Differential systems of Gaussian type

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e.g., $\{$monodr. $+$ Stokes$\}(M)$ defined over $\mathbb{Q}$,
$\Longrightarrow$ $\{$monodr. $+$ Stokes$\}(\widehat{M})$ ***defined over*** $\mathbb{Q}$

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- Similar answer for the Fourier transform of ℓ-adic sheaves on $\mathbb{A}^1_{\mathbb{F}_q}$

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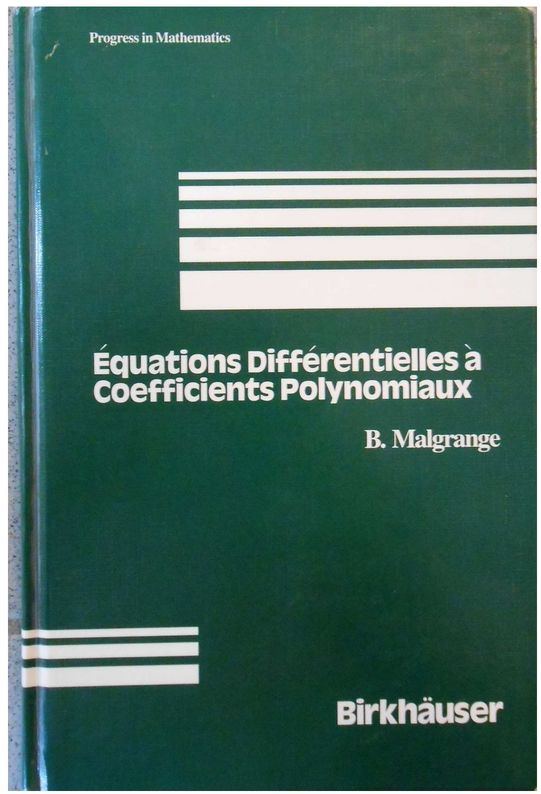

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\mathbb{C}[t]\langle\partial_t\rangle\text{-mod.} & \xleftrightarrow{\text{Laplace}} & \mathbb{C}[\tau]\langle\partial_\tau\rangle\text{-mod.} \\
{\scriptstyle\mathbf{RH}}\downarrow & & \downarrow{\scriptstyle\mathbf{RH}} \\
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- Work of T. Mochizuki giving an explicit family of path of integration for computing Laplace integrals.
- Goal: To understand these computations in the framework of complex and real algebraic geometry by using ***Asympt. Analysis in $\mathbb{C}$-dim. two***.

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$$\boxed{\mathbb{C}(\!(1/t)\!)\otimes M\simeq\bigoplus_{c\in C}\mathbb{C}(\!(1/t)\!)\otimes(E^{-ct^2/2}\otimes R_c)}$$

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- ***Formal stationary phase*** $\Longrightarrow$
  - $\widehat{M}$ of Gaussian type $\widehat{C}$
  - $\widehat{C} = \{\widehat{c} := -1/c \mid c \in C\}$
  - $R_{\widehat{c}} = R_c$.

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- ***Grading condition of type C***:

$$\boxed{\mathrm{gr}_c \mathscr{L} := \mathscr{L}_{\leqslant c}/\mathscr{L}_{<c} = \begin{cases} 0 & \text{if } c \notin C, \\ \text{Hor. sect. of } R_c & \text{if } c \in C. \end{cases}}$$

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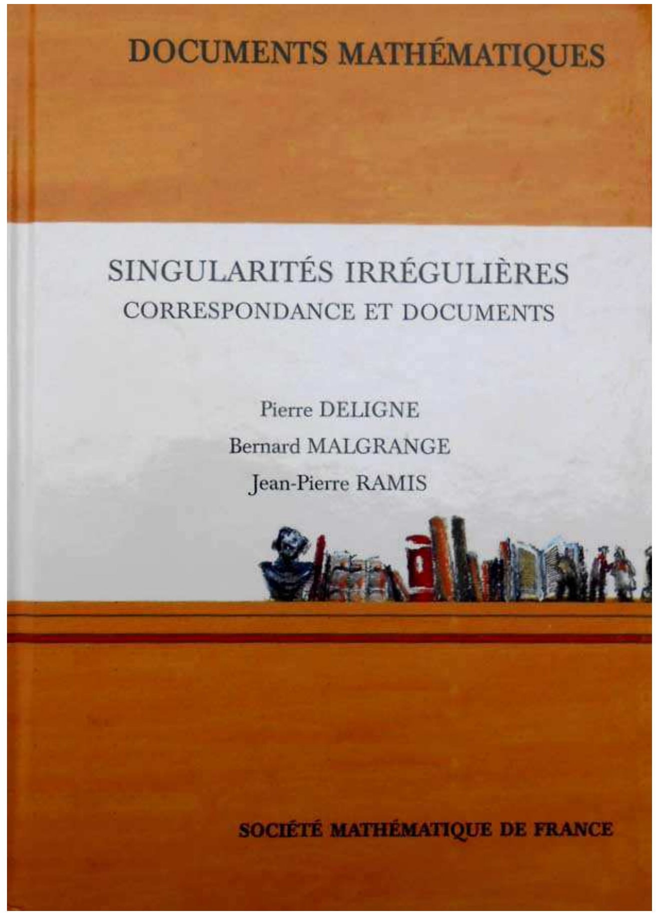

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THEOREM (Deligne, Malgrange): Equiv. of categories

M Gaussian type $C \longleftrightarrow (\mathscr{L},\mathscr{L}_\bullet)_{\mathbb{C}}$ Gaussian type C

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- ***Grading condition***. $\forall \nu \in \mathbb{Z}/4\mathbb{Z}$, ***oppositeness***:

$$\boxed{L = \bigoplus_{i=1}^{n} \big(L_i^{(\nu)} \cap L_i^{(\nu+1)}\big)}$$

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- $(L_i^{(\nu)})_{i=1,\dots,n} = \begin{cases} \text{increasing } (\nu \text{ even}) \\ \text{decreasing } (\nu \text{ odd}) \end{cases}$ filtr. of L
- ***Grading condition***. $\forall \nu \in \mathbb{Z}/4\mathbb{Z}$, ***oppositeness***:

$$\boxed{L = \bigoplus_{i=1}^{n} \big(L_i^{(\nu)} \cap L_i^{(\nu+1)}\big)}$$

$$(\mathscr{L}, \mathscr{L}_\bullet) \longleftrightarrow \begin{cases} L = \Gamma(\widetilde{\mathbb{P}}^1, \mathscr{L}) = \mathscr{L}_{\theta_o^{(\nu)}} \quad \forall \nu, \\ L_i^{(\nu)} = \mathscr{L}_{\leqslant c_i, \theta_o^{(\nu)}} \end{cases}$$

Laplace transf. for Gaussian type

$$
\begin{tikzcd}
M \text{ Gauss. type } C \arrow[r, "\text{Laplace}"] \arrow[d, mapsto, "\wr"'] & \widehat{M} \text{ Gauss. type } \widehat{C} \arrow[d, mapsto, "\wr"] \\
\big(C, \theta_o, (L, L_\bullet^{(\nu)}, \nu \in \mathbb{Z}/4\mathbb{Z})\big) \arrow[r, "?"] & \big(\widehat{C}, \underset{\pi-\theta_o}{\widehat{\theta}_o}, (\widehat{L}, \widehat{L}_\bullet^{(\nu)}, \nu \in \mathbb{Z}/4\mathbb{Z})\big)
\end{tikzcd}
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- Difficult in general:

$$\text{order of } C \text{ at } \theta_o \overset{?}{\longleftrightarrow} \text{order of } \widehat{C} \text{ at } \widehat{\theta}_o$$

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- REMARK: $\exists$ Braid group action on Gaussian type C.

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Sheaf-theoretic Laplace transf.

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Formally at $(\infty, \widehat{\infty})$:

$$\mathbb{C}((1/t, 1/\tau)) \otimes (p^+ M \otimes E^{-t\tau + \gamma\tau^2/2})$$
$$\simeq \bigoplus_{c \in C} \mathbb{C}((1/t, 1/\tau)) \otimes (E^{-t\tau + \gamma\tau^2/2 - ct^2/2} \otimes p^+ R_c)$$

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$$X \xrightarrow{\varepsilon} \mathbb{P}^1_t \times \mathbb{P}^1_\tau, \quad t' = uv,\ \tau' = v, \quad \frac{\gamma u^2 - 2u - c}{2u^2v^2}$$

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Local form at a zero: $\boxed{\text{unit}/v^2 \text{ or } w/v^2 \text{ or } w^2/v^2}$

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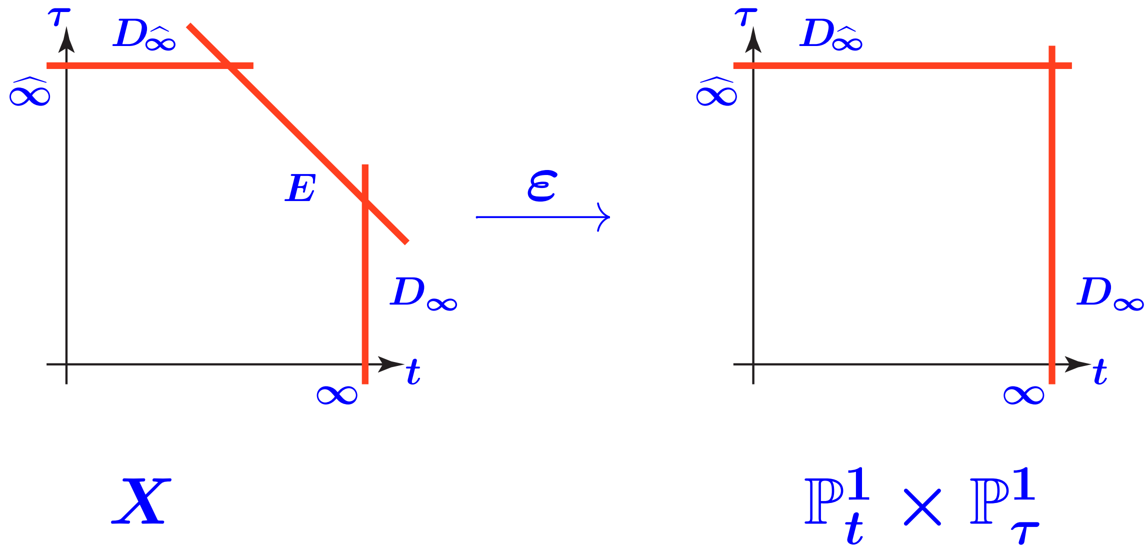

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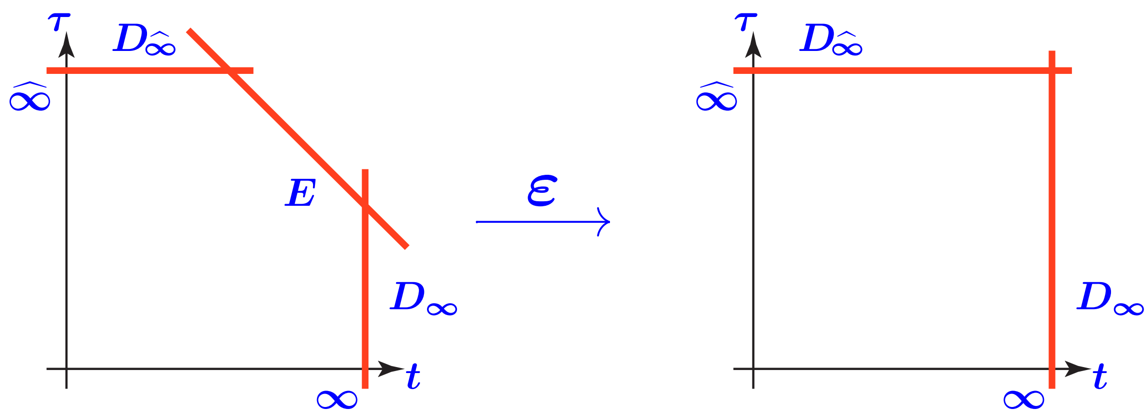

$$X \qquad\qquad\qquad \mathbb{P}^1_t \times \mathbb{P}^1_\tau$$

$$\widetilde{X} \xrightarrow{\ \widetilde{\varepsilon}\ } \mathbb{P}^1_t \times \widetilde{\mathbb{P}}^1_\tau$$

$\widetilde{X}$: real oriented blow-up of X along the compon. of D

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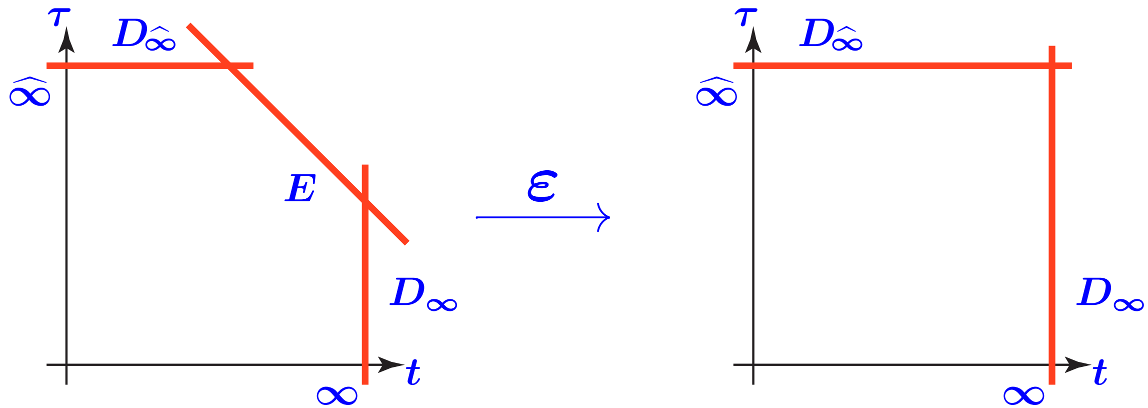

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THEOREM (Majima,...):
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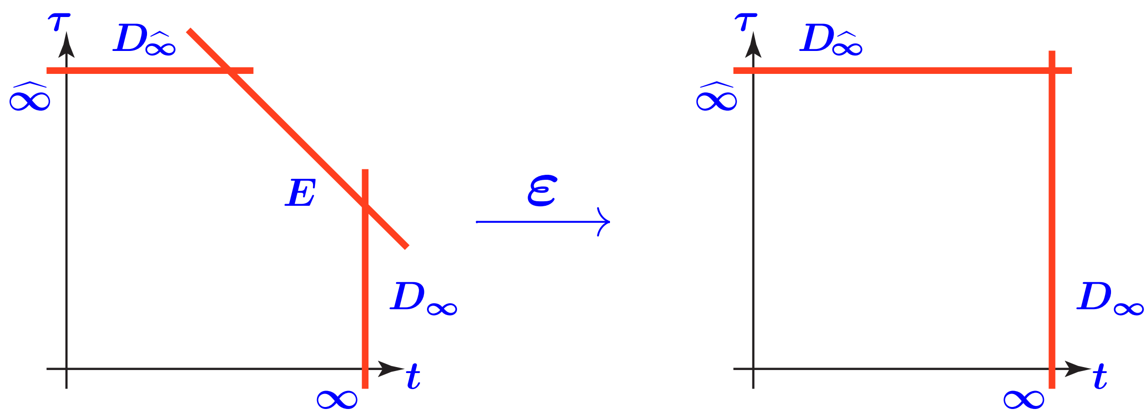

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$\mathscr{H}^0$ ***easy to compute sheaf-theor. in term of*** $(\mathscr{L}, \mathscr{L}_\bullet)$

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- $\widetilde{\widehat{q}} := \widetilde{\widehat{p}} \circ \widetilde{\varepsilon} : \widetilde{X} \longrightarrow \mathbb{P}^1_t \times \widetilde{\mathbb{P}}^1_\tau \longrightarrow \widetilde{\mathbb{P}}^1_\tau$

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- Same procedure may be applied in general to define the ***top. Laplace transf.***

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Assumption: $0 < c_1 < \cdots < c_n$, $\widehat{\theta}_o^{(\nu)} = \pi - \theta_o + \nu\pi/2$

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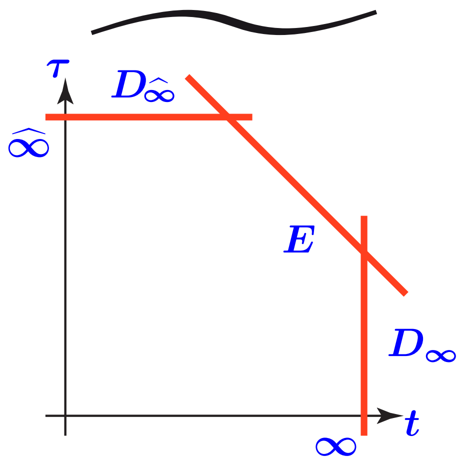

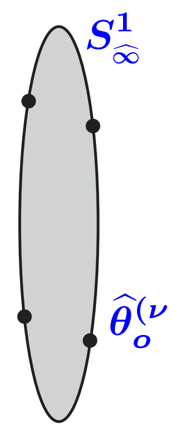

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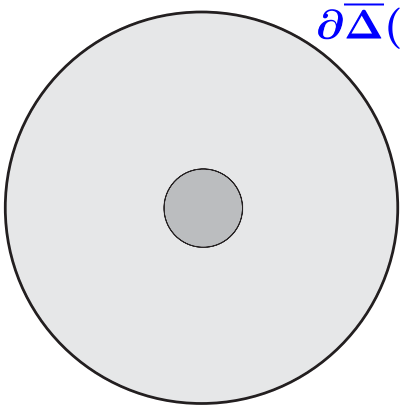

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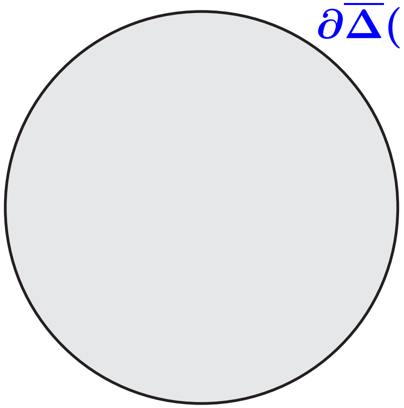

$\overline{\Delta}(\widehat{\theta}_o^{(\nu)})$ like $\widetilde{\mathbb{P}}^1_t$

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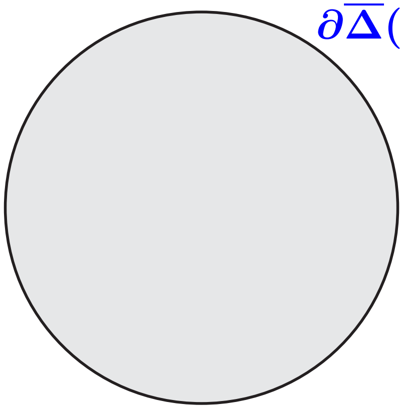

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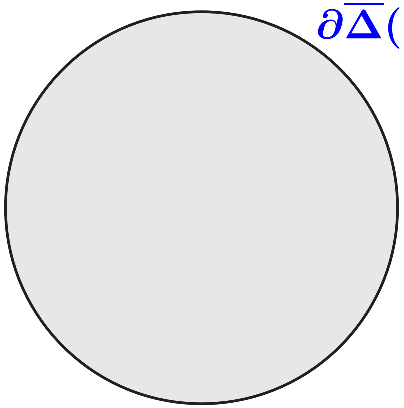

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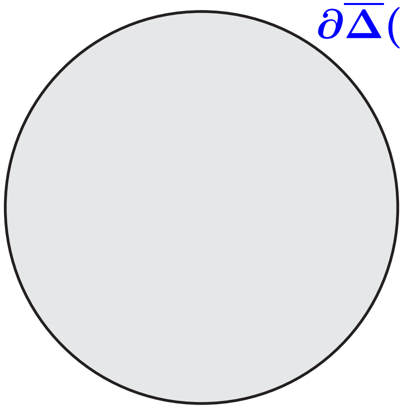

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$\tau_o \neq 0$ with $\arg(1/\tau) = \widehat{\theta}_o^{(\nu)}$

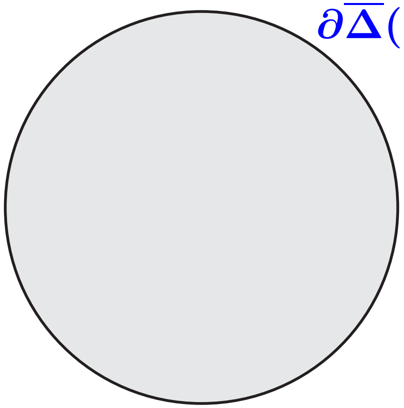
$$\widehat{\mathscr{L}}_{\widehat{\theta}_o^{(\nu)}} = H^1\big(\mathbb{P}^1_t, \mathrm{DR}(M \otimes E^{-t\tau_o})\big)$$
$$= H^1\big(\widetilde{\mathbb{P}}^1_t, \mathrm{DR}^{\mathrm{mod}\,\infty}(\mathcal{M} \otimes E^{-t\tau_o})\big)$$

Computation of $\widehat{L}$

Assumption: $0 < c_1 < \cdots < c_n$, $\widehat{\theta}_o^{(\nu)} = \pi - \theta_o + \nu\pi/2$

$$\widetilde{X} \xrightarrow{\ \widetilde{\widehat{q}}\ } \widetilde{\mathbb{P}}^1_\tau$$

Fibre $\widetilde{\widehat{q}}^{\,-1}(\widehat{\theta}_o^{(\nu)}) = \overline{\Delta}(\widehat{\theta}_o^{(\nu)})$:

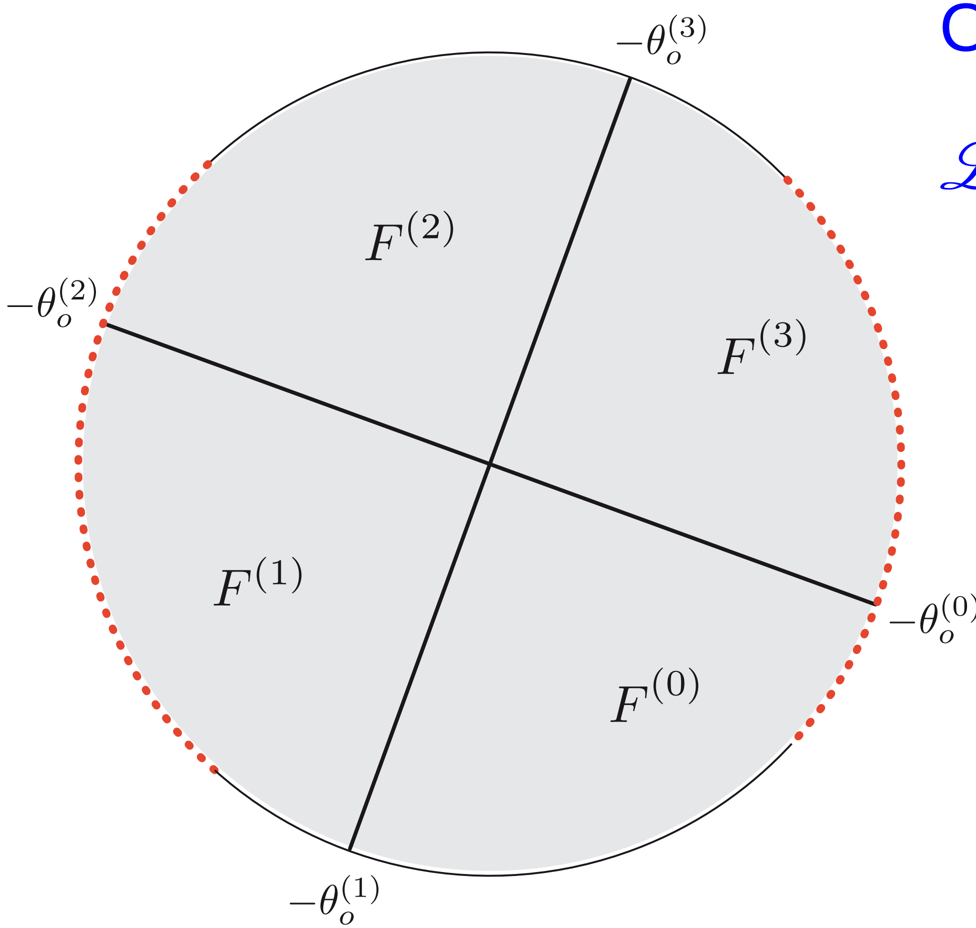


$\tau_o \neq 0$ with $\arg(1/\tau) = \widehat{\theta}_o^{(\nu)}$

$$\widehat{\mathscr{L}}_{\widehat{\theta}_o^{(\nu)}} = H^1\big(\mathbb{P}^1_t, \mathrm{DR}(M \otimes E^{-t\tau_o})\big)$$
$$= H^1\big(\widetilde{\mathbb{P}}^1_t, \mathrm{DR}^{\mathrm{mod}\,\infty}(\mathcal{M} \otimes E^{-t\tau_o})\big)$$
$$= H^1(\overline{\Delta}(\widehat{\theta}_o^{(\nu)}), \mathscr{L}_{\leqslant 0})$$

$(ct^2/2 + t\tau_o \sim ct^2/2$ if $t \to \infty)$

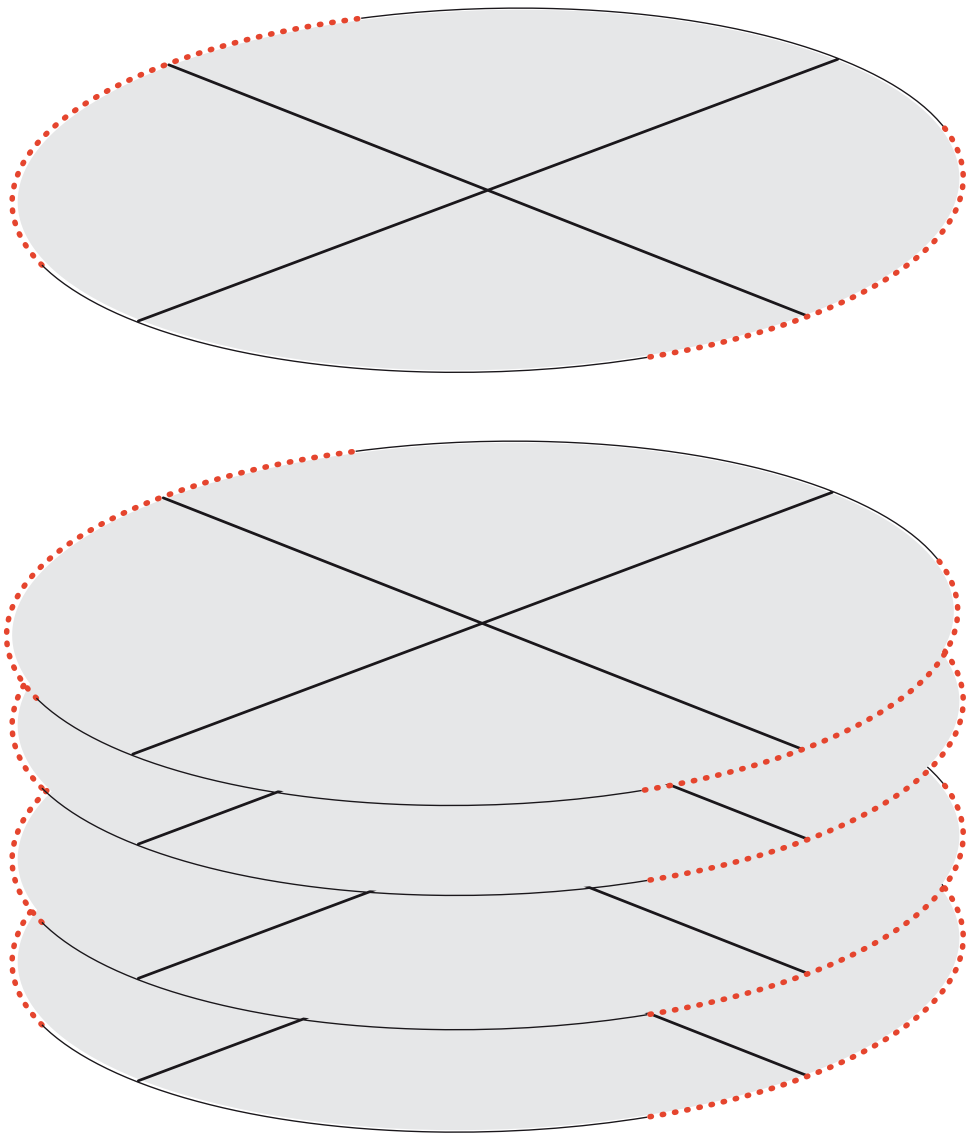
Computation of $\widehat{L}$



On each $F^{(\mu)}$

$$\mathscr{L}_{\leqslant 0} = \bigoplus_{c \in C} j_{c \leqslant 0,!}\, j^*_{c \leqslant 0}\, \mathrm{gr}_c\, \mathscr{L}$$

Computation of $\widehat{L}$



On each $F^{(\mu)}$

$$j_{c_n\leqslant 0,!}\, j^*_{c_n\leqslant 0}\, \mathrm{gr}_{c_n}\, \mathscr{L}$$

$$\oplus$$

$$\vdots$$

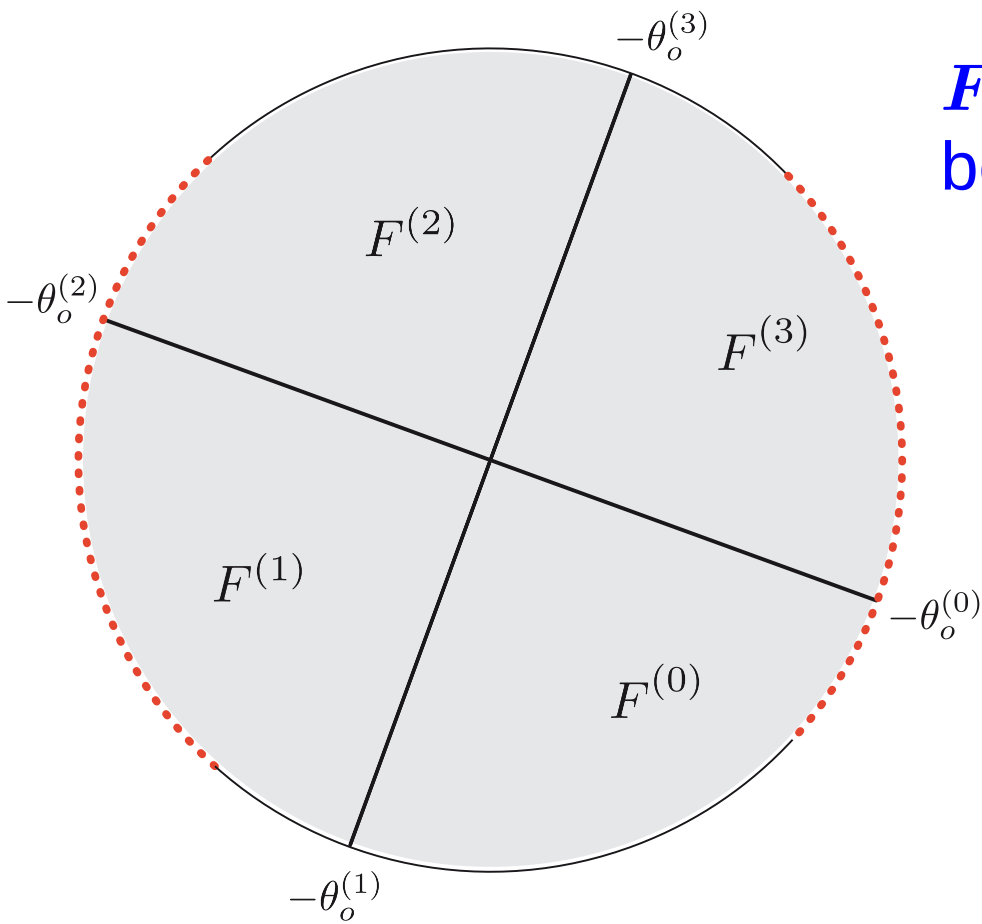
$$\oplus$$

$$j_{c_2\leqslant 0,!}\, j^*_{c_2\leqslant 0}\, \mathrm{gr}_{c_2}\, \mathscr{L}$$

$$\oplus$$

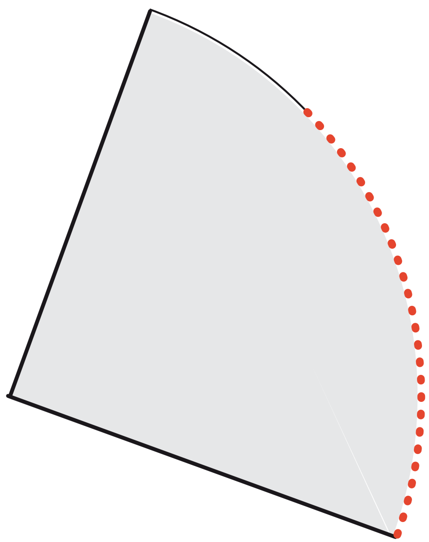
$$j_{c_1\leqslant 0,!}\, j^*_{c_1\leqslant 0}\, \mathrm{gr}_{c_1}\, \mathscr{L}$$

Leray covering



$\boldsymbol{F^{(\mu)}}$ Leray for $\mathscr{L}_{\leqslant \mathbf{0}}$ because

Leray covering



$H^j_c = 0 \ \forall j$

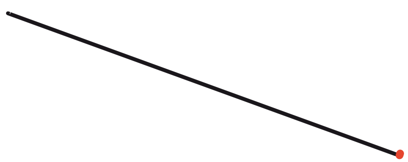
hence $\mathscr{C}^0(F^{(\bullet)}, \mathscr{L}_{\leqslant 0}) = 0$

Leray covering

$$H_c^j = 0 \; \forall j \geqslant 1$$

Leray covering

$$H_c^j = 0 \ \forall j$$



Leray covering

$$H_c^j = 0 \; \forall j \geqslant 1$$

•

Leray covering

$\Longrightarrow$

$$
\begin{aligned}
\widehat{L} = \widehat{\mathscr{L}}_{\widehat{\theta}_o^{(\nu)}} &= H^1\big(\overline{\Delta}(\widehat{\theta}_o^{(\nu)}), \mathscr{L}_{\leqslant 0}\big) \\
&\simeq \bigoplus\nolimits_{c\in C} H^0\Big(\Big/, \mathrm{gr}_c\,\mathscr{L}\Big) \\
&= \bigoplus\nolimits_{c\in C} \mathrm{gr}_c\,\mathscr{L}_{\theta_o^{(\nu)}} \\
&= L
\end{aligned}
$$

Computation of $\widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}_o^{(\nu)}}$

Computation of $\widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}_o^{(\nu)}}$

Assumption: ν odd,

$$\widehat{c}_n <_\nu \cdots < \widehat{c}_{k+1} <_\nu \gamma \leqslant_\nu \widehat{c}_k <_\nu \cdots <_\nu \widehat{c}_1$$

Computation of $\widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}_o^{(\nu)}}$

Assumption: ν odd,

$$\widehat{c}_n <_\nu \cdots < \widehat{c}_{k+1} <_\nu \gamma \leqslant_\nu \widehat{c}_k <_\nu \cdots <_\nu \widehat{c}_1$$

$$\widehat{\mathscr{L}}_{<\gamma} = R^1\widetilde{\widehat{q}}_* \underbrace{\mathscr{H}^0\,\mathrm{DR}^{\mathrm{rd}D}\,\varepsilon^+(p^+\mathcal{M}\otimes E^{-t\tau+\gamma\tau^2/2})}_{\mathscr{G}_{<\gamma}}$$

Computation of $\widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}_o^{(\nu)}}$

Assumption: ν odd,

$$\widehat{c}_n <_\nu \cdots < \widehat{c}_{k+1} <_\nu \gamma \leqslant_\nu \widehat{c}_k <_\nu \cdots <_\nu \widehat{c}_1$$

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$$\widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}_o^{(\nu)}} = H^1\big(\overline{\Delta}(\widehat{\theta}_o^{(\nu)}), \mathscr{G}_{<\gamma,\widehat{\theta}_o^{(\nu)}}\big)$$

Computation of $\widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}_o^{(\nu)}}$

Assumption: ν odd,

$$\widehat{c}_n <_\nu \cdots < \widehat{c}_{k+1} <_\nu \gamma \leqslant_\nu \widehat{c}_k <_\nu \cdots <_\nu \widehat{c}_1$$

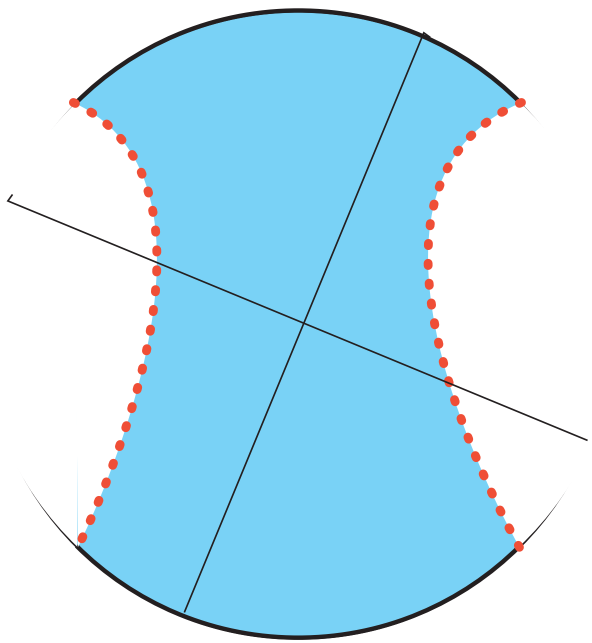
$$\widehat{\mathscr{L}}_{<\gamma} = R^1\widetilde{\widehat{q}}_* \underbrace{\mathscr{H}^0\,\mathrm{DR}^{\mathrm{rd}D}\,\varepsilon^+(p^+\mathcal{M}\otimes E^{-t\tau+\gamma\tau^2/2})}_{\mathscr{G}_{<\gamma}}$$

$$\widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}_o^{(\nu)}} = H^1\big(\overline{\Delta}(\widehat{\theta}_o^{(\nu)}), \mathscr{G}_{<\gamma,\widehat{\theta}_o^{(\nu)}}\big)$$

On each $F^{(\mu)}$,

$$\boxed{\mathscr{G}_{<\gamma,\widehat{\theta}_o^{(\nu)}} = \bigoplus_{c\in C} \mathscr{G}_{<\gamma,c,\widehat{\theta}_o^{(\nu)}}}$$

Computation of $\widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}_o^{(\nu)}}$

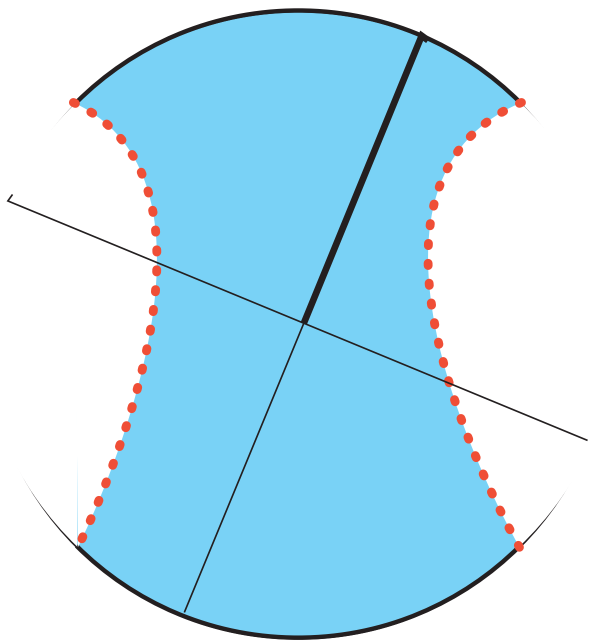


On each $F^{(\mu)}$

$\mathscr{G}_{<\gamma,c,\widehat{\theta}_o^{(\nu)}} = j_! j^* \operatorname{gr}_c \mathscr{L}$

Case $\hat{c} <_\nu \gamma$

Computation of $\widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}_o^{(\nu)}}$

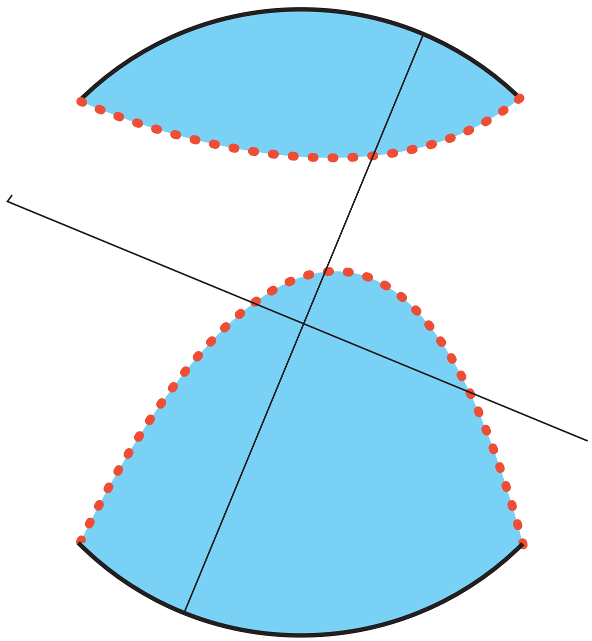


On each $F^{(\mu)}$

$\mathscr{G}_{<\gamma,c,\widehat{\theta}_o^{(\nu)}} = j_! j^* \operatorname{gr}_c \mathscr{L}$

Case $\hat{c} <_\nu \gamma$

Computation of $\widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}_o^{(\nu)}}$

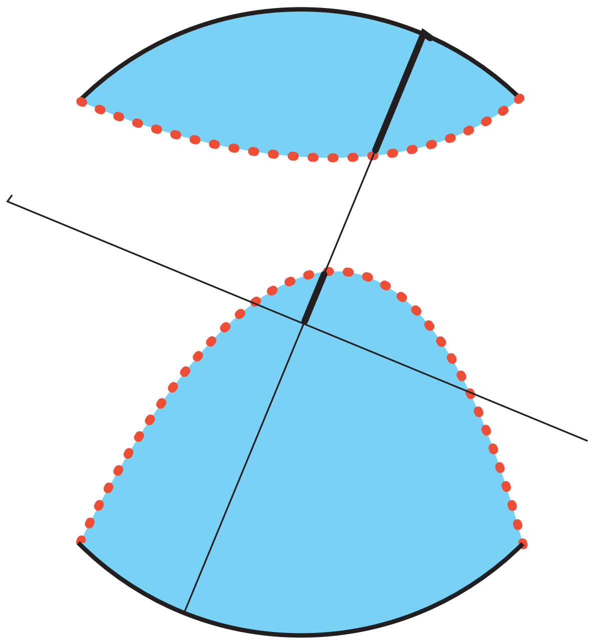


On each $F^{(\mu)}$

$\mathscr{G}_{<\gamma,c,\widehat{\theta}_o^{(\nu)}} = j_! j^* \operatorname{gr}_c \mathscr{L}$

Case $\gamma <_\nu \widehat{c}$

Computation of $\widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}_o^{(\nu)}}$

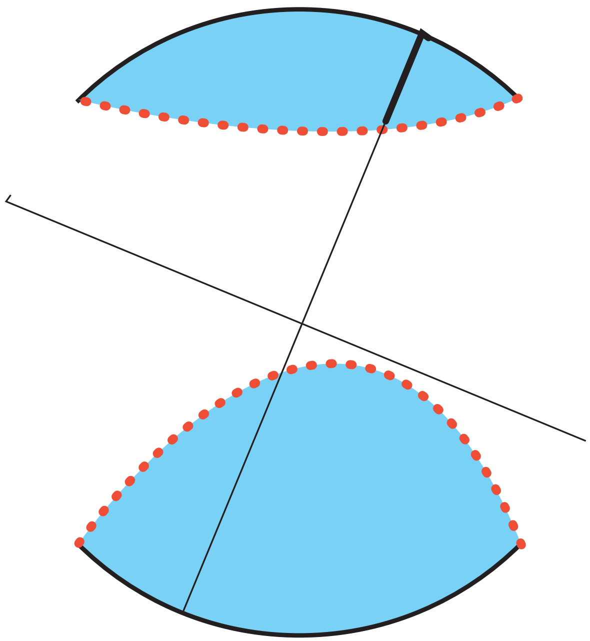


On each $F^{(\mu)}$

$\mathscr{G}_{<\gamma,c,\widehat{\theta}_o^{(\nu)}} = j_! j^* \, \mathrm{gr}_c \, \mathscr{L}$

Case $\gamma <_\nu \widehat{c}$

Computation of $\widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}_o^{(\nu)}}$

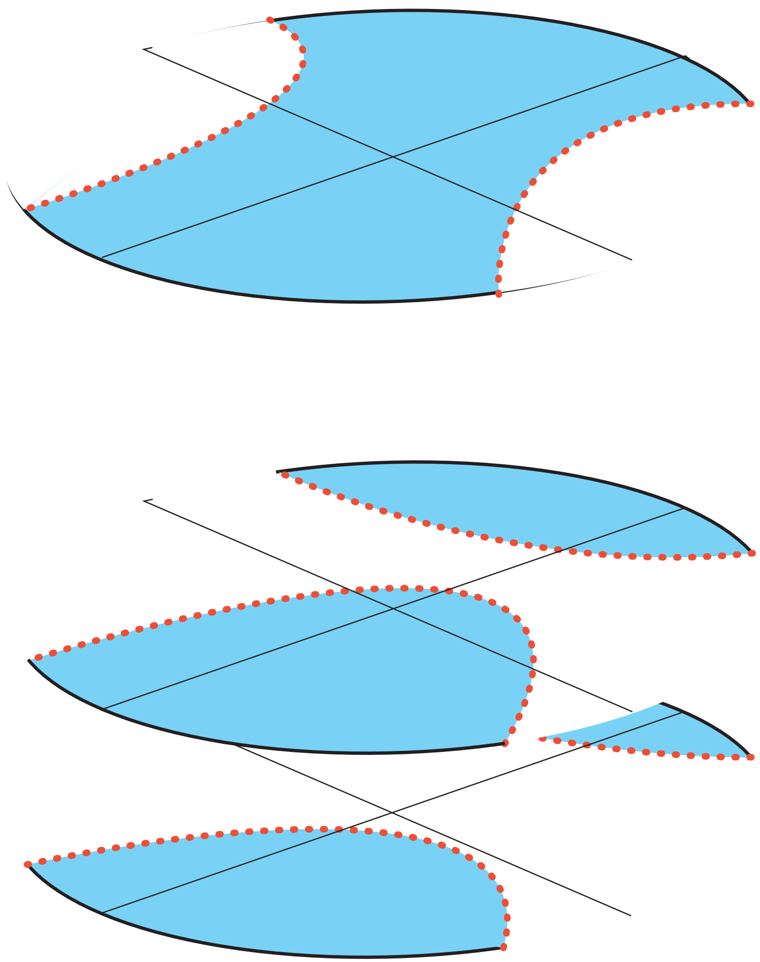


On each $F^{(\mu)}$

$\mathscr{G}_{<\gamma,c,\widehat{\theta}_o^{(\nu)}} = j_! j^* \operatorname{gr}_c \mathscr{L}$

Case $\gamma <_\nu \widehat{c}$

Computation of $\widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}_o^{(\nu)}}$



On each $F^{(\mu)}$

$$\mathscr{G}_{<\gamma,c_n,\widehat{\theta}_o^{(\nu)}}$$
$$\bigoplus$$
$$\vdots$$
$$\bigoplus$$
$$\mathscr{G}_{<\gamma,c_2,\widehat{\theta}_o^{(\nu)}}$$
$$\bigoplus$$
$$\mathscr{G}_{<\gamma,c_1,\widehat{\theta}_o^{(\nu)}}$$

Leray covering

$$\widehat{c}_n <_\nu \cdots < \widehat{c}_{k+1} <_\nu \gamma \leqslant_\nu \widehat{c}_k <_\nu \cdots <_\nu \widehat{c}_1$$

$$\Longrightarrow$$

$$\begin{aligned}
\widehat{L}^{(\nu)}_{<k} = \widehat{\mathscr{L}}_{<\gamma,\widehat{\theta}^{(\nu)}_o} &= H^1\big(\overline{\Delta}(\widehat{\theta}^{(\nu)}_o), \mathscr{G}_{<\gamma,\widehat{\theta}^{(\nu)}_o}\big) \\
&\simeq \bigoplus\nolimits_{j=k+1}^{n} H^0\Big(\;, \mathbf{gr}_{c_j} \mathscr{L}\Big) \\
&= \bigoplus\nolimits_{j=k+1}^{n} \mathbf{gr}_{c_j} \mathscr{L}_{\theta^{(\nu)}_o} \\
&= L^{(\nu)}_{<k}
\end{aligned}$$

$\square$