Irregular Hodge theory: Applications to arithmetic and mirror symmetry

Claude Sabbah

Centre de Mathématiques Laurent Schwartz
CNRS, École polytechnique, Institut Polytechnique de Paris
Palaiseau, France

Origins and motivations of irreg. Hodge theory

Deligne, 1984.

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- ***Griffiths' regularity theorem:***
  - (V,∇): alg. vect. bdle with connect. on a quasi-proj. curve.
  - (V,∇) underlies a PVHS $\implies$ ∇ has reg. sing. at ∞.
- E.g., regularity of the Gauss-Manin connection.
- Complex analogues of exponential sums over finite fields: (V,∇) with ***irreg. sing.*** at ∞.
- Is there a Hodge realization for such objects?
- Typical example: "e^x" on $\mathbb{A}^1 \xhookrightarrow{j} \mathbb{P}^1$, i.e., $(j_*\mathscr{O}_{\mathbb{A}^1}, \mathrm{d}+\mathrm{d}x)$.

- Deligne defines a $\searrow$ filtration $F^\bullet(j_*V)$ in many examples.
- $\rightsquigarrow$ Filtration of the de Rham complex

$$F^p\,\mathrm{DR}(j_*V,\nabla) := \{0 \to F^p(j_*V) \xrightarrow{\nabla} \Omega^1_{\mathbb{P}^1}\otimes F^{p-1}(j_*V) \to 0\}$$

- In these examples, ***degeneration at E^1***, i.e.,

$$\boldsymbol{H}^1(\mathbb{P}^1, F^p\,\mathrm{DR}(j_*V,\nabla)) \hookrightarrow \boldsymbol{H}^1(\mathbb{P}^1, \mathrm{DR}(j_*V,\nabla)).$$

- Filtration indexed by $p \in A+\mathbb{N}$, $A \subset [0,1)$ finite.

- *What could be the use of a "Hodge filtration" which does not lead to Hodge theory? A hope it that it imposes bounds to p-adic valuations of eigenvalues of Frobenius.*

Adolphson-Sperber, 1987–89.

- Lower bound of the p-adic Newton polygon of the L-function attached to a nondeg. Laurent pol. $f \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ given by a Newton polygon attached to f.
- $\rightsquigarrow$ Answers Deligne's hope, but no Hodge filtration.
- (Would like to interpret this as "Newton above Hodge".)

Simpson, 1990.

- Non abelian Hodge theory on curves. Correspondence between (V, ∇) with reg. sing. (tame) at ∞ and stable tame parabolic Higgs bdles.
- Simpson suggests it would be possible to extend this correspondence to (V, ∇) ***wild*** (i.e., with irreg. sing.).
- $\rightsquigarrow$ Positive answer on curves by CS and Biquard-Boalch (2000 $\pm\varepsilon$).
- Positive answer (any dimension) by T. Mochizuki (2011).
- ***Drawback:*** no Hodge filtration.

Mirror symmetry for Fano's.

- Need to consider a pair (X, f), $f : X \to \mathbb{A}^1$, X smooth quasi-proj., as possible mirror of a Fano mfld.
- $\rightsquigarrow$ Various cohomologies $H^\bullet(X, f)$ attached to (X, f), e.g.
  - dual of Betti homology (Lefschetz thimbles),
  - de Rham cohomology: hypercohom of $(\Omega_X^\bullet, \mathrm{d} + \mathrm{d}f)$,
  - Periodic cyclic homology,
  - Exponential motives.

Questions on the Hodge theory of Landau-Ginzburg models.

- If (X, f) is mirror of a Fano mfld Y, what is the Hodge filtration on $H^\bullet(X, f)$ corresponding to that of $H^\bullet(Y)$?
- If Y is a Fano orbifold (e.g. toric, like $\mathbb{P}(w_0, \ldots, w_n)$), $H^\bullet_{\mathrm{orb}}(Y)$ (Chen-Ruan) has rational exponents (corresponding to "twisted sectors"). Natural to expect that $F^\bullet$ for (X, f) is indexed by $A + \mathbb{N}$, $A \subset [0, 1) \cap \mathbb{Q}$.
- If Y is a Fano mfld, how to translate to $F^\bullet H^n(X, f)$ Hard Lefschetz for $c_1(TY)$?

E_1-degeneration

Hodge realization for a pair (X, f).

- X smooth quasi-proj.
- Choose a compact. $f : \overline{X} \to \mathbb{P}^1$ of f s.t. $D = \overline{X} \smallsetminus X$ ncd.
- $P := f^*(\infty)$, $\quad |P| \subset D$.

$$H^k_{\mathrm{dR}}(X, f) \simeq \begin{cases} \boldsymbol{H}^k(\overline{X}, (\Omega^\bullet_{\overline{X}}(*D), \mathrm{d} + \mathrm{d}f)), \\ \boldsymbol{H}^k(\overline{X}, (\Omega^\bullet_{\overline{X}}(\log D, f), \mathrm{d} + \mathrm{d}f)) \end{cases}$$

$$\Omega^k_{\overline{X}}(\log D, f) := \left\{\omega \in \Omega^k_{\overline{X}}(\log D) \mid \mathrm{d}f \wedge \omega \in \Omega^{k+1}_{\overline{X}}(\log D)\right\}$$
$$= \left\{\omega \in \Omega^k_{\overline{X}}(\log D) \mid (\mathrm{d} + \mathrm{d}f \wedge)\omega \in \Omega^{k+1}_{\overline{X}}(\log D)\right\}$$

- Quasi-isomorphic filtered complexes:
  - Yu: $\quad F^\bullet(\Omega^\bullet_{\overline{X}}(*D), \mathrm{d} + \mathrm{d}f)$,
  - K-K-P: $F^\bullet(\Omega^\bullet_{\overline{X}}(\log D, f), \mathrm{d} + \mathrm{d}f))$.

$$F^p(\Omega^\bullet_{\overline{X}}(\log D, f), \mathrm{d}) := \{0 \to \underset{p}{\Omega^p(\log D, f)} \to \cdots \to \Omega^n(\log D, f) \to 0\}$$

- Recall: for X quasi-projective (and $f \equiv 0$)

***Theorem* (Degeneration at E_1, Deligne (Hodge II, 1972)).**

$$\boldsymbol{H}^\bullet(\overline{X}, F^p(\Omega^\bullet_{\overline{X}}(\log D), \mathrm{d})) \hookrightarrow \boldsymbol{H}^\bullet(\overline{X}, (\Omega^\bullet_{\overline{X}}(\log D), \mathrm{d})) \simeq H^\bullet(X, \mathbb{C}).$$

***Theorem* (Esnault-S.-Yu, Katzarkov-Kontsevich-Pantev, M. Saito, T. Mochizuki).**

- *The spectral seq. for $F^\bullet(\Omega^\bullet_{\overline{X}}(*D), \mathrm{d} + \mathrm{d}f)$, equivalently for $F^\bullet(\Omega^\bullet_{\overline{X}}(\log D, f), \mathrm{d} + \mathrm{d}f))$, degenerates at E_1.*
- $\rightsquigarrow$ ***Irreg. Hodge filtr.*** $F^\bullet H^k_{\mathrm{dR}}(X, f)$.
- Four different proofs:
  - M. Saito uses a comparison with nearby cycles of f along $f^*(\infty)$ and Steenbrink/Schmid limit theorems.
  - K-K-P use reduction to char. p à la Deligne-Illusie. But need assumption that $f^*(\infty)$ is reduced.
  - E-S-Y use reduction to $X = \mathbb{A}^1$ by pushing forward by f and previous results on CS extending the original construction of Deligne on curves by means of ***twistor D-modules***.
  - T. Mochizuki uses the full strength of twistor D-modules in arbitrary dimensions.
- Can take into account multiplicities of $f^*(\infty)$ to refine $F^\bullet$ and index it by $A + \mathbb{N}$,

$$A = \left\{\ell/m_i \mid 0 \leqslant \ell < m_i,\ m_i = \text{mult. of a component of } f^*(\infty)\right\}.$$

Computation of Hodge numbers by means of irregular Hodge theory

- Standard course of calculus: often easier to compute convolution $f \star g$ by applying ***Fourier transformation***.
- Same idea for Hodge nbrs.
- Arithmetic motivation: Functional equation for the L-function attached to symmetric power moments of Kloosterman sums.
- Complex analogue of the Kloosterman sums: modified Bessel differential equation on $\mathbb{G}_{\mathrm{m}}$.
- $\mathrm{Kl}_2 : (\mathcal{O}^2_{\mathbb{G}_{\mathrm{m}}}, \nabla), \quad \nabla(v_0, v_1) = (v_0, v_1) \cdot \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} \cdot \frac{\mathrm{d}z}{z}$.
- For $k \geqslant 1$, want to consider $\mathrm{Sym}^k \mathrm{Kl}_2$:
  - free $\mathbb{C}[z, z^{-1}]$-mod. rk $k+1$ with connection,

  and its de Rham cohomology

$$H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) = \mathrm{coker}\Big[\nabla : \mathrm{Sym}^k \mathrm{Kl}_2 \longrightarrow \mathrm{Sym}^k \mathrm{Kl}_2 \otimes \frac{\mathrm{d}z}{z}\Big]$$

***Theorem* (Fresán-S-Yu).** *Assume k odd for simplicity.*

- *$H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ canonically endowed with a MHS of weights $k+1$ & $2k+2$.*
- $\dim H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)^{p,q} = 1$ *if $p+q = k+1$ and $p = 2, \dots, k-1$ or $p = q = k+1$, and 0 otherwise.*

Synopsis.

- ***Motivations.*** Series of papers by Broadhurst-Roberts: some Feynman integrals expressed as period integrals

$$\int_0^\infty I_0(t)^a K_0(t)^b t^c \mathrm{d}t \qquad (I_0, K_0 : \text{“modified Bessel functions”}).$$

  $\rightsquigarrow$ various conjectures on L fns of Kloosterman moments.
- On $\mathrm{Sym}^k \mathrm{Kl}_2$, ∇ has a regular sing. at $z = 0$, but an ***irregular*** one at ∞, hence ***does not*** underlie a PVHS (Griffiths th.).
- $H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ has a ***motivic*** interpretation: this explains the MHS.
- $\mathrm{Sym}^k \mathrm{Kl}_2$ underlies a ***variation of irregular Hodge structure*** (i.e., an irregular mixed Hodge module on $\mathbb{P}^1 \supset \mathbb{G}_{\mathrm{m}}$).
- $\Longrightarrow$ $H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ endowed with an ***irregular Hodge filtration***.
- We prove that this irreg. Hodge filtr. ***coincides*** with the Hodge filtr. of the MHS.
- We compute this irreg. Hodge filtration by toric methods of Adolphson-Sperber & Yu. (Irreg. analogue of Danilov-Khovanski computation for toric hypersurfaces).

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Motivic interpretation.

- (Kl_2, ∇) is the Gauss-Manin conn. of $\big(\mathscr{O}_{\mathbb{G}^2_{\mathrm{m}}}, \mathrm{d} + \mathrm{d}(x + z/x)\big)$ by the proj. $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}} \quad (x, z) \mapsto z$.
- $\big(\bigotimes^k \mathrm{Kl}_2, \nabla\big)$: G-M conn. of $\big(\mathscr{O}_{\mathbb{G}_{\mathrm{m}} \times \mathbb{G}^k_{\mathrm{m}}}, \mathrm{d} + \mathrm{d}(f_k)\big)$

$$f_k(x_1, \dots, x_k, z) = \textstyle\sum_i (x_i + z/x_i)$$

- Set $\widetilde{\mathrm{Kl}}_2 = [2]^* \mathrm{Kl}_2, \quad [2] : t \mapsto t^2$. Set $y_i = x_i/t$.
- Then $\big(\bigotimes^k \widetilde{\mathrm{Kl}}_2, \nabla\big)$: G-M conn. of $E^{t \cdot g_k} := \big(\mathscr{O}_{\mathbb{G}_{\mathrm{m}} \times \mathbb{G}^k_{\mathrm{m}}}, \mathrm{d} + \mathrm{d}(t \cdot g_k)\big)$

$$g_k(y_1, \dots, y_k) = \textstyle\sum_i (y_i + 1/y_i) : \mathbb{G}^k_{\mathrm{m}} \to \mathbb{A}^1.$$

- $$\begin{aligned} H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) &\simeq H^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \textstyle\bigotimes^k \widetilde{\mathrm{Kl}}_2)^{\mathfrak{S}_k \times \mu_2} \\ &\simeq \boxed{H^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}} \times \mathbb{G}^k_{\mathrm{m}}, t \cdot g_k)^{\mathfrak{S}_k \times \mu_2}} \end{aligned}$$

- General fact (Fresán-Jossen, F-S-Y): U smooth quasi-proj., $g : U \to \mathbb{A}^1$ regular, $H^n_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}} \times U, t \cdot g)$ underlies a Nori motive, hence endowed with a canonical MHS.
- Analogue of Fourier inversion formula for $h : \mathbb{R} \to \mathbb{R}$:

$$h(0) = \star \int_{\mathbb{R}} \widehat{h}(t)\,\mathrm{d}t = \star \int_{\mathbb{R}^2} e^{2\pi i\, t \cdot h(x)} \mathrm{d}t\,\mathrm{d}x.$$

- Set $\mathscr{K} = g_k^{-1}(0) \subset \mathbb{G}^k_{\mathrm{m}}$. Variant of what we want:

$$H^{k+1}(\mathbb{A}^1 \times \mathbb{G}^k_{\mathrm{m}}, t \cdot g_k) \simeq H^{k-1}_{\mathrm{c}}(\mathscr{K})^\vee(-k).$$

Irregular mixed Hodge structures

There exist various generalizations of a MHS on a k-vect. space ($k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$).

- ***Mixed twistor structure*** (Simpson, 1997).
  - $\rightsquigarrow$ mixed twistor D-module (T. Mochizuki, 2011).
- ***Semi-infinite pure Hodge structure*** (Barannikov, 2001).
  - $\rightsquigarrow$ Construction of Frobenius mfld structures.
- ***Pure TERP structure*** (Hertling, 2002).
  - $\rightsquigarrow$ tt^* geometry on Frobenius manifolds.
- ***Non-commutative Hodge structure*** (Katzarkov-Kontsevich-Pantev, 2008).
  - $\rightsquigarrow$ Hodge theory for periodic cyclic homology of some dg-algebras.
- ***Exponential mixed Hodge structure*** (Kontsevich-Soibelman, 2011).
  - $\rightsquigarrow$ Hodge theory for cohomological Hall algebras.
- ***Irregular Hodge structure*** (S-Yu, 2018).
  - $\rightsquigarrow$ General framework for the irregular Hodge filtration.
- ***Example.*** $H^k_{\mathrm{dR}}(X, f)$ "underlies" an exponential MHS, hence an irreg. MHS, $F^\bullet H^k_{\mathrm{dR}}(X, f)$ is the irreg. Hodge filtration.

Integrable mixed twistor structure.

- Object $\boxed{((\mathscr{T}, \nabla), W_\bullet)}$:
  - $\mathscr{T}$: hol. vect. bdle on $\mathbb{P}^1 = \mathbb{A}^1_u \cup \mathbb{A}^1_v$ (twistor structure),
  - ∇: merom. connection on $\mathscr{T}$, pole of order $\leqslant 2$ at 0 & ∞, no other pole (integrable twistor structure),
  - $W_\bullet$: $\nearrow$ filtr. of $(\mathscr{T}, \nabla)$ such that each $\mathrm{gr}^W_\ell(\mathscr{T}, \nabla)$ is ***pure of weight*** ℓ, i.e., $\mathrm{gr}^W_\ell \mathscr{T} \simeq \mathscr{O}_{\mathbb{P}^1}^{r_\ell}(\ell)$ (integr. mixed twistor str.).
- Can add: polarization (in the pure case), real or rational structure (on the local system $\ker \nabla$ on $\mathbb{C}^*$ + Stokes struct. at $0, \infty$).
- Associated vector space H: $\mathscr{T}_1$ = fibre at 1

Irregular Hodge filtration.

- $((\mathscr{T}, \nabla), W_\bullet) \rightsquigarrow \searrow$ filtration on H:
  - $\boxed{(\mathscr{M}, \nabla) = (\mathscr{T}, \nabla)|\mathbb{A}^{1\,\mathrm{an}}_u}$
  - $\forall \alpha \in [0, 1)$, $(\mathscr{M}^\alpha, \nabla)$: vect. bdle on $\mathbb{P}^1$, extending $(\mathscr{M}, \nabla)$ s.t. ∇ has a ***log. sing.*** at $v = 0$, with residues having real part in $[\alpha, \alpha + 1)$ (Deligne's extension).
  - $\mathrm{HN}^p(\mathscr{M}^\alpha)$: Harder-Narasimhan filtr.
  - $\boxed{F^{p-\alpha}_{\mathrm{irr}} H} := \mathrm{HN}^p(\mathscr{M}^\alpha)|_1 \subset H$.

Irregular Hodge structure.

Definition. Category IrrMHS: subcategory of integr. mixed twistor structures with good limit properties w.r.t. the rescaling $u \mapsto \lambda \cdot u$ ($\lambda \to \infty$).

Example.

- X smooth quasi-projective and $f : X \to \mathbb{A}^1$ ***proper*** or ***tame***.
- $H = H^k_{\mathrm{dR}}(X, f)$.

Theorem. *$H^k_{\mathrm{dR}}(X, f)$ underlies a pure object of* IrrMHS, *with*

$$(\mathscr{M}, \nabla_u) = (H^k_{\mathrm{dR,rel.}}(X \times \mathbb{A}^1_u, f/u), \nabla_u)$$

and

$$\boxed{F^\bullet_{\mathrm{irr}} H = F^\bullet H^k_{\mathrm{dR}}(X, f)}$$

- for $\omega \in \Omega^k_{X\times\mathbb{A}^1_u/\mathbb{A}^1_u}$:

$$\nabla_X \omega = e^{-f/u} \cdot \mathrm{d}_X \cdot e^{f/u}(\omega),$$

$$\nabla_u \omega = e^{-f/u} \cdot \frac{\partial}{\partial u} \cdot e^{f/u}(\omega) = \boxed{-\frac{f}{u^2}\,\omega} + \partial_u \omega.$$

- $\mathscr{M}$: hypercohomology on X of

$$\cdots \longrightarrow \Omega^{k-1}_X[u] \xrightarrow{\nabla_X} \Omega^k_X[u] \xrightarrow{\nabla_X} \Omega^{k+1}_X[u] \longrightarrow \cdots$$

Irregular Hodge-Tate structures

- $(\mathscr{T}, \nabla)$ pure irreg. MHS of some weight,
- $F^\bullet_{\mathrm{irr}} H$: irreg. Hodge filtr.
- Jumps of $F^\bullet_{\mathrm{irr}} H$ are integers $\iff$ ***unipotent*** monodromy on $\ker \nabla_{|\mathbb{C}^*}$.
- ***unipotent*** monodromy $\rightsquigarrow$ Jakobson-Morosov filtr. $M_\bullet H$ associated to its nilpotent part.

Definition. $(\mathscr{T}, \nabla)$ is ***irreg. Hodge-Tate*** if

$$\forall p, \quad \boxed{\dim \mathrm{gr}^M_{2p} H = \dim \mathrm{gr}^p_{F_{\mathrm{irr}}} H \quad \text{and} \quad \mathrm{gr}^M_{2p+1} H = 0}$$

***Conjecture* (K-K-P, 2017).** *If (X, f) is the Landau-Ginzburg model mirror to a projective Fano mfld Y, then the irreg. MHS $H^n(X, f)$ ($n = \dim X$) is pure and irregular Hodge-Tate.*

Many works on the conjecture.

- Lunts, Przyjalkowski, Harder
- Shamoto

The toric case.

- Lattices $M \subset \mathbb{R}^n$, $N = M^\vee$.
- $\Delta \subset \mathbb{R}^n$: reflexive simplicial polyhedron with vertices in M, s.t. 0 is the only integral point in Δ.
- Δ^*: dual polyhedron (vertices in N and of the same kind as Δ).
- Σ: fan dual to Δ, $=$ cone $(0, \Delta^*)$.
- $Y = \mathbb{P}_\Sigma$ assumed smooth, hence toric Fano (Batyrev).
- Chow ring $A^*(Y) \simeq H^{2*}(Y, \mathbb{Z})$ generated by div. classes D_v, $v \in \mathrm{Vertices}(\Delta^*) =: V(\Delta^*)$.
- $c_1(K_Y^\vee) = \sum_{v \in V(\Delta^*)} D_v$ satisfies Hard Lefschetz on $H^{2*}(Y, \mathbb{Q})$.

- Coordinates $x_1, \dots, x_n$ s.t. $\mathbb{C}[N] = \mathbb{C}[x, x^{-1}]$.

$$X := \mathrm{Spec}\, \mathbb{C}[x, x^{-1}],$$

$$f : X \longrightarrow \mathbb{A}^1, \qquad f(x) = \sum_{v \in V(\Delta^*)} x^v$$

$$H^n_{\mathrm{dR}}(X, f) = \Omega^n_X / (\mathrm{d} + \mathrm{d}f\wedge)\Omega^{n-1}_X \simeq \big[\mathbb{C}[x, x^{-1}]/(\partial f)\big] \cdot \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}$$

- Newton filtration $\mathcal{N}_\bullet$ on the Jacobian ring $\mathbb{Q}[x, x^{-1}]/(\partial f)$
- Borisov-Chen-Smith: $\boxed{H^{2*}(Y, \mathbb{Q}) \simeq \mathrm{gr}^{\mathcal{N}}_\bullet\big(\mathbb{Q}[x, x^{-1}]/(\partial f)\big)}$
- ***Hard Lefschetz*** $\Longrightarrow$ $\forall k$ s.t. $0 \leqslant k \leqslant n/2$,

$$\boxed{f^{n-2k} : \mathrm{gr}^{\mathcal{N}}_k\big(\mathbb{Q}[x, x^{-1}]/(\partial f)\big) \xrightarrow{\sim} \mathrm{gr}^{\mathcal{N}}_{n-k}\big(\mathbb{Q}[x, x^{-1}]/(\partial f)\big)}$$

- Idea of Varchenko from Singularity theory (Doklady, 1981): interpret multipl. by f as the nilpotent part of a monodromy operator.
- Adapt and apply this idea to $H^n_{\mathrm{dR}}(X, f)$
- $\Longrightarrow$ irreg. Hodge-Tate property. □