EXAMPLES OF FROBENIUS MANIFOLDS

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This is a joint work with Antoine Douai (Nice), written in [**1, 2**].

Introduction

Let $w_0, \dots, w_n$ ($n \geqslant 1$) be positive integers such that $\gcd(w_0, \dots, w_n) = 1$. In this talk, I will consider some properties of the function

$$f(u_0, \dots, u_n) = w_0 u_0 + \cdots + w_n u_n$$

restricted to the torus $U \subset \mathbf{C}^{n+1}$ defined by the equation

$$u_0^{w_0} \cdots u_n^{w_n} = 1.$$

This function has μ simple critical points, with $\mu = w_0 + \cdots + w_n$. The μ critical values $\mu e^{2i\pi k/\mu}$ ($k = 0, \dots, \mu - 1$) are distinct.

If $t \in \mathbf{C}$ is not a critical value, the relative cohomology $H^k(U, f^{-1}(t), \mathbf{Q})$ vanishes, unless $k = \dim U = n$. The "limit" when $t \to \infty$ is a vector space space $H_{\mathbf{Q}}$ equipped with an endomorphism $T : H_{\mathbf{Q}} \to H_{\mathbf{Q}}$, called the *monodromy at infinity*. It is also equipped with a *mixed Hodge structure*, according to Elzein, Steenbrink-Zucker, M. Saito.

The interest in such a polynomial f on U is that the data "at infinity" look very much like the *orbifold cohomology* of the weighted projective space $\mathbf{P}(w_0, \dots, w_n)$.

One can associate to a universal unfolding of f a canonical structure of *Frobenius manifold* on the germ $(\mathbf{C}^\mu, 0)$ of the parameter space.

On the other hand, the *quantum orbifold cohomology*, defined by Chen and Ruan, allows one to associate the structure of a Frobenius manifold on $H^*_{\mathrm{orb}}(\mathbf{P}(w_0,\dots,w_n))$. It is then natural that a "mirror symmetry phenomenon" holds, *i.e.*, that both Frobenius structures coincide.

When $w_i=1$ for all i, so that $\mathbf{P}(w_0,\dots,w_n)$ is the ordinary projective space $\mathbf{P}^n$, such a result has been guessed by A. Givental and proved by S. Barannikov.

In fact, $\mathbf{P}(w_0,\dots,w_n)$ has two different orbifold structures: the local one, which was referred to above, and the global one, with respect to which it is a global quotient. D. van Straten has conjectured that the same symmetry phenomenon holds between the global quotient $\mathbf{P}(w_0,\dots,w_n)$ and the Pham-Brieskorn polynomial

$$g(x_0,\dots,x_n)=\frac{1}{a_0}x_0^{a_0}+\cdots+\frac{1}{a_n}x_n^{a_n}$$

restricted to the torus $x_0\dots x_n=1$, where $w_i=\mathrm{lcm}(a_0,\dots,a_n)/a_i$.

1. The Gauss-Manin system

1.1. General results. The results in this paragraph apply to any regular function on an affine manifold U, satisfying some nonsingularity condition at infinity. The polynomial f considered above is such a function on the torus U.

The Gauss-Manin system G of the Laurent polynomial f is a module over the ring $\mathbf{C}[\tau,\tau^{-1}]$, where τ is a new variable, and comes equipped with a connection, that we view as a $\mathbf{C}$-linear morphism $\partial_\tau: G\to G$ satisfying Lebniz rule

$$\forall\, g\in G,\ \forall\,\varphi\in\mathbf{C}[\tau,\tau^{-1}],\qquad \partial_\tau(\varphi\cdot g)=\frac{\partial\varphi}{\partial\tau}\cdot g+\varphi\partial_\tau(g).$$

It will be convenient to put $\theta=\tau^{-1}$, and consider (τ,θ) as complementary coordinates on $\mathbf{P}^1$. Then G is a $\mathbf{C}[\theta,\theta^{-1}]$-module with connection, and $\partial_\tau=-\theta^2\partial_\theta$.

Let $\Omega^n(U)$ denote the space of algebraic differential forms of maximal degree on the torus U, and let $\Omega^n(U)[\tau,\tau^{-1}]$ be the space of Laurent polynomials in τ with coefficients in $\Omega^n(U)$.

The Gauss-Manin system is defined by the formula

$$G=\frac{\Omega^n(U)[\tau,\tau^{-1}]}{(d-\tau df\wedge)\Omega^{n-1}(U)[\tau,\tau^{-1}]}$$

$$(d-\tau df\wedge)\textstyle\sum_k\eta_k\tau^k=\sum_k(d\eta_k-df\wedge\eta_{k-1})\tau^k.$$

The action of the connection ∇_τ on G, *i.e.*, the $\mathbf{C}[\tau]\langle\partial_\tau\rangle$-module structure on G, is first defined on the image of $\Omega^n(U)$ by

$$\forall\,\omega\in\Omega^n(U),\quad \partial_\tau[\omega]=[f\omega],$$

and is then extended to G using the Leibniz rule

$$\forall\, p \in \mathbf{Z}, \quad \partial_\tau(\tau^p[\omega]) = p\tau^{p-1}[\omega] + \tau^p[f\omega].$$

In order to extend it as a rank μ vector bundle on $\mathbf{P}^1$ (coordinates τ and $\theta = \tau^{-1}$), one is led to study *lattices*, *i.e.*, $\mathbf{C}[\tau]$ or $\mathbf{C}[\theta]$ submodules which are free of rank μ.

The lattice in the chart θ. The Briekorn lattice

$$G_0 = \operatorname{image}(\Omega^n(U)[\theta] \to G) = \frac{\Omega^n(U)[\theta]}{(\theta d - df\wedge)\Omega^{n-1}(U)[\theta]}$$

is a free $\mathbf{C}[\theta]$-module of rank μ. It is stable by the action of ∂_τ, also acting as $-\theta^2\partial_\theta$. Therefore, ∂_θ is a connection on G_0 with a pole of order 2.

We will also consider the increasing exhaustive filtration $G_p = \tau^p G_0$ ($p \in \mathbf{Z}$) of G.

The lattice in the chart τ. In the chart τ, there are various natural lattices, indexed by $\mathbf{Q}$. We denote them by $V_\alpha(G)$, with $V_{\alpha-1}(G) = \tau V_\alpha(G)$. For any $\alpha \in \mathbf{Q}$, define $V_\alpha(G)$ as the set of $[\omega]$ in G satisfying a differential equation of the kind

$$\prod_{\beta \leqslant \alpha} (\tau\partial_\tau + \beta)^{\nu_\beta}[\omega] = \tau P(\tau, \tau\partial_\tau)[\omega].$$

On the quotient space $H_\alpha = V_\alpha(G)/V_{<\alpha}(G)$ exists a nilpotent endomorphism N induced by the action of $-(\tau\partial_\tau + \alpha)$. Put $T = \exp(2i\pi\alpha \operatorname{Id} + N)$. Consider also the decreasing filtration $F^p(H_\alpha) = G_{n-p}H_\alpha$.

Theorem. *The space $\oplus_{\alpha\in[0,1[} H_\alpha$ with its endomorphism T is naturally isomorphic to $(H_{\mathbf{C}}, T)$. Moreover, the filtration $\oplus_{\alpha\in[0,1[} F^p H_\alpha$ is the limit Hodge filtration on $H_{\mathbf{C}}$.*

A basic isomorphism. We will make use of the following isomorphism, which emphasizes either the chart τ (*i.e.*, the lattices $V_\bullet(G)$) or the chart θ (*i.e.*, the Brieskorn lattice), for any $\alpha \in \mathbf{Q} \cap [0, 1[$.

$$\begin{tikzcd}
\dfrac{G_p \cap V_\alpha}{G_{p-1} \cap V_\alpha + G_p \cap V_{<\alpha}} = & \mathrm{gr}_F^{n-p}(H_\alpha) \arrow[d, shift left, "\cdot\,\theta^p"] \\
\dfrac{V_{\alpha+p} \cap G_0}{V_{\alpha+p} \cap G_{-1} + V_{<\alpha+p} \cap G_0} = & \mathrm{gr}^V_{\alpha+p}(G_0/G_{-1}) \arrow[u, shift left, "\cdot\,\tau^p"]
\end{tikzcd}$$

The multiplicity of $\beta \in \mathbf{Q}$ in the *spectrum at infinity* of f is

$$\nu_\beta = \dim \operatorname{gr}^V_\beta (G_0/G_{-1}).$$

1.2. Some combinatorics. Consider the set

$$\mathcal{S}_w \overset{\text{def}}{=} \coprod_{i=0}^{n} \{\ell/w_i \mid \ell = 0, \dots, w_i - 1\}$$

and number its elements in an increasing way $s(0) \leqslant \cdots \leqslant s(\mu-1)$. Put $\sigma_w(k) = k - \mu s(k)$. The image of $\mathcal{S}_w$ in $\mathbf{Q}$ is $S_w = \bigcup_{i=0}^n \{\ell/w_i \mid \ell = 0, \dots, w_i - 1\}$. For $s \in S_w$, let $m(s)$ the number of its preimages in $\mathcal{S}_w$ (the number of ways of writing s as ℓ/w_i).

One has

$$\begin{aligned}
\prod_\beta (X+\beta)^{\nu_\beta} &= \prod_{k=0}^{\mu-1} (X + \sigma_w(k)) \quad \text{(proved in [2])} \\
&= \prod_{s \in S_w} \prod_{j=0}^{m(s)-1} \big(X + n - (j + \{sw_0\} + \cdots + \{sw_n\})\big) \quad \text{(É. Mann)}
\end{aligned}$$

1.3. A nice presentation of the Gauss-Manin system. Let ω_0 be the n-form on U defined by

$$\omega_0 = \frac{\frac{du_0}{u_0} \wedge \cdots \wedge \frac{du_n}{u_n}}{d\big(\prod_i u_i^{w_i}\big)}\Bigg|_{\prod_i u_i^{w_i} = 1}.$$

Define inductively, for $k \geqslant 0$,

$$[\omega_{k+1}] = -\frac{1}{\mu\tau}(\tau\partial_\tau + \sigma_w(k))[\omega_k].$$

Then

Proposition

- ω_0 *induces an isomorphism between the μ-dimensional vector spaces*

$$\mathbf{C}[u, u^{-1}] \Big/ \Big(u_0 \frac{\partial f}{\partial u_0}, \dots, u_n \frac{\partial f}{\partial u_n}\Big) \xrightarrow{\ \sim\ } G_0/G_{-1},$$

defining thus a ring structure on G_0/G_{-1}.

- $[\omega_\mu] = [\omega_0]$ *and* $[\omega_k] = [u^{a(k)}\omega_0]$ *for some multi-index $a(k)$ with $|a(k)| = k$.*
- *The classes $[\omega_0], [\omega_1], \dots, [\omega_{\mu-1}]$ form a $\mathbf{C}[\theta]$-basis $[\boldsymbol{\omega}]$ of G_0 and a $\mathbf{C}[\tau, \tau^{-1}]$-basis of G. In this basis, the connection is written as*

$$-\partial_\tau[\boldsymbol{\omega}] = [\boldsymbol{\omega}] \cdot \Big[A_0 + \frac{1}{\tau}A_\infty\Big], \quad \text{or} \quad \partial_\theta[\boldsymbol{\omega}] = [\boldsymbol{\omega}] \cdot \Big[\frac{1}{\theta^2}A_0 + \frac{1}{\theta}A_\infty\Big]$$

with

$$A_0=\mu\begin{pmatrix}0&0&\cdots&\cdots&1\\1&0&\cdots&\cdots&0\\0&1&0&\cdots&0\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&\cdots&0&1&0\end{pmatrix},\qquad A_\infty=\operatorname{diag}\big(\sigma_w(0),\dots,\sigma_w(\mu-1)\big).$$

- *The "sesquilinear" pairing* $S:G\otimes_{\mathbf{C}[\tau,\tau^{-1}]}G^{\mathrm{a}}\to\mathbf{C}[\tau,\tau^{-1}]$ *defined by*

$$S([\omega_k],[\omega_\ell])=\begin{cases}1 & \textit{if } k+\ell\equiv n \bmod \mu\\ 0 & \textit{otherwise}\end{cases}$$

is, up to a constant, the pairing coming from Poincaré duality on the fibres of f. *It is compatible with the connection.*

***Corollary* (Simplification of the structure)**

- *The* V*-order of* $[\omega_k]$ *is equal to* $\sigma_w(k)$ *and* $[\boldsymbol{\omega}]$ *induces a* $\mathbf{C}$*-basis* $\{\boldsymbol{\omega}\}$ *of* $\oplus_\beta\operatorname{gr}^V_\beta(G_0/G_{-1})$.
- *The graded space* $\oplus_{\beta\in\mathbf{Q}}\operatorname{gr}^V_\beta(G_0/G_{-1})$ *is a (rationally) graded space equipped with a graded commutative product* $\cup$ *such that*

$$\{\omega_k\}\cup\{\omega_\ell\}=\begin{cases}\{\omega_{k+\ell \bmod \mu}\} & \textit{if } \sigma_w(k+\ell \bmod \mu)=\sigma_w(k)+\sigma_w(\ell),\\ 0 & \textit{otherwise.}\end{cases}$$

and a nondegenerate pairing induced by S. *Both induce a symmetric trilinear form*

$$\big\langle\{\omega_j\},\{\omega_k\},\{\omega_\ell\}\big\rangle=S(\{\omega_j\},\{\omega_k\}\cup\{\omega_\ell\})=\begin{cases}1 & \textit{if } \begin{cases}j+k+\ell\equiv n \bmod \mu \textit{ and}\\ \sigma_w(j)+\sigma_w(k)+\sigma_w(\ell)=n,\end{cases}\\ 0 & \textit{otherwise.}\end{cases}$$

Remark. This graded algebra G_0/G_{-1}, with its nondegenerate pairing, looks very much like the orbifold cohomology algebra of $\mathbf{P}(w_0,\dots,w_n)$ with its Poincaré duality pairing.

Theorem. *The basis* $[\boldsymbol{\omega}]$ *is canonically attached to* f.

Explanation. The space $H_{\mathbf{C}}$ has a real structure. The conjugate filtration $\overline{F}^\bullet H_{\mathbf{C}}$ is in general not opposite to $F^\bullet H_{\mathbf{C}}$. It is opposite on a convenient graded space of $H_{\mathbf{C}}$ with respect to the *weight filtration* $W_\bullet H_{\mathbf{Q}}$. There is a canonical way of constructing an opposite filtration to $F^\bullet H_{\mathbf{C}}$ by putting

$$F'^\bullet H_{\mathbf{C}}=\sum_q\overline{F}^q\cap W_{n+q-\bullet}.$$

Lemma. *There is a* 1-1 *correspondence between opposite filtrations to* $F^{\bullet}H_{\mathbf{C}}$ *and free, rank* μ, $\mathbf{C}[\tau]$*-submodules* G'^{0} *of* G *on which the connection is logarithmic and such that* (G_0, G'^{0}) *defines a trivial vector bundle on* $\mathbf{P}^1$.

The statement of the theorem means that the G'^{0} coming from $F'^{\bullet}H_{\mathbf{C}}$ is

$$\mathbf{C}[\tau]\big\langle[\omega_0],\ldots,\tau^{[\sigma_w(k)]}[\omega_k],\ldots,\tau^{[\sigma_w(\mu-1)]}[\omega_{\mu-1}]\big\rangle. \qquad \square$$

2. The canonical Frobenius structure

2.1. General results. Let M be a complex manifold of dimension μ. A *Frobenius structure* on M, as defined by B. Dubrovin, or a *flat structure*, a defined by K. Saito, consists of 3 data on the tangent bundle TM of M:

(1) a homogeneity (Euler) holomorphic vector field $\mathfrak{E}$ on M,

(2) a commutative associative product $\circ$ with unit e on vector fields, depending holomorphically on the point in M,

(3) a nondegenerate bilinear form g on vector fields, depending holomorphically on the point in M, such that the associated torsionless connection $\bigtriangledown$ is *flat* (curvature 0).

These data have to be related by the following properties

- the 4-tensor

$$(\xi_1,\xi_2,\xi_3,\xi_4)\longmapsto \bigtriangledown_{\xi_1}g(\xi_2\circ\xi_3,\xi_4)$$

is symmetric in its arguments;

- $\bigtriangledown e = 0$,
- $\mathcal{L}_{\mathfrak{E}}e=-e$, $\mathcal{L}_{\mathfrak{E}}(\circ)=\circ$, $\mathcal{L}_{\mathfrak{E}}(g)=Dg$ for some $D\in\mathbf{C}$.

Theorem. *let* f *be any Laurent polynomial on a torus* $U=(\mathbf{C}^*)^n$, *which is convenient and nondegenerate with respect to its Newton polyhedron; in particular,*

$$\mu \overset{\text{def}}{=} \dim \mathbf{C}[v_1,v_1^{-1},\ldots,v_n,v_n^{-1}]/(v_1\partial f/\partial v_1,\ldots,v_n\partial f/\partial v_n)<+\infty.$$

Choose a family $\varphi_0=1,\varphi_1,\ldots,\varphi_{\mu-1}$ *inducing a basis of this vector space. Then there exists a canonical Frobenius structure locally on the space of parameters* $x_0,\ldots,x_{\mu-1}$ *of the unfolding* $F=f+\sum x_i\varphi_i$, *for which*

- $\partial_{x_i}\circ\partial_{x_j}=\delta_{ij}\partial_{x_i}$,
- $\mathfrak{E}=\sum_i x_i\partial_{x_i}$.

***Theorem* (Generation of a semisimple Frobenius structure, B. Dubrovin)**

Given $(A_0, A_\infty, \omega_0)$, *where*

- A_0, A_∞ *are* $\mu\times\mu$ *complex matrices,* A_0 *regular semisimple,* $A_\infty + {}^tA_\infty = w\,\mathrm{Id}$ *for some* $w\in\mathbf{Z}$,

• *ω_0 is an eigenvector of A_∞, corresponding to the eigenvalue α, with non zero component on the basis of eigenvectors of A_0,*

there exists a unique Frobenius structure on some neighbourhood of $x^o = \operatorname{Spec} A_0$ in $M = \mathbf{C}^\mu \smallsetminus$ diagonals such that, in the canonical basis $\partial_{x_1}, \dots, \partial_{x_n}$ of M,

• *A_0 is the matrix of $\xi \mapsto \mathfrak{E} \circ \xi$ at x^o,*
• *A_∞ is the matrix of $\xi \mapsto \nabla_\xi \mathfrak{E}$ at x^o,*
• *ω_0 is the vector of components of the unit field e at x^o,*
• *$2\alpha + 2 - w$ is the homogeneity constant D.*

We now come back to $f = \sum_{i=0}^n w_i u_i$ on U.

Theorem. *The data*

$$\omega_0 = \frac{\frac{du_0}{u_0} \wedge \cdots \wedge \frac{du_n}{u_n}}{d\big(\prod_i u_i^{w_i}\big)}\Big|_{\prod_i u_i^{w_i}=1},$$

$$A_0 = \mu \begin{pmatrix} 0 & 0 & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$A_\infty = \operatorname{diag}\big(\sigma_w(0), \dots, \sigma_w(\mu-1)\big)$$

generate the canonical Frobenius structure attached to any universal unfolding of $\sum_{i=0}^n w_i u_i$ on $\prod_i u_i^{w_i} = 1$.

References

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