Stokes phenomenon and real singularities

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Theorem (Levelt-Turrittin). Given A, $\exists$ a ***formal*** gauge transf. $\widehat{P} \in \mathrm{GL}_d(\mathbb{C}((z^{1/q})))$ s.t. $\widehat{B} = \widehat{P}^{-1}A\widehat{P} + \widehat{P}^{-1}\widehat{P}'$ is a normal form.

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$$0 \longrightarrow \mathscr{A}^{\text{rd}\,0}_{S^1} \longrightarrow \mathscr{A}_{S^1}[1/z] \longrightarrow \varpi^{-1}\mathbb{C}((z)) \longrightarrow 0$$

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On S^1, $\quad \mathrm{Re}\,\varphi = 0 \iff \theta = \dfrac{1}{m}(\arg u(0) + \pi/2) \mod \mathbb{Z}\pi/m.$

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Problem. To describe a stratif. of $\widetilde{X}$ adapted to this complex.

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  - ***Theorem***. The sheaf of rapid decay sols of ∇ is real constr. w.r.t. any stratification compatible with the projection of $\mathrm{St}(E, \nabla)$ on $\widetilde{X}$.
- ***Corollary***. If $\dim X = 2$, and ***any*** (E, ∇), then the complex of rapid decay sols of ∇ is real constructible on $\widetilde{X}$.