A panorama on irregular singularities

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A panorama on singular varieties:

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Setting for this talk

- X: cplx manifold, D: reduced divisor in X.
- V: vector bdle on X, $\nabla : V \to \Omega^1_X(*D) \otimes V$ ***integrable*** meromorphic connection.
- Loc. coordinates $x_1, \ldots, x_n$,
  - $\nabla = \mathrm{d} + \sum_{i=1}^n A_i(x)\,\mathrm{d}x_i, \quad A_i \in \mathrm{M}_n(\mathcal{O}_X(*D))$,

$$\boxed{\frac{\partial A_i(x)}{\partial x_j} - \frac{\partial A_j(x)}{\partial x_i} = [A_i(x), A_j(x)].}$$

- Enough to consider $\mathcal{V} := \mathcal{O}_X(*D) \otimes_{\mathcal{O}_X} V$: ***meromorphic vect. bdle***.

Basic examples

- $\mathcal{V} = \mathcal{O}_X(*D)$: trivial merom. bdle of rank one.
- φ: merom. fnct. with poles in D.
- ***Case 1:*** $\nabla = \boxed{\mathrm{d} + \alpha \dfrac{\mathrm{d}\varphi}{\varphi}}$, $\alpha \in \mathbb{C}^*$.
  - $\nabla v = 0 \Longrightarrow v = \varphi^{-\alpha}$, multi-valued hol. fnct ***with moderate growth all around*** D
  - $\Rightarrow$ ***regular singularity*** along D.

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- ***Case 1:*** $\nabla = \boxed{\mathrm{d} + \alpha \dfrac{\mathrm{d}\varphi}{\varphi}}, \quad \alpha \in \mathbb{C}^*$.
  - $\nabla v = 0 \Longrightarrow v = \varphi^{-\alpha}$, multi-valued hol. fnct ***with moderate growth all around*** D
  - $\Rightarrow$ ***regular singularity*** along D.
- ***Case 2:*** $\nabla = \boxed{\mathrm{d} + \alpha\, \mathrm{d}\varphi}, \quad \alpha \in \mathbb{C}^*$.
  - $\nabla v = 0 \Longrightarrow v = e^{-\alpha\varphi}$, hol. fnct. ***with moderate growth only on the semi-analytic domain***

$$\boxed{\mathrm{Re}(\alpha\varphi) > 0 \quad \text{in } X \smallsetminus D.}$$

  - $\Rightarrow$ ***irrregular singularity*** along D.

Reg. versus irreg. singularities

$(\mathcal{V}, \nabla)$: any merom. bdle with integrable connection.

- ∇ has ***reg. sing.*** along D if all local vectors v s. t. $\nabla v = 0$ have moderate growth all around D.
- ∇ has ***irreg. sing.*** along D ***otherwise***.

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  - To describe the subanalytic domains around D on which some components of v have moderate growth.
  - To describe the change that occurs when one moves from a subanalytic domain to another one (***Stokes phenomenon***, ***Wall crossing formulas***).

Case $\dim X = 1$

- $\mathcal{V} = \mathbb{C}(\{z\})^r$, $\widehat{\mathcal{V}} = \mathbb{C}((z)) \otimes \mathcal{V}$,
- $\nabla = \mathrm{d} + A(z)\mathrm{d}z$, $A \in \mathrm{End}(\mathbb{C}(\{z\})^r)$.

THEOREM (Levelt-Turrittin): Replace z with $\zeta = z^{1/q}$, then $\exists \widehat{M} \in \mathrm{GL}_r(\mathbb{C}((\zeta)))$,

$$\widehat{B}(\zeta)\,\mathrm{d}\zeta := \left(\widehat{M}^{-1}A(\zeta^q)\widehat{M} + \widehat{M}^{-1}\widehat{M}'_\zeta\right)\mathrm{d}\zeta$$

$$= \begin{pmatrix} \boxed{\mathrm{d}\varphi_1(\zeta)\,\mathrm{Id}_1 + C_1\dfrac{\mathrm{d}\zeta}{\zeta}} & & \\ & \ddots & \\ & & \boxed{\mathrm{d}\varphi_k(\zeta)\,\mathrm{Id}_k + C_k\dfrac{\mathrm{d}\zeta}{\zeta}} \end{pmatrix}$$

$\varphi_j \in \mathbb{C}[1/\zeta]$, $C_i =$ cst matrix.

Case $\dim X = 1$

- Basic invariant: ***Irreg. number.***
- $X = \mathbb{C}$, $D = \{0\}$, $j : X \setminus D \hookrightarrow X$,
- $(\mathcal{V}^{\mathrm{ess}}, \nabla) := (j_* j^* \mathcal{V}/\mathcal{V}, \nabla)$,
- $\mathrm{irr}(\mathcal{V}, \nabla, 0) := \chi\big(\mathcal{V}^{\mathrm{ess}} \xrightarrow{\nabla} \mathcal{V}^{\mathrm{ess}}\big) = \dim(\mathcal{V}^{\mathrm{ess}})^{\nabla}$.

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THEOREM: $\mathrm{irr}(\mathcal{V}, \nabla, 0)$ only depends on $(\widehat{\mathcal{V}}, \nabla)$, and more precisely,

$$\mathrm{irr}(\mathcal{V}, \nabla, 0) = \sum_{j=1}^{k} r_j p_j, \quad \begin{cases} p_j := \text{pole ord. of } \varphi_j, \\ r_j := \text{size of } C_j. \end{cases}$$

Case $\dim X = 1$

- Asympt. anal.: $\nabla v = 0 \Rightarrow v = \begin{pmatrix} e^{-\varphi_1} \cdot \text{mod. growth} \\ \vdots \\ e^{-\varphi_k} \cdot \text{mod. growth} \end{pmatrix}$
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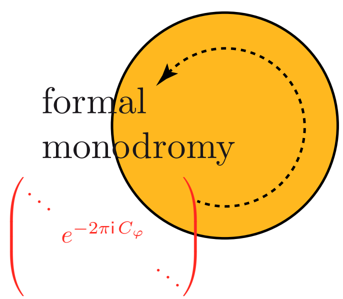

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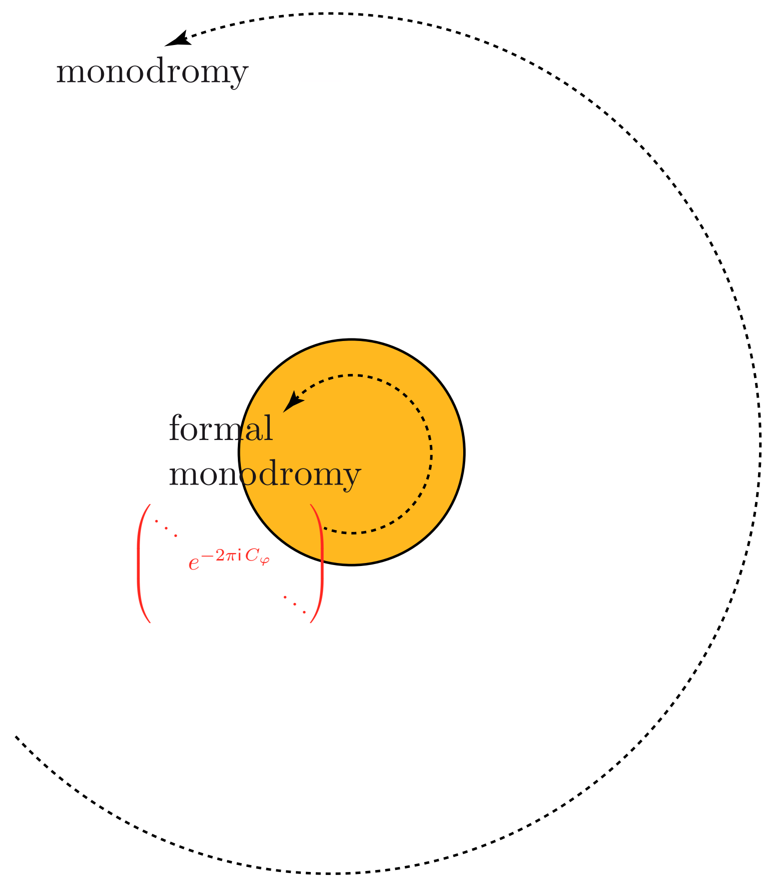

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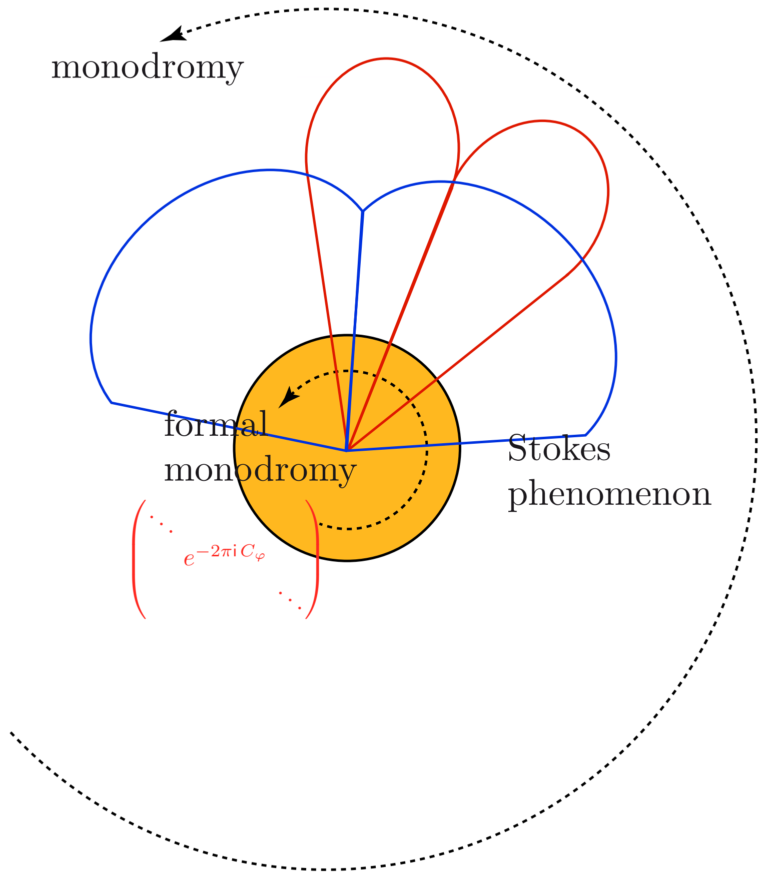

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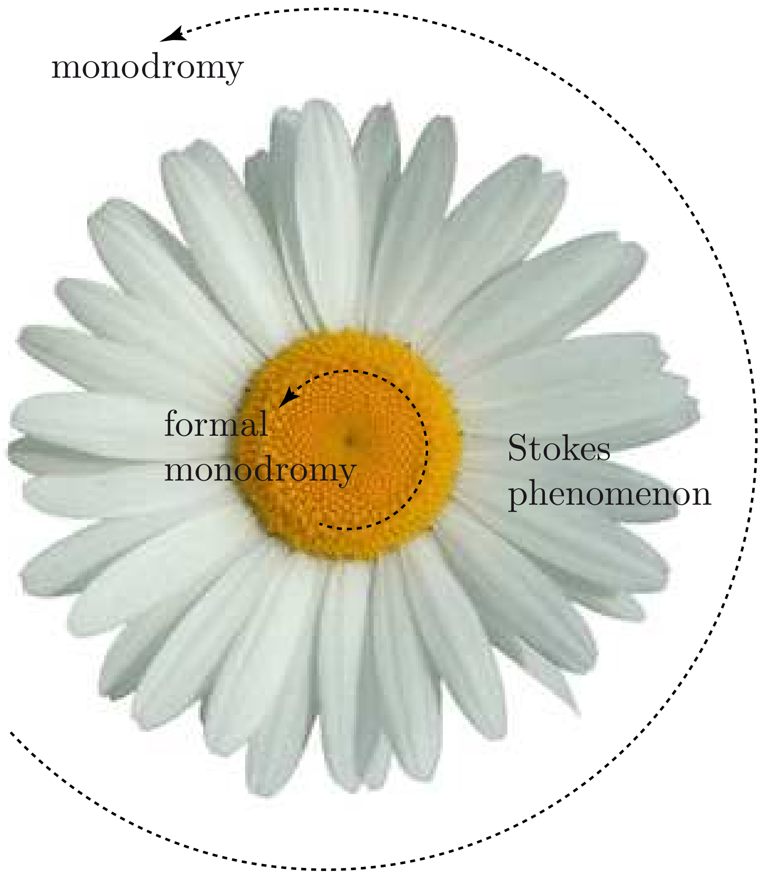

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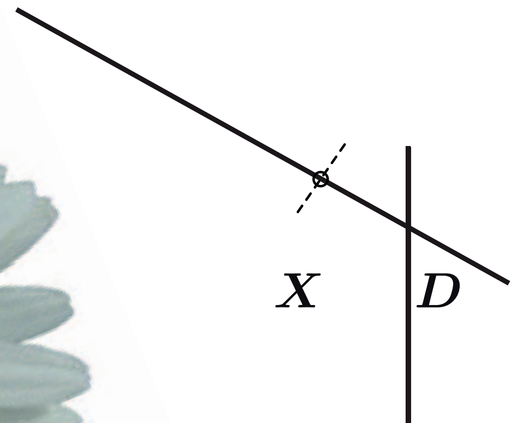

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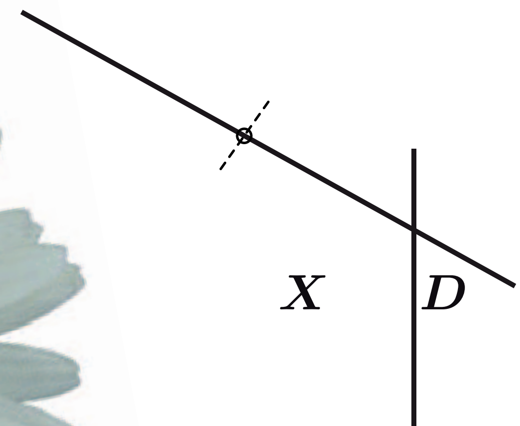

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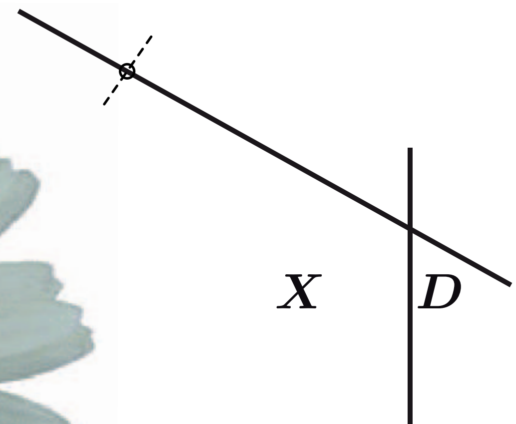

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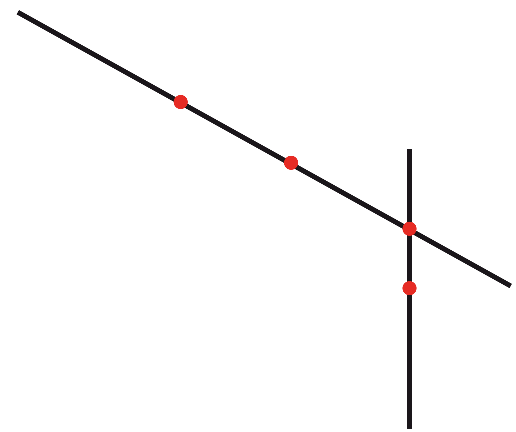

THEOREM (André 2007): The red points where the transverse structure changes drastically (and does not only rotate) form a ***closed cplx analytic subset***.

The irregularity complex

The de Rham complex of $\mathrm{DR}(\mathcal{V})$ is the complex

$$0 \longrightarrow \mathcal{V} \xrightarrow{\nabla} \Omega_X^1 \otimes \mathcal{V} \longrightarrow \cdots \xrightarrow{\nabla} \Omega_X^n \otimes \mathcal{V} \longrightarrow 0$$

and the cone $\mathrm{Irr}(\mathcal{V})[+1]$ (***irregularity complex***):

$$\mathrm{Irr}(\mathcal{V}) \longrightarrow \mathrm{DR}(\mathcal{V}) \longrightarrow Rj_*\big(\mathrm{DR}(\mathcal{V})_{X \smallsetminus D}\big) \xrightarrow{+1}$$

is a complex on D which basically computes flat sections of $\mathcal{V}$ with ***essential sing.*** along D.

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THEOREM (Mebkhout 1989): $\mathrm{Irr}(\mathcal{V})$ is perverse.

CONJECTURE (Teyssier 2013): Singularities of $\mathrm{Irr}(\mathcal{V})$ correspond to the red point locus.

The real oriented blow up along D

- If $D = (f)$, replace D with $D \times S^1$ and get

  $$\varpi : \widetilde{X}(D) \longrightarrow X, \quad \varpi^{-1}(D) = D \times S^1 = \partial \widetilde{X}(D).$$

  Real semi-analytic spaces. Can be globalized.
- $\mathcal{A}_{\widetilde{X}(D)}^{>\mathrm{mod} D}$: holom. fncts on $X \smallsetminus D$, modulo those having mod. growth near pts of $\partial \widetilde{X}(D)$.
- $\mathrm{DR}^{>\mathrm{mod} D}\, \mathcal{V}$: complex on $\partial \widetilde{X}(D)$.

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- $\mathrm{DR}^{>\mathrm{mod}D}\,\mathcal{V}$: complex on $\partial\widetilde{X}(D)$.

THEOREM: $\mathrm{DR}^{>\mathrm{mod}D}\,\mathcal{V}$ has $\mathbb{R}$-constr. cohom. and $R\varpi_*(\mathrm{DR}^{>\mathrm{mod}D}\,\mathcal{V}) \simeq \mathbf{Irr}(\mathcal{V})[+1]$.

The proof uses the reduction to ***formal normal form*** after cplx blow-up.

Formal normal form

- $(\mathcal{V}, \nabla)$ has a ***good formal structure along*** D if
  - D has normal crossings,
  - at ***each point*** of D, loc. coord. $x_1, \dots, x_n$,
    $D = \{x_1 \cdots x_\ell = 0\}$,
    $\nabla = \mathrm{d} + A = \mathrm{d} + \sum_i A_i(x)\mathrm{d}x_i$,
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    $A_i \in \mathbb{C}\{x\}[1/x_1 \cdots x_\ell]$,
  - up to ramif. around D: $\begin{cases} \xi_i = x_i^{1/q_i}, & i = 1, \dots, \ell, \\ \xi_i = x_i & i > \ell, \end{cases}$
  - $\exists \widehat{M} \in \mathbf{GL}_r(\mathbb{C}\{\xi\}[1/\xi_1 \cdots \xi_\ell])$ such that
  - $\widehat{B}(\xi)$ is block-diagonal, with blocks

$$\boxed{\mathrm{d}\varphi_j \, \mathrm{Id}_j + \sum_{i=1}^{\ell} C_{i,j} \frac{\mathrm{d}\xi_i}{\xi_i}}$$

$\varphi_j \in \mathbb{C}\{\xi\}[1/\xi_1 \cdots \xi_\ell]$, $\quad C_{i,j} =$ cst matrix,
$+$ “good condition” on the family $(\varphi_j)_j$.

Formal normal form

THEOREM (Kedlaya, Mochizuki, 2008-11): $\exists$ a finite sequence of complex blowing-ups after which the pull-back of $(\mathcal{V}, \boldsymbol{\nabla})$ has a good formal structure everywhere.

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THEOREM (Majima 1984, CS, Mochizuki): A good formal structure can be lifted in poly-sectors to an $\mathcal{A}^{\mathrm{mod}\,D}_{\widetilde{X}(D)}$-decomposition.

- $\widetilde{X}(D)$: real-oriented blow-up along the ***irred. comp.*** of D (polar coordinates w.r.t. $x_1, \dots, x_\ell$).
- Near each point $\widetilde{x}_o$ of $\partial\widetilde{X}(D)$, $\widehat{M}$ can be lifted to $M \in \mathrm{GL}_r(\mathcal{A}^{\mathrm{mod}\,D}_{\widetilde{X}(D),\widetilde{x}_o})$,
- $\rightsquigarrow \widetilde{B}_{x_o}$ has the same block diagonal form as $\widehat{B}$.

Formal normal form

- If $(\mathcal{V}, \nabla)$ has a good formal structure along D,
  $\rightsquigarrow$ ***Stokes structure*** along D well-understood:
  - "Rotating daisy struct." along each stratum of D,
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THEOREM (CS, sugg. by Teyssier): On any open stratum D_I of D, $\mathbf{Irr}(\mathcal{V}, \nabla)_{|D_I}$ only depends on the formal module $\mathcal{O}_{\widehat{D_I}} \otimes (\mathcal{V}, \nabla)$.

- i.e., each cohomology sheaf of $\mathbf{Irr}(\mathcal{V}, \nabla)_{|D_I}$ is a local system on D_I whose monodromy ***does not depend*** on the Stokes phenomenon transverse to D_I.
- Analogue of the one-dim. case.

Geometry of Riemann-Hilbert corresp.

- Assume $(\mathcal{V}, \nabla)$ has a good formal structure along D.
- $D_I :=$ stratum of D,
- $\widetilde{X}(D)_{|\mathrm{nb}(D_I)} = D_I \times (S^1)^\ell \times (\mathbb{R}_+)^\ell$,
- $\widetilde{X}(D)_{|D_I} = D_I \times (S^1)^\ell \times \{0\}$
- $\varphi_j \sim u_j(x_{\ell+1}, \ldots, x_n) \cdot \xi^{-p_{j,1}} \cdots \xi^{-p_{j,\ell}}, \quad u_j(0) \neq 0$,
- $\arg \varphi_j = \arg u_j(x) - \sum_{i=1}^{\ell} p_{j,i} \arg \xi_i$

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- $\arg \varphi_j = \arg u_j(x) - \sum_{i=1}^{\ell} p_{j,i} \arg \xi_i$
- $\varphi_j \overset{\text{def}}{\leqslant} \varphi_{j_o} \iff e^{-\varphi_j} = e^{-\varphi_{j_o}} \cdot$ mod. growth
  $\iff \mathrm{Re}(\varphi_j) > \mathrm{Re}(\varphi_{j_o})$
  $\iff \arg(\varphi_j) - \arg(\varphi_{j_o}) \in (\pi/2, 3\pi/2)$
- $\{\varphi_j \leqslant \varphi_{j_o}\}$: semi-analytic open set in $D_I \times (S^1)^\ell$.
- Global setting $\rightsquigarrow$ finite cover. Σ_I of $(S^1)^\ell_{D_I}(X)$

Riemann-Hilbert correspondence

- $\mathcal{L} := (j^*\mathcal{V})^\nabla$: local system on $X \smallsetminus D$,
- extends as a local system $\mathcal{L}$ on $\widetilde{X}(D)$
- $\rightsquigarrow$ local system $\mathcal{L}_I$ on Σ_I.
- $\forall \varphi_{j_o}$, $\mathcal{L}_{I,\leqslant\varphi_{j_o}}$: flat sect. $v = \begin{pmatrix} e^{-\varphi_1} \cdot \text{mod. growth} \\ \vdots \\ e^{-\varphi_k} \cdot \text{mod. growth} \end{pmatrix}$ whose entries are $\leqslant \varphi_{j_o}$
- $\mathcal{L}_{I,\leqslant\varphi_{j_o}}$: $\mathbb{R}$-construct. subsheaf of $\mathcal{L}_I$.
- $\rightsquigarrow$ ***Stokes-filtered local system*** $(\mathcal{L}, \mathcal{L}_{I,\leqslant\bullet})$ on Σ_I
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THEOREM (Deligne, Malgrange, Mochizuki, CS):
$(\mathcal{V}, \nabla)$ with good formal struct. along $D \longmapsto$
Stokes-filtered constr. sheaf on $\widetilde{X}(D)$
is an ***equivalence of categories***.

Riemann-Hilbert correspondence

- Drawback: What happens if $(\mathcal{V}, \nabla)$ does not have a good formal structure along D, e.g., D not ncd?
- Main problem: What kind of an object extends $\mathcal{L}$ on $X \smallsetminus D$.
- D'Agnolo-Kashiwara: Define a category of $\mathbb{R}$-constructible ***enhanced Ind-sheaves*** and functors

$$\begin{array}{ccc}(\mathcal{V}, \nabla) & \xmapsto{\mathrm{DR}^{\mathsf{E}}} & \mathrm{DR}^{\mathsf{E}}(\mathcal{V}, \nabla) \in \mathsf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_X) \\ & & \Big\downarrow \Psi^{\mathsf{E}}_X \\ & & \mathsf{D}^{\mathrm{b}}(\mathcal{D}_X)\end{array}$$

with an isom. $(\mathcal{V}, \nabla) \simeq \Psi^{\mathsf{E}}_X(\mathrm{DR}^{\mathsf{E}}(\mathcal{V}, \nabla))$

Riemann-Hilbert correspondence

- D'Agnolo-Kashiwara (2014):

$$\begin{array}{ccc} \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) & \xrightarrow{\ \mathrm{DR}^{\mathsf{E}}\ } & \mathsf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbb{C}_X) \\ & & \big\downarrow \Psi^{\mathsf{E}}_X \\ & & \mathsf{D}^{\mathrm{b}}(\mathcal{D}_X) \end{array}$$

  with an isom. $\Psi^{\mathsf{E}}_X \circ \mathrm{DR}^{\mathsf{E}} \simeq \mathrm{Id}$

- Essential image of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ by DR^{E}?
- Mochizuki (2016): Ess. image $=$ objects whose restriction to any germ of cplx analytic parametrized curve corresponds to Stokes-filtered loc. syst.