Irregular Hodge theory

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The Riemann existence theorem

- $P(z,\partial_z)=\sum_0^d a_k(z)\Big(\dfrac{\mathrm{d}}{\mathrm{d}z}\Big)^k,\quad a_k\in\mathbb{C}[z],\ a_d\not\equiv 0$
- $S=\{z\mid a_d(z)=0\}$ sing. set (assumed $\neq\varnothing$)
- Associated linear system

$$(*)\qquad \frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}u_1\\ \vdots\\ u_d\end{pmatrix}=A(z)\begin{pmatrix}u_1\\ \vdots\\ u_d\end{pmatrix},\qquad A(z)\in\mathrm{End}(\mathbb{C}(z)^d)$$

- $\rightsquigarrow$ ***Monodromy*** representation of the solution vectors by analytic continuation

$$\rho:\pi_1(\mathbb{C}\smallsetminus S,z_o)\longrightarrow\mathrm{GL}_d(\mathbb{C})$$

The Riemann existence theorem

- $\rho\iff(T_s\in\mathrm{GL}_d(\mathbb{C}))_{s\in S}$ (and $T_\infty:=\big(\prod_s T_s\big)^{-1}$)
- ***Conversely***, any ρ (any finite S) comes from a system $(*)$ s.t., $\forall s\in S\cup\infty$, $\exists$ ***formal*** merom. gauge transf. $\to$ ***at most simple pole*** (i.e., ***reg. sing.***):
  - $\exists\ M(z-s)\in\mathrm{GL}_d\,(\mathbb{C}(\!(z-s)\!))$ s.t.

$$(z-s)\cdot\big[M^{-1}AM+M^{-1}M_z'\big]\in\mathrm{End}(\mathbb{C}[\![z-s]\!]).$$

- Proof: Near $s\in S$, this amounts to finding $C_s\in\mathrm{End}(\mathbb{C}^d)$ s.t. $T_s=e^{-2\pi iC_s}$. Then $A(z):=C_s/(z-s)$ has monodromy T_s around s.
  Globalization: non-explicit procedure.

Rigid irreducible representations

- Assume ρ is ***irreducible***:

cannot put all T_s in a upper block-triang. form simultaneously

and ***rigid***:

if $T_s'\sim T_s\ \forall s\in S\cup\infty$, then $\rho'\sim\rho$

- and assume $\forall s\in S\cup\infty$,

$\forall\lambda$ eigenvalue of T_s, $\lvert\lambda\rvert=1$

- $\Rightarrow$ ***More structure*** on the solution to the Riemann existence th.

Variations of pol. Hodge structure

THEOREM (Deligne 1987, Simpson 1990):

$\exists!$ var. of polarized Hodge structure (wt. $=0$) adapted to ρ

- G_z: ***pos. def. Herm.*** $d\times d$ matrix, C^∞ w.r.t. $z\in\mathbb{C}\smallsetminus S$
- ***Hodge decomp.*** $\forall z\in\mathbb{C}\smallsetminus S$:

$$\mathbb{C}^d=\overset{\perp}{\bigoplus}_p H_z^p,\qquad H_z^{-p}=\overline{H_z^p}$$

- $z\mapsto H_z^p$: C^∞ & possibly ***not hol.*** but $z\mapsto F^pH_z:=\bigoplus_{p'\geqslant p}H_z^{p'}$ ***holomorphic*** and

$$\Big(\frac{\mathrm{d}}{\mathrm{d}z}+A\Big)\cdot F^pH_z\subset F^{p-1}H_z$$

- $\widetilde G_z$ s.t. $\widetilde G_{z|H^p}:=(-1)^pG_{|H^p}$, then

$$\partial_z\widetilde G_z\cdot\widetilde G_z^{-1}={}^t\!A(z).$$

Variations of pol. Hodge structure

THEOREM (Deligne 1987, Simpson 1990):

$\exists!$ var. of polarized Hodge structure (wt. $=0$) adapted to ρ

- $\Rightarrow$ Numbers $f^p=\mathrm{rk}\,F^pH_z$ attached to ρ.
- Moreover (Griffiths),

$$\mathbb{C}[z,(z-s)^{-1}_{s\in S}]^d=\mathscr{O}(\mathbb{C}\smallsetminus S)^d_{G\text{-mod. growth}}.$$

Hypergeom. differential eqns

- Given $\begin{cases}0\leqslant\alpha_1\leqslant\cdots\leqslant\alpha_d<1,\\ 0\leqslant\beta_1\leqslant\cdots\leqslant\beta_d<1,\end{cases}\quad\alpha_i\neq\beta_j\ \forall i,j.$

$$\boxed{P(z,\partial_z):=\prod_{i=1}^d\Big(z\frac{\mathrm{d}}{\mathrm{d}z}-\alpha_i\Big)-z\prod_{j=1}^d\Big(z\frac{\mathrm{d}}{\mathrm{d}z}-\beta_j\Big)}$$

$S=\{0,1\}$.

- Beukers & Heckman: ρ is ***irreducible rigid***, with $\lambda=e^{-2\pi i\alpha}$ or $e^{2\pi i\beta}$.
- Set $\ell_j=\#\{i\mid\alpha_i\leqslant\beta_j\}-j$

THEOREM (R. Fedorov, 2015):

$f^p=\#\{j\mid\ell_j\geqslant p\}$

- mixed: $F^1=0$, $F^0=\mathscr{O}(\mathbb{C}\smallsetminus S)^d\Rightarrow$ unitary conn.
- unmixed: $0=F^d\subset\cdots\subset F^0=\mathscr{O}(\mathbb{C}\smallsetminus S)^d$.

Confluent hypergeom. diff. eqns

$$\boxed{P(z,\partial_z):=\prod_{i=1}^{d'}\Big(z\frac{\mathrm{d}}{\mathrm{d}z}-\alpha_i\Big)-z\prod_{j=1}^d\Big(z\frac{\mathrm{d}}{\mathrm{d}z}-\beta_j\Big)}$$

with $d'<d\Rightarrow S=0$ and 0 is an ***irreg. sing.*** ($\infty=$ reg. sing).

- Riemann existence th. breaks down for irreg. sing.
- Need ***Stokes data*** to reconstruct the differential eqn from sols.
- $\rightsquigarrow$ Riemann-Hilbert-Birkhoff correspondence.

Confluent hypergeom. diff. eqns

$$P(z,\partial_z) := \prod_{i=1}^{d'}\Big(z\frac{\mathrm d}{\mathrm dz}-\alpha_i\Big) - z\prod_{j=1}^{d}\Big(z\frac{\mathrm d}{\mathrm dz}-\beta_j\Big)$$

with $d' < d$.

- Same condition on α,β's $\Rightarrow$ ***irreducible*** and ***rigid***:
  - ***irreducible***: Cannot split $P(z,\partial_z) = P_1(z,\partial_z)\cdot P_2(z,\partial_z)$ in $\mathbb C(z)\langle\partial_z\rangle$ with $\deg P_1, \deg P_2 \geqslant 1$.
  - ***rigid***: Any other linear diff. syst. (sings at $S\cup\infty$) which is ***gauge-equiv. over*** $\mathbb C(\!(z-s)\!)$ at each $s\in S\cup\infty$ to the given system is ***gauge-equiv. over*** $\mathbb C(z)$ to the given system.
- But: ***Cannot find*** a var. of pol. Hodge struct. s.t. the sol. to R-H-B exist. th. given by $\mathscr O(\mathbb C\smallsetminus S)^d_{G\text{-mod. growth}}$.

Harmonic metrics

Given:

- a diff. system $\frac{\mathrm d}{\mathrm dz} + A(z)$, $A(z)\in\mathrm{End}(\mathbb C(z)^d)$, pole set $= S\subset\mathbb C$.
- G_z: any pos. def. Herm. mtrx, C^∞ w.r.t. $z\in\mathbb C\smallsetminus S$.
- Then $\exists!$ A'_{G_z}, A''_{G_z} $d\times d$, C^∞ w.r.t. z, s.t.

(compatibility with G)
$$\partial_z G_z = {}^t\!A'_{G_z}\cdot G_z + G_z\cdot\overline{A''_{G_z}}$$
$$\overline\partial_z G_z = {}^t\!A''_{G_z}\cdot G_z + G_z\cdot\overline{A'_{G_z}}$$
$$\underbrace{-A''_{G_z}}_{\theta''_z} = \underbrace{(A-A'_{G_z})}_{\theta'_z}{}^*.$$

- G is ***harmonic w.r.t.*** A if

$$\boxed{\overline\partial_z\theta'_z + [\theta'_z,\theta'^*_z] = 0}$$

Harmonic metrics

THEOREM (Simpson 1990, CS 1998, Biquard-Boalch 2004, T. Mochizuki 2011):

- If A is ***irreducible***, $\exists!$ harmonic metric G w.r.t. A s.t.
  - Coefs of θ' have ***mod. growth*** at $S\cup\infty$,
  - $\mathbb C[z,((z-s)^{-1})_{s\in S}]^d = (\mathscr O(\mathbb C\smallsetminus S)^d)_{G\text{-mod. growth}}$.
- E.g., the Hodge metric of a var. pol. Hodge structure is harmonic w.r.t. the reg. sing. conn. A.
- If A is ***irreg.***, what about ***rigid*** irreducible A?
- Answer in the last slide of the talk.

The irregular Hodge filtration

Deligne (2007):

"The analogy between vector bundles with integrable connection having irregular singularities at infinity on a complex algebraic variety U and ℓ-adic sheaves with wild ramification at infinity on an algebraic variety of characteristic p, leads one to ask how such a vector bundle with integrable connection can be part of a system of realizations analogous to what furnishes a family of motives parametrized by U...

In the 'motivic' case, any de Rham cohomology group has a natural Hodge filtration. Can we hope for one on $H^i_{\mathrm{dR}}(U,\nabla)$ for some classes of (V,∇) with irregular singularities?"

The irregular Hodge filtration

"The reader may ask for the usefulness of a "Hodge filtration" not giving rise to a Hodge structure. I hope that it forces bounds to p-adic valuations of Frobenius eigenvalues. That the cohomology of '$e^{-z}z^\alpha$' ($0<\alpha<1$) has Hodge degree $1-\alpha$ is anlogous to formulas giving the p-adic valuation of Gauss sums."

The irregular Hodge filtration

- Ex.: $U=\mathbb C^*$, $f: z\mapsto -z$, $\nabla = \mathrm d + \mathrm df + \alpha\,\mathrm dz/z$

$$\begin{array}{ccccc}
\mathbb C[z,z^{-1}] & \xrightarrow{\ \nabla\ } & \mathbb C[z,z^{-1}]\cdot\frac{\mathrm dz}{z} & \longrightarrow & H^1_{\mathrm{dR}}(U,\nabla)\\
{\scriptstyle e^{-z}z^\alpha}\downarrow\wr & & \uparrow\wr\,{\scriptstyle e^{z}z^{-\alpha}} & & \uparrow\wr\\
\mathbb C[z,z^{-1}]e^{-z}z^\alpha & \xrightarrow{\ \mathrm d\ } & \mathbb C[z,z^{-1}]\cdot e^{-z}z^\alpha\frac{\mathrm dz}{z} & \longrightarrow & \mathbb C\cdot\Big[e^{-z}z^\alpha\frac{\mathrm dz}{z}\Big]
\end{array}$$

period: $\displaystyle\int_0^\infty e^{-z}z^\alpha\frac{\mathrm dz}{z} = \Gamma(\alpha)$

$$\overset{?}{\Rightarrow}\ [e^{-z}z^\alpha\mathrm dz/z]\in F^{1-\alpha}H^1_{\mathrm{dR}}(U,\nabla).$$

The Hodge filtration in dim $\geqslant 1$

- Setting:
  - U: smooth cplx quasi-proj. var. (e.g. $U=(\mathbb C^*)^n$).
  - Choose (according to Hironaka) any X such that
    - X: smooth cplx proj. variety,
    - D: reduced divisor with normal crossings in X locally, $D=\{x_1\cdots x_\ell=0\}$
    - $U = X\smallsetminus D$.
- THEOREM (Deligne 1972):

$$H^k(U,\mathbb C)\simeq H^k(X,(\Omega^\bullet_X(\log D),\mathrm d))$$

and $\forall p$, (E_1-***degeneration***)

$$H^k(X,\sigma^{\geqslant p}(\Omega^\bullet_X(\log D),\mathrm d))\longrightarrow H^k(X,(\Omega^\bullet_X(\log D),\mathrm d))$$

is ***injective***, its image defining the Hodge filtration $F^pH^k(U,\mathbb C)$.

- $\rightsquigarrow$ Mixed Hodge structure on $H^k(U,\mathbb C)$.

Twisted de Rham cohomology

- Setting:
  - U: smth cplx quasi-proj. var., $f: U\to\mathbb C$ alg. fnct.
- ***Twisted de Rham cohomology*** $H^k_{\mathrm{dR}}(U,\mathrm d+\mathrm df)$: Cohomology of the alg. de Rham cplx. E.g. $U=\mathbb C^n$:

$$0\to\mathbb C[x]\to\bigoplus_i\mathbb C[x]\mathrm dx_i\to\cdots\to\bigoplus_i\mathbb C[x]\widehat{\mathrm dx_i}\to\mathbb C[x]\mathrm dx\to 0$$

$$g(x)\longmapsto\textstyle\sum_i(g'_{x_i}+gf'_{x_i})\mathrm dx_i$$

$$\textstyle\sum_i h_i\widehat{\mathrm dx_i}\longmapsto\Big[\sum_i(-1)^{i-1}((h_i)'_{x_i}+h_if'_{x_i})\Big]\mathrm dx$$

Good compactification

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- Choose (according to Hironaka) any X such that
  - X: smooth cplx proj. variety,
  - D: reduced divisor with normal crossings in X locally, $D = \{x_1 \cdots x_\ell = 0\}$
  - $U = X \smallsetminus D$.
  - s.t. f extends as an hol. map

$$f : X \longrightarrow \mathbb{P}^1 = \mathbb{C} \cup \infty, \quad f^{-1}(\infty) \subset D. \quad P := f^*(\infty).$$

The Kontsevich complex

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- For $\alpha \in [0,1) \cap \mathbb{Q}$,
  - $\Omega_X^k(\log D)([\alpha P])$: forms with log pole along $D - P$ and pole at most "$\log + [\alpha P]$" along $f^{-1}(\infty)$. (e.g. $\mathrm{d}f = f \cdot \mathrm{d}f/f \in \Omega_X^1(\log D)(P)$.)
  - Define $\Omega_f^k(\alpha)$ as

$$\left\{\omega \in \Omega_X^k(\log D)([\alpha P]) \mid \mathrm{d}f \wedge \omega \in \Omega_X^{k+1}(\log D)([\alpha P])\right\}$$

- Significant α's: ℓ/m, $m =$ mult. of a component of P, $\ell = 0, \ldots, m-1$.
- $\rightsquigarrow$ Kontsevich complex $(\Omega_f^\bullet(\alpha), \mathrm{d} + \mathrm{d}f)$.
- $\boxed{H^k(X, (\Omega_f^\bullet(\alpha), \mathrm{d} + \mathrm{d}f)) \simeq H_{\mathrm{dR}}^k(U, \mathrm{d} + \mathrm{d}f)}$

The irreg. Hodge filtration in dim $\geqslant 1$

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THEOREM (Kontsevich, Esnault-CS-Yu 2014, M. Saito 2014, T. Mochizuki 2015):

- $\forall p$, (E_1*-degeneration*)

$$H^k(X, \sigma^{\geqslant p}(\Omega_f^\bullet(\alpha), \mathrm{d}+\mathrm{d}f)) \longrightarrow H^k(X, (\Omega_f^\bullet(\alpha), \mathrm{d}+\mathrm{d}f))$$

  is ***injective***, its image defining the irregular Hodge filtration $F^{p-\alpha} H_{\mathrm{dR}}^k(U, \mathrm{d} + \mathrm{d}f)$.
- $\lambda \geqslant \mu \in \mathbb{Q} \Rightarrow$

$$\boxed{F^\lambda H_{\mathrm{dR}}^k(U, \mathrm{d} + \mathrm{d}f) \subset F^\mu H_{\mathrm{dR}}^k(U, \mathrm{d} + \mathrm{d}f)}$$

- Jumps at most at $\lambda = \ell/m + p$, $p \in \mathbb{Z}$, $\ell = 0, \ldots, m-1$, $m =$ mult. component of P.

History of the result, dim. one

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- Deligne (1984, IHÉS seminar notes).

$A \in \mathrm{GL}_d(\mathbb{C}(z))$ with ***reg sing***. on $S \cup \infty$, and ***unitary***. $f \in \mathbb{C}(z)$. Defines a filtr. ($\lambda \in \mathbb{R}$)

$$F^\lambda \mathbb{C}[z, (z-s)_{s \in S}]^d \xrightarrow{\mathrm{d} + A + \mathrm{d}f} F^{\lambda - 1} \mathbb{C}[z, (z-s)_{s \in S}]^d \mathrm{d}z$$

an proves E_1-degeneration.

- Deligne (2006). Adds more explanations and publication in the volume "Correspondance Deligne-Malgrange-Ramis" (SMF 2007).
- CS (2008). Same as Deligne, with A underlying a ***pol. var. of Hodge structure***. Uses harmonic metrics through the theory of var. of twistor structures (Simpson, Mochizuki, CS).

History of the result, dim > 1

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- J.-D. Yu (2012): defines $F^\lambda H_{\mathrm{dR}}^k(U, \mathrm{d} + \mathrm{d}f)$ + many properties and E_1-degeneration in some cases.
- Esnault-CS-Yu (2013): E_1-degeneration by reducing to (CS, 2008) (push-forward by f).
- Kontsevich (2012), letters to Katzarkov and Pantev, arXiv 2014: defines the Kontsevich complex and proves E_1-degeneration if $P = P_{\mathrm{red}}$, by the method of Deligne & Illusie (reduction to char. p). ***Does not extend if*** $P \neq P_{\mathrm{red}}$. Motivated by mirror symmetry of Fano manifolds.
- M. Saito (2013): E_1-degeneration by comparing with limit mixed Hodge structure of f at ∞.
- T. Mochizuki (2015): E_1-degeneration by using the theory of mixed twistor $\mathscr{D}$-modules.

Rigid irreducible diff. eqns

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- Given diff. operator $\dfrac{\mathrm{d}}{\mathrm{d}z} + A(z)$, $A(z) \in \mathrm{End}(\mathbb{C}(z)^d)$, pole set $= S \subset \mathbb{C}$.
- Assume it is ***irreducible*** and ***rigid***.
- Assume eigenvalues λ of $\widehat{T}_s$ ($s \in S \cup \infty$) s.t. $|\lambda| = 1$.

THEOREM (CS 2015): $\exists$ ***canonical*** filtration

$$F^\lambda \mathbb{C}[z, ((z-s)^{-1})_{s \in S}]^d \quad (\lambda \in \mathbb{R})$$

by free $\mathbb{C}[z, ((z-s)^{-1})_{s \in S}]$-modules attached to $A(z)$, s.t.

$$\Big(\frac{\mathrm{d}}{\mathrm{d}z} + A(z)\Big) F^\lambda \subset F^{\lambda - 1}.$$

Rigid irreducible diff. eqns

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- Needs the construction of a category of ***Irregular mixed Hodge modules*** between the category of mixed Hodge modules (M. Saito) and that of mixed twistor $\mathscr{D}$-modules (T. Mochizuki). Use of the Arinkin-Deligne's algorithm similar to Katz' algorithm.
- QUESTION: For confluent hypergeom. eqns, how to compute the ***jumping indices*** and the ***rank*** of the Hodge bundles?
- Recent work of Castaño Dominguez and Sevenheck on some confluent hypergeometric diff. eqns.
- Other interesting examples: rigid irregular connections of Gross-Frenkel.