A conjecture by M. Kashiwara

- X, Y: Projective varieties over $\mathbb{C}$
- $\mathcal{F}$: A perverse sheaf of $\mathbb{C}$-vector spaces on X
- $f : X \longrightarrow Y$: A morphism

***Conjecture* (Regular case)**

*Fix an ample line bundle on X. Assume that $\mathcal{F}$ is **semisimple**. Then*

- *the relative Hard Lefschetz Theorem holds for the perverse cohomology sheaves ${}^{p}\mathcal{H}^{i}(Rf_{*}\mathcal{F})$ of the direct image;*
- *the direct image complex $Rf_{*}\mathcal{F}$ decomposes (maybe non canonically) as the direct sum of its perverse cohomology sheaves:*

$$Rf_{*}\mathcal{F} \simeq \bigoplus_{i} {}^{p}\mathcal{H}^{i}(Rf_{*}\mathcal{F})[-i];$$

- *each perverse cohomology sheaf ${}^{p}\mathcal{H}^{i}(Rf_{*}\mathcal{F})$ is **semisimple**.*

***Theorem* (V. Drinfeld)**

Kashiwara's conjecture (regular case) is implied by a conjecture of ***de Jong*** *on rigidity of irreducible representations of* π_1 *in characteristic* p.

***Corollary* (de Jong, V. Drinfeld)**

Kashiwara's conjecture is true for perverse sheaves of generic rank $\leqslant 2$.

***Main Theorem* (C. S.)**

The conjecture is true if X *is* ***smooth*** *and* $\mathcal{F}$ *is a* ***semisimple local system*** *on* X.

3 main sources of ideas:

- the theory of ***polarizable Hodge modules*** of M. Saito,
- the notion of a ***variation of twistor structure***, introduced by C. Simpson (after a suggestion of Deligne),
- the use of ***distributions*** and of ***Hermitian forms*** on $\mathcal{D}$-modules, after D. Barlet et M. Kashiwara.

Harmonic metrics and Higgs bundles

Theorem **(K. Corlette 1988, C. Simpson)**
Let (V, ∇) *be a holomorphic vector bundle equipped with a flat connection on a compact Kähler manifold* X*. Then, the locally constant sheaf* $\mathcal{F}$ *of its horizontal sections is* ***semisimple*** *if and only if* (V, ∇) *has a* ***harmonic metric*** h.

harmonic metric:

$$D_V = D'_V + D''_V$$

the flat connection on the bundle

$$H := \mathcal{C}^\infty_X \bigotimes_{\mathcal{O}_X} V$$

obtained from ∇, so that

$$(V, \nabla) = (\mathrm{Ker}\, D''_V, D'_V).$$

Let h be a metric on H. One may then find a unique connection

$$D_E = D'_E + D''_E \quad \text{on} \quad H$$

which preserves the metric h in such a way that, if one puts

$$\theta'_E := D'_E - D'_V \qquad \text{(type } (1,0) \text{ with values in } \mathrm{End}(H)\text{)}$$

$$\theta''_E := D''_E - D''_V \qquad \text{(type } (0,1) \text{ with values in } \mathrm{End}(H)\text{)}$$

θ''_E is the h-***adjoint*** of θ'_E.

Definition

The metric h *is* ***harmonic*** *relatively to the flat holomorphic bundle* (V, ∇) *if*

$$(D''_E + \theta'_E)^2 = 0$$

that is,

$$D''^2_E = 0, \qquad D''_E(\theta'_E) = 0, \qquad \theta'_E \wedge \theta'_E = 0.$$

$$E := \mathrm{Ker}\, D''_E : H \to H$$

E is a ***holomorphic bundle*** equipped with a 1-form θ'_E with values in $\mathrm{End}(E)$, which satisfies

$$\theta'_E \wedge \theta'_E = 0$$

θ'_E is a ***Higgs field*** for E.

For any fixed $z_o \in \mathbb{C}$, the operator

$$D''_E + z_o\theta''_E$$

is a complex structure on H.

$$V_{z_o} = \mathrm{Ker}(D''_E + z_o\theta''_E) \qquad \nabla_{z_o} = D'_E + \frac{1}{z_o}\theta'_E$$

Family of holomorphic bundles if $z_o \neq 0$, ***flat holomorphic connection***

If h is harmonic, the identities of Kähler geometry apply to the mixed operators

$$\mathcal{D}_\infty = D'_E + \theta''_E, \quad \mathcal{D}_0 = D''_E + \theta'_E, \quad D_V = \mathcal{D}_\infty + \mathcal{D}_0,$$

$$\mathcal{D}_{z_o} = (z_o D'_E + \theta'_E) + (D''_E + z_o \theta''_E) = z_o \mathcal{D}_\infty + \mathcal{D}_0$$

$$\Delta_{D_V} = 2\Delta_{\mathcal{D}_\infty} = 2\Delta_{\mathcal{D}_0}, \quad \Delta_{\mathcal{D}_{z_o}} = (1 + |z_o|^2)\Delta_{\mathcal{D}_0}.$$

Simpson deduces from them the ***Hard Lefschetz Theorem*** and all locally constant sheaves $\mathcal{F}_{z_o} := \mathrm{Ker}\, \nabla_{z_o}$, $z_o \neq 0$, have the same cohomology.

Variations of polarized Hodge structures of weight w

H a vector bundle C^∞ equipped with $\begin{cases} \text{a decomposition } H = \bigoplus_{p\in\mathbb{Z}} H^{p,w-p} \ (w \in \mathbb{Z}), \\ \text{a flat connection } D_V = D'_V + D''_V, \\ \text{a nondegenerate Hermitian form } k \end{cases}$

such that

- the decomposition is ***k-orthogonal***,
- $(-1)^p k$ on $H^{p,w-p}$ is ***positive definite***,

and (Griffiths' ***transversality relations***)

$$D'_V(H^{p,w-p}) \subset \left(H^{p,w-p} \oplus H^{p-1,w-p+1}\right) \otimes_{\mathcal{O}_X} \Omega^1_X$$
$$D''_V(H^{p,w-p}) \subset \left(H^{p,w-p} \oplus H^{p+1,w-p-1}\right) \otimes_{\mathcal{O}_{\overline{X}}} \Omega^1_{\overline{X}}.$$

Define $\quad h = (-1)^p k$ on $H^{p,w-p}$.

Griffiths' transversality relations give $\quad D'_V = D'_E + \theta'_E, \quad D''_V = D''_E + \theta''_E.$

The metric h is ***harmonic***.

Variations of polarized twistor structures

C. Simpson presents this notion by stating the

Meta theorem **(C. Simpson)**

If the words ***"Hodge structure"*** *are replaced with* ***"twistor structure"*** *in the assumptions and conclusions of any theorem in Hodge theory, one still gets a true statement, the proof of which is analogous to that of its model.*

The notion of a twistor structure is a

"deshomogeneization"

of that of a Hodge structure, which has a notion of ***weight***.

The Hodge graduation on a bundle on $\boldsymbol{X}$ is replaced with the extension of this bundle as a bundle on $\boldsymbol{X} \times \mathbb{P}^1$.

The conjugation $\boldsymbol{H}^{q,p} = \overline{\boldsymbol{H}^{p,q}}$ is replaced with a geometric conjugation.

Geometric conjugation

Let $f(x)$ be a holomorphic function on an open set of X. Its conjugate

$$\overline{f}(x) := \overline{f(x)}$$

is a holomorphic function on the ***complex conjugate manifold*** $\overline{X}$.

Let $g(z)$ be a holomorphic function on an open set U of $\mathbb{P}^1$. Its "conjugate"

$$\text{“}\overline{g}(z)\text{”} := \overline{g(-1/\overline{z})}$$

is a holomorphic function on the ***"geometric conjugate set"*** "$\overline{U}$".

U_0: Affine chart centered at 0, $U_\infty = \overline{U}_0$: Affine chart centered at ∞.

Mix these two notions to define $\overline{f}(x, z)$.

If $\mathcal{F}$ is a $\mathcal{O}_{X \times U}$-module, then $\overline{\mathcal{F}}$ is a $\mathcal{O}_{\overline{X} \times \overline{U}}$-module.

Polarized twistor structure

A ***twistor structure*** $\mathcal{T} = (\mathcal{H}', \mathcal{H}'', C)$ ***of weight*** w ***on*** H consists of

- two $\mathcal{O}_{U_0}$-modules $\mathcal{H}', \mathcal{H}''$, locally free of rank d
- and of a ***glueing*** between $\mathcal{H}'^*$ and $\overline{\mathcal{H}}''$ on an annulus $\mathbf{A}$ invariant under ***geometric conjugation***

$$C : \Gamma(\mathbf{A}, \mathcal{H}') \underset{\mathcal{O}(\mathbf{A})}{\otimes} \overline{\Gamma(\mathbf{A}, \mathcal{H}'')} \longrightarrow \mathcal{O}(\mathbf{A}).$$

The bundle $\widetilde{\mathcal{H}}$ on $\mathbb{P}^1$ defined by this gluing: ***isomorphic to*** $\overline{H} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(w)$.

Tate twist: $\mathcal{T}(k) = (\mathcal{H}', \mathcal{H}'', (iz)^{-2k} C), k \in \frac{1}{2}\mathbb{Z}$.

Hermitian duality: $\mathcal{T}^* = (\mathcal{H}'', \mathcal{H}', C^*)$ with $C^*(x, \overline{y}) = \overline{C(y, \overline{x})}$.

$$w(\mathcal{T}^*) = -w(\mathcal{T}), \quad w(\mathcal{T}(k)) = w(\mathcal{T}) - 2k, \quad \mathcal{T}(k)^* = \mathcal{T}^*(-k).$$

Polarization:

- If $\mathcal{T}$ has weight $\mathbf{0}$, a polarization is an isomorphism

$$S : \mathcal{H}'' \xrightarrow{\sim} \mathcal{H}'$$

such that

$$C \circ (S \otimes \mathrm{Id}) : \Gamma(\mathrm{A}, \mathcal{H}'') \underset{\mathcal{O}(\mathrm{A})}{\otimes} \overline{\Gamma(\mathrm{A}, \mathcal{H}'')} \longrightarrow \mathcal{O}(\mathrm{A})$$

induces a ***positive definite Hermitian form*** h on $H \underset{\mathbb{C}}{\otimes} \overline{H}$.

- If $\mathcal{T}$ has weight w,

$$\mathcal{S} : \mathcal{T} \longrightarrow \mathcal{T}^*(-w),$$

$$\mathcal{S} = (S', S''), \quad S', S'' : \mathcal{H}'' \longrightarrow \mathcal{H}'$$

such that

$$\mathcal{S}^* = (-1)^w \mathcal{S}, \quad S' = (-1)^w S''$$

and S'': a polarization of $\mathcal{T}(w/2)$.

Theorem **(C. Simpson)**

Variation of polarized twistor structures of weight $\mathbf{0} \Leftrightarrow$ *harmonic metric on* H.

Corollary

If X *is compact Kähler, The restriction to* $z = 1$ *induces an equivalence between the category of variations of polarized twistor structures of weight* $\mathbf{0}$ *and that of semisimple representations of* $\pi_1(X)$.

Hodge-Simpson Theorem

If X *is compact Kähler,* L *the Lefschetz operator, and if*

$$\mathcal{T} = (\mathcal{H}', \mathcal{H}'', C)$$

is a variation of polarized twistor structure of weight w *on* X*, then for any* $k \geqslant \mathbf{0}$*, the primitive part of the* k*th de Rham cohomology is a polarized twistor structure of weight* $w + k$.

Twistor $\mathcal{D}_X$-modules

$$\mathcal{R}_{X\times U_0} \simeq \mathcal{O}_{X\times U_0}\langle z\partial_{x_1}, \dots, z\partial_{x_n}\rangle \quad \textit{locally}$$

Polarized twistor $\mathcal{D}_X$-module: $(\mathcal{M}', \mathcal{M}'', C, S)$

- Two ***strict holonomic*** $\mathcal{R}_{X\times U_0}$-modules $\mathcal{M}', \mathcal{M}''$,
- and a $\mathcal{R} \otimes_{\mathcal{O}(\mathrm{A})} \overline{\mathcal{R}}$-linear pairing

$$C : \pi_{\mathrm{A}*}\mathcal{M}' \underset{\mathcal{O}(\mathrm{A})}{\otimes} \overline{\pi_{\mathrm{A}*}\mathcal{M}''} \longrightarrow \mathfrak{Db}^{(\mathrm{A})}_{X_\mathbb{R}}.$$

- A polarization is an isomorphism of $\mathcal{R}_{X\times U_0}$-modules $S : \mathcal{M}'' \xrightarrow{\sim} \mathcal{M}'$.

Inductive Requirements: $\mathcal{M}', \mathcal{M}''$ are ***specializable*** along any germ of holomorphic function and the specialized modules are objects of the same kind.

Initial Requirements: If $\dim X = 0$, a ***polarized twistor structure of weight*** $w + \cdots$.

Main Theorem

*The category of regular holonomic $\mathcal{D}_X$-modules equipped with a polarized twistor structure is **semisimple**.*

*If $f : X \to Y$ is a morphism between smooth projective manifolds, the direct image of a regular holonomic module equipped with a polarized twistor structure decomposes in **direct sum** of its cohomology modules, which are regular holonomic $\mathcal{D}_Y$-modules equipped with a polarized twistor structure (the weight is obtained in the usual way).*

Conjecture

*If X is smooth projective, the restriction functor to $z = 1$ is an equivalence between the category of regular holonomic $\mathcal{D}_X$-modules **equipped with a polarized twistor structure of weight** 0 and that of **semisimple** regular holonomic $\mathcal{D}_X$-modules.*

C. Simpson's and O. Biquard's work: true in the following cases:

- The ***smooth*** $\mathcal{D}_X$-modules and the locally constant sheaves.
- X is a ***compact Riemann surface***.