Irregular Hodge theory

Claude Sabbah

Centre de Mathématiques Laurent Schwartz
École polytechnique, CNRS, Université Paris-Saclay
Palaiseau, France

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The Riemann existence theorem

- $P(z, \partial_z) = \sum_0^d a_k(z) \Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^k, \quad a_k \in \mathbb{C}[z],\ a_d \not\equiv 0$
- $S = \{z \mid a_d(z) = 0\}$ sing. set (assumed $\neq \varnothing$)
- Associated linear system

$$(*) \qquad \frac{\mathrm{d}}{\mathrm{d}z} \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} = A(z) \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix}, \quad A(z) \in \mathrm{End}(\mathbb{C}(z)^d)$$

- $\rightsquigarrow$ ***Monodromy*** representation of the solution vectors by analytic continuation

$$\rho : \pi_1(\mathbb{C} \smallsetminus S, z_o) \longrightarrow \mathrm{GL}_d(\mathbb{C})$$

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- ***Conversely***, any ρ (any finite S) comes from a system $(*)$ s.t., $\forall s \in S \cup \infty$, $\exists$ ***formal*** merom. gauge transf. $\longrightarrow$ ***at most simple pole*** (i.e., ***reg. sing.***):
  - $\exists\, M(z-s) \in \mathrm{GL}_d\big(\mathbb{C}(\!(z-s)\!)\big)$ s.t.

$$(z-s)\cdot\big[M^{-1}AM + M^{-1}M'_z\big] \in \mathrm{End}(\mathbb{C}[\![z-s]\!]).$$

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- Proof: Near $s \in S$, this amounts to finding $C_s \in \mathrm{End}(\mathbb{C}^d)$ s.t. $T_s = e^{-2\pi i C_s}$. Then $A(z) := C_s/(z-s)$ has monodromy T_s around s.
  Globalization: non-explicit procedure.

Rigid irreducible representations

- Assume ρ is ***irreducible***:

cannot put all T_s in a upper block-triang. form simultaneously

and ***rigid***:

if $T_s' \sim T_s \ \forall s \in S \cup \infty$, then $\rho' \sim \rho$

- and assume $\forall s \in S \cup \infty$,

$\forall \lambda$ eigenvalue of T_s, $|\lambda| = 1$

- $\Rightarrow$ ***More structure*** on the solution to the Riemann existence th.

Variations of pol. Hodge structure

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- $z \mapsto H_z^p$: C^∞ & possibly ***not hol.*** but
  $z \mapsto F^p H_z := \bigoplus_{p' \geqslant p} H_z^{p'}$ ***holomorphic*** and

$$\Big(\frac{\mathrm{d}}{\mathrm{d}z} + A\Big) \cdot F^p H_z \subset F^{p-1} H_z$$

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- $\widetilde{G}_z$ s.t. $\widetilde{G}_{z|H^p} := (-1)^p G_{|H^p}$, then

$$\partial_z \widetilde{G}_z \cdot \widetilde{G}_z^{-1} = {}^t\!A(z).$$

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- $\Rightarrow$ Numbers $f^p = \mathrm{rk}\, F^p H_z$ attached to ρ.
- Moreover (Griffiths),

$$\mathbb{C}[z, (z-s)^{-1}_{s \in S}]^d = \mathscr{O}(\mathbb{C} \smallsetminus S)^d_{G\text{-mod. growth}}.$$

Hypergeom. differential eqns

- Given $\begin{cases} 0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_d < 1, \\ 0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_d < 1, \end{cases}$ $\alpha_i \neq \beta_j \; \forall i, j$.

$$P(z, \partial_z) := \prod_{i=1}^{d} \Big(z\frac{\mathrm{d}}{\mathrm{d}z} - \alpha_i\Big) - z \prod_{j=1}^{d} \Big(z\frac{\mathrm{d}}{\mathrm{d}z} - \beta_j\Big)$$

  $S = \{0, 1\}$.

- Beukers & Heckman: ρ is ***irreducible rigid***, with $\lambda = e^{-2\pi \mathrm{i}\alpha}$ or $e^{2\pi \mathrm{i}\beta}$.

- Set $\ell_j = \#\{i \mid \alpha_i \leqslant \beta_j\} - j$

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  - $\alpha_1 \leqslant \beta_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \beta_d \Rightarrow \ell_j = 0 \ \forall j$
  - $\alpha_d \leqslant \beta_1 \Rightarrow \ell_j = d - j$.

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- mixed: $F^1 = 0$, $F^0 = \mathscr{O}(\mathbb{C} \smallsetminus S)^d \Rightarrow$ unitary conn.
- unmixed: $0 = F^d \subset \cdots \subset F^0 = \mathscr{O}(\mathbb{C} \smallsetminus S)^d$.

Confluent hypergeom. diff. eqns

$$P(z,\partial_z) := \prod_{i=1}^{d'}\left(z\frac{\mathrm{d}}{\mathrm{d}z} - \alpha_i\right) - z\prod_{j=1}^{d}\left(z\frac{\mathrm{d}}{\mathrm{d}z} - \beta_j\right)$$

with $d' < d \Rightarrow S = 0$ and 0 is an ***irreg. sing.***
($\infty =$ reg. sing).

- Riemann existence th. breaks down for irreg. sing.
- Need ***Stokes data*** to reconstruct the differential eqn from sols.
- $\rightsquigarrow$ Riemann-Hilbert-Birkhoff correspondence.

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- Same condition on α, β's $\Rightarrow$ ***irreducible*** and ***rigid***:
  - ***irreducible***: Cannot split $P(z,\partial_z) = P_1(z,\partial_z)\cdot P_2(z,\partial_z)$ in $\mathbb{C}(z)\langle\partial_z\rangle$ with $\deg P_1, \deg P_2 \geqslant 1$.
  - ***rigid***: Any other linear diff. syst. (sings at $S\cup\infty$) which is ***gauge-equiv. over*** $\mathbb{C}((z-s))$ at each $s\in S\cup\infty$ to the given system is ***gauge-equiv. over*** $\mathbb{C}(z)$ to the given system.

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- But: ***Cannot find*** a var. of pol. Hodge struct. s.t. the sol. to R-H-B exist. th. given by $\mathscr{O}(\mathbb{C}\smallsetminus S)^d_{G\text{-mod. growth}}$.

Harmonic metrics

Given:

- a diff. system $\frac{\mathrm{d}}{\mathrm{d}z} + A(z)$, $A(z) \in \mathrm{End}(\mathbb{C}(z)^d)$, pole set $= S \subset \mathbb{C}$.
- G_z: any pos. def. Herm. mtrx, C^∞ w.r.t. $z \in \mathbb{C} \smallsetminus S$.
- Then $\exists!$ A'_{G_z}, A''_{G_z} $d \times d$, C^∞ w.r.t. z, s.t.

(compatibility with G)

$$\partial_z G_z = {}^t\!A'_{G_z} \cdot G_z + G_z \cdot \overline{A''_{G_z}}$$

$$\overline{\partial}_z G_z = {}^t\!A''_{G_z} \cdot G_z + G_z \cdot \overline{A'_{G_z}}$$

$$\underbrace{-A''_{G_z}}_{\theta''_z} = (\underbrace{A - A'_{G_z}}_{\theta'_z})^*.$$

- G is ***harmonic w.r.t.*** A if

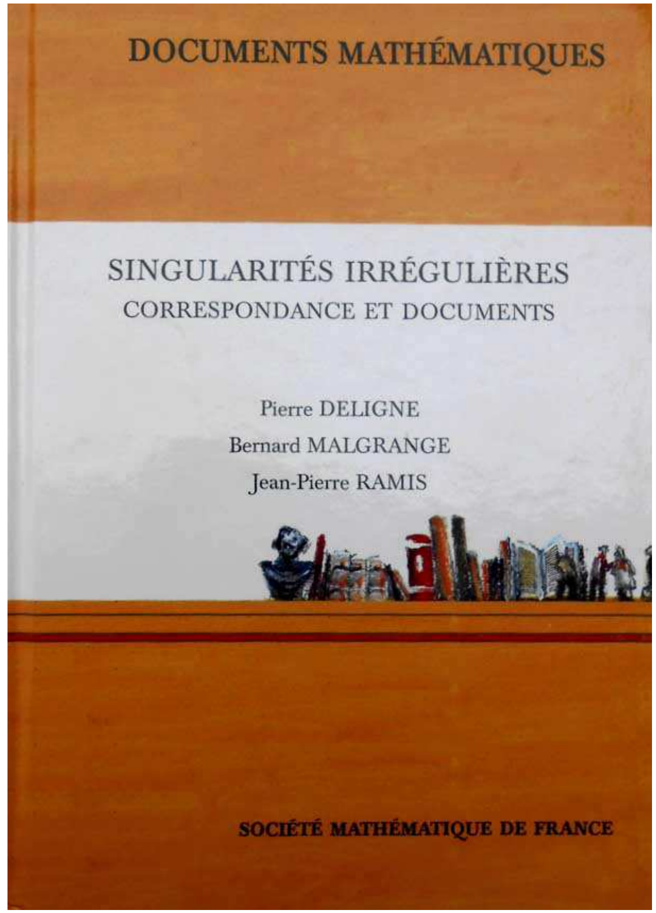
$$\boxed{\overline{\partial}_z \theta'_z + [\theta'_z, \theta'^*_z] = 0}$$

Harmonic metrics

THEOREM (Simpson 1990, CS 1998, Biquard-Boalch 2004, T. Mochizuki 2011):

- If A is ***irreducible***, $\exists!$ harmonic metric G w.r.t. A s.t.
  - Coefs of $\operatorname{Char}\theta'$ have ***mod. growth*** at $S \cup \infty$,
  - $\mathbb{C}[z, ((z-s)^{-1})_{s\in S}]^d = (\mathscr{O}(\mathbb{C} \smallsetminus S)^d)_{G\text{-mod. growth}}$.
- E.g., the Hodge metric of a var. pol. Hodge structure is harmonic w.r.t. the reg. sing. conn. A.
- If A is ***irreg.***, what about ***rigid*** irreducible A?
- Answer in the last slide of the talk.

The irregular Hodge filtration



The irregular Hodge filtration

Deligne (2007):

“The analogy between vector bundles with integrable connection having irregular singularities at infinity on a complex algebraic variety U and ℓ-adic sheaves with wild ramification at infinity on an algebraic variety of characteristic p, leads one to ask how such a vector bundle with integrable connection can be part of a system of realizations analogous to what furnishes a family of motives parametrized by U...

In the ‘motivic’ case, any de Rham cohomology group has a natural Hodge filtration. Can we hope for one on $H^i_{\mathrm{dR}}(U, \nabla)$ for some classes of (V, ∇) with irregular singularities?”

The irregular Hodge filtration

“The reader may ask for the usefulness of a “Hodge filtration” not giving rise to a Hodge structure. I hope that it forces bounds to p-adic valuations of Frobenius eigenvalues. That the cohomology of ‘$e^{-z}z^{\alpha}$’ ($0 < \alpha < 1$) has Hodge degree $1 - \alpha$ is anlogous to formulas giving the p-adic valuation of Gauss sums.”

The irregular Hodge filtration

- Ex.: $U = \mathbb{C}^*$, $f : z \mapsto -z$, $\nabla = \mathrm{d} + \mathrm{d}f + \alpha \mathrm{d}z/z$

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$$\begin{array}{ccccc}
\mathbb{C}[z,z^{-1}] & \xrightarrow{\nabla} & \mathbb{C}[z,z^{-1}]\cdot\dfrac{\mathrm{d}z}{z} & \longrightarrow & H^1_{\mathrm{dR}}(U,\nabla) \\
{\scriptstyle e^{-z}z^{\alpha}}\big\downarrow\wr & & \wr\big\uparrow{\scriptstyle e^{z}z^{-\alpha}} & & \wr\big\uparrow \\
\mathbb{C}[z,z^{-1}]e^{-z}z^{\alpha} & \xrightarrow{\mathrm{d}} & \mathbb{C}[z,z^{-1}]\cdot e^{-z}z^{\alpha}\dfrac{\mathrm{d}z}{z} & \longrightarrow & \mathbb{C}\cdot\left[e^{-z}z^{\alpha}\dfrac{\mathrm{d}z}{z}\right]
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e^{-z}z^{\alpha}\Big\downarrow\wr & & \wr\Big\uparrow e^{z}z^{-\alpha} & & \wr\Big\uparrow \\
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\end{array}$$

period: $\displaystyle\int_0^\infty e^{-z}z^{\alpha}\frac{\mathrm{d}z}{z} = \Gamma(\alpha)$

$$\overset{?}{\Longrightarrow} [e^{-z}z^{\alpha}\mathrm{d}z/z] \in F^{1-\alpha}H^1_{\mathrm{dR}}(U,\nabla).$$

The Hodge filtration in dim $\geqslant 1$

- Setting:
  - U: smooth cplx quasi-proj. var. (e.g. $U = (\mathbb{C}^*)^n$).
  - Choose (according to Hironaka) any X such that
    - X: smooth cplx proj. variety,
    - D: reduced divisor with normal crossings in X locally, $D = \{x_1 \cdots x_\ell = 0\}$
    - $U = X \smallsetminus D$.

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$$H^k(U, \mathbb{C}) \simeq H^k\big(X, (\Omega_X^\bullet(\log D), \mathrm{d})\big)$$

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$$\left[\begin{array}{l} \Omega_X^1(\log D) \overset{\text{loc.}}{=} \sum_{i=1}^{\ell} \mathscr{O}_X \dfrac{\mathrm{d}x_i}{x_i} + \sum_{j>\ell} \mathscr{O}_X \, \mathrm{d}x_j, \\ \Omega_X^k(\log D) = \overset{k}{\wedge} \Omega_X^1(\log D) \end{array}\right]$$

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and $\forall p$, (E_1***-degeneration***)

$$H^k\big(X, \sigma^{\geqslant p}(\Omega_X^\bullet(\log D), \mathrm{d})\big) \longrightarrow H^k\big(X, (\Omega_X^\bullet(\log D), \mathrm{d})\big)$$

is ***injective***, its image defining the Hodge filtration $F^p H^k(U, \mathbb{C})$.

- $\rightsquigarrow$ Mixed Hodge structure on $H^k(U, \mathbb{C})$.

Twisted de Rham cohomology

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- ***Twisted de Rham cohomology*** $H^k_{\mathrm{dR}}(U, \mathrm{d} + \mathrm{d}f)$: Cohomology of the alg. de Rham cplx. E.g. U affine:

$$0 \longrightarrow \mathscr{O}(U) \xrightarrow{\mathrm{d} + \mathrm{d}f} \cdots \xrightarrow{\mathrm{d} + \mathrm{d}f} \Omega^n(U) \longrightarrow 0$$

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$$g(x) \longmapsto \textstyle\sum_i (g'_{x_i} + g f'_{x_i})\mathrm{d}x_i$$

$$\textstyle\sum_i h_i \mathrm{d}\widehat{x_i} \longmapsto \Big[\sum_i (-1)^{i-1}((h_i)'_{x_i} + h_i f'_{x_i})\Big]\mathrm{d}x$$

Good compactification

- Choose (according to Hironaka) any X such that
  - X: smooth cplx proj. variety,
  - D: reduced divisor with normal crossings in X locally, $D = \{x_1 \cdots x_\ell = 0\}$
  - $U = X \smallsetminus D$.
  - s.t. f extends as an hol. map

$$f : X \longrightarrow \mathbb{P}^1 = \mathbb{C} \cup \infty, \quad f^{-1}(\infty) \subset D. \quad P := f^*(\infty).$$

The Kontsevich complex

- For $\alpha \in [0,1) \cap \mathbb{Q}$,
  - $\Omega^k_X(\log D)([\alpha P])$: forms with log pole along $D - P$ and pole at most "$\log + [\alpha P]$" along $f^{-1}(\infty)$. (e.g. $\mathrm{d}f = f \cdot \mathrm{d}f/f \in \Omega^1_X(\log D)(P)$.)
  - Define $\Omega^k_f(\alpha)$ as

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$$\Big\{\omega \in \Omega^k_X(\log D)([\alpha P]) \mid \mathrm{d}f \wedge \omega \in \Omega^{k+1}_X(\log D)([\alpha P])\Big\}$$

- Significant α's: ℓ/m, $\quad m =$ mult. of a component of P, $\quad \ell = 0, \dots, m-1$.
- $\rightsquigarrow$ Kontsevich complex $(\Omega^\bullet_f(\alpha), \mathrm{d} + \mathrm{d}f)$.
- $$\boxed{H^k\big(X, (\Omega^\bullet_f(\alpha), \mathrm{d} + \mathrm{d}f)\big) \simeq H^k_{\mathrm{dR}}(U, \mathrm{d} + \mathrm{d}f)}$$

The irreg. Hodge filtration in dim $\geqslant 1$

THEOREM (Kontsevich, Esnault-CS-Yu 2014, M. Saito 2014, T. Mochizuki 2015):

- $\forall p$, (E_1-***degeneration***)

$$H^k\big(X, \sigma^{\geqslant p}(\Omega_f^\bullet(\alpha), \mathrm{d}+\mathrm{d}f)\big) \longrightarrow H^k\big(X, (\Omega_f^\bullet(\alpha), \mathrm{d}+\mathrm{d}f)\big)$$

  is ***injective***, its image defining the irregular Hodge filtration $F^{p-\alpha}H^k_{\mathrm{dR}}(U, \mathrm{d}+\mathrm{d}f)$.

- $\lambda \geqslant \mu \in \mathbb{Q} \Rightarrow$

$$\boxed{F^\lambda H^k_{\mathrm{dR}}(U, \mathrm{d}+\mathrm{d}f) \subset F^\mu H^k_{\mathrm{dR}}(U, \mathrm{d}+\mathrm{d}f)}$$

- Jumps at most at $\lambda = \ell/m + p$, $p \in \mathbb{Z}$, $\ell = 0, \ldots, m-1$, $m =$ mult. component of P.

History of the result, dim. one

- Deligne (1984, IHÉS seminar notes).

$A \in \mathrm{GL}_d(\mathbb{C}(z))$ with ***reg sing***. on $S \cup \infty$, and ***unitary***.
$f \in \mathbb{C}(z)$. Defines a filtr. ($\lambda \in \mathbb{R}$)

$$F^{\lambda}\mathbb{C}[z,(z-s)_{s\in S}]^d \xrightarrow{\mathrm{d}+A+\mathrm{d}f} F^{\lambda-1}\mathbb{C}[z,(z-s)_{s\in S}]^d\mathrm{d}z$$

an proves E_1-degeneration.

- Deligne (2006). Adds more explanations and publication in the volume “Correspondance Deligne-Malgrange-Ramis” (SMF 2007).
- CS (2008). Same as Deligne, with A underlying a ***pol. var. of Hodge structure***. Uses harmonic metrics through the theory of var. of twistor structures (Simpson, Mochizuki, CS).

History of the result, dim > 1

- J.-D. Yu (2012): defines $F^\lambda H^k_{\mathrm{dR}}(U, \mathrm{d} + \mathrm{d}f)$ + many properties and E_1-degeneration in some cases.
- Esnault-CS-Yu (2013): E_1-degeneration by reducing to (CS, 2008) (push-forward by f).
- Kontsevich (2012), letters to Katzarkov and Pantev, arXiv 2014: defines the Kontsevich complex and proves E_1-degeneration if $P = P_{\mathrm{red}}$, by the method of Deligne & Illusie (reduction to char. p). ***Does not extend if $P \neq P_{\mathrm{red}}$***. Motivated by mirror symmetry of Fano manifolds.
- M. Saito (2013): E_1-degeneration by comparing with limit mixed Hodge structure of f at ∞.
- T. Mochizuki (2015): E_1-degeneration by using the theory of mixed twistor $\mathscr{D}$-modules.

Rigid irreducible diff. eqns

- Given diff. operator $\dfrac{\mathrm{d}}{\mathrm{d}z} + A(z)$, $A(z) \in \mathrm{End}(\mathbb{C}(z)^d)$, pole set $= S \subset \mathbb{C}$.
- Assume it is ***irreducible*** and ***rigid***.
- Assume eigenvalues λ of $\widehat{T}_s$ ($s \in S \cup \infty$) s.t. $|\lambda| = 1$.

THEOREM (CS 2015): $\exists$ ***canonical*** filtration

$$F^\lambda \mathbb{C}[z, ((z-s)^{-1})_{s\in S}]^d \quad (\lambda \in \mathbb{R})$$

by free $\mathbb{C}[z, ((z-s)^{-1})_{s\in S}]$-modules attached to $A(z)$, s.t.

$$\Big(\frac{\mathrm{d}}{\mathrm{d}z} + A(z)\Big) F^\lambda \subset F^{\lambda-1}.$$

Rigid irreducible diff. eqns

- Needs the construction of a category of ***Irregular mixed Hodge modules*** between the category of mixed Hodge modules (M. Saito) and that of mixed twistor $\mathscr{D}$-modules (T. Mochizuki). Use of the Arinkin-Deligne's algorithm similar to Katz' algorithm.
- QUESTION: For confluent hypergeom. eqns, how to compute the ***jumping indices*** and the ***rank*** of the Hodge bundles?
- Recent work of Castaño Dominguez and Sevenheck on some confluent hypergeometric diff. eqns.
- Other interesting examples: rigid irregular connections of Gross-Frenkel.