RECENT ADVANCES ON HOLONOMIC $\mathscr{D}$-MODULES

STOCKHOLM, MAY 2010

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1. Introduction

Let $u \in \mathscr{S}'(\mathbb{C}^n)$ be a tempered distribution on $\mathbb{C}^n$ which is the solution of differential equations $P_1, \ldots, P_r$, where each P_i takes the form $P = \sum_{\alpha\in\mathbb{N}^n} a_\alpha(x)\partial_x^\alpha$ and $a_\alpha(x) \in \mathbb{C}[x_1, \ldots, x_n]$. In 1986, M. Kashiwara made the following conjecture (in a more general setting, however):

Conjecture. *Assume that the system of differential equations induced by* $(P_1, \ldots, P_r)$ *is holonomic and that* $P_i u = 0$ *(*$i = 1, \ldots, r$*). Then there exists a holonomic system* $(Q_1, \ldots, Q_s)$ *of differential equations of the same kind such that* $\overline{Q_j} u = 0$ *(*$j = 1, \ldots, s$*).*

Kashiwara proved this result (in a more general setting) in the case the system $(P_1, \ldots, P_r)$ has *regular singularities* on $\mathbb{C}^n$ and included at infinity.

Example. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$.

(1) Set $u = |f|^{2\lambda}$. Then u is a tempered distribution for suitable $\lambda \in \mathbb{C}$, and is solution of a holonomic system of differential equations as well as a anti-holonomic system. Both have regular singularities.

(2) Set $u = e^{f-\overline{f}}$. Then u is a tempered distribution. It is a solution of a holonomic system and a anti-holonomic system, and both have an irregular singularity at infinity.

The question of holonomic and regular holonomic distributions occupies a whole chapter in Björk's book on analytic $\mathscr{D}$-modules. In this talk I will report on the following result:

Theorem. *The conjecture is true if* $n \leqslant 2$*.*

In fact, I had given a proof of this theorem in 2000, but the proof relied on conjectural properties of holonomic $\mathscr{D}$-modules when $n=2$. Recent results of T. Mochizuki on the one hand, and K. Kedlaya, on the other hand, solve these conjectural properties and open the way to the proof of the conjecture for any n.

2. Good formal structure

I consider now the local analytic setting. Let X be a complex manifold and let $\mathscr{D}_X$ be the sheaf of holomorphic differential operators. Fix a divisor D in X. An example of a holonomic $\mathscr{D}_X$-module is $\mathscr{O}_X(*D)$ equipped with its usual differential, or more generally the case of locally free $\mathscr{O}_X(*D)$-module $\mathscr{M}$ (called a meromorphic bundle) together with a flat meromorphic connection ∇ (the holonomicity follows from a theorem of Kashiwara on the b-function). By a theorem of Malgrange, each such flat meromorphic bundle is produced by a holomorphic bundle on X with a flat meromorphic connection, which is a more understandable object.

Assume that D is a normal crossing divisor and ∇ has regular singularities along D. Then there exists a local simple model (called Deligne meromorphic extension) for any such $(\mathscr{M},\nabla)$. This is an essential point in the proof of the conjecture by Kashiwara when regular singularities are assumed. However, up to recently, a similar statement was not known in general.

Let me recall the Turrittin-Levelt theorem in dimension one. It gives the existence of a formal model for $(\mathscr{M},\nabla)$. Usually, the corresponding formal isomorphism is not convergent. The Hukuhara-Turrittin theorem gives a holomorphic isomorphism in small sectors around the origin.

***Theorem* (Turrittin-Levelt).** *If $X=\Delta$ and $D=\{0\}$ then, given $(\mathscr{M},\nabla)$, up to a finite ramification w.r.t. x, there exists a finite set $\Phi\subset\mathscr{O}_{X,0}(*0)/\mathscr{O}_{X,0}=x^{-1}\mathbb{C}[x^{-1}]$ such that, setting $\widehat{\mathscr{M}}=\mathbb{C}[\![x]\!]\otimes_{\mathbb{C}\{x\}}\mathscr{M}$,*

$$(\widehat{\mathscr{M}},\widehat{\nabla})\xrightarrow[\sim]{\widehat{\lambda}}\bigoplus_{\varphi\in\Phi}\Big(\mathscr{O}_{X,0}(*0)^{d_\varphi},d+d\varphi\operatorname{Id}_{d_\varphi}+C_\varphi\frac{dx}{x}\Big),\qquad C_\varphi\in\mathrm{M}_{d_\varphi}(\mathbb{C}).$$

Let now $\varpi:\widetilde{X}=S^1\times[0,1)\to X$ $((e^{i\theta},\rho)\mapsto x=\rho e^{i\theta})$ be the real blow-up of X at the origin (space of polar coordinates). It is equipped with a sheaf of "holomorphic functions" $\mathscr{A}_{\widetilde{X}}=\ker\big[\overline{x}\partial_{\overline{x}}:\mathscr{C}^\infty_{\widetilde{X}}\to\mathscr{C}^\infty_{\widetilde{X}}\big]$. There is a surjective morphism (Borel-Ritt) $\mathscr{A}_{\widetilde{X}}\to\varpi^{-1}\mathbb{C}[\![x]\!]$.

***Theorem* (Hukuhara-Turrittin).** *For each $e^{i\theta}\in S^1$, the formal isomorphism $\widehat{\lambda}$ can be locally lifted as an isomorphism*

$$\mathscr{A}_{\widetilde{X}}\otimes_{\varpi^{-1}\mathscr{O}_X}(\mathscr{M},\nabla)\xrightarrow[\sim]{\lambda_\theta}\bigoplus_{\varphi\in\Phi}\Big(\mathscr{A}_{\widetilde{X},e^{i\theta}}(*0)^{d_\varphi},d+d\varphi\operatorname{Id}_{d_\varphi}+C_\varphi\frac{dx}{x}\Big).$$

Let now $X=\Delta^n$, $D=\{x_1\cdots x_\ell=0\}$, and $(\mathscr{M},\nabla)$ a meromorphic bundle with flat connection (poles are along D).

Definition. A finite set $\Phi \subset \mathscr{O}_{X,0}(*D)/\mathscr{O}_{X,0}$ is said to be *good* if for any pair $\varphi, \psi \in \Phi$ with $\varphi \neq \psi$, the divisor of zeros of $\varphi - \psi$ is empty near the origin.

For instance, $\Phi = \{0, x/y\}$ is not good.

Definition. We say that $(\mathscr{M}, \nabla)$ has a *good formal decomposition* at $0 \in \Delta^n$ if there exists a *good* finite set $\Phi \subset \mathscr{O}_{X,0}(*D)/\mathscr{O}_{X,0}$ such that, for any subset $I \subset \{1, \ldots, \ell\}$, we have on some neighbourhood U of 0,

$$(\mathscr{M}, \nabla)_{|\widehat{D}_I^o} \simeq \bigoplus_{\varphi_I \in \Phi_I} \Big(\mathscr{O}_{X,0}(*0)^{d_{\varphi_I}}, d + d\varphi_I \operatorname{Id}_{d_{\varphi_I}} + \sum_{i \in I} C_{\varphi_I, i} \frac{dx_i}{x_i}\Big), \quad C_{\varphi_I, i} \in \mathrm{M}_{d_{\varphi_i}}(\mathbb{C}).$$

Remark. The various decompositions hold on disjoint subsets D_I^o. No compatibility between the decompositions with respect to the various I is assumed, but the set Φ does not depend on I. The question of compatibility is quite subtle and needs the introduction of *good lattices*, which will not be explained in this talk.

As in dimension one, the idea to get analytic objects is to work on multi-sectors. Now $\varpi : \widetilde{X} \simeq (S^1)^\ell \times [0,1)^\ell \times \Delta^{n-\ell} \to X$ is the real blow-up space along the components $D_1, \ldots, D_\ell$, and $\mathscr{A}_{\widetilde{X}}$ is the corresponding sheaf of "holomorphic functions" on it.

***Theorem* (Hukuhara-Turrittin-Sibuya-Majima-C.S.-Mochizuki)**
Let $(\mathscr{M}, \nabla)$ be a meromorphic connection with poles along D. Assume that $\mathscr{M}$ has a good lattice with set of exponential factors $\Phi \subset \mathscr{O}_{X,0}(*D)/\mathscr{O}_{X,0}$. Then, for any $e^{i\theta} \in \varpi^{-1}(0)$, the previous decomposition for $I = \{1, \ldots, \ell\}$ can be lifted as a decomposition over $\mathscr{A}_{\widetilde{X},\theta}$.

3. Existence of a good formal structure

The Hukuhara-Turrittin-Majima theorem is essential in order to prove the conjecture of Kashiwara. However, it relies on the assumption of the existence of a good lattice (a strong version of a good formal decomposition).

***Conjecture* (C.S.).** *Assume* $\dim X = 2$ *and* $(\mathscr{M}, \nabla)$ *is a meromorphic bundle with flat connection having poles along a divisor D. There exists a finite sequence of point blowing-ups $e : X' \to X$ such that $D' := e^{-1}(D)$ has only normal crossings and $e^*(\mathscr{M}, \nabla)$ has a good formal structure (i.e., good formal decomposition after al local ramification around D') at each point of D'.*

Steps in the proof of this conjecture ($\dim X = 2$).

- C.S., 2000: proof if $\operatorname{rk} \mathscr{M} \leqslant 5$ and other particular cases.
- Y. André, 2007: proof of some consequence of the conjecture (Malgrange's conjecture on semi-continuity of irregularity).
- T. Mochizuki, 2008: proof when X is projective and $(\mathscr{M}, \nabla)$ is a rational connection.
- K. Kedlaya, 2009: proof in general.

***Remark* (on the proofs of Mochizuki and Kedlaya).** The difficult point is to guess the possible values of φ entering in the decomposition after blowing-up. The proof of Mochizuki is valid in the algebraic case because it uses reduction modulo a big prime number. Assume for instance that the matrix of the original connection is defined over $\mathbb{Z}[\![x,y]\!][x^{-1},y^{-1}]$, then Mochizuki reduces the coefficients modulo p for p large. The possible φ appear as eigenvalues of the p-curvature operator attached to the connection.

On the other hand, Kedlaya analyses the connection from a point of view coming from p-adic analysis of differential equations. A whole family of invariants is attached to a connection, parametrized by valuations considered as point on the Berkovich disc. These invariants define some functions on the Berkovich disc, whose properties (called sub-harmonicity) is essential for the proof.

The higher dimensional case. Soon after his proof for surfaces, T. Mochizuki proved a similar result in arbitrary dimension, still with the assumption that the connection is algebraic on an algebraic variety. The proof is completely different from the case of dimension two.

***Theorem* (T. Mochizuki).** *Let $(\mathscr{M},\nabla)$ be a meromorphic bundle with flat connection having poles along a divisor D on a smooth projective variety X. Then there exists a finite sequence of blowing-ups $e: X'\to X$ such that $D':=e^{-1}(D)$ has only normal crossings and $e^*(\mathscr{M},\nabla)$ has a good good lattice.*

More recently, Kedlaya has given a proof of the following:

***Theorem* (K. Kedlaya).** *Let $(\mathscr{M},\nabla)$ be a meromorphic bundle with flat connection having poles along a divisor D. Then, for any point $x\in D$ there exists an open neighbourhood $U\ni x$ and a finite sequence of blowing-ups $e: U'\to U$ such that $D':=e^{-1}(D)$ has only normal crossings and $e^*(\mathscr{M},\nabla)$ has a good formal structure (i.e., good formal decomposition after al local ramification around D') at each point of D'.*

However, it does not seem that Kedlaya proved the existence of a good lattice after blowing up, a result which is needed in dimension $\geqslant 3$ for proving Kashiwara's theorem. Note also that the algebraic variant of Kedlaya's theorem is similar to that of Mochizuki's theorem, with a good formal structure instead of a good lattice however.

4. Application to holonomic distributions ($\dim X=2$)

When $\dim X=2$, the existence of a good formal structure along a normal crossing divisor implies the existence of a good lattice (a result due to T. Mochizuki), so we can apply the Hukuhara-Turrittin-Majima theorem after point blowing-ups, according to the results of Mochizuki (algebraic case) or Kedlaya (local analytic case).

Various reductions. According to there results of Mochizuki and Kedlaya, there one can use various reductions which are also used in the regular case by Kashiwara (and also explained in Björk's book), which reduce to the following local statement.

Proposition. *Set $X = \Delta^2$ with coordinates (x_1, x_2), $D = x_1 x_2 = 0$, and let $(\mathscr{M}, \nabla)$ be a meromorphic bundle with flat connection having poles on D which has a good formal decomposition at $x_1 = x_2 = 0$ (with a good set Φ of exponential factors such that $\Phi \cup \{0\}$ is also good). Then $C_X \mathscr{M} := \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathfrak{Db}_X^{\mathrm{mod}\,D})$ is a anti-meromorphic bundle with flat connection, and $\mathscr{E}xt^i_{\mathscr{D}_X}(\mathscr{M}, \mathfrak{Db}_X^{\mathrm{mod}\,D}) = 0$ for $i > 0$.*

Now the Hukuhara-Turrittin-Majima theorem almost gives the result, namely it gives it on the real blow-up space $\widetilde{X}$ (space of polar coordinates with respect to x_1, x_2).

***Corollary* (of Hukuhara-Turrittin-Majima Theorem).** *Under the assumptions of the proposition, $C_{\widetilde{X}} \mathscr{M} := \mathscr{H}om_{\varpi^{-1}\mathscr{D}_X}(\varpi^{-1}\mathscr{M}, \mathfrak{Db}_{\widetilde{X}}^{\mathrm{mod}\,D})$ is a locally free $\mathscr{A}_{\widetilde{X}}(*D)$-module with flat connection, and $\mathscr{E}xt^i_{\varpi^{-1}\mathscr{D}_X}(\varpi^{-1}\mathscr{M}, \mathfrak{Db}_{\widetilde{X}}^{\mathrm{mod}\,D}) = 0$ for $i > 0$.*

The question that remains is to show that $\overline{C_{\widetilde{X}} \mathscr{M}} = \mathscr{A}_{\widetilde{X}}(*D) \otimes_{\varpi^{-1}\mathscr{O}_X(*D)} \varpi^{-1}\mathscr{N}$ for some meromorphic bundle $\mathscr{N}$ with connection. For this, one needs a control of the Stokes matrices by the conjugation functor. This argument is specific to the case with irregular singularities.

5. Real structures on holonomic $\mathscr{D}$-modules

Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. The $\mathscr{D}_X$-module $\overline{C_X \mathscr{M}} := \mathscr{H}om_{\mathscr{D}_{\overline{X}}}(\overline{\mathscr{M}}, \mathfrak{Db}_X)$ plays the role of the Hermitian dual of $\mathscr{M}$, and is hopefully holonomic. A real structure on $\mathscr{M}$ can be defined as an isomorphism $\iota : \mathscr{M} \xrightarrow{\sim} \overline{C_X \mathscr{M}^\vee}$ of $\mathscr{D}_X$-modules such that $\iota \circ \overline{C_X \iota^\vee} = \mathrm{Id}$.

Such a real structure induces a real structure on the associated de Rham complex, which is a $\mathbb{C}$-perverse sheaf. Conversely, a real structure on a $\mathbb{C}$-perverse sheaf determines a real structure on $\mathscr{M}$ if $\mathscr{M}$ has regular singularities, but this is not enough in general if $\mathscr{M}$ has irregular singularities.

Very recently, T. Mochizuki has defined the notion of a real structure (or a $\boldsymbol{k}$-structure, for any subfield $\boldsymbol{k}$ of $\mathbb{C}$) on any holonomic $\mathscr{D}$-module, which satisfies all the expected functoriality properties.

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