THE IRREGULAR HODGE FILTRATION

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Abstract. Given a regular function f on a smooth complex quasi-projective variety, we generalize the construction of Deligne (1984) of the "irregular Hodge filtration" on the hypercohomology of the de Rham complex with twisted differential d+df and we show a E_1-degeneracy property. One proof uses the theory of twistor D-modules. We will also report on another approach by M. Kontsevich, and the corresponding proofs by Kontsevich (by characteristic p methods) and Morihiko Saito (by Steenbrink-Hodge theoretical methods). This is a joint work with Hélène Esnault (Berlin) and Jeng-Daw Yu (Taipei).

Contents

1. Introduction et motivations

Since it is not usual to work with irregular singularities in Algebraic geometry, I will first insist on motivations for that.

1.a. Meromorphic connections with regular singularities and Hodge theory (curve case)

Let S be a smooth complex algebraic curve (possibly not projective) and let V be an algebraic vector bundle on S equipped with a connection $\nabla : V \to \Omega^1_S \otimes V$. The connection may have regular or irregular singularities at ∞.

Examples 1.1.

(1) Let $\omega \in H^0(S, \Omega^1_S)$ be an algebraic one-form. Then $\mathrm{d} + \omega$ defines a connection on the trivial bundle $\mathscr{O}_S$. If ω has poles of order $\geqslant 2$ in some (or any) smooth compactification $\overline{S}$ of S, the connection has irregular singularities.

(2) Set $j; S \hookrightarrow \overline{S}$, $\infty = \overline{S} \smallsetminus S$ and extend V as locally free $j_*\mathscr{O}_X$-module j_*V of finite rank, and for each $s \in \infty$, we get a free $(j_*\mathscr{O}_S)_s$-module $(j_*V)_s$. The connection extends in a natural way. It has regular singularities at s if for some $(j_*\mathscr{O}_S)_s$-basis of $(j_*V)_s$, the matrix of the connection has a pole of order at most one. Otherwise, it has regular singularities.

(3) Let $f : X \to S$ be a smooth proper morphism of complex algebraic varieties. A celebrated theorem asserts that the Gauss-Manin connection on the vector bundle $R^k f_* \mathscr{O}_X$ has regular singularities at infinity.

(4) Assume that $(V^{\mathrm{an}}, \nabla^{\mathrm{an}})$ is a holomorphic vector bundle with connection on S^{an} which underlies a polarized variation of Hodge structure, with Hodge filtration $F^\bullet V^{\mathrm{an}}$. Then the subsheaf on $\overline{S}^{\mathrm{an}}$ consisting of sections of V^{an} whose norm has moderate growth near $\infty = \overline{S} \smallsetminus S$ is a locally free $\mathscr{O}_{\overline{S}^{\mathrm{an}}}(*\infty)$-module, and the connection ∇^{an} extends as a meromorphic connection $\overline{\nabla}^{\mathrm{an}}$. Moreover, this connection has regular singularities at infinity (theorem of Griffiths-Schmid, [**Sch73**, Th. 4.13(a)]). By using a GAGA-type result on the projective curve $\overline{S}$, one gets a bundle (V, ∇) with regular singularities at infinity.

1.b. Arithmetic analogies (Deligne). The motivations of Deligne in 1984 came from arithmetic.

"The analogy between vector bundles with integrable connection having irregular singularities on a complex algebraic variety X and ℓ-adic sheaves with wild ramification on an algebraic variety of characteristic p, leads one to ask how such a vector bundle with integrable connection can be part of a system of realizations analogous to what furnishes a family of motives parametrized by X...

In the "motivic" case, any de Rham cohomology group has a natural Hodge filtration. Can we hope for one on $H^i_{\mathrm{dR}}(U, V)$ for some classes of (V, ∇) with irregular singularities?"

Deligne hopes (1984) that such is the case when $(V, \nabla^{\mathrm{reg}})$ underlies a variation of polarized Hodge structure and $(V, \nabla) = (V, \nabla^{\mathrm{reg}} + \mathrm{d}f \otimes \mathrm{Id}_V)$.

1.c. Irregular Hodge theory and L-functions (Adolphson-Sperber)

Let f be a Laurent polynomial in n variables with coefficients in $\mathbb{Z}$. Assume that f is convenient and nondegenerate. For p a prime number away from a finite set, one considers the exponential sums

$$S_i(f) = \sum_{x \in \mathbb{G}_m(\mathbb{F}_{p^i})^n} \Psi \circ \mathrm{Tr}_{\mathbb{F}_{p^i}/\mathbb{F}_p}(f(x))$$

and the corresponding L function $L(f, t) = \exp\big(\sum_i S_i(f) t^i / i\big)$. It is known that, with these assumptions on f, $L(f, t)^{(-1)^{n-1}}$ is a polynomial, with coefficients in $\mathbb{Q}(\zeta_p)$. A polygon is attached to such a polynomial by considering the p-adic valuation of the coefficients. On the other hand, a polygon is attached to f by means of the Newton polyhedron defined by f, so is not difficult to compute. Adolphson and Sperber (1989) prove that the L-polygon lies above the Newton polygon. This is motivated by an analogous result of Mazur for the zeta function of a smooth algebraic variety (Mazur-Katz theorem). In the latter situation, the Newton polygon is replaced by a Hodge polygon, so it is natural to ask for an interpretation of the Newton polygon attached to f in terms of "Hodge data" attached to the connection $\mathrm{d} + \mathrm{d}f$. This was also a motivation of Deligne, *cf.* below.

1.d. Extended motivic exponential $\mathscr{D}$-modules (Kontsevich)

In [**Kon09**], M. Kontsevich defines the category of extended motivic-exponential $\mathscr{D}$-modules on smooth algebraic varieties over a field k of characteristic zero as the minimal class which is closed under extensions, sub-quotients, push-forwards and pull-backs, and contain all $\mathscr{D}_X$-modules of type $(\mathscr{O}_X, \mathrm{d} + \mathrm{d}f)$ for $f \in \mathscr{O}(X)$. Recall that a $\mathscr{D}_X$-module is a $\mathscr{O}_X$-module equipped with an integrable connection.

The natural question which arises, in the case the field k is equal to $\mathbb{C}$, is to endow the objects of this category of a "Hodge filtration".

1.e. Mirror symmetry (Kontsevich). For some Fano manifolds (or orbifolds), one looks for the mirror object as regular function $f : X \to \mathbb{A}^1$. Kontsevich conjectures that the Hodge numbers of the Fano manifolds can be read on a "Hodge filtration" on the cohomology $H^k_{\mathrm{dR}}(X, \mathrm{d} + \mathrm{d}f)$. This fitltration is nothing but the irregular Hodge filtration as suggested by Deligne and extended in arbitrary dimension by J.-D. Yu.

2. Some previous results

2.a. P. Deligne (1984) [Del07]. Deligne starts from a bundle with regular connection $(V,\nabla^{\mathrm{reg}})$ on an affine curve S such that $(V^{\mathrm{an}},\nabla^{\mathrm{reg,an}})$ corresponds to a unitary representation of $\pi_1(S)$, and considers the bundle with connection (V,∇), with $\nabla=\nabla^{\mathrm{reg}}+\mathrm{d}f$, for some $f\in\mathscr{O}(S)$. The problem is to endow $H^1_{\mathrm{dR}}(S,(V,\nabla)):=H^1(\Omega^\bullet(S)\otimes V(S),\nabla)$ with a "Hodge" filtration. More precisely, consider the complex of sheaves $j_*(\Omega^\bullet_S\otimes V,\nabla)$ on $\overline{S}$. Deligne defines a filtration of this complex:

$$F^p_{\mathrm{Del}}j_*(\Omega^\bullet_S\otimes V,\nabla)$$

and shows a E_1-degeneration result:

$$\boldsymbol{H}^1\big(\overline{S},F^p_{\mathrm{Del}}j_*(\Omega^\bullet_S\otimes V,\nabla)\big)\hookrightarrow\boldsymbol{H}^1\big(\overline{S},j_*(\Omega^\bullet_S\otimes V,\nabla)\big)=\boldsymbol{H}^1\big(S,(\Omega^\bullet_S\otimes V,\nabla)\big).$$

In such a definition of F_{Del} the exponents may be rational or even real, but there is only a finite set of jumps. I will not give the precise formula for F_{Del} now, but I will explain why Deligne expects rational or real indices on a simple example.

Consider $S=\mathbb{A}^1$ (coord. t) and $f(t)=t^2$. One has $\dim H^1_{\mathrm{dR}}(\mathbb{A}^1,\mathrm{d}-\mathrm{d}(t^2))=1$, generated by the class of $\mathrm{d}t\otimes e^{-t^2}$. The period of this class along the 1-cycle with closed support $\mathbb{R}$ is equal to $\pi^{1/2}$. Deligne explains that this suggests that $\mathrm{d}t\otimes e^{-t^2}$ has Hodge type 1/2, that is, $H^1=F^{1/2}_{\mathrm{Del}}H^1=\mathrm{gr}^{1/2}_{F_{\mathrm{Del}}}H^1$.

On the other hand, consider the family of Laurent polynomials "$e^{z+\lambda/z}$" on $\mathbb{C}^*$ parametrized by λ and the corresponding family of "Hodge" filtrations. Deligne shows that, in such a family, there exists λ for which the "Hodge filtration" is not opposed to the conjugate filtration. And Deligne gives this comment:

"The reader may ask what a "Hodge" filtration which does not give rise to a Hodge structure is good for. One hope is that it forces some bounds for p-adic valuations of eigenvalues of Frobenius. That the cohomology of "$e^{-x}x^{\alpha}$" $(0<\alpha<1)$ is purely of Hodge type $1-\alpha$ is analogous to formulas giving the p-adic valuation of Gauss sums." ([**Del07**]).

2.b. C.S. (2010) [Sab10]. I have given a generalization of Deligne's construction in the case where $(V,\nabla^{\mathrm{reg}})$ underlies a polarized variation of Hodge structure and where one considers, as Deligne does, a connection of the form $\nabla^{\mathrm{reg}}+\mathrm{d}f$. I have proved the E_1-degeneration of the corresponding spectral sequence. The new tool is Fourier transformation of $\mathscr{D}$-modules with a twistor structure (polarizable twistor $\mathscr{D}$-modules).

2.c. J.-D. Yu (2012) [Yu12]. More recently, J.-D. Yu has proposed the following formula, in the case where $(V,\nabla^{\mathrm{reg}})=(\mathscr{O},\mathrm{d})$ on a smooth complex quasi-projective variety U and $f\in\Gamma(U,\mathscr{O}_U)$, which is compactified as $f:X\to\mathbb{P}^1$, $X=U\cup D$ where D is a NCD. We denote by P the (possibly non reduced) divisor of the poles of $f:X\to\mathbb{P}^1$ and by ∇ the connection $\mathrm{d}+\mathrm{d}f$.

Then Yu considers the following filtration on the meromorphic de Rham complex $(\Omega^\bullet_X(*D),\nabla)$:

$$F^\lambda_{\mathrm{Yu}}(\Omega^\bullet_X(*D),\nabla):=\Big\{\mathscr{O}_X([-\lambda P])_+\xrightarrow{\nabla}\Omega^1_X(\log D)([(1-\lambda)P])_+\xrightarrow{\nabla}\cdots\\ \xrightarrow{\nabla}\Omega^n_X(\log D)([(n-\lambda)P])_+\Big\},$$

where, for $\mu\in\mathbb{Q}$, one sets

$$\Omega^k_X(\log D)([\mu P])_+=\begin{cases}\Omega^k_X(\log D)([\mu P]) & \text{if } \mu\geqslant 0,\\ 0 & \text{otherwise.}\end{cases}$$

Yu has shown the E_1-degeneration of the corresponding spectral sequence in various particular cases. This is generalized below.

3. E_1-degeneration: various approaches

3.a. Esnault-Sabbah-Yu (2013) [ESY13].

Theorem. *In the setting of §2.c, for each $\alpha\in[0,1)$, the spectral sequence*

$$E_1^{p,q}=\boldsymbol{H}^{p+q}\big(X,F^{-\alpha+p}_{\mathrm{Yu}}(\Omega^\bullet_X(*D),\nabla)\big)\Longrightarrow\boldsymbol{H}^{p+q}\big(X,(\Omega^\bullet_X(*D),\nabla)\big)=H^{p+q}_{\mathrm{dR}}(U,\nabla)$$

degenerates at E_1, that is, for each $\alpha\in[0,1)$, $k\in\mathbb{N}$ and $p\in\mathbb{Z}$, the natural morphism

$$\boldsymbol{H}^k\big(X,F^{-\alpha+p}_{\mathrm{Yu}}(\Omega^\bullet_X(*D),\nabla)\big)\longrightarrow\boldsymbol{H}^k\big(X,(\Omega^\bullet_X(*D),\nabla)\big)$$

is injective.

The proof consists in reducing to the case $X=\mathbb{P}^1$ treated in [**Sab10**] (§2.b) by pushing-forward by the projection. Results coming from the theory of twistor $\mathscr{D}$-modules are used.

3.b. The f-logarithmic complex (2012) [Kon12a, Kon12b]. By Deligne's mixed Hodge theory on U, the Hodge filtration of the mixed Hodge structure $H^k(U,\mathbb{C})$ satisfies:

$$\mathrm{gr}^p_F H^q(U,\mathbb{C})\simeq H^q(X,\Omega^p_X(\log D)).$$

We now consider a similar property for Yu's filtration. In the setting considered by Yu, Kontsevich has introduced the following sheaves:

$$\Omega_f^p = \bigl\{\omega \in \Omega_X^p(\log D) \mid \mathrm{d}f \wedge \omega \in \Omega_X^{p+1}(\log D)\bigr\}.$$

In particular:

$$\Omega_f^0 = \mathscr{O}_X(-P), \quad \Omega_f^n = \Omega_X^n(\log D).$$

The sheaves Ω_f^p are $\mathscr{O}_X$-locally free, and one can make a complex with differential $u\mathrm{d} + v\mathrm{d}f$ for each $(u,v) \in \mathbb{C}^2$.

More generally, for each $\alpha \in [0,1)$, one can define

$$\Omega_f^p(\alpha) = \bigl\{\omega \in \Omega_X^p(\log D)([\alpha P]) \mid \mathrm{d}f \wedge \omega \in \Omega_X^{p+1}(\log D)([\alpha P])\bigr\}.$$

Corollary. *For $\alpha \in [0,1)$ fixed, we have*

$$\mathrm{gr}^p_{F_{\mathrm{Yu},\alpha}} H^{p+q}(U,\nabla) = H^q(X, \Omega_f^p(\alpha)).$$

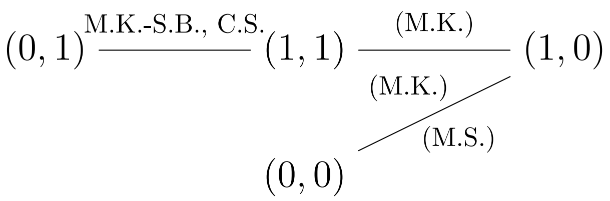
The f-logarithmic complexes are interesting in their own right. I will focus on the case $\alpha = 0$ for the sake of simplicity.

Theorem. *For each k, the dimension of $\boldsymbol{H}^k\bigl(X, (\Omega_f^\bullet, u\mathrm{d} + v\mathrm{d}f)\bigr)$ is independent of u, v and is equal to $\dim H^k_{\mathrm{dR}}(U,\nabla)$. In other words, for each pair $(u,v) \neq (0,0)$, the spectral sequence*

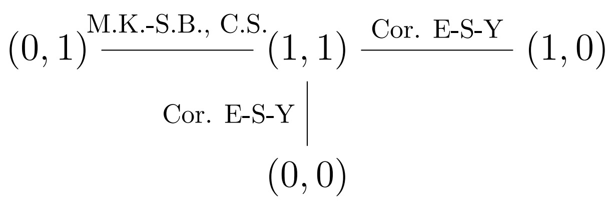
$$E_1^{p,q} = H^q(X, \Omega_f^p) \Longrightarrow \boldsymbol{H}^{p+q}\bigl(X, (\Omega_f^\bullet, u\mathrm{d} + v\mathrm{d}f)\bigr)$$

degenerates at E_1.

There are at least tow methods to prove this theorem:



Method of Kontsevich



Method of E-S-Y

The degeneration $(0,0) \Rightarrow (1,0)$ is shown by Kontsevich when the divisor P is reduced, by the method of Deligne-Illusie which reduces to characteristic p.

The degeneration $(0,0) \Rightarrow (1,1)$ is a consequence of the degeenration of Yu's filtration (in the case $\alpha = 0$) since we have:

$$\sigma^p(\Omega_f^\bullet, \nabla) \xrightarrow{\sim} F^p_{\mathrm{Yu}}(\Omega_X^\bullet(*D), \nabla), \quad \forall\, p.$$

3.c. M. Saito (2012), Appendix to [ESY13]. In an appendix to [**ESY13**], Morihiko Saito proposes a different approach for the degeneration $(0,0) \Rightarrow (1,0)$, relying on old results by Steenbrink [**Ste76, Ste77**] concerning the limit of the Hodge structure of fibres of a holomorphic function. The idea consists in considering the exact sequence of complexes (with differentials d)

$$0 \longrightarrow \Omega_f^\bullet \longrightarrow \Omega^\bullet(\log D) \longrightarrow Q^\bullet \longrightarrow 0.$$

The quotient complex Q is supported on the divisor P_{red}, so that one can work locally analytically in the neighbourhood of P_{red} and consider the function $g = 1/f$. Then $Q^\bullet$ is identified with the restriction to P of the relative logarithmic complex:

$$Q^\bullet = \Omega^\bullet_{X'/\Delta}(\log D)\Big/ g\Omega^\bullet_{X_\Delta/\Delta}(\log D)$$

which has been considered by Steenbrink, where Δ is a neighbourhood of ∞ in $\mathbb{P}^1$ et $X_\Delta = f^{-1}(\Delta)$. Steenbrink defines the limit Hodge filtration of the morphism g by using the stupid filtration of this complex. M. Saito uses the corresponding degeneration for the stupid filtration on $Q^\bullet$ and that (due to Deligne) for the stupid filtration on $\Omega^\bullet(\log D)$ in order to deduce the degeneration for the stupid filtration on $\Omega_f^\bullet$.

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