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A GLOBAL POINT OF VIEW ON SINGULARITIES OF FUNCTIONS

SEOUL, JANUARY 22–24, 2024

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Abstract. We illustrate the notion of irregular Hodge theory on the example of pairs (U, f) formed by a smooth quasi-projective variety with a regular function. Such a datum can be sometimes seen as a mirror image of a projective Fano manifold (or orbifold). After indicating Deligne's (1984) initial motivation for such a theory, we introduce the notion of exponential mixed Hodge structure (Kontsevich-Soibelman), which we illustrate for pairs (U, f) by introducing the notion of space of global vanishing cycles and its irregular Hodge filtration. We conclude with some general results on the theory of irregular mixed Hodge modules, including a Kodaira-Saito-type vanishing theorem for irregular Hodge bundles.

LECTURE 1

1.1. Introduction

Our main object of study in these lectures consists of pairs (U, f) formed by a smooth quasi-projective variety with a regular function on it. Techniques of Singularity theory and of Algebraic geometry can be simultaneously applied

- by regarding (U, f) as analogous to $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$,
- and by considering the particular case $U = (U, 0)$.

Mirror symmetry for Fano manifolds or orbifolds can produce such pairs, known as Landau-Ginzburg models.

- Singularity theory leads us to define spaces of *global vanishing cycles* $H^k(U, f)$ together with a monodromy structure.
- Algebraic geometry leads us to consider simultaneously
  - the Betti (topological) aspects, defined over $\mathbb{Q}$,
  - the de Rham aspects (over $\mathbb{C}$, or $\mathbb{Q}$ if (U, f) is defined over $\mathbb{Q}$),
  - the Hodge aspects (which Hodge theory is suitable for such pairs?),
  - the motivic aspects (make clear the relations between these various aspects, e.g. develop the notion of *exponential period*).

Irregular Hodge theory (Deligne, 1984). To $f \in \mathbb{Z}[x_1, \dots, x_n]$ one can associate for each $q = p^k$ an exponential sum $\sum_{x\in\mathbb{F}_q^n} \exp\bigl(2\pi i(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} f(x))/p\bigr)$, and then, varying p, an L-function, which, in good cases (e.g. f is tame) is a polynomial or the inverse of a polynomial with coefficients in $\mathbb{Q}(\zeta_p)$. Deligne introduced a notion of irregular Hodge filtration as an invariant which would allow one to bound the p-adic valuation of the coefficients of this polynomial.

Running example. Let M be the standard lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Let $\Delta \subset \mathbb{R} \otimes M$ be a convex polyhedron with set $V(\Delta)$ of vertices contained in M and having 0 in its interior. We say that Δ is

- simplicial is each face is a simplex,

• reflexive if the equation of any hyperplane containing a codimension-one face takes the form $L(x)+1=0$ for some integral linear form L. (In such a case, 0 is the unique integral point in the interior of Δ.)

Let $\Delta^\vee = \{y \in (\mathbb{R}^n)^\vee \mid \langle y, x\rangle \geqslant -1\}$ be the dual polyhedron. Then $V(\Delta^\vee) \subset N = M^\vee$, and $\Delta^\vee$ is also simplicial reflexive with 0 in its interior. The fan $\Sigma \subset N$ dual to Δ, is also the cone on $\Delta^\vee$ with apex 0. We assume that Σ is the fan of nonsingular toric variety X_Σ of dimension n, that is, each set of vertices of the same $(n-1)$-dimensional face of $\partial\Delta^\vee$ is a $\mathbb{Z}$-basis of N. It is known (Batyrev) that X_Σ is Fano and any smooth toric Fano variety is obtained like this.

Let us fix coordinates $x=(x_1,\dots,x_n)$ such that $\mathbb{Q}[N]=\mathbb{Q}[x,x^{-1}]$. Our running example will be the Laurent polynomial $f(x_1,\dots,x_n)=\sum_{v\in V(\Delta^\vee)} x^v$.

The space of global vanishing cycles, Betti/de Rham aspects

For each $k\in\mathbb{N}$, we set

$$H^k_{\text{Betti}}(U,f) := \lim_{|t|\to\infty} H^k(U^{\text{an}}, f^{-1}(t);\mathbb{Q}) \quad \text{(provisional definition)},$$
$$H^k_{\text{dR}}(U,f) := H^k(U,(\Omega^\bullet_U, \mathrm{d}+\mathrm{d}f)).$$

Theorem. *The space of global vanishing cycles $H^k(U,f)$ admits a canonical irregular mixed Hodge structure.*

1.2. Exponential Hodge theory and the irregular Hodge filtration

1.2.a. Admissible variations of MHS on the punctured line. Recall that a (graded-polarizable) MHS H consists of the data

- $(H_\mathbb{C}, F^\bullet)$: $\searrow$ filtered finite-dim. $\mathbb{C}$-vector space,
- $(H_\mathbb{Q}, W_\bullet)$: $\nearrow$ filtered finite-dim. $\mathbb{Q}$-vector space,
- a comparison isomorphism comp : $H_\mathbb{C} \simeq H_\mathbb{Q}\otimes\mathbb{C}$

such that $\mathrm{gr}^W_\ell H$ is a polarizable Hodge structure of weight ℓ for any $\ell\in\mathbb{Z}$. This is a neutral Tannakian category with respect to $\otimes$, and the Betti (resp. de Rham) fiber functor is the underlying $\mathbb{Q}$- (resp. $\mathbb{C}$-) vector space.

For a variation H of MHS (e.g. on a curve C),

- $(H_\mathbb{C}, \nabla, F^\bullet)$ is a flat holomorphic bundle filtered by sub-bundles satisfying the Griffiths transversality property,
- $(H_\mathbb{Q}, W_\bullet)$ is a $\mathbb{Q}$-local system on C filtered by sub-local systems,
- there is given a comparison isomorphism comp : $(H_\mathbb{C})^\nabla \simeq H_\mathbb{Q}\otimes\mathbb{C}$,
- the polarization on $\mathrm{gr}^W_\ell H$ is flat.

This is a neutral Tannakian category with respect to $\otimes$ and each point of C leads to a fiber functor. Note that the fiber is more than a vector space: it is a MHS. One usually adds the condition of *admissibility at infinity on C*, which corresponds to the meromorphicity of the period map.

1.2.b. Exponential mixed Hodge structures. Assume that C is a punctured affine line $\mathbb{C} \smallsetminus \{p_1, \dots, p_r\}$. In order to consider ($W$-filtered) objects on $\mathbb{C}$ and not only on $\mathbb{C} \smallsetminus \{p_1, \dots, p_r\}$, one replaces ($W$-filtered) a $\mathbb{Q}$-local system with a $\mathbb{Q}$-perverse sheaf on $\mathbb{C}$ (coordinate t) with possible singularities at the punctures, and the filtration $W_\bullet$ is a filtration by $\mathbb{Q}$-perverse subsheaves (in the abelian category $\mathrm{Perv}(\mathbb{C})$). The category of such perverse sheaves is not Tannakian with respect to $\otimes$, but the group structure on $\mathbb{C}$ allows one to replace $\otimes$ with the *convolution product* $\mathcal{F} \star \mathcal{G} = Rs_*(\mathcal{F} \boxtimes \mathcal{G})$ (up to a suitable shift), where $s : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is the sum map $(t_1, t_2) \mapsto t_1 + t_2$.

The category $\mathrm{Perv}_0(\mathbb{C})$**.** The full subcategory $\mathrm{Perv}_0(\mathbb{C}) \subset \mathrm{Perv}(\mathbb{C})$ of perverse sheaves with vanishing global cohomology on $\mathbb{C}$ is stable by convolution. Firstly, one notices that the convolution of two constructible complexes remains constructible and if one of them has zero global cohomology, so has the convolution. This is because of the isomorphism

$$R\Gamma(\mathbb{C}, Rs_*(\mathcal{F} \boxtimes \mathcal{G})) \simeq R\Gamma(\mathbb{C}, \mathcal{F}) \otimes R\Gamma(\mathbb{C}, \mathcal{G}).$$

On the other hand, if $\mathcal{F}$ and $\mathcal{G}$ are perverse and $\mathcal{G} \in \mathrm{Perv}_0$, then $\mathcal{F} \star \mathcal{G}$ is also perverse, hence in Perv_0 according to the previous identification. This can be seen in two ways: either by giving a precise description of perverse sheaves on $\mathbb{C}$, or by using the Fourier transformation of constructible complexes and the property (as expected) that it transforms the convolution product into the tensor product; the Fourier transform of a perverse sheaf $\mathcal{F}$ on $\mathbb{C}$ is a perverse sheaf on the dual line $\mathbb{C}^\vee$ which is a shifted locally constant sheaf $\mathcal{L}[1]$ on $(\mathbb{C}^\vee)^*$, and $\mathcal{F} \in \mathrm{Perv}_0(\mathbb{C})$ iff its Fourier transform takes the form $Ri_*\mathcal{L}[1]$, with $i : (\mathbb{C}^\vee)^* \hookrightarrow \mathbb{C}^\vee$.

The projector $\Pi : \mathrm{Perv}(\mathbb{C}) \to \mathrm{Perv}_0(\mathbb{C})$**.** An example of an object of $\mathrm{Perv}_0(\mathbb{C})$ is the perverse sheaf $j_!\mathbb{Q}_{\mathbb{C}^*}[1]$, where $j : \mathbb{C}^* \hookrightarrow \mathbb{C}$ is the inclusion. For $\mathcal{F} \in \mathrm{Perv}(\mathbb{C})$, $\Pi(\mathcal{F}) := \mathcal{F} \star (j_!\mathbb{Q}_{\mathbb{C}^*}[1])$ is thus an object of $\mathrm{Perv}_0(\mathbb{C})$. The functor Π is exact. The projector Π has the effect of killing any constant perverse sheaf, hence, for $\mathcal{F} \in \mathrm{Perv}(\mathbb{C})$ any constant subquotient of $\mathcal{F}$ is annihilated by Π.

Tannakian property of $\mathrm{Perv}_0(\mathbb{C})$**.** Let us consider the compactification of $\mathbb{C}$ as a closed disc and let $\widehat{\mathbb{C}}$ denote the open subset which is the union of $\mathbb{C}$ and the open half-interval $\mathrm{Re}(t) > 0$ in its boundary. We let $\alpha : \mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$ denote the open inclusion of $\mathbb{C}$ into $\widehat{\mathbb{C}}$.

Proposition. *The category* $\mathrm{Perv}_0(\mathbb{C})$ *of perverse sheaves with vanishing global cohomology on* $\mathbb{C}$ *is a neutral Tannakian category with respect to* $\star$*, with fiber functor given by* $\mathcal{F} \mapsto H^0_c(\widehat{\mathbb{C}}, R\alpha_*\mathcal{F})$.

Weight filtration. If $\mathcal{F} \in \mathrm{Perv}(\mathbb{C})$ is equipped with a filtration $W_\bullet\mathcal{F}$ (it will be the weight filtration when considering mixed Hodge modules), then $\Pi\mathcal{F}$ acquires a filtration $W^{\mathrm{EMHS}}_\bullet(\Pi\mathcal{F}) := \Pi(W_\bullet\mathcal{F})$. Note that $\Pi\mathcal{F}$ can be pure of weight w without $\mathcal{F}$ being pure.

Caveat. The perverse sheaf $j_!\mathbb{Q}_{\mathbb{C}^*}[1]$ has a weight filtration with weights $1, 2$, an one can define a weight filtration on $\Pi\mathcal{F} = \mathcal{F} \star (j_!\mathbb{Q}_{\mathbb{C}^*}[1]) = Rs_*(\mathcal{F} \boxtimes j_!\mathbb{Q}_{\mathbb{C}^*}[1])$ in the usual way. However, the latter is not in general equal to $W_\bullet^{\mathrm{EMHS}}(\Pi\mathcal{F})$.

The de Rham side. On the other hand, the flat algebraic bundle $(H_{\mathbb{C}}, \nabla)$ on $\mathbb{A}^1 \setminus \{p_1, \dots, p_r\}$ is replaced with a quasi-coherent $\mathcal{O}_{\mathbb{A}^1}$-module $\mathcal{M}$ with connection ∇ (in fact, a holonomic $\mathcal{D}_{\mathbb{A}^1}$-module) and a filtration $F^\bullet\mathcal{M}$ by coherent subsheaves. We obtain the category $\mathsf{MHM}(\mathbb{A}^1)$ of *mixed Hodge modules* by imposing various supplementary constraints to $M = ((\mathcal{M}, F^\bullet), (\mathcal{F}, W_\bullet), \mathrm{comp})$ at the punctures. Any admissible variation H of MHS on $\mathbb{A}^1 \setminus \{p_1, \dots, p_r\}$ can be extended to an object of $\mathsf{MHM}(\mathbb{A}^1)$. Conversely, the restriction of an object of $\mathsf{MHM}(\mathbb{A}^1)$ to the complement of its singularities is and admissible variation of MHS.

Shortcuts. $\mathcal{M} \otimes \mathcal{E}^t = (\mathcal{M}, \nabla + \mathrm{d}t)$ and ${}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}) = (\Omega^\bullet_{\mathbb{A}^1} \otimes \mathcal{M}, \nabla)[1]$.

Definition. The category EMHS of exponential mixed Hodge structure is the subcategory of $\mathsf{MHM}(\mathbb{A}^1)$ consisting of objects $M = ((\mathcal{M}, F^\bullet), (\mathcal{F}, W_\bullet), \mathrm{comp})$ such that $\mathcal{F} \in \mathrm{Perv}_0(\mathbb{C})$ and $\mathrm{comp} : {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} \xrightarrow{\sim} \mathcal{F} \otimes \mathbb{C}$ is an isomorphism.

***Theorem* (Kontsevich-Soibelman).** *The category* EMHS *is a neutral Tannakian category with respect to* $\star$ *with Betti fiber functor being that of* Perv_0*, and de Rham functor defined as* $\mathcal{M} \mapsto H^0(\mathbb{A}^1, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M} \otimes \mathcal{E}^t))$.

Remarks

(1) For a mixed Hodge module M on $\mathbb{A}^1$, the comparison isomorphism comp induces an isomorphism between the de Rham and the complex Betti fibers: $H^0(\mathbb{A}^1, {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M} \otimes \mathcal{E}^t)) \simeq H^0_{\mathrm{c}}(\widehat{\mathbb{C}}, R\alpha_*\mathcal{F}_{\mathbb{C}})$.

(2) The projector Π extends as a functor $\mathsf{MHM}(\mathbb{A}^1) \to \mathsf{MHM}(\mathbb{A}^1)$ with essential image EMHS.

(3) MHS is indeed a full subcategory of EMHS by the following trick: we identify it with the subcategory of EMHS consisting of objects M such that the underlying perverse sheaf $\mathcal{F}$ is the constant local system (shifted by one) on $\mathbb{C} \setminus \{0\}$. One recovers a usual MHS by taking vanishing cycles at the origin. Furthermore, the weight filtration in MHS and that in EMHS coincide.

Question. The Betti fiber of an object in EMHS is naturally equipped with the induced weight filtration $W_\bullet H^0_{\mathrm{c}}(\widehat{\mathbb{C}}, R\alpha_*\mathcal{F}_{\mathbb{C}}) := H^0_{\mathrm{c}}(\widehat{\mathbb{C}}, R\alpha_* W_\bullet\mathcal{F}_{\mathbb{C}})$. In the case of an admissible variation of MHS, we also get a Hodge filtration on the de Rham fiber. Is there any kind of Hodge filtration on the de Rham fiber of an exponential mixed Hodge structure?

LECTURE 2

2.1. EMHS for (U, f)

Set $n = \dim U$. Recall that the constant sheaf ${}^{\mathrm{p}}\mathbb{Q}_U = \mathbb{Q}[n]$ is a perverse sheaf on U. For each $j \in \mathbb{Z}$, one can consider the perverse cohomology sheaves $\mathcal{F}^j = {}^{\mathrm{p}}R^j f_* {}^{\mathrm{p}}\mathbb{Q}_U$ on $\mathbb{C}$. They do not belong to $\mathrm{Perv}_0(\mathbb{C})$ in general, so one considers their projection $\Pi(\mathcal{F}^j) \in \mathrm{Perv}_0(\mathbb{C})$. Such a construction can be lifted to mixed Hodge modules as we only use standard functors for it. Starting from the pure Hodge module ${}^{\mathrm{p}}\mathbb{Q}_U^{\mathrm{H}} = ((\mathcal{O}_U, F_\bullet), {}^{\mathrm{p}}\mathbb{Q}_U)$ of weight n, we obtain in this way the object $H^{j+n}_{\mathrm{EMHS}}(U, f) \in \mathsf{EMHS}$.

Proposition. *The Betti fiber of $H^k_{\mathrm{EMHS}}(U, f)$ is isomorphic to $H^k_{\mathrm{Betti}}(U, f)$ and its de Rham fiber is isomorphic to $H^k_{\mathrm{dR}}(U, f)$.*

Corollary. *The space $H^k_{\mathrm{Betti}}(U, f)$ comes equipped with a canonical weight filtration.*

***Example:* $f = tg$.** Assume that $U = V \times \mathbb{A}^1{}_t$ and $f = t \cdot g$, with $g : V \to \mathbb{A}^1$. Then for each k, $H^k_{\mathrm{EMHS}}(U, f)$ belongs to MHS. In other words, for each $j \in \mathbb{Z}$, the perverse sheaf ${}^{\mathrm{p}}R^j f_* {}^{\mathrm{p}}\mathbb{Q}_U$ is constant (shifted by one) on $\mathbb{C}^*$.

***Running example* $f = \sum_{v \in V(\Delta^\vee)} x^v$.** In this case, f is a tame function on $U = (\mathbb{C}^*)^n$. Then $H^k_{\mathrm{EMHS}}(U, f) = 0$ for $k \neq n$ and $H^n_{\mathrm{EMHS}}(U, f)$ is pure of weight n. Considering the projective Fano variety X_Σ, one has $\dim H^n_{\mathrm{Betti/dR}}(U, f) = \dim H^*(X_\Sigma)$. We will see later how to recover the grading of $H^*(X_\Sigma)$.

2.2. Irregular Hodge filtration on $H^\bullet_{\mathrm{dR}}(U, f)$

Let us fix a compactification $f : X \to \mathbb{P}^1$ of $f : U \to \mathbb{A}^1$ such that $D = X \smallsetminus U$ is a ncd. Recall that the filtration by “stupid” truncation (“filtration bête”) of the logarithmic de Rham complex $(\Omega^\bullet_X(\log D), \mathrm{d})$ is the filtration

$$F^p(\Omega^\bullet_X(\log D), \mathrm{d}) = \big\{0 \to \cdots \to 0 \to \Omega^p_X(\log D) \to \cdots \to \Omega^n_X(\log D) \to 0\big\},$$

and that, for each p and k, the natural morphism

$$H^k\big(X, F^p(\Omega^\bullet_X(\log D), \mathrm{d})\big) \longrightarrow H^k_{\mathrm{dR}}(U)$$

is injective, with image defining the decreasing filtration $F^\bullet H^k_{\mathrm{dR}}(U)$.

Let P denote the (non-reduced) divisor $f^*(\infty)$ with support P_{red}. For any $a\in\mathbb{Q}$, we can consider the integral part $\lfloor aP\rfloor$, that is, if $P=\sum_i m_iP_i$ with P_i reduced, we set $\lfloor aP\rfloor=\sum_i\lfloor am_i\rfloor P_i$. The family of divisors $\lfloor aP\rfloor$ with $a\in\mathbb{Q}$ is increasing, and there exists a finite set $\mathcal{A}$ of rational numbers in $[0,1)$ such that the jumps occur at most for $a\in\mathcal{A}+\mathbb{Z}$ (since the jumps occur at most when the denominator of a divides some m_i). Multiplication by f sends $\mathcal{O}_X(\lfloor aP\rfloor)$ to $\mathcal{O}_X(\lfloor(a+1)P\rfloor)$. On noting that $\mathrm{d}f=f\cdot\mathrm{d}f/f$ and that both d and $\mathrm{d}f/f$ preserve logarithmic poles along D, we can consider for each $\alpha\in\mathcal{A}$ the subsheaf of $\Omega^k_X(\log D)(\lfloor(\alpha P\rfloor)$:

$$\Omega^k(\log D,f,\alpha)=\ker\Big[\mathrm{d}f:\Omega^k_X(\log D)(\lfloor(\alpha P\rfloor)\to\Omega^{k+1}_X(*D)/\Omega^{k+1}_X(\log D)(\lfloor\alpha P\rfloor)\Big]$$

and consider the filtered Kontsevich-Yu complex $(\Omega^\bullet(\log D,f,\alpha),\mathrm{d}+\mathrm{d}f)$ with $F^p(\Omega^\bullet(\log D,f,\alpha),\mathrm{d}+\mathrm{d}f)$ given by:

$$\big\{0\to\cdots\to0\to\Omega^p(\log D,f,\alpha)\to\cdots\to\Omega^n(\log D,f,\alpha)\to0\big\},$$

We omit α in the notation if $\alpha=0$. The properties previously recalled for the logarithmic de Rham complex extend to the Kontsevich complex.

Theorem.

(1) *For each $\alpha\in\mathcal{A}$, the inclusion*

$$(\Omega^\bullet(\log D,f),\mathrm{d}+\mathrm{d}f)\hookrightarrow(\Omega^\bullet(\log D,f,\alpha),\mathrm{d}+\mathrm{d}f)$$

is a quasi-isomorphism, leading to an identification

$$H^k\big(X,(\Omega^\bullet(\log D,f),\mathrm{d}+\mathrm{d}f)\big)=H^k\big(X,(\Omega^\bullet(\log D,f,\alpha),\mathrm{d}+\mathrm{d}f)\big).$$

(2) *Furthermore, for each $\alpha\in\mathcal{A}$, $p\geqslant0$ and $k\geqslant0$, the natural morphism*

$$H^k\big(X,F^p(\Omega^\bullet(\log D,f,\alpha),\mathrm{d}+\mathrm{d}f)\big)\longrightarrow H^k_{\mathrm{dR}}(U,f)$$

is injective, with image defining the decreasing filtration $F^\bullet_{\mathrm{irr},\alpha}H^k_{\mathrm{dR}}(U,f)$. For each k, we have a decomposition

$$H^k_{\mathrm{dR}}(U,f)\simeq\bigoplus_{p\geqslant0}\mathrm{gr}^p_{F_{\mathrm{irr},\alpha}}H^k_{\mathrm{dR}}(U,f)\simeq\bigoplus_{p+q=k}H^q(X,\Omega^p(\log D,f,\alpha)).$$

(3) *A similar result holds for the Kontsevich-Yu complex "with compact support" $(\Omega^\bullet(\log D,f,\alpha)(-D),\mathrm{d}+\mathrm{d}f)$.*

Complement. The irregular Hodge filtration can also be defined from the meromorphic de Rham complex. In such a way, by looking at the order of the poles, one finds an inclusion, for $\alpha,\beta\in[0,1)$:

$$p-\alpha\geqslant q-\beta\implies F^p_{\mathrm{irr},\alpha}H^k_{\mathrm{dR}}(U,f)\subset F^q_{\mathrm{irr},\beta}H^k_{\mathrm{dR}}(U,f).$$

One can thus consider the irregular Hodge filtration as indexed by $\mathsf{p}\in-\mathcal{A}+\mathbb{Z}$: if $\mathsf{p}=p-\alpha$, then one sets $F^{\mathsf{p}}_{\mathrm{irr}}H^k_{\mathrm{dR}}(U,f)=F^p_{\mathrm{irr},\alpha}H^k_{\mathrm{dR}}(U,f)$.

2.3. The Kontsevich-Yu bundles and the Brieskorn-Deligne bundles

Theorem. *The dimension of the twisted de Rham complex*

$$H^k\big(X, (\Omega^\bullet(\log D, f, \alpha), u_o\mathrm{d} + v_o\mathrm{d}f)\big)$$

is independent of $(u_o, v_o) \in \mathbb{C}^2$.

It follows that we can define a vector bundle $\mathcal{K}^k_\alpha$ on $\mathbb{P}^1$, called the *Kontsevich bundle with index* α, by decomposing $\mathbb{P}^1 = \mathbb{A}^1_v \cup \mathbb{A}^1_u$ with $u = v^{-1}$ on $\mathbb{G}_\mathrm{m} : \mathbb{A}^1_v \cap \mathbb{A}^1_u$, and setting

$$\begin{aligned}
\mathcal{K}^k_\alpha|_{\mathbb{A}^1_v} &= H^k\big(X, (\Omega^\bullet(\log D, f, \alpha)[v], \mathrm{d} + v\mathrm{d}f)\big),\\
\mathcal{K}^k_\alpha|_{\mathbb{A}^1_u} &= H^k\big(X, (\Omega^\bullet(\log D, f, \alpha)[u], u\mathrm{d} + \mathrm{d}f)\big)\\
&\simeq H^k\big(X, (u^{-\bullet}\Omega^\bullet(\log D, f, \alpha)[u], \mathrm{d} + \mathrm{d}f/u)\big),\\
\mathcal{K}^k_\alpha|_{\mathbb{G}_\mathrm{m}} &= H^k\big(X, (\Omega^\bullet(\log D, f, \alpha)[u, u^{-1}], \mathrm{d} + \mathrm{d}f/u)\big)\\
&\simeq H^k\big(X, (\Omega^\bullet(\log D, f, \alpha)[v, v^{-1}], \mathrm{d} + v\mathrm{d}f)\big),
\end{aligned}$$

In a way analogous to that of theorem of the previous section, the filtration by the stupid truncation of each of these twisted de Rham complexes degenerates at E^1 and induces a filtration $F^\bullet_{\mathrm{irr},\alpha}\mathcal{K}^k_\alpha$ indexed by $\mathbb{N}$. Its restriction at $u = 1$ is the filtration considered in that theorem.

Proposition. *The filtration* $F^\bullet_{\mathrm{irr},\alpha}\mathcal{K}^k_\alpha$ *is the Harder-Narasimhan filtration of* $\mathcal{K}^k_\alpha$*, that is,*

$$\forall\, p \in \mathbb{N},\ \exists\, h^p_\alpha \in \mathbb{N}, \quad \mathrm{gr}^p_{F_{\mathrm{irr},\alpha}}\mathcal{K}^k_\alpha \simeq \mathcal{O}_{\mathbb{P}^1}(p)^{h^p_\alpha}.$$

Behavior on $\mathbb{G}_\mathrm{m}$. One checks, in a way analogous to that of the theorem of the previous section, that the natural morphism $\mathcal{K}^k_\beta|_{\mathbb{G}_\mathrm{m}} \to \mathcal{K}^k_\alpha|_{\mathbb{G}_\mathrm{m}}$ is an isomorphism for $\beta \leqslant \alpha \in \mathcal{A}$ and we set $H^k_{\mathrm{dR}}(U, f/u) = H^k\big(X, (\Omega^\bullet(\log D, f)[u, u^{-1}], \mathrm{d} + \mathrm{d}f/u)\big)$ the common value of these $\mathbb{C}[u, u^{-1}]$-modules.

By restricting to $\mathbb{G}_\mathrm{m}$ the Harder-Narasimhan filtration of each $\mathcal{K}^k_\alpha$, we obtain a family of filtrations $F^\bullet_{\mathrm{irr},\alpha}H^k_{\mathrm{dR}}(U, f/u)$ indexed by $\alpha \in \mathcal{A}$. Furthermore, for $\beta \leqslant \alpha \in \mathcal{A}$, the quasi-isomorphism

$$(\Omega^\bullet(\log D, f, \beta)[u, u^{-1}], \mathrm{d} + \mathrm{d}f/u) \longrightarrow (\Omega^\bullet(\log D, f, \alpha)[u, u^{-1}], \mathrm{d} + \mathrm{d}f/u)$$

is filtered with respect to the filtration by stupid truncation, so that we obtain, for all $p \in \mathbb{N}$, the inclusion

$$F^p_{\mathrm{irr},\beta}H^k_{\mathrm{dR}}(U, f/u) \subset F^p_{\mathrm{irr},\alpha}H^k_{\mathrm{dR}}(U, f/u).$$

We can thus regard the irregular Hodge filtration as indexed by $-\mathcal{A} + \mathbb{Z} \subset \mathbb{Q}$ by setting, for $\mathsf{p} = p - \alpha \in -\mathcal{A} + \mathbb{Z}$,

$$F^{\mathsf{p}}_{\mathrm{irr}}H^k_{\mathrm{dR}}(U, f/u) := F^p_{\mathrm{irr},\alpha}H^k_{\mathrm{dR}}(U, f/u).$$

Proposition. *The irregular Hodge filtration $F_{\mathrm{irr}}^{\mathsf{p}} H_{\mathrm{dR}}^k(U, f/u)$ is a filtration by subbundles, that is, each $\mathbb{C}[u, u^{-1}]$-module*

$$\mathrm{gr}_{F_{\mathrm{irr}}}^{\mathsf{p}} H_{\mathrm{dR}}^k(U, f/u) := F_{\mathrm{irr}}^{\mathsf{p}} H_{\mathrm{dR}}^k(U, f/u) / F_{\mathrm{irr}}^{>\mathsf{p}} H_{\mathrm{dR}}^k(U, f/u)$$

is free, where $>\mathsf{p}$ denotes the successor of p in $-\mathcal{A} + \mathbb{Z}$.

The connection on **$H_{\mathrm{dR}}^k(U, f/u)$.** The complex $(\Omega^\bullet(\log D, f)[u, u^{-1}], \mathrm{d} + \mathrm{d}f/u)$ comes equipped with an action ∇_{∂_u} commuting with $\mathrm{d} + \mathrm{d}f/u$: $u\nabla_{\partial_u}(\eta u^k) = (k - f/u)\eta u^k$. This connection descends to $H_{\mathrm{dR}}^k(U, f/u)$ and extends as a meromorphic connection on each $\mathcal{K}_\alpha^k$ with a pole of order two at $u = 0$ and a regular singularity at $u = \infty$. The irregular Hodge filtration satisfies the Griffiths transversality property $\nabla(F_{\mathrm{irr},\alpha}^p \mathcal{K}_\alpha^k|_{\mathbb{G}_{\mathrm{m}}}) \subset \Omega_{\mathbb{G}_{\mathrm{m}}}^1 \otimes F_{\mathrm{irr},\alpha}^{p-1} \mathcal{K}_\alpha^k|_{\mathbb{G}_{\mathrm{m}}}$ for all $p \in \mathbb{N}$.

Limiting properties. At the limit $u \to \infty$ (i.e., $v \to 0$) where the connection ∇ has a regular singularity, the filtration $F_{\mathrm{irr}}^{\bullet} H_{\mathrm{dR}}^k(U, f/u)$ behaves in the same way as a the Hodge filtration of a polarizable Hodge module in the sense of M. Saito. Furthermore, one can show that, for each $\alpha \in \mathcal{A}$, the *limit filtration* of $F_{\mathrm{irr},\alpha}^{\bullet} H_{\mathrm{dR}}^k(U, f/u)$ is the Hodge filtration of a mixed Hodge structure, isomorphic to the limiting Hodge filtration of the mixed Hodge structure on $H^k(U, f^{-1}(t); \mathbb{C})$ when $t \to \infty$ on the generalized eigenspace corresponding the eigenvalue $\exp(-2\pi \mathtt{i} \alpha)$ of the monodromy.

On the other hand, the limit at $u = 0$ is much less understood.

Link with the spectrum at infinity. Since $H_{\mathrm{dR}}^k(U, f/u)$ has a connection with a regular singularity at $u = \infty$, we can consider for each $a \in \mathbb{Q}$ the Deligne canonical lattices $V^a(U, f)$ which are free $\mathbb{C}[u^{-1}]$-module with logarithmic connection whose residues has eigenvalues in $[a, a+1)$. On the other hand, the Brieskorn lattice $G_k(U, f) = H^k\big(X, (u^{-\bullet}\Omega^\bullet(\log D, f)[u], \mathrm{d} + \mathrm{d}f/u)\big)$ is a free $\mathbb{C}[u]$-module, according to the previous theorem. Gluing both for $a = -\alpha$ with $\alpha \in [0, 1) \cap \mathbb{Q}$ leads to the *Brieskorn-Deligne bundle* $\mathcal{H}_\alpha^k$.

Theorem. *For each $\alpha \in [0, 1) \cap \mathbb{Q}$, the bundles $\mathcal{K}_\alpha^k$ and $\mathcal{H}_\alpha^k$ are isomorphic.*

Definition. The *spectrum at infinity in degree k* of (U, f) is the set of pairs $(-\mathsf{p}, \nu_{\mathsf{p}}^k)$ with $a \in \mathbb{Q}$ and

$$\nu_{\mathsf{p}}^k = \dim \frac{V^{\mathsf{p}} \cap G_k}{[V^{>\mathsf{p}} \cap G_k] + [V^{\mathsf{p}} \cap uG_k]}.$$

Proposition **(Spectrum at infinity).** *For each $\mathsf{p} \in \mathbb{Q}$, we have:*

$$\mathrm{rk}\, \mathrm{gr}_{F_{\mathrm{irr}}}^{\mathsf{p}} H_{\mathrm{dR}}^k(U, f/u) = \nu_{\mathsf{p}}^k.$$

LECTURE 3

APPLICATIONS

3.1. A conjecture of K-K-P in the toric case

I will state and sketch the proof of this conjecture in the case of the running example (toric case). Other cases of the conjecture are proved in the literature, but the general case is still open. Before stating the result, let me add some properties of the irregular Hodge filtration in this example, where $U = (\mathbb{C}^*)^n$ and $f(x) = \sum_{v \in V(\Delta^\vee)} x^v$.

The irregular Hodge filtration of (U, f) ***jumps at integers only.*** It is enough to prove that the spectral numbers (i.e., $a \in \mathbb{Q}$ such that $\nu_a^n \neq 0$) are integers. The spectral numbers can be computed as the jumping numbers of the Newton filtration on the Jacobian quotient $\mathbb{C}[x, x^{-1}]/(\partial f)$, according to a result of Douai-Sabbah, analogous to a result of Varchenko in Singularity theory.

Since Δ and $\Delta^\vee$ are simplicial, reflexive and with vertices in $\mathbb{Z}^n$, the jumps of the Newton filtration are integers.

The monodromy at infinity of f ***is unipotent.*** This property follows from the relation between the spectral numbers and the eigenvalues of the monodromy of $H^n(U, f^{-1}(t))$ around infinity: these eigenvalues are of the form $\exp(2\pi\mathtt{i}$ spectral numbers$)$.

Statement of the conjecture. The only nonvanishing irregular Hodge numbers are denoted $h^{p,q}(f)$:

$$h^{p,q}(f) := \dim \operatorname{gr}^p_{F_{\mathrm{irr}}} H^{p+q}_{\mathrm{DR}}(U, \mathrm{d} + \mathrm{d}f).$$

In the case of the running example, we then have $h^{p,q}(f) = 0$ unless $p + q = n$.

Let $W_\bullet$ be the monodromy filtration of the nilpotent part of the monodromy around $t = \infty$ on $H^n(U, f^{-1}(t))$ centered at n. The conjecture of K-K-P that we consider is the equality

$$h^{p,n-p}(f) = \dim \operatorname{gr}^W_{2p} H^n(U, f^{-1}(t)).$$

Theorem. *The above equality holds true for the Laurent polynomial $f(x)$.*

Sketch of proof. One knows that X is a smooth Fano projective variety (i.e., the canonical bundle K_X is anti-ample). In particular, by anti-ampleness of K_X, the cup product with $c_1(TX)$ satisfies the Hard Lefschetz property on the $\mathbb{Q}$-Chow ring of X. Due to a formula of Borisov-Chen-Smith, this Chow ring is identified with the graded ring with respect to the Newton filtration:

$$\mathrm{gr}^{\mathcal{N}}_{\bullet}(\mathbb{Q}[x, x^{-1}]/(\partial f).$$

Then one identifies multiplication by $c_1(TX) = f$ on this ring tensored by $\mathbb{C}$ with the action of the monodromy on $H^n_{\mathrm{dR}}(U, f)$, by an argument similar to that used by Varchenko. The Hard Lefschetz property implies then the desired equality. □

3.2. What is an irregular Hodge structure?

General procedure to construct an irregular Hodge filtration. Let H be a finite-dimensional $\mathbb{C}$-vector space and let u be a new variable. Assume that the free $\mathbb{C}[u]$-module $G = H[u]$ is equipped with an algebraic connection ∇ having a pole of order two at the origin and no other pole. It defines a local system on $\mathbb{C}^*$. Assume that the eigenvalues of the monodromy of this local system have absolute value equal to one (or are roots of the unity). The connection defines, for each $\mathsf{p} \in \mathbb{R}$ (or $\mathbb{Q}$), a Deligne extension V^{p} which is a free $\mathbb{C}[u^{-1}]$-module such that the connection ∇ has a simple pole at $u^{-1} = 0$ and the eigenvalues of its residue belong to $[\mathsf{p}, \mathsf{p}+1)$. We regard both G and V^{p} as contained in $H[u, u^{-1}]$ and we identify H with $G/(u-1)G$. Then we set

$$F^{\mathsf{p}}_{\mathrm{irr}}(H) = \frac{V^{\mathsf{p}} \cap G}{(V^{\mathsf{p}} \cap G) \cap (u-1)G}.$$

Without any other assumption, such a filtration does not deserve the name of 'Hodge'. For example, given a morphism of $\mathbb{C}[u]$-modules with connection $\varphi : (G, \nabla) \to (G', \nabla')$, the morphism induced mod $(u-1)$, $H \to H'$, preserves the filtrations $F^{\bullet}_{\mathrm{irr}}$, but *need not be strict*, a property that would be expected for Hodge filtrations.

Irregular mixed Hodge structures. The definition is given in a few steps. We let $\sigma : \mathbb{P}^1 \to \overline{\mathbb{P}}^1$ denote the anti-linear morphism $u \mapsto -1/\overline{u}$.

(1) (Pure twistor structure) As (G, ∇) is the source of the irregular Hodge filtration, a condition replacing oppositeness in Hodge theory has to hold for (G, ∇). A pure integrable twistor structure $\mathcal{T}$ of weight w consists of the data (G, ∇, γ), where γ is a gluing $(G, \nabla)|_{\mathbb{C}^*} \xrightarrow{\sim} \sigma^*(\overline{G}, \overline{\nabla})|_{\mathbb{C}^*}$, so that the corresponding bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(w)^{\oplus r}$.

(2) There is a natural notion of polarization for such objects.

(3) There is a natural notion of weight filtration. This leads to the notion of mixed twistor structure (Simpson).

(4) A kind of admissibility condition is to be added to obtain good properties of the irregular Hodge filtration. For that purpose, one considers the rescaling operation,

that rescales the parameter: $u \mapsto u\tau$, $\tau \in \mathbb{C}^*$. The pullback of the given mixed twistor structure $(\mathcal{T}, W_\bullet)$ by $(z,\tau) \mapsto u = z/\tau$ can be regarded as a variation of mixed twistor structure parametrized by $\tau \in \mathbb{C}^*$. An integrable mixed twistor structure is an *irregular MHS* if this variation satisfies suitable extendability conditions at $\tau = 0$.

Theorem.

(1) *The category* IrrMHS *is an abelian category and any morphism is strictly compatible with both the weight and the irregular Hodge filtration.*

(2) *The de Rham fiber of an object of* EMHS *is canonically equipped with an irregular mixed Hodge structure.*

(3) *The (already defined) irregular Hodge filtration* $F_{\mathrm{irr}}^\bullet H_{\mathrm{dR}}^k(U,f)$ *is the irregular Hodge filtration of* $H_{\mathrm{dR}}^k(U,f)$ *as the de Rham fiber of* $H_{\mathrm{EMHS}}^k(U,f)$*.*

This notion can be extended with a base of arbitrary dimension and leads to the category $\mathsf{IrrMHM}(X)$ of irregular mixed Hodge modules on a complex manifold X.

3.3. Vanishing theorems

Let X be a smooth projective variety of dimension n. Let $\mathcal{T}$ be an object of $\mathsf{IrrMHM}(X)$ and let $\mathcal{M}$ be the underlying holonomic (left) $\mathcal{D}_X$-module, with its irregular Hodge filtration indexed by $-A+\mathbb{Z}$ for some subset $A \subset [0,1)$. Given $\mathsf{p} \in -A+\mathbb{Z}$, the shifted holomorphic de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is filtered by setting

$$F_{\mathrm{irr}}^{\mathsf{p}}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} = \{0 \to F_{\mathrm{irr}}^{\mathsf{p}}\mathcal{M} \to \Omega_X^1 \otimes F_{\mathrm{irr}}^{\mathsf{p}-1}\mathcal{M} \to \cdots \to \Omega_X^n \otimes \underset{\bullet}{F_{\mathrm{irr}}^{\mathsf{p}-n}}\mathcal{M} \to 0\},$$

where $\bullet$ indicates the term in degree zero.

Theorem **(Kodaira-Saito vanishing).** *Let* L *be an ample line bundle on* X *and let* $\mathcal{T}$, $\mathcal{M}$ *and* A *be as above. Then we have*

$$H^k(X, \mathrm{gr}_{F^{\mathrm{irr}}}\,{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}) \otimes L) = 0 \quad \textit{for } k > 0,$$
$$H^k(X, \mathrm{gr}_{F_{\mathrm{irr}}}\,{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}) \otimes L^{-1}) = 0 \quad \textit{for } k < 0.$$

Corollary. *For* $\mathcal{T}$, $\mathcal{M}$ *and* A *as above, let* $\mathsf{p}_o \in -A+\mathbb{Z}$ *be such that* $F_{\mathrm{irr}}^{>\mathsf{p}_o}\mathcal{M} = 0$*, and let us set* $\omega_X = \Omega_X^n$*. Then we have the vanishing*

$$H^k(X, \omega_X \otimes F_{\mathrm{irr}}^{\mathsf{p}}(\mathcal{M}) \otimes L) = 0 \quad \forall\, k > 0,\ \textit{and}\ \forall\, \mathsf{p} \in (\mathsf{p}_o - 1, \mathsf{p}_o].$$

Note that we can replace $F_{\mathrm{irr}}^{\mathsf{p}}(\mathcal{M})$ by $\mathrm{gr}_{F_{\mathrm{irr}}}^{\mathsf{p}}(\mathcal{M})$ in this corollary. We also have the analogue of Kollár's vanishing theorem:

Corollary **(Kollár vanishing for the irregular Hodge filtration)**

Let $\mathcal{T}$, $\mathcal{M}$, A *and* p_o *be as above. Let* $f : X \to Y$ *be a projective morphism to a* *smooth projective variety Y and let L be an ample line bundle on Y. Then we have the vanishing*

$$H^k(Y, R^j f_*(\omega_X \otimes \mathrm{gr}^{\mathsf{p}}_{F_{\mathrm{irr}}} \mathcal{M}) \otimes L) = 0 \quad \forall\, j,\ \forall\, k > 0,\ \textit{and}\ \forall\, \mathsf{p} \in (\mathsf{p}_o - 1, \mathsf{p}_o].$$

3.4. Geometric consequences

Let us emphasize an example where the vanishing theorem applies. Let L be an ample line bundle on X and let $D \subset X$ be a divisor with normal crossings. Recall that the Kodaira-Norimatsu vanishing theorem: for each integer $p \geqslant 0$,

$$H^q(X, \Omega^p_X(\log D) \otimes L) = 0 \quad \text{for } p + q > n,$$
$$H^q(X, \Omega^p_X(\log D) \otimes L^{-1}) = 0 \quad \text{for } p + q < n.$$

The theorem of Section 3.2 enables us to extend this vanishing result when D contains the support of the pole divisor P of a morphism $f : X \to \mathbb{P}^1$. Recall the definition of $\Omega^k(\log D, f, \alpha) \subset \mathcal{O}_X(\lfloor \alpha P \rfloor) \otimes_{\mathcal{O}_X} \Omega^k_X(\log D)$ for each $\alpha \in [0,1) \cap \mathbb{Q}$ and each $k \geqslant 0$. For example, $\Omega^0(\log D, f, \alpha) = \mathcal{O}_X(-P)$ and $\Omega^n(\log D, f, \alpha) = \omega_X(D + \lfloor \alpha P \rfloor)$.

Corollary. *With the above assumptions and notations, for each $p \geqslant 0$, the sheaves $\Omega^p(\log D, f, \alpha)$ ($\alpha \in [0,1) \cap \mathbb{Q}$) satisfy the Kodaira-Saito vanishing property*

$$H^q(X, \Omega^p_X(\log D, f, \alpha) \otimes L) = 0 \quad \textit{for } p + q > n,$$
$$H^q(X, \Omega^p_X(\log D, f, \alpha) \otimes L^{-1}) = 0 \quad \textit{for } p + q < n.$$

In particular, we obtain the vanishing $H^k(X, \omega_X(D + \lfloor \alpha P \rfloor) \otimes L)$ for $k > 0$.

Corollary. *Let $f : X \to \mathbb{P}^1$ be a projective morphism and set $P = f^*(\infty)$. Assume that the support of P is contained in a (reduced) divisor with normal crossings D in X. Let $f : X \to Y$ be a projective morphism to a smooth projective variety Y and let L be an ample line bundle on Y. Then for each $\alpha \in A$ we have the vanishing property*

$$H^k(Y, R^j f_* \omega_X(D + \lfloor \alpha P \rfloor) \otimes L) = 0 \quad \textit{for all } k > 0 \textit{ and all } j.$$