# Introduction

*Joint work with Antoine Douai (Nice)*

- $w_0, \dots, w_n$ ($n \geqslant 1$): positive integers such that $\gcd(w_0, \dots, w_n) = 1$.
- $f(u_0, \dots, u_n) = w_0 u_0 + \cdots + w_n u_n$ restricted to the ***torus*** $U \subset \mathbb{C}^{n+1}$: $u_0^{w_0} \cdots u_n^{w_n} = 1$.
- $\mu = w_0 + \cdots + w_n$, $f$ has $\mu$ simple critical points.
- Critical values: $\mu e^{2i\pi k/\mu}$ ($k = 0, \dots, \mu - 1$).
- If $t \in \mathbb{C}$ is not a critical value,

$$\dim H^k(U, f^{-1}(t), \mathbb{Q}) = \begin{cases} 0 & \text{for } k \neq \dim U = n, \\ \mu & \text{for } k = n. \end{cases}$$

- $H_{\mathbb{Q}} \overset{\text{def}}{=} \underset{t \to \infty}{\text{"limit"}} H^n(U, f^{-1}(t), \mathbb{Q})$, $\quad T_\infty : H_{\mathbb{Q}} \to H_{\mathbb{Q}}$: ***monodromy at infinity***.
- ***Mixed Hodge structure*** (Elzein, Steenbrink-Zucker, M. Saito).

- The interest in such a polynomial $f$ on $U$ is that the data "at infinity" look very much like the ***orbifold cohomology*** of the weighted projective space $\mathbf{P}(w_0, \dots, w_n)$.
- One can associate to any universal unfolding of $f$ a ***canonical*** structure of ***Frobenius manifold*** on the germ $(\mathbf{C}^\mu, 0)$ of the parameter space.
- On the other hand, the ***quantum orbifold cohomology***, defined by Chen and Ruan, allows one to associate the structure of a Frobenius manifold on the germ $(H^*_{\mathrm{orb}}(\mathbf{P}(w_0, \dots, w_n)), 0)$. It is then natural to expect a ***"mirror symmetry phenomenon"***, *i.e.*, that both Frobenius structures coincide.
- When $w_i = 1$ for all $i$, so that $\mathbf{P}(w_0, \dots, w_n)$ is the ordinary projective space $\mathbf{P}^n$, such a result has been guessed by A. Givental and proved by S. Barannikov.

# The Gauss-Manin system: general results

- $f$: any regular function on an ***affine*** manifold $U$, satisfying some ***nonsingularity condition at infinity***.
- $\Omega^n(U)$: the space of algebraic differential forms of maximal degree on $U$.
- $\tau$: a new variable. $\theta = \tau^{-1}$
- Gauss-Manin system: $G = \dfrac{\Omega^n(U)[\tau, \tau^{-1}]}{(d - \tau df\wedge)\Omega^{n-1}(U)[\tau, \tau^{-1}]}$

  with

$$(d - \tau df\wedge) \sum_k \eta_k \tau^k = \sum_k (d\eta_k - df \wedge \eta_{k-1})\tau^k.$$

- Connection: $\begin{cases} \forall\, \omega \in \Omega^n(U), \quad \partial_\tau[\omega] = [f\omega], \\ \forall\, p \in \mathbf{Z}, \quad \partial_\tau(\tau^p[\omega]) = p\tau^{p-1}[\omega] + \tau^p[f\omega]. \end{cases}$

# Lattices

- The ***Briekorn lattice*** (chart $\theta$):
  - $G_0 = \mathrm{image}(\Omega^n(U)[\theta] \to G) = \dfrac{\Omega^n(U)[\theta]}{(\theta d - df\wedge)\Omega^{n-1}(U)[\theta]}$
  - free $\mathbf{C}[\theta]$-module of rank $\mu = \dim \Omega^n(U)/df \wedge \Omega^{n-1}(U)$,
  - stable by the action of $\partial_\tau$, hence acted on by $\partial_\theta = -\dfrac{1}{\theta^2}\partial_\tau$ with a pole of order $2$.
- ***Malgrange-Kashiwara lattices*** ($\alpha \in [0,1[$, chart $\tau$):
  - $V_\alpha(G)$: set of $[\omega]$ in $G$ such that $\prod_{\beta\leqslant\alpha}(\tau\partial_\tau+\beta)^{\nu_\beta}[\omega] = \tau P(\tau,\tau\partial_\tau)[\omega]$.
  - free $\mathbf{C}[\tau]$-module of rank $\mu$, stable by the action of $\tau\partial_\tau$.
  - $H_\alpha = V_\alpha(G)/V_{<\alpha}(G)$, $N$ induced by $-(\tau\partial_\tau+\alpha)$, $T = \exp(2i\pi\alpha\,\mathrm{Id} + N)$.
- $F^p(H_\alpha) = \tau^{n-p}G_0H_\alpha$.

***Theorem***

*The space* $\bigoplus_{\alpha\in[0,1[}H_\alpha$ *with its endomorphism* $T$ *is naturally isomorphic to* $(H_{\mathbb{C}}, T)$. *Moreover, the filtration* $\bigoplus_{\alpha\in[0,1[}F^pH_\alpha$ *is the* ***limit Hodge filtration*** *on* $H_{\mathbb{C}}$.

***A basic isomorphism***

$$\frac{F^{n-p}\cap V_\alpha}{F^{n-p+1}\cap V_\alpha + F^{n-p}\cap V_{<\alpha}} = \mathrm{gr}_F^{n-p}(H_\alpha)$$

$$\cdot\,\tau^p \uparrow \quad \downarrow \cdot\,\theta^p$$

$$\frac{V_{\alpha+p}\cap G_0}{V_{\alpha+p}\cap \theta G_0 + V_{<\alpha+p}\cap G_0} = \mathrm{gr}^V_{\alpha+p}(G_0/\theta G_0)$$

***Spectrum at infinity*** of $f$: $\{(\beta, \nu_\beta) \mid \beta\in\mathbb{Q},\ \nu_\beta = \dim \mathrm{gr}^V_\beta(G_0/\theta G_0)\}$.

# A nice presentation of the Gauss-Manin system

- $\mathcal{S}_w \stackrel{\text{def}}{=} \coprod_{i=0}^{n} \{\ell/w_i \mid \ell = 0, \ldots, w_i - 1\} \longrightarrow \mathbb{Q}$.
  For $s \in \mathbb{Q}$, $m(s) =$ ***number of preimages*** of $s$ in $\mathcal{S}_w$.
- $\mathcal{S}_w = \{s(0) \leqslant s(1) \leqslant \cdots \leqslant s(\mu - 1)\}$
  $\sigma_w(k) = k - \mu s(k)$, $k = 0, \ldots, \mu - 1$.
- $\omega_0 = \left.\dfrac{\frac{du_0}{u_0} \wedge \cdots \wedge \frac{du_n}{u_n}}{d\left(\prod_i u_i^{w_i}\right)}\right|_{\prod_i u_i^{w_i} = 1}$, $\quad [\omega_{k+1}] = -\dfrac{1}{\mu\tau}(\tau\partial_\tau + \sigma_w(k))[\omega_k]$.
- $[\omega_0 \wedge] : \mathcal{O}(U)\Big/\big(\mathrm{Jac}(f)\big) \xrightarrow{\sim} G_0/\theta G_0$
  $\Longrightarrow$ ***ring structure*** on $G_0/\theta G_0$.
- $[\omega_\mu] = [\omega_0]$ and $[\omega_k] = [u^{a(k)}\omega_0]$ for some multi-index $a(k)$ with $|a(k)| = k$.

- ***Spectral polynomial***

$$\prod_{\beta}(X+\beta)^{\nu_\beta} = \begin{cases} \prod_{k=0}^{\mu-1}(X+\sigma_w(k)) \\ \prod_{s\in Q}\prod_{j=0}^{m(s)-1}\left(X+n-(j+\{sw_0\}+\cdots+\{sw_n\})\right) \end{cases}$$

- The classes $[\underline{\omega}]$ form a $\mathbb{C}[\theta]$-basis $G_0$ and a $\mathbb{C}[\tau,\tau^{-1}]$-basis of $G$.
- ***Connection in this basis***:

$$-\partial_\tau[\underline{\omega}] = [\underline{\omega}]\cdot\left[A_0+\frac{1}{\tau}A_\infty\right], \quad \text{or} \quad \partial_\theta[\underline{\omega}] = [\underline{\omega}]\cdot\left[\frac{1}{\theta^2}A_0+\frac{1}{\theta}A_\infty\right]$$

with

$$A_0 = \mu\begin{pmatrix} 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \qquad A_\infty = \operatorname{diag}\left(\sigma_w(0),\ldots,\sigma_w(\mu-1)\right).$$

- The "sesquilinear" ***pairing*** $S : G \otimes_{\mathbb{C}[\tau,\tau^{-1}]} G^{\mathrm{a}} \to \mathbb{C}[\tau,\tau^{-1}]$ defined by

$$S([\omega_k],[\omega_\ell]) = \begin{cases} 1 & \text{if } k+\ell \equiv n \bmod \mu \\ 0 & \text{otherwise} \end{cases}$$

  is, up to a constant, the pairing coming from ***Poincaré duality*** on the fibres of $f$. It is compatible with the connection.

- The graded algebra $\oplus_{\beta\in\mathbb{Q}} \mathrm{gr}^V_\beta(G_0/\theta G_0)$, with its nondegenerate pairing, looks very much like the orbifold cohomology algebra

$$\oplus_\beta H^{2(n-\beta)}_{\mathrm{orb}}(\mathbb{P}(w_0,\dots,w_n))$$

  with its Poincaré duality pairing.

***Corollary*** **(Simplification of the structure)**

- *The* $\boldsymbol{V}$*-order of* $[\omega_k]$ *is equal to* $\boldsymbol{\sigma_w(k)}$ *and* $[\underline{\omega}]$ *induces a* $\mathbb{C}$*-basis* $\{\underline{\omega}\}$ *of* $\bigoplus_\beta \mathrm{gr}^V_\beta(G_0/\theta G_0)$.
- *The graded space* $\bigoplus_{\beta\in\mathbb{Q}} \mathrm{gr}^V_\beta(G_0/\theta G_0)$ *is a (rationally) graded space equipped with a* ***graded commutative product*** $\cup$ *such that*

$$\{\omega_k\}\cup\{\omega_\ell\} = \begin{cases} \{\omega_{k+\ell \bmod \mu}\} & \textit{if } \sigma_w(k+\ell \bmod \mu) = \sigma_w(k) + \sigma_w(\ell), \\ 0 & \textit{otherwise}. \end{cases}$$

*and a nondegenerate pairing induced by* $\boldsymbol{S}$. *Both induce a* ***symmetric trilinear form***

$$S(\{\omega_j\},\{\omega_k\}\cup\{\omega_\ell\}) = \begin{cases} 1 & \textit{if } \begin{cases} j+k+\ell \equiv n \bmod \mu & \textit{and} \\ \sigma_w(j)+\sigma_w(k)+\sigma_w(\ell) = n, \end{cases} \\ 0 & \textit{otherwise}. \end{cases}$$

# Frobenius structure (Dubrovin) or flat structure (K. Saito)

***Data***:

**(0)** $M$: complex manifold of dimension $\mu$

**(1)** $\circ$: ***commutative associative product*** with ***unit*** $e$ on vector fields, depending holomorphically on the point in $M$,

**(2)** $g$: ***nondegenerate bilinear form*** on vector fields, depending holomorphically on the point in $M$.

**(3)** $\mathfrak{E}$: homogeneity (Euler) holomorphic vector field on $M$,

***Constraints***:

- ***Symmetry*** of the 4-tensor $(\xi_1, \xi_2, \xi_3, \xi_4) \longmapsto \nabla_{\xi_1} g(\xi_2 \circ \xi_3, \xi_4)$;
- $\nabla$ (***torsionless*** connection associated to $g$) is ***flat*** and $\nabla e = 0$;
- $\mathcal{L}_{\mathfrak{E}} e = -e$, $\quad \mathcal{L}_{\mathfrak{E}}(\circ) = \circ$, $\quad \mathcal{L}_{\mathfrak{E}}(g) = Dg$ for some $D \in \mathbb{C}$.

***Theorem***

*let* $f$ *be any Laurent polynomial on a torus* $U = (\mathrm{C}^*)^n$, *which is* ***convenient and nondegenerate*** *with respect to its Newton polyhedron; in particular,*

$$\mu \stackrel{\mathrm{def}}{=} \dim \mathrm{C}[v_1, v_1^{-1}, \dots, v_n, v_n^{-1}]/(v_1\partial f/\partial v_1, \dots, v_n\partial f/\partial v_n) < +\infty.$$

*Choose a family* $\varphi_0 = 1, \varphi_1, \dots, \varphi_{\mu-1}$ *inducing a basis of this vector space. Then there exists a* ***canonical Frobenius structure*** *locally on the space of parameters* $x_0, \dots, x_{\mu-1}$ *of the unfolding* $F = f + \sum x_i\varphi_i$, *for which*

- $\partial_{x_i} \circ \partial_{x_j} = \delta_{ij}\partial_{x_i}$,
- $\mathfrak{E} = \sum_i x_i\partial_{x_i}$.

***Theorem*** **(Generation of a semisimple Frobenius structure, B. Dubrovin)**

*Given* $(A_0, A_\infty, \omega_0)$, *where*

- $A_0, A_\infty$ *are* $\mu \times \mu$ *complex matrices,* $A_0$ ***regular semisimple***, $A_\infty + {}^tA_\infty = w\,\mathrm{Id}$ *for some* $w \in \mathbf{Z}$,
- $\omega_0$ *is an* ***eigenvector*** *of* $A_\infty$, *with eigenvalue* $\alpha$, *with* ***no component equal to*** $\mathbf{0}$ *on the basis of eigenvectors of* $A_0$,

*there exists a* ***unique Frobenius structure*** *on some neighbourhood of* $x^o = \mathrm{Spec}\,A_0$ *in* $M = \mathbf{C}^\mu \smallsetminus$ ***diagonals*** *such that, in the canonical basis* $\partial_{x_1}, \ldots, \partial_{x_n}$ *of* $M$,

- $A_0$ *is the matrix of* $\xi \mapsto \mathfrak{E} \circ \xi$ *at* $x^o$,
- $A_\infty$ *is the matrix of* $\xi \mapsto \nabla_\xi \mathfrak{E}$ *at* $x^o$,
- $\omega_0$ *is the vector of components of the* ***unit field*** $e$ *at* $x^o$,
- $2\alpha + 2 - w$ *is the homogeneity constant* $D$.

$$f = w_0 u_0 + \cdots + w_n u_n \text{ on } U = \{\prod_i u_i^{w_i} = 1\}.$$

***Theorem***

*The data*

$$A_0 = \mu \begin{pmatrix} 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \qquad A_\infty = \mathrm{diag}\left(\sigma_w(0), \ldots, \sigma_w(\mu-1)\right),$$

$$\omega_0 = \frac{\frac{du_0}{u_0} \wedge \cdots \wedge \frac{du_n}{u_n}}{d\left(\prod_i u_i^{w_i}\right)}\Big|_{\prod_i u_i^{w_i}=1}$$

*generate the* ***canonical Frobenius structure*** *attached to any universal unfolding of $f$ on $U$.*