $P_1, \ldots, P_m$ are zero.*

Proof. It is the same proof as in the commutative case. □

Remark 6.2.6. As in commutative case, this proposition allows one to give a natural algorithm for computing a Gröbner basis of I from a family $P_1, \ldots, P_s \in I$ generating $K \otimes_A I$.

Let $\mathfrak{p}$ be a prime ideal of A, $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ the residual field of $A_{\mathfrak{p}}$. Denote by ε the natural evaluation map $A \to \kappa(\mathfrak{p})$ and the naturel map

$$\mathscr{B} \longrightarrow \kappa(\mathfrak{p}) \otimes_A \mathscr{B}, \quad P = \textstyle\sum a_\alpha z^\alpha \longmapsto \varepsilon(P) = \textstyle\sum \varepsilon(a_\alpha) z^\alpha.$$

If I is a left ideal of $\mathscr{B}$, we denote by $I \cdot (\kappa(\mathfrak{p}) \otimes_A \mathscr{B})$ the ideal $\kappa(\mathfrak{p}) \otimes_A \mathscr{B}$ generated by $\varepsilon(I)$.

Proposition 6.2.7. *Let I be a left ideal of $\mathscr{B}$, $P_1, \dots, P_m \in \mathscr{B}$ a Gröbner basis of I, $\boldsymbol{a}$ the product of $c(P_i)$ and $\mathfrak{p}$ a prime ideal of A such that $\boldsymbol{a} \notin \mathfrak{p}$. Then, $(\varepsilon(P_1), \dots, \varepsilon(P_m))$ is a Gröbner basis of $I \cdot (\kappa(\mathfrak{p}) \otimes_A \mathscr{B})$.*

Proof. Let be $P \in \mathscr{B}$. Remark that, if R is the remainder of P in the division by $P_1, \dots, P_m$, then $\varepsilon(R)$ is, modulo the product by a power of $\varepsilon(\boldsymbol{a})$, the remainder of $\varepsilon(P)$ in the division by $\varepsilon(P_1), \dots, \varepsilon(P_m)$. On the other hand, the $\varepsilon(P_i)$ generate $I \cdot (\kappa(\mathfrak{p}) \otimes_A \mathscr{B})$ and $\varepsilon(S(P_i, P_j)) = S(\varepsilon(P_i), \varepsilon(P_j))$. Proposition 6.2.5 allows one to conclude. ☐

Let q be an integer smaller than r. We identify $\mathbb{N}^q$ with $\mathbb{N}^q \times \{0\} \subset \mathbb{N}^{\mathrm{right}}$.

Assumption 4. *$\mathscr{C} = \sum_{\beta \in \mathbb{N}^q} A z^\beta$ is a subalgebra of $\mathscr{B}$.*

The restriction to $\mathbb{N}^q$ of our well-ordering is a well-ordering and thus $\mathscr{C}$ satisfies Assumptions 1, 2 and 3. In order to eliminate the $(z_j)_{j>q}$, we make the following assumption on our well-ordering:

Assumption 5. *The well-ordering of $\mathbb{N}^{\mathrm{right}}$ satisfies $d(z_j) > d(z^\beta)$ for every $j > q$ and $\beta \in \mathbb{N}^q$.*

Proposition 6.2.8. *Let I be a left ideal in $\mathscr{B}$ and $P_1, \dots, P_m \in \mathscr{B}$ a Gröbner basis of I. The subfamily $\mathscr{C} \cap \{P_1, \dots, P_m\}$ is a Gröbner basis of the left ideal $I \cap \mathscr{C}$ of $\mathscr{C}$.*

Proof. Same as in the commutative case. ☐

We deduce from Proposition 6.2.5 and 6.2.8 that specializing at a generic point of the scheme of the coefficient ring commutes with the intersection:

Corollary 6.2.9. *Let I be a left ideal of $\mathscr{B}$ et $P_1, \dots, P_m \in \mathscr{B}$ a Gröbner basis of I. If $\mathfrak{p}$ is a prime ideal in A that does not contain the product of $c(P_k)$, then we have*

$$(I \cap \mathscr{C}) \cdot (\kappa(\mathfrak{p}) \otimes_A \mathscr{C}) = (I \cdot (\kappa(\mathfrak{p}) \otimes_A \mathscr{B})) \cap (\kappa(\mathfrak{p}) \otimes_A \mathscr{C}).$$

***Example 6.2.10* (Polynomial ring).** The ring $A[Z_1, \dots, Z_r]$ of polynomials with coefficients in A satisfies Assumptions 1, 2 et 3 for any total well-ordering compatible with addition.

Example 6.2.11 **(Differential operators).** Let $A\langle Z_1,\ldots,Z_r,\partial/\partial Z_1,\ldots,\partial Z_r\rangle$ be the subring of

$$\mathrm{Hom}_A(A[Z_1,\ldots,Z_r],A[Z_1,\ldots,Z_r])$$

generated by $\mathrm{Der}_A(A[Z_1,\ldots,Z_r])$ (derivations of the A-module $A[Z_1,\ldots,Z_r]$) and by the products by elements of $A[Z_1,\ldots,Z_r]$. This ring can be called the ring of differential operators with coefficients in $A[Z_1,\ldots,Z_r]$. It satisfies Assumptions 1, 2 and 3 for every well-ordering compatible with addition.

Example 6.2.12 **(Enveloping algebra of a free finite type Lie A-algebra)**

Let $\mathfrak{g}$ be a Lie A-algebra. That is, a A-algebra where the product, denoted by $(x,y)\mapsto[x,y]$, satisfies, for every $x,y,z\in\mathfrak{g}$,

(1) $[x,x]=0$

(2) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ (Jacobi identity)

We assume that $\mathfrak{g}$ is free and of finite type as a A-module. Let $z_1,\ldots,z_r$ be a basis of $\mathfrak{g}$. The enveloping algebra $\mathscr{U}(\mathfrak{g})$ of $\mathfrak{g}$ is the quotient of the tensor algebra of the A-module $\mathfrak{g}$:

$$T=A\oplus\mathfrak{g}\oplus(\mathfrak{g}\otimes_A\mathfrak{g})\oplus(\mathfrak{g}\otimes_A\mathfrak{g}\otimes_A\mathfrak{g})\oplus\cdots$$

by the two-sided ideal generated by the tensors $x\otimes y-y\otimes x-[x,y]$, where $x,y\in\mathfrak{g}$. It follows from the Poincaré-Birkhoff-Witt theorem that the enveloping algebra $\mathscr{U}(\mathfrak{g})$ is a free A-module with basis $(z^\alpha)_{\alpha\in\mathbb{N}^{\mathrm{right}}}$ and that $\mathfrak{g}$ is naturally identified with a submodule of $\mathscr{U}(\mathfrak{g})$ [**Bou71**, Th. 1 p. 30 and Cor. 2 p. 33]. As for every i,j, the commutator $z_iz_j-z_jz_i$ is in $\mathfrak{g}$, then, using on $\mathbb{N}^{\mathrm{right}}$ the graded lexicographic ordering, one has $d(z_iz_j-z_jz_i)<d(z_iz_j)$. Therefore, $\mathfrak{g}$ satisfies Assumptions 1, 2 and 3.

Another ordering can also be suitable. It depends on the expression of the commutators $[z_i,z_j]$. Let $a_{i,j,\ell}\in A$ such that

$$[z_i,z_j]=\sum_{\ell=1}^{\mathrm{right}}a_{i,j,l}z_\ell.$$

The algebra $\mathscr{U}(\mathfrak{g})$ is the solution to a universal problem. It is naturally isomorphic to a quotient of the associative free algebra built on the alphabet $\{Z_1,\ldots,Z_r\}$ by the two-sided ideal generated by the elements $Z_iZ_j-Z_jZ_i-\sum_{l=1}^{\mathrm{right}}a_{i,j,l}Z_\ell$ where $1\leqslant i<j\leqslant r$.

Example 6.2.13 **(Quotient of a words algebra).** Let $\{Z_1,\ldots,Z_r\}$ be a set of cardinal r. Denote by $\mathscr{A}$ the associative free A-algebra built on the words of the alphabet $\{Z_1,\ldots,Z_r\}$. For every $1\leqslant i<j\leqslant r$ and $0\leqslant\ell\leqslant r$, let us give elements $a_{i,j,\ell}$ of $\mathscr{A}$ and set $R_{i,j}=\sum_{\ell=1}^{\mathrm{right}}a_{i,j,\ell}Z_\ell+a_{i,j,0}$, and $R_{i,i}=0$ and $R_{j,i}=-R_{i,j}$ (for $j<i$).

We assume that $R_{i,j}$ satisfy the so-called Jacobi relations: for every $1\leqslant i\leqslant j\leqslant k\leqslant r$,

$$\sum_{\ell=1}^{\mathrm{right}}a_{j,k,\ell}R_{i,\ell}+\sum_{\ell=1}^{\mathrm{right}}a_{k,i,\ell}R_{j,\ell}+\sum_{\ell=1}^{\mathrm{right}}a_{i,j,\ell}R_{k,\ell}=0.$$

Let $\mathscr{R}$ be the two-sided ideal of $\mathscr{A}$ generated by the elements

$$\rho_{i,j} = Z_iZ_j - Z_jZ_i - R_{i,j}.$$

It is easy to deduce from the previous example that the A-quotient algebra $\mathscr{A}/\mathscr{R}$ satisfies Assumptions 1,2,3.

6.3. Some applications

6.3.a. Computation of the characteristic variety. Let $A_n(\boldsymbol{k})$ be the Weyl algebra with n variables $\boldsymbol{k}\langle Z_1, \dots, Z_n, \partial/\partial Z_1, \dots, \partial/\partial Z_n\rangle$. Let us consider the following ordering of $\mathbb{N}^n \times \mathbb{N}^n$: $(\alpha, \beta) < (\alpha', \beta')$ if $|\beta| < |\beta'|$ or $|\beta| = |\beta'|$ and (α, β) smaller than (α', β') for the lexicographic ordering.

Let us consider a left ideal J of $A_n(\boldsymbol{k})$. If $P_1, \dots, P_r$ is a Gröbner basis of J, then the principal symbols $\sigma(P_1), \dots, \sigma(P_r)$ generate the ideal of $\boldsymbol{k}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ generated by the principal symbols of the elements of J.

If $\boldsymbol{k} = \mathbb{C}$, the zero set of $\sigma(P_1), \dots, \sigma(P_r)$ is the characteristic variety of $A_n(\mathbb{C})/J$.

6.3.b. Algorithmic computation of the Bernstein-Sato polynomial, [BM02]

Let m be a section of a left $A_n(\boldsymbol{k})$-module M and let $F \in \boldsymbol{k}[X_1, \dots, X_n]$ be a polynomial. We set $A_n(\boldsymbol{k})[s] = \boldsymbol{k}[s] \otimes_{\boldsymbol{k}} A_n(\boldsymbol{k})$, which has a natural structure of $\boldsymbol{k}$-algebra. We denote by $M[1/F, s]F^s$ the $\boldsymbol{k}[X, s, 1/F]$-module naturally isomorphic to $M[1/F, s]$, with the action of $A_n(\boldsymbol{k})[s]$-module shifted from the natural one by F^s:

$$\forall\, m \in M[1/F, s],\ \forall\, i = 1, \dots, n, \quad \frac{\partial}{\partial X_i}(mF^s) = \Big(\frac{\partial}{\partial X_i} m\Big) F^s + \frac{s}{F} \frac{\partial F}{\partial X_i} mF^s.$$

We then consider the following ideal of $\boldsymbol{k}[s]$:

$$I(m, F) = \{b(s) \in \boldsymbol{k}[s] \mid b(s)mF^s \in A_n(\boldsymbol{k})[s]mF^{s+1}\},$$

where $F^{s+1} = F \cdot F^s$.

We want to describe an algorithm to compute the ideal $I(m, F)$. We will make the following assumptions.

(A1) We assume that we know a system of generators $(P_1, \dots, P_r)$ of the left ideal

$$\operatorname{ann}_{A_n(\boldsymbol{k})} A_n(\boldsymbol{k})m := \{P \in A_n(\boldsymbol{k}) \mid Pm = 0\}.$$

(A2) We assume (for simplicity) that $A_n(\boldsymbol{k})m$ has no F-torsion.

In order to get this algorithm, let us consider the algebra $\mathscr{E} = A_n(\boldsymbol{k})[s]\langle \partial/\partial T\rangle$, which is the quotient of the free $A_n(\boldsymbol{k})[s]$-algebra generated by $\partial/\partial T$ by the only supplementary non trivial relation

$$s(\partial/\partial T) - (\partial/\partial T)s = (\partial/\partial T).$$

The formalism of Gröbner basis can be applied to $A_n(\boldsymbol{k})[s]$ and to $\mathscr{E}$. We will consider the lexicographic order such that:

$$\frac{\partial}{\partial T} > \frac{\partial}{\partial X_1} > \cdots > \frac{\partial}{\partial X_n} > s.$$

If $P(X, s, \partial/\partial X) \in A_n(\boldsymbol{k})[s]$, we set

$$\widetilde{P} = P\Big(X, s, \frac{\partial}{\partial X_1} + \frac{\partial F}{\partial X_1}\frac{\partial}{\partial T}, \dots, \frac{\partial}{\partial X_n} + \frac{\partial F}{\partial X_n}\frac{\partial}{\partial T}\Big) \in \mathscr{E}.$$

Let us describe now the algorithm. We start with the set of generators $\sigma(P_1), \dots, \sigma(P_r)$ of $\operatorname{ann}_{A_n(\boldsymbol{k})} A_n(\boldsymbol{k})m$. By the algorithm giving a Gröbner basis, we construct a Gröbner basis $\mathscr{G}$ of the left ideal in $\mathscr{E}$ generated by

$$\widetilde{P_1}, \dots, \widetilde{P_r}, s + F\frac{\partial F}{\partial T}.$$

Next, we consider $\widetilde{\mathscr{G}} = \mathscr{G} \cap A_n(\boldsymbol{k})[s]$. Still using the algorithm giving a Gröbner basis, we construct a Gröbner basis $\mathscr{L}$ of the left ideal in $\mathscr{E}$ generated by $\widetilde{\mathscr{G}}, F$.

Assertion. *$I(m, F)$ is generated by $\mathscr{L} \cap \boldsymbol{k}[s]$.*

In order to prove this assertion, notice that $M[1/F, s]F^s$ has a natural structure of left $\mathscr{E}$-module which extends the structure of left $A_n(\boldsymbol{k})[s]$-module:

$$\frac{\partial}{\partial T} a(s, X)mF^s = -a(s-1, X)smF^{s-1}.$$

Notice that

$$\Big(s\frac{\partial}{\partial T} - \frac{\partial}{\partial T}s\Big)a(s, X)mF^s = \frac{\partial}{\partial T}a(s, X)mF^s.$$

Lemma 6.3.1. *$\widetilde{P_1}, \dots, \widetilde{P_r}, s + F\partial F/\partial T$ is a set of generators of* $\operatorname{ann}_{\mathscr{E}} \mathscr{E} mF^s$

Proof. The proof is left to the reader. Be careful, the assumption "no F torsion" is used here. ☐

Then, we remark that

$$\operatorname{ann}_{\mathscr{E}} \mathscr{E} mF^s \cap A_n(\boldsymbol{k})[s] = \operatorname{ann}_{A_n(\boldsymbol{k})[s]} A_n(\boldsymbol{k})[s]mF^s.$$

Lemma 6.3.2. $\operatorname{ann}_{A_n(\boldsymbol{k})[s]} A_n(\boldsymbol{k})[s]mF^s + A_n(\boldsymbol{k})[s]F$ *is the ideal annihilating the class of mF^s in the quotient $A_n(\boldsymbol{k})[s]mF^s / A_n(\boldsymbol{k})[s]mF^{s+1}$.*

Proof. Left to the reader (easy). ☐

Finally, the assertion follows from the lemmas and from Proposition 6.2.8.

If M is a holonomic $A_n(\boldsymbol{k})$-module, Bernstein has proved in 1972 that, for any section m of M, the ideal $I(m, F)$ is not reduced to $\{0\}$. The generator of $I(m, F)$ is called the *Bernstein polynomial* associated to m, F (see also Lecture 7).

LECTURE 7

SPECIALIZABLE $\mathscr{D}_X$-MODULES

In the theory of sheaves of vector spaces over a field, there are standard functors which allow one to restrict a sheaf on X to a closed subset $Y \xhookrightarrow{i} X$, namely the inverse image, denoted by i^{-1}, and the extraordinary inverse image, denoted by $i^!$. For constructible sheaves on complex analytic spaces, other functors (nearby/vanishing cycles) may be used.

The notion of *specialization* of sheaves (more precisely the functors called "nearby cycles" and "vanishing cycles") has been introduced by P. Deligne. It is a powerful tool as it generalizes the "cohomology of the Milnor fibre" attached to a function on a manifold. A "dual" notion, that of *microlocalization* of sheaves, has been introduced later by M. Kashiwara and P. Schapira (although some aspects of it were present in the work of M. Sato). These functors often replace the inverse image functor with respect to the inclusion of a hypersurface in the ambient manifold. One of their main advantages is that they preserve the property for a sheaf of being perverse. The reason for this is essentially that they commute with the duality functor for constructible sheaves (Poincaré-Verdier duality), a theorem due originally to O. Gabber.

Except in the noncharacteristic situation (see § 2.3), the restriction of a $\mathscr{D}_X$-module to a smooth closed subvariety Y is in general not $\mathscr{D}_Y$-coherent. Given a submanifold $Y \subset X$, it is natural to consider the category of $\mathscr{D}_X$-modules for which the restriction to Y is $\mathscr{D}_Y$-coherent. This is related to complex boundary value problems in the theory of partial differential equations. However, even if the restriction of $\mathscr{M}$ to Y is $\mathscr{D}_Y$-coherent, the restriction is usually not a single $\mathscr{D}_Y$-module but a complex with many cohomology modules. Introducing the notion of specialization (also called "moderate nearby cycles" and "moderate vanishing cycles", in analogy with sheaves) allows one to understand better each cohomology module of the restricted object.

One is therefore led to give an analytic or algebraic version of the topological specialization functors (for instance, these functors happen to be one of the main tools in the theory of polarized Hodge Modules [**Sai88**]). This is has been done for $\mathscr{D}$-modules by B. Malgrange in an important special case, and then by M. Kashiwara in general. Their interest was motivated by a topological theorem of M. Goresky and

R.D. MacPherson, which says that “the sheaf of nearby cycles of a perverse sheaf is perverse”. Moreover, the notion of specialization for $\mathscr{D}$-modules is often more useful than the restriction.

The basic idea in the construction is to replace the notion of monodromy and its minimal polynomial, which is inherent to vanishing cycles of sheaves, with the notion of Bernstein polynomial and Malgrange-Kashiwara filtration, denoted by V.

The purpose of this lecture is to introduce the latter notions and to consider their behaviour under direct images. We will present them with some details. The computational aspect has yet been considered in § 6.3.b.

Holonomic $\mathscr{D}_X$-modules (see Chapter 5) are basic objects, when applying the theory of partial differential equations to algebraic geometry. They may be defined in various ways. The geometrical definition is by the dimension of the characteristic variety. The algebraic definition is by the vanishing of $\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M},\mathscr{D}_X)$ for $k \neq n$. One of the main result says that *holonomic $\mathscr{D}$-modules are specializable along any hypersurface*. This is Bernstein theory in the algebraic situation, and this is a result of M. Kashiwara in the analytic setting.

There is a third definition of holonomic $\mathscr{D}$-modules by using the notion of specialization. Consider the category $\mathscr{H}_n$ of coherent $\mathscr{D}_X$-modules $\mathscr{M}$ with $\dim \operatorname{Supp} \mathscr{M} \leqslant n$, which are specializable along any germ of hypersurface cutting the support in codimension one, and such that their moderate nearby or vanishing cycles are in $\mathscr{H}_{n-1}$. This defines inductively the category $\mathscr{H}_n$. One can prove that this category is nothing but the category of holonomic $\mathscr{D}_X$-modules with $\dim \operatorname{Supp} \mathscr{M} \leqslant n$.

It is not much more difficult to analyze the specialization of $\mathscr{D}$-modules equipped with a good filtration. This explains the following notation in this lecture: $\widetilde{\mathscr{D}}_X$ denotes the sheaf $\mathscr{D}_X$ or the Rees sheaf of rings $R_F\mathscr{D}_X$ introduced in § 1.5, from which we keep notation. In the first case, we set $z = 1$ in the formulas below, and $\eth = \partial$. Similarly, we denote by $\widetilde{\mathscr{O}}_X$ the sheaf $\mathscr{O}_X$ or the sheaf $R_F\mathscr{O}_X = \mathscr{O}_X[z]$.

The reader who wants to forget about good filtrations should replace $\widetilde{\mathscr{D}}_X$ with $\mathscr{D}_X$ and, in the formulas below, should set $z = 1$ and $\eth = \partial$. Moreover, the notion of strictness is then tautological.

The filtered approach may be considered as an introduction to some parts of [**Sai88**].

7.1. The V-filtration

Let $Y \subset X$ be a smooth hypersurface [1] with ideal $\mathscr{I}_Y \subset \mathscr{O}_X$. Put $\widetilde{\mathscr{I}}_Y := \mathscr{I}_Y \cdot \widetilde{\mathscr{O}}_X \subset \widetilde{\mathscr{O}}_X$. Denote by $V_\bullet\widetilde{\mathscr{D}}_X$ the increasing filtration indexed by $\mathbb{Z}$ associated with Y: for any $x \in X$, one sets

$$V_k\widetilde{\mathscr{D}}_{X,x} := \{P \in \widetilde{\mathscr{D}}_{X,x} \mid P \cdot \widetilde{\mathscr{I}}^j_{Y,x} \subset \widetilde{\mathscr{I}}^{j-k}_{Y,x}\ \forall\, j \in \mathbb{Z}\}$$

---

[1] The same definition applies to any closed analytic submanifold of X.

where we set $\widetilde{\mathscr{I}^\ell} = \widetilde{\mathscr{O}}_X$ if $\ell \leqslant 0$. Let P be a germ in $\widetilde{\mathscr{D}}_{X,x}$. In any local coordinate system $(t, x_2, \dots, x_n) = (t, x')$ of X centered in x for which $Y = \{t = 0\}$, one has

$$P \in V_0\widetilde{\mathscr{D}}_{X,x} \iff P = \sum_{(\alpha,j)\in\mathbb{N}^n} a_{\alpha,j}\eth_{x'}^{\alpha}(t\eth_t)^j, \quad a_{\alpha,j} \in \widetilde{\mathscr{O}}_{X,x}$$

and, for any $k \in \mathbb{N}$,

$$\begin{aligned} P \in V_{-k}\widetilde{\mathscr{D}}_{X,x} &\iff P = t^kQ \quad \text{with } Q \in V_0\widetilde{\mathscr{D}}_{X,x}, \\ P \in V_k\widetilde{\mathscr{D}}_{X,x} &\iff P = \sum_{0\leqslant \ell\leqslant k} Q_j\eth_t^\ell \quad \text{with } Q_\ell \in V_0\widetilde{\mathscr{D}}_{X,x}. \end{aligned}$$

For $k \in \mathbb{Z}$, set $V_k\widetilde{\mathscr{O}}_X = V_k\widetilde{\mathscr{D}}_X \cap \widetilde{\mathscr{O}}_X$. This is nothing but the $\widetilde{\mathscr{I}_Y}$-adic filtration on $\widetilde{\mathscr{O}}_X$ viewed as an increasing filtration.

Exercise 7.1.1. Show that

- for any k, $V_k\widetilde{\mathscr{D}}$ is a locally free $V_0\widetilde{\mathscr{D}}$-module.
- $\widetilde{\mathscr{D}}_X = \cup_k V_k\widetilde{\mathscr{D}}_X$ (the filtration is exhaustive),
- $V_k\widetilde{\mathscr{D}}_X \cdot V_\ell\widetilde{\mathscr{D}}_X \subset V_{k+\ell}\widetilde{\mathscr{D}}_X$ with equality for $k, \ell \leqslant 0$ or $k, \ell \geqslant 0$,
- $V_k\widetilde{\mathscr{D}}_{X\smallsetminus Y} = \widetilde{\mathscr{D}}_{X\smallsetminus Y}$ for any $k \in \mathbb{Z}$,
- $(\cap_k V_k\widetilde{\mathscr{D}}_X)_{|Y} = \{0\}$.

***Exercise 7.1.2* (Euler vector field)**

(1) Show that the class E of $t\eth_t$ in $\mathrm{gr}_0^V\widetilde{\mathscr{D}}_X$ does not depend on the choice of the local coordinate system (t, x') as above.

(2) Show that $\mathrm{gr}_0^V\mathscr{D}_X$ is a sheaf of rings and that E belongs to its center.

(3) Show that, locally on X, one has an isomorphism $\mathrm{gr}_0^V\mathscr{D}_X \simeq \mathscr{D}_Y[E]$.

Remarks 7.1.3

(1) It is straightforward to develop the theory below in the case of right $\widetilde{\mathscr{D}}_X$-modules. If $U_\bullet(\mathscr{M})$ is a V-filtration of the left module $\mathscr{M}$, then $U_\bullet(\omega_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}) := \omega_X \otimes_{\widetilde{\mathscr{O}}_X} U_\bullet(\mathscr{M})$ is the corresponding filtration of the corresponding right module. This correspondence is compatible with taking the graded object with respect to $U_\bullet$. The operator $-\eth_t t$ (acting on the left) corresponds to $t\eth_t$ (acting on the right).

(2) In the literature, one also finds decreasing V-filtrations, by analogy with the $\widetilde{\mathscr{I}_Y}$-adic filtration. One uses the following rule for going from increasing to decreasing filtrations: given an increasing filtration $U_\bullet\mathscr{M}$ (lower indices), the associated decreasing filtration (upper indices) is defined by $U^k = U_{-k-1}$.

7.2. Coherence

Coherence of the Rees sheaf of rings. Introduce the Rees sheaf of rings $R_V\widetilde{\mathscr{D}}_X = \oplus_k V_k\widetilde{\mathscr{D}}_X \cdot \tau^k$, where τ is a new variable, and similarly $R_V\widetilde{\mathscr{O}}_X = \oplus_k V_k\widetilde{\mathscr{O}}_X \cdot \tau^k$, which is naturally a $\widetilde{\mathscr{O}}_X$-module.

Exercise 7.2.1 **(Coherence of $R_V\widetilde{\mathscr{O}}_X$)**

(1) Let K be a compact polycylinder in X. Show that $R_V\widetilde{\mathscr{O}}_X(K) = R_V(\widetilde{\mathscr{O}}_X(K))$ is Noetherian. [Hint: it is the Rees ring of the $\widetilde{\mathscr{I}}_Y$-adic filtration on the Noetherian ring $\widetilde{\mathscr{O}}_X(K)$.]

(2) Show that the ring $(R_V\widetilde{\mathscr{O}}_X)_x = R_V\widetilde{\mathscr{O}}_X(K) \otimes_{\widetilde{\mathscr{O}}_X(K)} \widetilde{\mathscr{O}}_{X_x}$ is flat over $R_V\widetilde{\mathscr{O}}_X(K)$.

(3) Show that $R_V\widetilde{\mathscr{O}}_X$ is coherent. [Hint: let U be any open set in X and let $\varphi : (R_V\widetilde{\mathscr{O}}_X)^q_{|U} \to (R_V\widetilde{\mathscr{O}}_X)^p_{|U}$ be any morphism; let K be a compact polycylinder contained in U; then, $\operatorname{Ker}\varphi(K)$ is finitely generated over $R_V\widetilde{\mathscr{O}}_X(K)$ by noetherianity and we have $\operatorname{Ker}\varphi_{|K} = \operatorname{Ker}\varphi(K) \otimes_{R_V\widetilde{\mathscr{O}}_X(K)} (R_V\widetilde{\mathscr{O}}_X)_{|K}$ by flatness; so $\operatorname{Ker}\varphi_{|K}$ is finitely generated.]

Exercise 7.2.2 **(Coherence of $R_V\widetilde{\mathscr{D}}_X$)**

(1) *A simple situation.* Let A be the ring $\mathbb{C}[t,\theta]$, on which one considers the filtration $V_\bullet A$ for which t has degree -1 and θ has degree 1. Show that the Rees ring $R_VA = \oplus_k V_kA\tau^k$ is isomorphic to the quotient ring $B = \mathbb{C}[t,u,v,w,\tau]/(vw-u, \tau w - t)$. Conclude that R_VA is Noetherian. [Hint: decompose B as $\mathbb{C}[t,u,w] \oplus \mathbb{C}[t,u,v,\tau]$.]

(2) Consider the sheaf $\widetilde{\mathscr{O}}_X[\theta,\xi_2,\dots,\xi_n]$ equipped with the V-filtration for which θ has degree 1, $\xi_2,\dots,\xi_n$ have degree 0, and inducing the V-filtration on $\widetilde{\mathscr{O}}_X$. Show that, if $K \subset X$ is any polycylinder, then $R_V\widetilde{\mathscr{O}}_X[\theta,\xi_2,\dots,\xi_n](K)$ is Noetherian.

(3) Show that, if K is any sufficiently small polycylinder, then $R_V\widetilde{\mathscr{D}}_X(K)$ is Noetherian. [Hint: $R_V\widetilde{\mathscr{D}}_X$ can be filtered (by the degree of the operators) in such a way that, locally on X, $\mathrm{gr}R_V\widetilde{\mathscr{D}}_X$ is isomorphic to $R_V(\widetilde{\mathscr{O}}_X[\theta,\xi_2,\dots,\xi_n])$.]

(4) Conclude that $R_V\widetilde{\mathscr{D}}_X$ is coherent. [Hint: apply the same arguments as in Theorem 2.1.3.]

Good V-filtrations. Let $(\mathscr{M}, U_\bullet\mathscr{M})$ be a V-filtered $\widetilde{\mathscr{D}}_X$-module. The filtration $U_\bullet\mathscr{M}$ is said to be *good* if any $U_\ell\mathscr{M}$ is $V_0\widetilde{\mathscr{D}}_X$-coherent and, for any compact set $K \subset X$, there exists $k_0 \geqslant 0$ such that, in some neighbourhood of K we have, for all $k \geqslant k_0$,

$$U_{-k}\mathscr{M} = t^{k-k_0}U_{-k_0}\mathscr{M} \quad \text{and} \quad U_k\mathscr{M} = \sum_{0\leqslant j\leqslant k-k_0} \eth_t^j U_{k_0}\mathscr{M}.$$

Exercise 7.2.3. Let $U_\bullet(\mathscr{M})$ and $U'_\bullet(\mathscr{M})$ be two good V-filtrations of a coherent $\widetilde{\mathscr{D}}_X$-module $\mathscr{M}$. Show that, locally, there exist two integers $k_1, k_2 \in \mathbb{Z}$ such that:

$$\forall\, k \in \mathbb{Z} \quad U_{k_1+k}(\mathscr{M}) \subset U'_k(\mathscr{M}) \subset U_{k_2+k}(\mathscr{M}).$$

Exercise 7.2.4. Let $\mathscr{M}$ be a coherent $\widetilde{\mathscr{D}}_X$-module. Show that the filtration $U_\bullet\mathscr{M}$ is good if and only if one of the following equivalent properties is satisfied:

(1) the Rees module $\oplus_k U_k\mathscr{M} \cdot \tau^k$ is coherent over $R_V\widetilde{\mathscr{D}}_X$,

(2) there exists, locally on X, a surjective morphism $\widetilde{\mathscr{D}}_X^a \to \mathscr{M} \to 0$, inducing for each $k \in \mathbb{Z}$ a surjective morphism $U_k\widetilde{\mathscr{D}}_X^a \to U_k\mathscr{M} \to 0$, where the filtration on the free module $\widetilde{\mathscr{D}}_X^a$ is obtained by suitably shifting $V_\bullet\widetilde{\mathscr{D}}_X$ on each summand.

In particular, we get:

Lemma 7.2.5. *Locally on X, there exists k_0 such that, for any $k \leqslant k_0$, $t : U_{-k}\mathscr{M} \to U_{-k-1}\mathscr{M}$ is bijective.*

Proof. Indeed, using a presentation of $\mathscr{M}$ as above, it is enough to show the lemma for $\widetilde{\mathscr{D}}_X^a$ with a filtration as above, and we are reduced to consider each summand $\widetilde{\mathscr{D}}_X$ with a shifted standard V-filtration. There, we may choose k_0 such that $U_{k_0}\widetilde{\mathscr{D}}_X = V_0\widetilde{\mathscr{D}}_X$. ☐

Lemma 7.2.6 **(Artin-Rees).** *If $\mathscr{N}$ is a coherent $\widetilde{\mathscr{D}}_X$-submodule of $\mathscr{M}$ and $U_\bullet\mathscr{M}$ is a good filtration of $\mathscr{M}$, then $U_\bullet\mathscr{N} := \mathscr{N} \cap U_\bullet\mathscr{M}$ is also good.*

Proof. Analogous to that of Corollary 2.2.13. ☐

Exercise 7.2.7. Let $\varphi : \mathscr{M} \to \mathscr{N}$ be a surjective morphism of coherent $\widetilde{\mathscr{D}}_X$-modules. Let $U_\bullet\mathscr{M}$ be a good V-filtration. Show that $U_\bullet\mathscr{N} := \varphi(U_\bullet\mathscr{M})$ is a good V-filtration of $\mathscr{N}$.

Exercise 7.2.8. Let $\mathscr{U}$ be a coherent $V_0\widetilde{\mathscr{D}}_X$-module and let $\mathscr{T}$ be its t-torsion subsheaf, i.e., the subsheaf of local sections locally killed by a power of t. Then, locally on X, there exists ℓ such that $\mathscr{T} \cap t^\ell\mathscr{U} = 0$. [Hint: Consider the t-adic filtration on $V_0\widetilde{\mathscr{D}}_X$, i.e., the filtration $V_j\widetilde{\mathscr{D}}_X$ with $j \leqslant 0$; remark that the filtration $t^{-j}\mathscr{U}$ is good with respect to it, and locally there is a surjective morphism $(V_0\widetilde{\mathscr{D}}_X)^n \to \mathscr{U}$ which is strict with respect to the V-filtration; its kernel $\mathscr{K}$ is coherent and comes equipped with the induced V-filtration, which is good; in particular, locally on X, there exists $j_0 \leqslant 0$ such that $V_{j+j_0}\mathscr{K} = t^{-j}V_{j_0}\mathscr{K}$ for any $j \leqslant 0$; show that, for any $j \leqslant 0$, there is locally an exact sequence

$$(V_j\widetilde{\mathscr{D}}_X)^m \longrightarrow (V_{j+j_0}\widetilde{\mathscr{D}}_X)^n \longrightarrow t^{-(j+j_0)}\mathscr{U} \longrightarrow 0;$$

as $t : V_k\widetilde{\mathscr{D}}_X \to V_{k-1}\widetilde{\mathscr{D}}_X$ is bijective for $k \leqslant 0$, conclude that $t : t^{-j_0}\mathscr{U} \to t^{-j_0+1}\mathscr{U}$ is so, hence $\mathscr{T} \cap t^{-j_0}\mathscr{U} = 0$.]

7.3. Y-specializable $\mathscr{D}_X$-modules

We now consider the non-filtered situation. We will denote by E any local lifting in $V_0(\mathscr{D}_X)$ of the global section E of $\mathrm{gr}_0^V(\mathscr{D}_X)$ (see Exercise 7.1.2).

***Definition 7.3.1* (Bernstein polynomial).** Let $U_\bullet\mathscr{M}$ be a good V-filtration of a coherent $\mathscr{D}_X$-module $\mathscr{M}$. We say that this filtration has a Bernstein polynomial if there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ satisfying $b(E+k)U_k(\mathscr{M}) \subset U_{k-1}(\mathscr{M})$, for any $k \in \mathbb{Z}$. Any such polynomial is called a Bernstein polynomial for $U_\bullet\mathscr{M}$.

Proposition 7.3.2. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. The following properties are equivalent:*

(1) *in the neighbourhood of any point of X, there exists a good V-filtration $U_\bullet(\mathscr{M})$ having a Bernstein polynomial,*

(2) *any good V-filtration $U_\bullet(\mathscr{M})$ has locally a Bernstein polynomial,*

(3) *for every finite system of local generators $(m_i)_{i=1,\dots,\ell}$ of $\mathscr{M}$, there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that $b(E)m_i \in \sum_{j=1}^{\ell} V_{-1}(\mathscr{D}_X)m_j$,*

(4) *for any local section m of $\mathscr{M}$, there exists a nonzero polynomial $b(E) \in \mathbb{C}[s]$ such that $b(s)m \in V_{-1}(\mathscr{D}_X)m$.*

Proof. (1) implies (2). Given two good filtrations $U_\bullet\mathscr{M}$ and $U'_\bullet\mathscr{M}$, let $b(s)$ be a Bernstein polynomial for $U_\bullet\mathscr{M}$. Use Exercise 7.2.3 to get

$$b(E+k+k_1)\cdots b(E+k+k_2-1)b(E+k+k_2)U'_k(\mathscr{M}) \subset U'_{k-1}(\mathscr{M}).$$

(2) implies (3). After Exercise 7.2.4, $U_k(\mathscr{M}) = \sum_i V_k(\mathscr{D}_X)m_i$, $k \in \mathbb{Z}$, is a good V-filtration of $\mathscr{M}$.

(3) implies (1). Let us consider the good V-filtration $U_k(\mathscr{M}) = \sum_i V_k(\mathscr{D}_X)m_i$, $k \in \mathbb{Z}$. It follows from the commutation relations that one has

$$b(E+k)V_k(\mathscr{D}_X) \subset V_k(\mathscr{D}_X)b(E) + V_{k-1}(\mathscr{D}_X)$$

for any $k \in \mathbb{Z}$, hence (1).

(4) implies (3). Clear.

(1) implies (4). Use Proposition 7.3.9 below. □

***Definition 7.3.3* (Specializable $\mathscr{D}_X$-modules)**

(1) A coherent $\mathscr{D}_X$-module $\mathscr{M}$ is said to be *specializable along* Y or Y-*specializable* if it satisfies one of the equivalent properties of Proposition 7.3.2.

(2) Let $f : X \to \mathbb{C}$ be a holomorphic function. A coherent $\mathscr{D}_X$-module $\mathscr{M}$ is said to be f-*specializable* if its direct image $i_+\mathscr{M}$ by the graph inclusion $i : X \hookrightarrow X \times \mathbb{C}$ of f is specializable along $X \times \{0\}$.

Exercise 7.3.4. Show that both definitions are consistent, namely that, if f is smooth, then $\mathscr{M}$ is f-specializable if and only if it is specializable along $f^{-1}(0)$. [Hint: work in local coordinates; then, starting from a good V-filtration of $\mathscr{M}$, construct a good V-filtration of $i_+\mathscr{M}$.]

Exercise 7.3.5. Let $\mathscr{M}$ be a Y-specializable $\mathscr{D}_X$-module and let $U_\bullet\mathscr{M}$ be any good V-filtration. Show that, after a local identification of $\mathrm{gr}_0^V\mathscr{D}_X$ with $\mathscr{D}_Y[E]$ (see Exercise 7.1.2), each $\mathrm{gr}_k^U\mathscr{M}$ is a coherent $\mathscr{D}_Y$-module.

Exercise 7.3.6 **(Basic examples to keep in mind).** We assume here that X is a disc with coordinate t and that Y is equal to the origin of the disc.

(1) Show that, for any *nonzero* differential operator P on X, the $\mathscr{D}_X$-module $\mathscr{M}=\mathscr{D}_X/\mathscr{D}_X\cdot P$ is specializable at 0.

(2) Choose $P=t^2\partial_t+1$ (irregular singularity at 0). Show that the constant filtration $V_k\mathscr{M}=\mathscr{M}$ for all k is a good V-filtration and that the corresponding Bernstein polynomial is 1. Deduce that, for any good V-filtration $U_\bullet\mathscr{M}$, we have $\mathrm{gr}_k^U\mathscr{M}=0$ for all k.

(3) What can be the usefulness of the V-filtration in such cases? [Hint: see Proposition 7.4.4.]

Definition 7.3.7 **(Bernstein-Sato polynomial).** Assume that $\mathscr{M}$ is specializable along Y.

(1) Let $U_\bullet(\mathscr{M})$ be some good V-filtration of $\mathscr{M}$. Let K be a compact set of X on which this filtration is defined. The unitary polynomial $b(s)\in\mathbb{C}[s]$ with smallest degree such that

$$\forall\, k\in\mathbb{Z},\quad b(E+k)U_k(\mathscr{M})_{|K}\subset U_{k-1}(\mathscr{M})_{|K}$$

is called *the Bernstein-Sato polynomial* of the good V-filtration $U_\bullet(\mathscr{M})_{|K}$.

Let m a local section of Y-specializable module $\mathscr{M}$. The unitary polynomial of smallest degree satisfying:

$$b(E)m\in V_{-1}(\mathscr{D}_X)m$$

is called *the Bernstein-Sato polynomial* of the local section m. We denote by $b_m(s)\in\mathbb{C}[s]$ this polynomial.

Exercise 7.3.8. Let $\mathscr{M}$ be a $\mathscr{D}_X$-module which is specializable along Y and let K be a compact set in X. Show that there exists a finite set $A\subset\mathbb{C}$ such that, for any $x\in K$, any germ $m\in\mathscr{M}_x$ and any good V-filtration $U_\bullet\mathscr{M}_x$, the roots of the Bernstein-Sato polynomial of m and of $U_\bullet\mathscr{M}_x$ belong to $A+\mathbb{Z}$.

Proposition 7.3.9 **(Stability by extension).** *Let $0\to\mathscr{M}_1\to\mathscr{M}\to\mathscr{M}_2\to 0$ be an exact sequence of coherent $\mathscr{D}_X$-modules. The module $\mathscr{M}$ is specializable along Y if and only of the modules $\mathscr{M}_1$ and $\mathscr{M}_2$ are so.*

Proof. Assume that $\mathscr{M}$ is Y-specializable. We denote by $b(s)$ the Bernstein-Sato polynomial associated to a good V-filtration $U_\bullet(\mathscr{M})$. After Proposition 7.2.6 and Exercise 7.2.7, the induced filtration and the image filtration are V-good. It is then clear that $b(s)$ is a Bernstein polynomial associated with these filtrations (in the sense of Definition 7.3.1). Therefore, $\mathscr{M}_1$ and $\mathscr{M}_2$ are Y-specializable.

Conversely, let $b_1(s)$ (resp. $b_2(s)$) be a Bernstein polynomial of the induced (resp. image) good V-filtration $U_\bullet(\mathscr{M})$ of $\mathscr{M}$. It is easy to see that product $b_1(s)b_2(s)$ is a Bernstein polynomial for the V-filtration $U_\bullet(\mathscr{M})$. □

The category of the Y-specializable $\mathscr{D}_X$-modules is then an abelian subcategory, stable by extension, of the category of coherent $\mathscr{D}_X$-modules.

Proposition 7.3.10. *Let $\sigma : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ be a section of the natural projection $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$. Let $\mathscr{M}$ be a coherent Y-specializable $\mathscr{D}_X$-module. Then there exists a unique good V-filtration, denoted by $V_\bullet^\sigma(\mathscr{M})$, such that its Bernstein-Sato polynomial $b^\sigma(s) \in \mathbb{C}[s]$ has roots in the image of σ.*

Proof. Let $b(s)$ be the Bernstein-Sato polynomial of a good V-filtration $U_\bullet(\mathscr{M})$. Modulo a shift of this filtration, we can assume that the real part of all roots α of $b(s)$ satisfy $\operatorname{Re}\alpha \leqslant \operatorname{Re}\sigma\pi(\alpha)$. Then, let $\lambda \in \mathbb{C}$ be a zero of $b(s)$ with multiplicity $\ell \in \mathbb{N}$ and satisfying $\operatorname{Re}\lambda < \operatorname{Re}\sigma\pi(\lambda)$. We write $b(s) = (s-\lambda)^\ell b_1(s)$. Therefore, the filtration defined by

$$\forall\, k \in \mathbb{Z}, \quad U'_k(\mathscr{M}) = U_{k-1}(\mathscr{M}) + (E+k-\lambda)^\ell U_k(\mathscr{M})$$

is a good V-filtration of $\mathscr{M}$. Notice that $(s-\lambda-1)^\ell b_1(s)$ is a Bernstein polynomial associated to this filtration. In this way, one can construct step by step a good V-filtration for which the Bernstein-Sato polynomial has its roots in the image of the section σ.

Consider now two good V-filtrations $U_\bullet(\mathscr{M})$ and $V_\bullet(\mathscr{M})$ such that their Bernstein-Sato polynomials $b_U(s)$ and $b_V(s)$ have their roots in the image of the section σ. It is enough to show that $U_\bullet(\mathscr{M}) \subset V_\bullet(\mathscr{M})$. After Exercise 7.2.3, there exists an integer $\ell \in \mathbb{Z}$ such that for every $k \in \mathbb{Z}$, $U_k(\mathscr{M}) \subset V_{k+\ell}(\mathscr{M})$. The inclusion $U_\bullet(\mathscr{M}) \subset V_\bullet(\mathscr{M})$ being clear when $\ell \leqslant 0$, let us treat the case $\ell \in \mathbb{N} \smallsetminus \{0\}$. Notice that, the polynomials $b_U(s+k)$ and $b_V(s+k+\ell)$ being coprime, there exist two polynomials p and q such that $1 = p(s)b_U(s+k) + q(s)b_V(s+k+\ell)$. In particular, for any local section m of $U_k(\mathscr{M})$, we have

$$m = p(E)b_U(E+k)m + q(E)b_V(E+k+\ell)m \in U_{k-1}(\mathscr{M}) + V_{k+\ell-1}(\mathscr{M}) \subset V_{k+\ell-1}(\mathscr{M}).$$

Iterating this process, we obtain the inclusion $U_\bullet(\mathscr{M}) \subset V_\bullet(\mathscr{M})$. □

Corollary 7.3.11. *Let $0 \to \mathscr{M}_1 \to \mathscr{M} \to \mathscr{M}_2 \to 0$ be an exact sequence of $\mathscr{D}_X$-modules which are specializable along Y. Let $\sigma : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ be a section of the natural projection $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$.*

(1) *For every $k \in \mathbb{Z}$, the sequences of $V_0(\mathscr{D}_X)$-modules*

$$0 \longrightarrow V_k^\sigma(\mathscr{M}_1) \longrightarrow V_k^\sigma(\mathscr{M}) \longrightarrow V_k^\sigma(\mathscr{M}_2) \longrightarrow 0$$

are exact.

(2) *The sequence of graded $\mathrm{gr}^V(\mathscr{D}_X)$-modules*

$$0 \longrightarrow \mathrm{gr}^{V^\sigma}(\mathscr{M}_1) \longrightarrow \mathrm{gr}^{V^\sigma}(\mathscr{M}) \longrightarrow \mathrm{gr}^{V^\sigma}(\mathscr{M}_2) \longrightarrow 0$$

is exact.

Proof. Remark that the filtration $V_\bullet^\sigma(\mathscr{M})$ induces on $\mathscr{M}_1$ and $\mathscr{M}_2$ good filtrations, the Bernstein-Sato polynomials of which divide $b^\sigma(s)$. After Proposition 7.3.10, these induced filtrations are the filtrations $V_\bullet^\sigma(\mathscr{M}_1)$ and $V_\bullet^\sigma(\mathscr{M}_2)$. This implies the exactness property of the sequences. □

Last, if $\varphi : \mathscr{M} \to \mathscr{N}$ is a morphism between Y-specializable $\mathscr{D}_X$-modules, it follows from Corollary 7.3.11 that, for any $k \in \mathbb{Z}$, we have

$$\varphi(V_k^\sigma(\mathscr{M})) = V_k^\sigma(\mathscr{N}) \cap \varphi(\mathscr{M}).$$

In other words, any morphism is *strict* for the $V_\bullet^\sigma$ filtrations. Hence, for every $k \in \mathbb{Z}$, $\mathscr{M} \to V_k^\sigma(\mathscr{M})$ defines a functor from the category of Y-specializable $\mathscr{D}_X$-modules to the category of coherent $V_0(\mathscr{D}_X)$-modules.

Exercise 7.3.12

(1) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module supported on Y. Show that $\mathscr{M}$ is specializable along Y and that the set A of Exercise 7.3.8 may be chosen equal to $\{0\}$. Show that $V_\bullet^\sigma\mathscr{M}$ does not depend on σ, provided that $\sigma(0) = 0$. We denote it by $V_\bullet\mathscr{M}$. Show that the roots of the Bernstein-Sato polynomial of $V_\bullet\mathscr{M}$ belong to $\mathbb{N}$.

(2) If $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module which is noncharacteristic with respect to Y, show that $\mathscr{M}$ is specializable along Y and that the set A of Exercise 7.3.8 may be chosen equal to $\{0\}$. Show that $V_\bullet^\sigma\mathscr{M}$ does not depend on σ, provided that $\sigma(0) = 0$. We denote it by $V_\bullet\mathscr{M}$. Show that $V_k\mathscr{M} = \mathscr{M}$ if $k \geqslant -1$ and $V_{-k}\mathscr{M} = \mathscr{I}_Y^{k-1}\mathscr{M}$ if $k \geqslant 1$. Conclude that the roots of the Bernstein-Sato polynomial of $V_\bullet\mathscr{M}$ belong to $-\mathbb{N}^*$.

***Exercise 7.3.13* (The canonical V-filtration).** Let $\mathscr{M}$ be a Y-specializable $\mathscr{D}_X$-module. For any compact set $K \subset X$, show that there exists a finite set $A \subset \mathbb{C}$ and, for any $a \in \mathrm{Re}(A)$ a unique good V-filtration $V_{a+\bullet}\mathscr{M}$ indexed by $\mathbb{Z}$ such that

- $V_{a+k}\mathscr{M} \subset V_{a'+\ell}\mathscr{M}$ if $a + k \leqslant a' + \ell$ in $\mathbb{R}$,
- for any $a \in \mathrm{Re}(A) + \mathbb{Z}$, the roots α of the minimal polynomial of $-\partial_t t$ on $\mathrm{gr}_a^V\mathscr{M} := V_a\mathscr{M}/V_{<a}\mathscr{M}$ satisfy $\mathrm{Re}(\alpha) = a$ [by definition, $< a = \max\{a' \in \mathrm{Re}(A) + \mathbb{Z} \mid a' < a\}$]. [Hint: if such filtrations exist, estimate the roots of the Bernstein-Sato polynomial of any local section m of $V_a\mathscr{M}$ for any $a \in \mathrm{Re}(A)+\mathbb{Z}$.]

Show that the canonical V-filtration satisfies the following properties:

(1) for any $a < 0$ in $\mathrm{Re}(A) + \mathbb{Z}$, $t : V_a\mathscr{M} \to V_{a-1}\mathscr{M}$ is an isomorphism of $\mathscr{O}_X$-modules [Hint: use Lemma 7.2.5],

(2) for any $a \neq -1$, $\partial_t : \mathrm{gr}_a^V\mathscr{M} \to \mathrm{gr}_{a+1}^V\mathscr{M}$ is bijective.

Exercise 7.3.14 **(Invariance by embedding).** Let $i : X \hookrightarrow X'$ be a closed embedding of complex manifolds. Let Y' be a smooth hypersurface in X' which is transverse to X, i.e., such that the (scheme-theoretical) intersection $Y = Y' \cap X$ is a smooth hypersurface in X. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module.

Show that $\mathscr{M}$ is specializable along Y if and only if $i_+\mathscr{M}$ is specializable along Y'. Still denoting by i the inclusion $Y \hookrightarrow Y'$, show that, with notation of Exercise 7.3.13, for any $a \in \mathbb{R}$ we have (after defining the right-hand term) $\mathrm{gr}_a^V i_+\mathscr{M} = i_+\mathrm{gr}_a^V\mathscr{M}$.

7.4. Localization and restriction of specializable $\mathscr{D}_X$-modules

If $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module and $Y \xhookrightarrow{i} X$ is a smooth hypersurface, the inverse image $i^+\mathscr{M}$ (see Exercise 4.2.2) is not $\mathscr{D}_Y$-coherent in general (e.g. take $\mathscr{M} = \mathscr{D}_X$). Similarly, the localization $\mathscr{O}_X(*Y) \otimes_{\mathscr{O}_X} \mathscr{M}$ does not remain $\mathscr{D}_X$-coherent in general. The category of Y-specializable $\mathscr{D}_X$-modules is the right subcategory of $\mathsf{M}_{\mathrm{coh}}(\mathscr{D}_X)$ where coherence is preserved under these two functors, as we will see below. We will work locally, and we fix local coordinates (t, x) on X so that Y is defined by $t = 0$.

If $\mathscr{M}$ is an $\mathscr{O}_X$-module, we denote by $\mathscr{M}[1/t]$ the localized $\mathscr{O}_X$-module $\mathscr{O}_X(*Y) \otimes_{\mathscr{O}_X} \mathscr{M}$. As $\mathscr{O}_X(*Y)$ ha a natural structure of a left $\mathscr{D}_X$-module, we know from Exercise 1.2.6 that, if $\mathscr{M}$ is a left $\mathscr{D}_X$-module, then so is $\mathscr{M}[1/t]$.

Let s be a new variable. Consider the sheaf $\mathscr{D}_X[s]$ of differential operators with coefficients in $\mathscr{O}_X[s]$ and set $\mathscr{M}[1/t, s] = \mathscr{D}_X[s] \otimes_{\mathscr{D}_X} \mathscr{M}[1/t]$. This is a left $\mathscr{D}_X[s]$-module. We will now twist this structure, keeping fixed however the underlying $\mathscr{O}_X[1/t, s]$-structure.

Lemma 7.4.1. *The following rule defines a left $\mathscr{D}_X[s]$-module structure on the $\mathscr{O}_X[s]$-module $\mathscr{M}[1/t, s]$: for any $\ell \in \mathbb{N}$ and any local section m of $\mathscr{M}[1/t]$,*

$$\begin{aligned} \partial_{x_j} s^\ell m &= s^\ell \partial_{x_j} m, \\ \partial_t s^\ell m &= s^\ell[\partial_t m + st^{-1}m]. \end{aligned}$$

Proof. Use Lemma 1.2.1. ☐

It will be convenient to denote by $\mathscr{M}[1/t, s]t^s$ the $\mathscr{O}_X[1/t, s]$-module $\mathscr{M}[1/t, s]$ equipped with this twisted structure. That is, we formally write the new connection as $t^{-s} \circ \nabla \circ t^s$. Be careful however that “t^s” is nothing but a symbol which allows one to remember, through the Leibniz rule, the left $\mathscr{D}_X[s]$ structure.

***Exercise 7.4.2* (Specialization to $s = k$).** Let k be any integer.

(1) Show that $t^{-k}\nabla t^k$ defines a left $\mathscr{D}_X$-structure on the $\mathscr{O}_X[1/t]$-module $\mathscr{M}[1/t]$.

(2) Show that $(\mathscr{M}[1/t], t^{-k}\nabla t^k) \simeq \mathscr{M}[1/t, s]t^s\big/(s-k)\mathscr{M}[1/t, s]t^s$.

***Exercise 7.4.3* (Bernstein's functional equation).** Let m be a local section of $\mathscr{M}[1/t]$ and let $b(s) \in \mathbb{C}[s]$. Show that the following conditions are equivalent:

(1) $b(E)m \in V_{-1}(\mathscr{D}_X)m$,

(2) $b(-s-1)mt^s \in \mathscr{D}_X[s]mt^{s+1}$.

[Hint: show, for any local section m of $\mathscr{M}[1/t]$, the identity $(t\partial_t m)t^s = -(s+1)mt^s + \partial_t(mt^{s+1})$ and then, for any integer k, $((t\partial_t)^k m)t^s - (-s-1)^k mt^s \in \mathscr{D}_X[s]mt^{s+1}$.]

Proposition 7.4.4. *Let $\mathscr{M}$ be a Y-specializable $\mathscr{D}_X$-module. The $\mathscr{D}_X$-module $\mathscr{M}(*Y)$ is specializable along Y (in particular coherent).*

Proof. Let $\mathscr{M}$ be a Y-specializable $\mathscr{D}_X$-module. Let us first show the coherence of $\mathscr{M}(*Y)$. This is a local problem; moreover, by induction on the cardinal of a generators system of $\mathscr{M}$, we can assume that $\mathscr{M}$ is generated by one section $m \in \mathscr{M}$. After Exercise 7.4.3, there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that $b(s)mt^s \in \mathscr{D}_X[s]mt^{s+1}$.

Let $k_0 \in \mathbb{N}$ be an integer, such that $b(-k) \neq 0$ for any $k \geqslant k_0 + 1$. Then $mt^{-k} \in \mathscr{D}_X mt^{-k_0}$, for $k \geqslant k_0 + 1$. From the identity $(\partial_t m)t^{-k} = \partial_t(mt^{-k}) + kmt^{-k-1}$, we get $\mathscr{M}[1/t] = \mathscr{D}_X \cdot mt^{-k_0}$. The filtration $F_\ell\mathscr{D}_X \cdot mt^{-k_0}$ ($\ell \in \mathbb{N}$) is a good filtration (see Exercise 2.2.3), hence the $\mathscr{D}_X$-module $\mathscr{M}[1/t]$ is coherent.

Let m' be a local section of $\mathscr{M}(*Y)$. It can be written as $m' = m/t^k$ for some local section m of $\mathscr{M}$. As $\mathscr{M}$ is Y-specializable, there exists a nonzero polynomial $b(s)$ such that $b(E)m \in V_{-1}(\mathscr{D}_X)m$. From this, we deduce a Bernstein's identity for $m' \in \mathscr{M}(*Y)$:

$$b(E+k)t^{-k}m \in V_{-1}(\mathscr{D}_X)t^{-k}m.$$

Therefore, $\mathscr{M}(*Y)$ is specializable along Y. □

Corollary 7.4.5. *Under the same assumptions, the natural morphism of $V_0(\mathscr{D}_X)$-modules*

$$V_{<0}(\mathscr{M}) \longrightarrow V_{<0}(\mathscr{M}(*Y))$$

is an isomorphism.

Proof. Let $T(\mathscr{M}) = \Gamma_{[Y]}\mathscr{M}$ be the $\mathscr{D}_X$-submodule in $\mathscr{M}$ of sections supported by Y. We have the exact sequence:

$$0 \longrightarrow T(\mathscr{M}) \longrightarrow \mathscr{M} \longrightarrow \mathscr{M}(*Y) \longrightarrow C(\mathscr{M}) \longrightarrow 0.$$

The modules $\mathscr{M}$ and $\mathscr{M}(*Y)$ are specializable along Y. It follows from Proposition 7.3.9 that the $\mathscr{D}_X$-modules $T(\mathscr{M})$ and $C(\mathscr{M})$ are so. On the other hand, these modules are supported by Y; we have seen that the roots of Bernstein's polynomials of their sections are strictly negative integers (see Exercise 7.3.12). Then $V_{<0}(T(\mathscr{M})) = 0$ and $V_{<0}(C(\mathscr{M})) = 0$ and we deduce from Corollary 7.3.11 the natural isomorphism:

$$V_{<0}(\mathscr{M}) \longrightarrow V_{<0}(\mathscr{M}(*Y)). \qquad \square$$

Exercise 7.4.6. Give a proof of the corollary using Lemma 7.2.5.

Proposition 7.4.7. *Let $\mathscr{M}$ be a Y-specializable $\mathscr{D}_X$-module. We denote by $i : Y \hookrightarrow X$ the inclusion. If the hypersurface Y has a global reduced equation $f = 0$ (or after restricting X), the complex $\boldsymbol{L}i^+\mathscr{M}$ is functorially isomorphic to the complex of $\mathscr{D}_Y$-modules*

$$0 \longrightarrow \operatorname{gr}_0^V(\mathscr{M})_{|Y} \xrightarrow{f} \operatorname{gr}_{-1}^V(\mathscr{M})_{|Y} \longrightarrow 0.$$

In particular, the cohomology sheaves of $\boldsymbol{L}i^+\mathscr{M}$ are $\mathscr{D}_Y$-coherent.

Proof. Let us consider the multiplication by f:

$$V_{<0}(\mathscr{M}[1/f]) \xrightarrow{\phi} V_{<-1}(\mathscr{M}[1/f]), \quad m \longmapsto fm.$$

As the multiplication by f in $\mathscr{M}[1/f]$ is bijective, the map ϕ is injective. Let us show its surjectivity. Let be $m \in V_{<-1}(\mathscr{M}[1/f])$ and $m' = (m/f) \in \mathscr{M}[1/f]$. From a Bernstein-Sato's equation of m, we get $f\, b_m(E+1)m' \in V_{-2}(\mathscr{D}_X)m'$. After division by f, we deduce that $b_m(s+1)$ is a multiple of Bernstein-Sato's polynomial of m'. Hence, $m' \in V_{<0}(\mathscr{M}[1/f])$ and ϕ is surjective and then finally bijective. We deduce from Proposition 7.4.4 that the morphism by $f : V_{<0}(\mathscr{M}) \to V_{<-1}(\mathscr{M})$ is an isomorphism (one can also use Exercise 7.3.13). The complexes

$$0 \to \operatorname{gr}_0^V(\mathscr{M})_{|Y} \xrightarrow{f} \operatorname{gr}_{-1}^F(\mathscr{M})_{|Y} \to 0 \quad \text{and} \quad 0 \to V_0(\mathscr{M})_{|Y} \xrightarrow{f} V_{-1}(\mathscr{M})_{|Y} \to 0$$

are therefore isomorphic. But, under the assumption of the proposition, the complex $\boldsymbol{L}i^+\mathscr{M}$ is represented by the complex of $\mathscr{D}_Y$-modules

$$0 \longrightarrow \mathscr{M}_{|Y} \xrightarrow{f} \mathscr{M}_{|Y} \longrightarrow 0.$$

In order to prove the proposition, it is enough to show that the morphism

$$\mathscr{M}/V_0(\mathscr{M}) \xrightarrow{\phi'} \mathscr{M}/V_{-1}(\mathscr{M}), \quad m \longmapsto fm,$$

is an isomorphism.

Recall that, for any integer $k \in \mathbb{N}$, the endomorphism E of $V_k(\mathscr{M})/V_{k-1}(\mathscr{M})$ is bijective. By induction on k, any section of $\mathscr{M} = \cup_{k\in\mathbb{N}} V_k(\mathscr{M})$ is a section of $f\mathscr{M}$ modulo $V_0(\mathscr{M})$. It follows that, for $k \in \mathbb{N}$, any section of $V_k(\mathscr{M})$ is a section of fM modulo $V_0(\mathscr{M})$. Hence, ϕ' is onto.

We have now to prove the injectivity of ϕ'. Let $m \in \mathscr{M}$ be such that $fm \in V_{-1}(\mathscr{M})$. The polynomial $b_{fm}(s)$ has positive roots and satisfies $b_{fm}(E)fm \in V_{-1}(\mathscr{D}_X)fm$. Apply the operator ∂_t to this identity to get $b_{fm}(E+1)(E+1)m \in V_{-1}(\mathscr{D}_X)m$. Hence, m belongs to $V_0(\mathscr{M})$ and ϕ' is injective.

Now, the coherence of the cohomology sheaves of $\boldsymbol{L}i^{+}\mathscr{M}$ follows from Exercise 7.3.5. $\square$

Theorem 7.4.8. *Any holonomic $\mathscr{D}_X$-module $\mathscr{M}$ is specializable along any hypersurface Y. Moreover $\mathscr{M}(*Y)$ and $\boldsymbol{L}^j f^{+}\mathscr{M}$ (see §4.2) are still holonomic, and so are the graded modules $\mathrm{gr}_k^V\mathscr{M}$, after any local identification of $\mathrm{gr}_0^V\mathscr{D}_X$ with $\mathscr{D}_Y[E]$.* $\square$

Exercise 7.4.9. Consider the monomial function $f(x,y)=x^2y^3$ and the $\mathscr{D}_{\mathbb{C}^3}$-module

$$\mathscr{M}=\mathscr{O}_{\mathbb{C}^3}[1/(t-f)]\Big/\mathscr{O}_{\mathbb{C}^3}.$$

Denote by $\delta(t-f)$ the class of $1/(t-f)$ in $\mathscr{M}$.

(1) Show that $\mathscr{M}$ is $\mathscr{D}_X$-coherent and that $\delta(t-f)$ is a generator.

(2) Compute the characteristic variety of $\mathscr{M}$.

(3) Compute a Bernstein polynomial for any local section $x^ky^\ell\delta(t-f)$, $k,\ell\in\mathbb{N}$.

(4) Show that $\mathscr{M}$ is specializable along $\{t=0\}$.

(5) Determine a set A of Exercise 7.3.8 in $]-1,0[\,\cap\,\mathbb{Q}$.

(6) Compute the various graded pieces $\mathrm{gr}_k^V\mathscr{M}$ and the action of E on them.

7.5. V-filtration and direct images

The purpose of this section is to establish the compatibility between taking a direct image and taking a graded part of a V-filtered $\widetilde{\mathscr{D}}_X$-module. We will give an analogue of Proposition 3.3.17 of [**Sai88**] and of its proof. Another proof, which only applies to holonomic $\mathscr{D}$-modules, is given in [**MS89**, § 4.8]. Let us first introduce a definition.

Definition 7.5.1. Let $Y\subset X$ be a smooth hypersurface, let $V_\bullet\widetilde{\mathscr{D}}_X$ be the corresponding V-filtration and let $\mathscr{M}$ be a left $\widetilde{\mathscr{D}}_X$-module equipped with an increasing filtration $U_\bullet\mathscr{M}$ indexed by $\mathbb{Z}$ such that $V_k\widetilde{\mathscr{D}}_X\cdot U_\ell\mathscr{M}\subset U_{k+\ell}\mathscr{M}$ for any $k,\ell\in\mathbb{Z}$. We say that $(\mathscr{M},U_\bullet\mathscr{M})$ (or also $\mathrm{gr}^U\mathscr{M}$) is *monodromic* if, locally on X, there exists a monic polynomial $b(s)=s^d+\sum_{i=0}^{d-1}a_i(z)s^i\in\mathbb{C}[z][s]$ such that

(1) $b(-(\eth_t t+kz))\cdot\mathrm{gr}_k^U\mathscr{M}=0$ for all $k\in\mathbb{Z}$,

(2) $\gcd(b(s-kz),b(s-\ell z))\in\mathbb{C}[z]\smallsetminus\{0\}$ for all $k\neq\ell$.

In the non-filtered case, one may forget about z. Then, any Y-specializable $\mathscr{D}_X$-module, when equipped with $V_\bullet^\sigma\mathscr{M}$ (for any choice of σ), is monodromic.

Theorem 7.5.2. *Let $f:X\to X'$ be holomorphic map between complex analytic manifolds and let $t\in\mathbb{C}$ be a new variable. Put $F=f\times\mathrm{Id}:X\times\mathbb{C}\to X'\times\mathbb{C}$. Let $\mathscr{M}$ be a right $\widetilde{\mathscr{D}}_{X\times\mathbb{C}}$-module equipped with a V-filtration $U_\bullet\mathscr{M}$ (relative to the hypersurface $Y=X\times\{0\}$). Then $U_\bullet\mathscr{M}$ defines canonically and functorially a V-filtration $U_\bullet\mathscr{H}^i(F_\dagger\mathscr{M})$.*

Assume that F is proper on the support of $\mathscr{M}$.

(1) *If $\mathscr{M}$ is good and $U_\bullet\mathscr{M}$ is a good V-filtration, then $U_\bullet\mathscr{H}^i(F_\dagger\mathscr{M})$ is a good V-filtration.*

(2) *If moreover $\mathrm{gr}^U\mathscr{M}$ is monodromic and $f_\dagger\mathrm{gr}^U\mathscr{M}$ is strict, then one has a canonical and functorial isomorphism of $\widetilde{\mathscr{D}}_{X'}$-modules ($k \in \mathbb{Z}$)*

$$\mathrm{gr}_k^U\big(\mathscr{H}^i F_\dagger\mathscr{M}\big) = \mathscr{H}^i\big(f_\dagger\mathrm{gr}_k^U\mathscr{M}\big),$$

and $\mathrm{gr}^U\big(\mathscr{H}^i F_\dagger\mathscr{M}\big)$ is monodromic and strict.

Remark 7.5.3. In the last assertion, we view $\mathrm{gr}_k^U\mathscr{M}$ as a right $\widetilde{\mathscr{D}}_X$-module. By functoriality, the action of $t\eth_t$ descends to $\mathscr{H}^i(f_\dagger\mathrm{gr}_k^U\mathscr{M})$.

Corollary 7.5.4. *Let $f : X \to X'$ be a morphism of complex manifolds and let $\varphi : X' \to \mathbb{C}$ be a holomorphic function. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module which is specializable along $\varphi \circ f = 0$. Assume that f is proper on* $\mathrm{Supp}\,\mathscr{M}$. *Then the cohomology modules $\mathscr{H}^i f_\dagger\mathscr{M}$ are specializable along $\varphi = 0$.* □

Proof. We will use the isomorphism $F_\dagger = f_\dagger$ for $\mathscr{M}$ (see Remark 4.3.4(3)), i.e., we take the direct image viewing $\mathscr{M}$ as a $\widetilde{\mathscr{D}}_{X\times\mathbb{C}/\mathbb{C}}$-module equipped with a compatible action of $\eth_t$. Put $\mathscr{N}^\bullet = f_\dagger\mathscr{M}$. This complex is naturally filtered by $U_\bullet\mathscr{N}^\bullet := f_\dagger U_\bullet\mathscr{M}$. Therefore, we define the filtration on its cohomology by

$$U_\bullet\mathscr{H}^i(F_\dagger\mathscr{M}) = U_\bullet\mathscr{H}^i(f_\dagger\mathscr{M}) := \mathrm{image}\big[\mathscr{H}^i(f_\dagger U_\bullet\mathscr{M}) \to \mathscr{H}^i(f_\dagger\mathscr{M})\big].$$

Notice that, for any j, $f_\dagger U_j\mathscr{M}$ is the direct image of $U_j\mathscr{M}$ viewed as a $\widetilde{\mathscr{D}}_{X\times\mathbb{C}/\mathbb{C}}$-module, on which we put the natural action of $t\eth_t$.

The relation with the Rees construction is given by the following lemma:

Lemma 7.5.5. *Let $(\mathscr{N}^\bullet, U_\bullet\mathscr{N}^\bullet)$ be a V-filtered complex of $\widetilde{\mathscr{D}}_{X'\times\mathbb{C}}$-modules. Put*

$$U_j\mathscr{H}^i(\mathscr{N}^\bullet) := \mathrm{image}\big[\mathscr{H}^i(U_j\mathscr{N}^\bullet) \to \mathscr{H}^i(\mathscr{N}^\bullet)\big].$$

Then we have

$$\mathscr{H}^i(R_U\mathscr{N}^\bullet)/\tau\text{-torsion} = R_U\mathscr{H}^i(\mathscr{N}^\bullet).$$

In particular, if $R_U\mathscr{N}^\bullet$ has $\widetilde{\mathscr{D}}_{X'\times\mathbb{C}}$-coherent cohomology, then $U_\bullet\mathscr{H}^i(\mathscr{N}^\bullet)$ is a good V-filtration.

Proof. By definition, one has a surjective morphism of graded modules $\mathscr{H}^i(R_U\mathscr{N}^\bullet) \to R_U\mathscr{H}^i(\mathscr{N}^\bullet)$, and this morphism induces an isomorphism after tensoring with $\mathbb{C}[\tau,\tau^{-1}]$. □

Lemma 7.5.6. *If $\mathscr{M}$ is good, then any coherent $V_0\widetilde{\mathscr{D}}_X$-submodule is good.*

Proof. As a coherent $V_0\widetilde{\mathscr{D}}_X$-submodule of $\mathscr{M}$ induces on any subquotient of $\mathscr{M}$ a coherent $V_0\widetilde{\mathscr{D}}_X$-submodule, we may reduce to the case where $\mathscr{M}$ has a good filtration. It is then enough to prove that any coherent $V_0\widetilde{\mathscr{D}}_X$-submodule $\mathscr{N}$ of $\mathscr{M}$ is contained in such a submodule having a good filtration. If $\mathscr{F}$ is a $\widetilde{\mathscr{O}}_X$-coherent submodule of $\mathscr{M}$ which generates $\mathscr{M}$, then $\mathscr{N}$ is contained in $V_k\widetilde{\mathscr{D}}_X \cdot \mathscr{F}$ for some k, hence the result. $\square$

This lemma allows one to apply Grauert's coherence theorem to each U_j, in order to get that each $f_\dagger U_j\mathscr{M}$ has $V_0\widetilde{\mathscr{D}}_X$-coherent cohomology under the properness assumption. We conclude that for each i, j, $U_j\mathscr{H}^i f_\dagger\mathscr{M}$ is $V_0\widetilde{\mathscr{D}}_X$-coherent.

In order to end the proof of (1), we need to prove that each $U_\bullet\mathscr{H}^i f_\dagger\mathscr{M}$ is a good V-filtration. We will compute directly the Rees module associated with this filtration, in order to get its coherence. Let us first consider the analogue of Lemma 7.5.6.

Keep notation of § 7.2. The graded ring $R_V\widetilde{\mathscr{D}}_X$ is filtered by the degree in the derivatives $\tau\partial_t, \partial_{x_j}$ and the degree-zero term of the filtration is $R_V\widetilde{\mathscr{O}}_X$, with $V_k\widetilde{\mathscr{O}}_X = \widetilde{\mathscr{O}}_X$ for $k \geqslant 0$ and $= t^{-k}\widetilde{\mathscr{O}}_X$ for $k \leqslant 0$.

Let $(\mathscr{M}, U_\bullet\mathscr{M})$ be a V-filtered right $\widetilde{\mathscr{D}}_X$-module and let $R_U\mathscr{M}$ be the associated Rees module. We therefore have the notion of a good filtration on $R_U\mathscr{M}$ (by coherent graded $R_V\widetilde{\mathscr{O}}_X$-submodules). If $R_U\mathscr{M}$ has a good filtration (or if $R_U\mathscr{M}$ is generated by a coherent graded $R_V\widetilde{\mathscr{O}}_X$-module), it is $R_V\widetilde{\mathscr{D}}_X$-coherent and has a left resolution by coherent "induced" graded $R_V\widetilde{\mathscr{D}}_X$-modules, of the form $G \otimes_{R_V\widetilde{\mathscr{O}}_X} R_V\widetilde{\mathscr{D}}_X$, where G is graded $R_V\widetilde{\mathscr{O}}_X$-coherent. We may even assume (by killing the τ-torsion) that each term $G \otimes_{R_V\widetilde{\mathscr{O}}_X} R_V\widetilde{\mathscr{D}}_X$ has no τ-torsion, or in other words that it takes the form $R_U(L \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X)$, where L is $\widetilde{\mathscr{O}}_X$-coherent, having support contained in $\operatorname{Supp}\mathscr{M}$, and equipped with a good V-filtration (i.e., a good $\mathscr{I}_Y$-adic filtration) and $U_\bullet(L \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X)$ is defined in the usual way.

We say that $R_U\mathscr{M}$ is *good* if, in the neighbourhood of any compact set $K \subset X$, $R_U\mathscr{M}$ is a finite successive extension of graded $R_V\widetilde{\mathscr{D}}_X$-modules having a good filtration.

Lemma 7.5.7. *Assume that $\mathscr{M}$ is a good $\widetilde{\mathscr{D}}_X$-module and let $U_\bullet\mathscr{M}$ be a good V-filtration of $\mathscr{M}$. Then $R_U\mathscr{M}$ is a good graded $R_V\widetilde{\mathscr{D}}_X$-module.*

Proof. Fix a compact set $K \subset X$. First, it is enough to prove the lemma when $\mathscr{M}$ has a good filtration in some neighbourhood of K, because a good V-filtration $U_\bullet\mathscr{M}$ induces naturally on any subquotient $\mathscr{M}'$ of $\mathscr{M}$ a good V-filtration, so that $R_U\mathscr{M}'$ is a subquotient of $R_U\mathscr{M}$.

Therefore, assume that $\mathscr{M}$ is generated by a coherent $\widetilde{\mathscr{O}}_X$-module $\mathscr{F}$, i.e., $\mathscr{M} = \widetilde{\mathscr{D}}_X \cdot \mathscr{F}$. Consider the V-filtration $U'_\bullet\mathscr{M}$ generated by $\mathscr{F}$, i.e., $U'_\bullet\mathscr{M} = V_\bullet\widetilde{\mathscr{D}}_X \cdot \mathscr{F}$. Then, clearly, $R_V\widetilde{\mathscr{O}}_X \cdot \mathscr{F} = \oplus_k V_k\widetilde{\mathscr{O}}_X \cdot \mathscr{F}\tau^k$ is a coherent graded $R_V\widetilde{\mathscr{O}}_X$-module which generates $R_{U'}\mathscr{M}$.

If the filtration $U''_\bullet\mathscr{M}$ is obtained from $U'_\bullet\mathscr{M}$ by a shift by $-\ell \in \mathbb{Z}$, i.e., if $R_{U''}\mathscr{M} = \tau^\ell R_{U'}\mathscr{M} \subset \mathscr{M}[\tau, \tau^{-1}]$, then $R_{U''}\mathscr{M}$ is generated by the $R_V\widetilde{\mathscr{O}}_X$-coherent submodule $\tau^\ell R_V\widetilde{\mathscr{O}}_X \cdot \mathscr{F}$.

On the other hand, let $U''_\bullet\mathscr{M}$ be a good V-filtration such that $R_{U''}\mathscr{M}$ has a good filtration. Then any good V-filtration $U_\bullet\mathscr{M}$ such that $U_k\mathscr{M} \subset U''_k\mathscr{M}$ for any k satisfies the same property, because $R_U\mathscr{M}$ is thus a coherent graded submodule of $R_{U''}\mathscr{M}$, so a good filtration on the latter induces a good filtration on the former.

As any good V-filtration $U_\bullet\mathscr{M}$ is contained, in some neighbourhood of K, in the good V-filtration $U'_\bullet\mathscr{M}$ suitably shifted, we get the lemma. $\square$

To end the proof of Part (1), it is therefore enough to prove it for induced modules $\mathscr{M} = \mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$, with $\mathscr{L}$ coherent over $\widetilde{\mathscr{O}}_X$ and $F_{|\operatorname{Supp}\mathscr{L}}$ proper. We will indicate it when $f : X = Y \times Z \to Y$ is the projection. We then have

$$\begin{aligned}U_j(\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_{X\times\mathbb{C}}} \widetilde{\mathscr{D}}_{X\times\mathbb{C}}) &= U_j\Big[(\mathscr{L} \otimes_{f^{-1}\widetilde{\mathscr{O}}_{Y\times\mathbb{C}}} f^{-1}\widetilde{\mathscr{D}}_{Y\times\mathbb{C}}) \underset{f^{-1}\widetilde{\mathscr{O}}_{Y\times\mathbb{C}}}{\otimes} \widetilde{\mathscr{D}}_{X\times\mathbb{C}/Y\times\mathbb{C}}\Big]\\ &= U_j(\mathscr{L} \otimes_{f^{-1}\widetilde{\mathscr{O}}_{Y\times\mathbb{C}}} f^{-1}\widetilde{\mathscr{D}}_{Y\times\mathbb{C}}) \underset{f^{-1}\widetilde{\mathscr{O}}_{Y\times\mathbb{C}}}{\otimes} \widetilde{\mathscr{D}}_{X\times\mathbb{C}/Y\times\mathbb{C}},\end{aligned}$$

because the V-filtration on $\widetilde{\mathscr{D}}_{X\times\mathbb{C}/Y\times\mathbb{C}}$ is nothing but the t-adic filtration. Now, we have

$$\begin{aligned}f_\dagger U_j(\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_{X\times\mathbb{C}}} \widetilde{\mathscr{D}}_{X\times\mathbb{C}}) &= \boldsymbol{R}f_*U_j(\mathscr{L} \otimes_{f^{-1}\widetilde{\mathscr{O}}_{Y\times\mathbb{C}}} f^{-1}\widetilde{\mathscr{D}}_{Y\times\mathbb{C}})\\ &= U_j(\boldsymbol{R}f_*\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_{Y\times\mathbb{C}}} \widetilde{\mathscr{D}}_{Y\times\mathbb{C}}),\end{aligned}$$

if we filter the complex $\boldsymbol{R}f_*\mathscr{L}$ by subcomplexes $\boldsymbol{R}f_*U_j(\mathscr{L})$ and we filter the tensor product as usual. By Grauert's theorem applied to coherent $R_V\widetilde{\mathscr{O}}_{X\times\mathbb{C}}$-sheaves, $\boldsymbol{R}f_*R_U\mathscr{L}$ is $R_V\widetilde{\mathscr{O}}_{Y\times\mathbb{C}}$-coherent, hence $f_\dagger R_U(\mathscr{L}\otimes_{\widetilde{\mathscr{O}}_X}\widetilde{\mathscr{D}}_X)$ is $R_V\widetilde{\mathscr{D}}_{Y\times\mathbb{C}}$-coherent. After Lemma 7.5.5, we get 7.5.2(1). $\square$

In order to get Part (2) of the theorem, we will first prove:

Proposition 7.5.8. *Let $(\mathscr{N}^\bullet, U_\bullet\mathscr{N}^\bullet)$ be a V-filtered complex of $\widetilde{\mathscr{D}}_{Y\times\mathbb{C}}$-modules. Assume that*

(1) *the complex $\operatorname{gr}^U\mathscr{N}^\bullet$ is strict and monodromic,*

(2) *there exists j_0 such that for all $j \leqslant j_0$ and all i, the left multiplication by t induces an isomorphism $t : U_j\mathscr{N}^i \xrightarrow{\sim} U_{j-1}\mathscr{N}^i$,*

(3) *There exists $i_0 \in \mathbb{Z}$ such that, for all $i \geqslant i_0$ and any j, one has $\mathscr{H}^i(U_j\mathscr{N}^\bullet) = 0$.*

Then for any i,j the morphism $\mathscr{H}^i(U_j\mathscr{N}^\bullet) \to \mathscr{H}^i(\mathscr{N}^\bullet)$ is injective. Moreover, the filtration $U_\bullet\mathscr{H}^i(\mathscr{N}^\bullet)$ defined by

$$U_j\mathscr{H}^i(\mathscr{N}^\bullet) = \operatorname{image}\big[\mathscr{H}^i(U_j\mathscr{N}^\bullet) \to \mathscr{H}^i(\mathscr{N}^\bullet)\big]$$

satisfies $\operatorname{gr}^U\mathscr{H}^i(\mathscr{N}^\bullet) = \mathscr{H}^i(\operatorname{gr}^U\mathscr{N}^\bullet)$.

Proof. It will have three steps.

First step. This step proves a formal analogue of the conclusion of the proposition. Put

$$\widehat{U_j\mathscr{N}^\bullet}=\varprojlim_{\ell} U_j\mathscr{N}^\bullet/U_\ell\mathscr{N}^\bullet \quad\text{and}\quad \widehat{\mathscr{N}^\bullet}=\varinjlim_{j}\widehat{U_j\mathscr{N}^\bullet}.$$

Under the assumption of Proposition 7.5.8, we will prove the following:

(a) For all $k\leqslant j$, $\widehat{U_k\mathscr{N}^\bullet}\to\widehat{U_j\mathscr{N}^\bullet}$ is injective (hence, for all j, $\widehat{U_j\mathscr{N}^\bullet}\to\widehat{\mathscr{N}^\bullet}$ is injective) and $\widehat{U_j\mathscr{N}^\bullet}/\widehat{U_{j-1}\mathscr{N}^\bullet}=U_j\mathscr{N}^\bullet/U_{j-1}\mathscr{N}^\bullet$.

(b) For any $k\leqslant j$, $\mathscr{H}^i(U_j\mathscr{N}^\bullet/U_k\mathscr{N}^\bullet)$ is strict.

(c) $\mathscr{H}^i(\widehat{U_j\mathscr{N}^\bullet})=\varprojlim_\ell \mathscr{H}^i(U_j\mathscr{N}^\bullet/U_\ell\mathscr{N}^\bullet)$.

(d) $\mathscr{H}^i(\widehat{U_j\mathscr{N}^\bullet})\to\mathscr{H}^i(\widehat{\mathscr{N}^\bullet})$ is injective.

(e) $\mathscr{H}^i(\widehat{\mathscr{N}^\bullet})=\varinjlim_j \mathscr{H}^i(\widehat{U_j\mathscr{N}^\bullet})$.

Define $U_j\mathscr{H}^i(\widehat{\mathscr{N}^\bullet})=\text{image}\Big[\mathscr{H}^i(\widehat{U_j\mathscr{N}^\bullet})\to\mathscr{H}^i(\widehat{\mathscr{N}^\bullet})\Big]$. Then the statements (a) and (d) imply that

$$\operatorname{gr}^U_j\mathscr{H}^i(\widehat{\mathscr{N}^\bullet})=\mathscr{H}^i(\widehat{U_j\mathscr{N}^\bullet}/\widehat{U_{j-1}\mathscr{N}^\bullet})=\mathscr{H}^i(\operatorname{gr}^U_j\mathscr{N}^\bullet).$$

For $\ell<k<j$ consider the exact sequence of complexes

$$0\longrightarrow U_k\mathscr{N}^\bullet/U_\ell\mathscr{N}^\bullet\longrightarrow U_j\mathscr{N}^\bullet/U_\ell\mathscr{N}^\bullet\longrightarrow U_j\mathscr{N}^\bullet/U_k\mathscr{N}^\bullet\longrightarrow 0.$$

As the projective system $(U_j\mathscr{N}^\bullet/U_\ell\mathscr{N}^\bullet)_\ell$ trivially satisfies the Mittag-Leffler condition (ML), the sequence remains exact after passing to the projective limit, so we get an exact sequence of complexes

$$0\longrightarrow\widehat{U_k\mathscr{N}^\bullet}\longrightarrow\widehat{U_j\mathscr{N}^\bullet}\longrightarrow U_j\mathscr{N}^\bullet/U_k\mathscr{N}^\bullet\longrightarrow 0,$$

hence (a).

Let us show by induction on $n\geqslant 1$ that, for all i and j,

$(\mathrm{b})_n$ $\mathscr{H}^i(U_j\mathscr{N}^\bullet/U_{j-n}\mathscr{N}^\bullet)$ is strict (hence (b));

Indeed, $(\mathrm{b})_1$ follows from Assumption 7.5.8(1). Remark also that, by induction on $n\geqslant 1$, 7.5.8(1) implies that, for any n,ℓ,i, $\mathscr{H}^i(U_\ell/U_{\ell-n})$ is killed by $\prod_{k=\ell-n+1}^{\ell}b(\eth_tt+kz)$.

For $n\geqslant 2$, consider the exact sequence

$$\cdots\longrightarrow\mathscr{H}^i(U_{j-1}/U_{j-n})\longrightarrow\mathscr{H}^i(U_j/U_{j-n})\longrightarrow\mathscr{H}^i(U_j/U_{j-1})\xrightarrow{\psi}\mathscr{H}^{i+1}(U_{j-1}/U_{j-n})\longrightarrow\cdots$$

Any local section of $\operatorname{Im}\psi$ is then killed by $b(\eth_tt+jz)$ and $\prod_{k=j-n+1}^{j-1}b(\eth_tt+kz)$, hence by a nonzero holomorphic function of z. By strictness $(\mathrm{b})_{n-1}$ applied to $\mathscr{H}^{i+1}(U_{j-1}/U_{j-n})$, this implies that $\psi=0$, so the previous sequence of $\mathscr{H}^i$ is exact and $\mathscr{H}^i(U_j/U_{j-n})$ is also strict, hence $(\mathrm{b})_n$.

By the same argument, we get an exact sequence, for all $\ell < k < j$,

(7.5.9)
$$0 \longrightarrow \mathscr{H}^i(U_k\mathscr{N}^\bullet/U_\ell\mathscr{N}^\bullet) \longrightarrow \mathscr{H}^i(U_j\mathscr{N}^\bullet/U_\ell\mathscr{N}^\bullet) \longrightarrow \mathscr{H}^i(U_j\mathscr{N}^\bullet/U_k\mathscr{N}^\bullet) \longrightarrow 0.$$

Consequently, the projective system $(\mathscr{H}^i(U_j\mathscr{N}^\bullet/U_\ell\mathscr{N}^\bullet))_\ell$ satisfies (ML), so we get (c) (see e.g. [**KS90**, Prop. 1.12.4]). Moreover, taking the limit on ℓ in the previous exact sequence gives, according to (ML), an exact sequence

$$0 \longrightarrow \mathscr{H}^i(\widehat{U_k\mathscr{N}^\bullet}) \longrightarrow \mathscr{H}^i(\widehat{U_j\mathscr{N}^\bullet}) \longrightarrow \mathscr{H}^i(U_j\mathscr{N}^\bullet/U_k\mathscr{N}^\bullet) \longrightarrow 0,$$

hence (d). Now, (e) is clear.

Second step. For any i, j, denote by $\mathscr{T}_j^i \subset \mathscr{H}^i(U_j\mathscr{N}^\bullet)$ the t-torsion subsheaf of $\mathscr{H}^i(U_j\mathscr{N}^\bullet)$. We will now prove that it is enough to show that there exists j_0 such that, for each i and each $j \leqslant j_0$,

$$\mathscr{T}_j^i = 0. \tag{7.5.10}$$

Assume that (7.5.10) is proved (step 3). Let $j \leqslant j_0$ and let $\ell \geqslant j$. Then, by definition of a V-filtration, $t^{\ell-j}$ acts by 0 on $U_\ell\mathscr{N}^\bullet/U_j\mathscr{N}^\bullet$, so that the image of $\mathscr{H}^{i-1}(U_\ell\mathscr{N}^\bullet/U_j\mathscr{N}^\bullet)$ in $\mathscr{H}^i(U_j\mathscr{N}^\bullet)$ is contained in $\mathscr{T}_j^i$, and thus is zero. We therefore have an exact sequence for any i:

$$0 \longrightarrow \mathscr{H}^i(U_j\mathscr{N}^\bullet) \longrightarrow \mathscr{H}^i(U_\ell\mathscr{N}^\bullet) \longrightarrow \mathscr{H}^i(U_\ell\mathscr{N}^\bullet/U_j\mathscr{N}^\bullet) \longrightarrow 0.$$

Using (7.5.9), we get for any ℓ the exact sequence

$$0 \longrightarrow \mathscr{H}^i(U_{\ell-1}\mathscr{N}^\bullet) \longrightarrow \mathscr{H}^i(U_\ell\mathscr{N}^\bullet) \longrightarrow \mathscr{H}^i(\mathrm{gr}_\ell^U\mathscr{N}^\bullet) \longrightarrow 0.$$

This implies that that $\mathscr{H}^i(U_\ell\mathscr{N}^\bullet) \to \mathscr{H}^i(\mathscr{N}^\bullet)$ is injective. Put

$$U_\ell\mathscr{H}^i(\mathscr{N}^\bullet) = \mathrm{image}\left[\mathscr{H}^i(U_\ell\mathscr{N}^\bullet) \to \mathscr{H}^i(\mathscr{N}^\bullet)\right].$$

We thus have, for any $i, \ell \in \mathbb{Z}$,

$$\mathrm{gr}_\ell^U\mathscr{H}^i(\mathscr{N}^\bullet) = \mathscr{H}^i(\mathrm{gr}_\ell^U\mathscr{N}^\bullet).$$

Third step: proof of (7.5.10). Remark first that, according to 7.5.8(2), the multiplication by t induces an isomorphism $t : \widehat{U_j\mathscr{N}^\bullet} \to \widehat{U_{j-1}\mathscr{N}^\bullet}$ for $j \leqslant j_0$, and that (d) in Step one implies that, for all i and all $j \leqslant j_0$, the multiplication by t on $\mathscr{H}^i(\widehat{U_j\mathscr{N}^\bullet})$ is injective.

The proof of (7.5.10) is done by decreasing induction on i. It clearly hods for $i \geqslant i_0$ (given by 7.5.8(3)). Assume that, for any $j \leqslant j_0$, we have $\mathscr{T}_j^{i+1} = 0$. We have (after 7.5.8(2)) an exact sequence of complexes, for any $\ell \geqslant 0$,

$$0 \longrightarrow U_j\mathscr{N}^\bullet \xrightarrow{t^\ell} U_j\mathscr{N}^\bullet \longrightarrow U_j\mathscr{N}^\bullet/U_{j-\ell}\mathscr{N}^\bullet \longrightarrow 0.$$

As $\mathscr{T}_j^{i+1} = 0$, we have, for any $\ell \geqslant 0$ an exact sequence

$$\mathscr{H}^i(U_j\mathscr{N}^\bullet) \xrightarrow{t^\ell} \mathscr{H}^i(U_j\mathscr{N}^\bullet) \longrightarrow \mathscr{H}^i(U_j\mathscr{N}^\bullet/U_{j-\ell}\mathscr{N}^\bullet) \longrightarrow 0,$$

hence, according to Step one,

$$\mathscr{H}^i(\widehat{U_j\mathscr{N}^\bullet})/\mathscr{H}^i(\widehat{U_{j-\ell}\mathscr{N}^\bullet}) = \mathscr{H}^i(U_j\mathscr{N}^\bullet/U_{j-\ell}\mathscr{N}^\bullet) = \mathscr{H}^i(U_j\mathscr{N}^\bullet)/t^\ell\mathscr{H}^i(U_j\mathscr{N}^\bullet).$$

According to Exercise 7.2.8, for ℓ big enough (locally on X), the map

$$\mathscr{T}_j^i \longrightarrow \mathscr{H}^i(U_j\mathscr{N}^\bullet)/t^\ell\mathscr{H}^i(U_j\mathscr{N}^\bullet)$$

is injective. It follows that $\mathscr{T}_j^i \to \mathscr{H}^i(\widehat{U_j\mathscr{N}^\bullet})$ is injective too. But we know that t is injective on $\mathscr{H}^i(\widehat{U_j\mathscr{N}^\bullet})$ for $j \leqslant j_0$, hence $\mathscr{T}_j^i = 0$, thus concluding Step 3. ☐

We apply the proposition to $\mathscr{N}^\bullet = f_\dagger\mathscr{M}$ equipped with $U_\bullet\mathscr{N}^\bullet = f_\dagger U_\bullet\mathscr{M}$ to get 7.5.2(2). That Assumption (1) in the proposition is satisfied follows from the assumptions in 7.5.2(2). Assumption (2) is a consequence of the fact that $U_\bullet\mathscr{M}$ is a good V-filtration and Lemma 7.2.5. Last, Assumption (3) is satisfied because f has finite cohomological dimension. ☐

BIBLIOGRAPHY

[BBDG82] A.A. Beilinson, J.N. Bernstein, P. Deligne & O. Gabber – "Faisceaux pervers", in *Analyse et topologie sur les espaces singuliers*, Astérisque, vol. 100, Société Mathématique de France, Paris, 1982, ed. 2018, p. 1–180.

[Ber72] J.N. Bernstein – "The analytic continuation of generalized functions with respect to a parameter", *Funct. Anal. Appl.* **6** (1972), p. 273–285.

[Bjö79] J.-E. Björk – *Rings of differential operators*, North Holland, Amsterdam, 1979.

[Bjö93] ———, *Analytic $\mathscr{D}$-modules and applications*, Kluwer Academic Publisher, Dordrecht, 1993.

[Bor87] A. Borel (ed.) – *Algebraic $\mathscr{D}$-modules*, Perspectives in Math., vol. 2, Boston, Academic Press, 1987.

[Bou71] N. Bourbaki – *Groupes et algèbres de Lie, chapitre 1*, Diffusion C.C.L.S., Paris, 1971.

[BM02] J. Briançon & Ph. Maisonobe – "Remarques sur l 'idéal de Bernstein associé à des polynômes", prépublication n° 650 de l'Université de Nice, mai 2002.

[CJ93] F.J. Castro-Jiménez – "Exercices sur le complexe de de Rham et l'image directe des $\mathscr{D}$-modules", in *Éléments de la théorie des systèmes différentiels* [**MS93b**], p. 15–45.

[Cou95] S.C. Coutinho – *A primer of algebraic $\mathscr{D}$-modules*, London Mathematical Society Student Texts, Cambridge University Press, 1995.

[Del07] P. Deligne – "Lettre à B. Malgrange du 19/4/1978", in *Singularités irrégulières, Correspondance et documents*, Documents mathématiques, vol. 5, Société Mathématique de France, Paris, 2007, p. 25–26.

[Fri67] J. Frisch – "Points de platitude d'un espace analytique", *Invent. Math.* **4** (1967), p. 118–138.

[Gab81] O. Gabber – "The integrability of the characteristic variety", *Amer. J. Math.* **103** (1981), p. 445–468.

[GGM85] A. Galligo, J.-M. Granger & Ph. Maisonobe (eds.) – *Systèmes différentiels et singularités (Luminy, 1983)*, Astérisque, vol. 130, Paris, Société Mathématique de France, 1985.

[God64] R. Godement – *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1964.

[GM93] M. Granger & Ph. Maisonobe – "A basic course on differential modules", in *Éléments de la théorie des systèmes différentiels* [**MS93a**], p. 103–168.

[Gri70a] P.A. Griffiths – "Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping", *Publ. Math. Inst. Hautes Études Sci.* **38** (1970), p. 125–180.

[Gri70b] ______, "Periods of integrals on algebraic manifolds: summary and discussion of open problems", *Bull. Amer. Math. Soc.* **76** (1970), p. 228–296.

[Har75] R. Hartshorne – "On the de Rham cohomology of algebraic varieties", *Publ. Math. Inst. Hautes Études Sci.* **45** (1975), p. 5–99.

[Kas76] M. Kashiwara – "B-functions and holonomic systems", *Invent. Math.* **38** (1976), p. 33–53.

[Kas78] ______, "On the holonomic systems of differential equations II", *Invent. Math.* **49** (1978), p. 121–135.

[Kas86] ______, "Regular holonomic $\mathscr{D}$-modules and distributions on complex manifolds", in *Complex analytic singularities*, Adv. Stud. Pure Math., vol. 8, North-Holland, Amsterdam, 1986, p. 199–206.

[Kas95] ______, *Algebraic study of systems of partial differential equations*, Mém. Soc. Math. France (N.S.), vol. 63, Société Mathématique de France, Paris, 1995, English translation of the Master thesis, Tokyo, 1970.

[Kas03] ______, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, R.I., 2003.

[KS90] M. Kashiwara & P. Schapira – *Sheaves on Manifolds*, Grundlehren Math. Wiss., vol. 292, Springer-Verlag, Berlin, Heidelberg, 1990.

[KO68] N. Katz & T. Oda – "On the differentiation of de Rham cohomology classes with respect to a parameter", *J. Math. Kyoto Univ.* **1** (1968), p. 199–213.

[Ked10] K. KEDLAYA – "Good formal structures for flat meromorphic connections, I: surfaces", *Duke Math. J.* **154** (2010), no. 2, p. 343–418.

[Ked11] ______, "Good formal structures for flat meromorphic connections, II: excellent schemes", *J. Amer. Math. Soc.* **24** (2011), no. 1, p. 183–229.

[Lau85] G. LAUMON – "Transformation canonique et spécialisation pour les $\mathscr{D}$-modules filtrés", in *Systèmes différentiels et singularités* [**GGM85**], p. 56–129.

[MN04] PH. MAISONOBE & L. NARVÁEZ MACARRO (eds.) – *Éléments de la théorie des systèmes différentiels géométriques, Cours du C.I.M.P.A., École d'été de Séville (1996)*, Séminaires & Congrès, vol. 8, Paris, Société Mathématique de France, 2004.

[MS93a] PH. MAISONOBE & C. SABBAH (eds.) – *$\mathscr{D}$-modules cohérents et holonomes*, Les cours du CIMPA, Travaux en cours, vol. 45, Paris, Hermann, 1993.

[MS93b] ______ (eds.) – *Images directes et constructibilité*, Les cours du CIMPA, Travaux en cours, vol. 46, Paris, Hermann, 1993.

[Maj84] H. MAJIMA – *Asymptotic analysis for integrable connections with irregular singular points*, Lect. Notes in Math., vol. 1075, Springer-Verlag, 1984.

[Mal78] B. MALGRANGE – "L'involutivité des caractéristiques des systèmes différentiels et microdifférentiels", in *Séminaire Bourbaki*, Lect. Notes in Math., vol. 710, Springer-Verlag, 1978, p. Exp. n° 522.

[Mal91] ______, *Équations différentielles à coefficients polynomiaux*, Progress in Math., vol. 96, Birkhäuser, Basel, Boston, 1991.

[Mal93] ______, "De Rham Complex and Direct Images of $\mathscr{D}$-Modules", in *Éléments de la théorie des systèmes différentiels* [**MS93b**], p. 1–13.

[Mal94a] ______, "Connexions méromorphes", in *Congrès Singularités (Lille, 1991)*, Cambridge University Press, 1994, p. 251–261.

[Mal94b] ______, "Filtration des modules holonomes", in *Analyse algébrique des perturbations singulières* (L. Boutet de Monvel, ed.), Travaux en cours, vol. 48, no. 2, Hermann, Paris, 1994, p. 35–41.

[Mal96] ______, "Connexions méromorphes, II: le réseau canonique", *Invent. Math.* **124** (1996), p. 367–387.

[Meb89] Z. MEBKHOUT – *Le formalisme des six opérations de Grothendieck pour les $\mathscr{D}$-modules cohérents*, Travaux en cours, vol. 35, Hermann, Paris, 1989.

[MN93] Z. Mebkhout & L. Narváez Macarro – "Le théorème de constructibilité de Kashiwara", in *Éléments de la théorie des systèmes différentiels* [**MS93b**], p. 47–98.

[MS89] Z. Mebkhout & C. Sabbah – "§ III.4 $\mathscr{D}$-modules et cycles évanescents", in *Le formalisme des six opérations de Grothendieck pour les $\mathscr{D}$-modules cohérents* [**Meb89**], p. 201–239.

[Moc07] T. Mochizuki – *Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules*, vol. 185, Mem. Amer. Math. Soc., no. 869-870, American Mathematical Society, Providence, R.I., 2007.

[Moc09] ______, "Good formal structure for meromorphic flat connections on smooth projective surfaces", in *Algebraic Analysis and Around (Kyoto, June 2007)*, Advanced Studies in Pure Math., vol. 54, Math. Soc. Japan, Tokyo, 2009, p. 223–253, `arXiv:0803.1346`.

[Moc11a] ______, *Wild harmonic bundles and wild pure twistor D-modules*, Astérisque, vol. 340, Société Mathématique de France, Paris, 2011.

[Moc11b] ______, "Stokes structure of a good meromorphic flat bundle", *J. Inst. Math. Jussieu* **10** (2011), no. 3, p. 675–712.

[Sab93] C. Sabbah – "Équations différentielles à points singuliers irréguliers en dimension 2", *Ann. Inst. Fourier (Grenoble)* **43** (1993), p. 1619–1688.

[Sab97] ______, "Classes caractéristiques et théorèmes d'indice: point de vue microlocal", in *Séminaire Bourbaki*, Astérisque, vol. 241, Société Mathématique de France, Paris, 1997, p. 381–409.

[Sab00] ______, *Équations différentielles à points singuliers irréguliers et phénomène de Stokes en dimension* 2, Astérisque, vol. 263, Société Mathématique de France, Paris, 2000.

[Sab05] ______, *Polarizable twistor $\mathscr{D}$-modules*, Astérisque, vol. 300, Société Mathématique de France, Paris, 2005, Errata: `sabbah_ast_300_err.pdf`.

[Sab09] ______, "Wild twistor $\mathscr{D}$-modules", in *Algebraic Analysis and Around (Kyoto, June 2007)*, Advanced Studies in Pure Math., vol. 54, Math. Soc. Japan, Tokyo, 2009, p. 293–353.

[Sab13] ______, *Introduction to Stokes structures*, Lect. Notes in Math., vol. 2060, Springer-Verlag, 2013.

[MHMP] C. Sabbah & Ch. Schnell – "The MHM project", `https://perso.pages.math.cnrs.fr/users/claude.sabbah/MHMProject/mhm.html`.

[Sai88] M. Saito – "Modules de Hodge polarisables", *Publ. RIMS, Kyoto Univ.* **24** (1988), p. 849–995.

[Sai89a] ______, "Induced $\mathscr{D}$-modules and differential complexes", *Bull. Soc. math. France* **117** (1989), p. 361–387.

[Sai89b] ______, "On the structure of Brieskorn lattices", *Ann. Inst. Fourier (Grenoble)* **39** (1989), p. 27–72.

[Sai90] ______, "Mixed Hodge Modules", *Publ. RIMS, Kyoto Univ.* **26** (1990), p. 221–333.

[SKK73] M. Sato, T. Kawai & M. Kashiwara – "Microfunctions and pseudo-differential equations", in *Hyperfunctions and pseudo-differential equations (Katata, 1971)*, Lect. Notes in Math., vol. 287, Springer-Verlag, 1973, p. 265–529.

[Sch85] P. Schapira – *Microdifferential systems in the complex domain*, Grundlehren Math. Wiss., vol. 269, Springer-Verlag, 1985.

[SS94] P. Schapira & J.-P. Schneiders – *Index theorem for elliptic pairs*, Astérisque, vol. 224, Société Mathématique de France, Paris, 1994.