Stokes phenomenon and real singularities

Claude Sabbah

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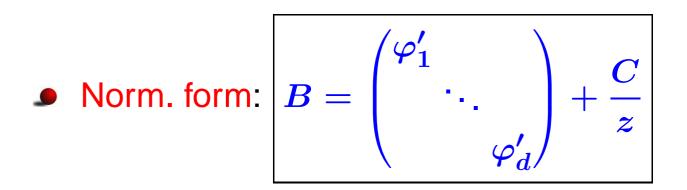
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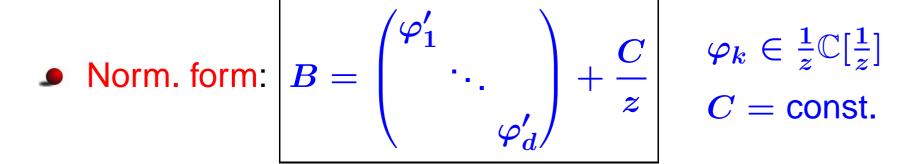
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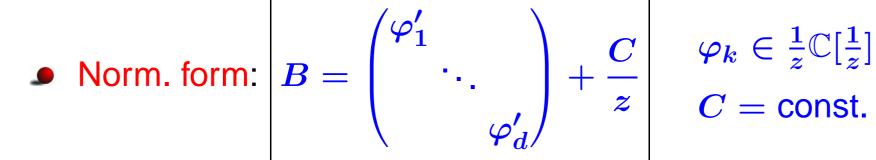
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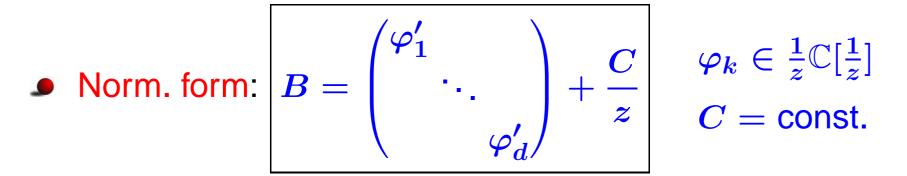
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Theorem (Levelt-Turrittin). Given A, \exists a **formal** gauge transf. $\hat{P} \in \operatorname{GL}_d(\mathbb{C}((z^{1/q})))$ s.t. $\hat{B} = \hat{P}^{-1}A\hat{P} + \hat{P}^{-1}\hat{P}'$ is a normal form.

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Basic exact sequence:

$$0\longrightarrow \mathscr{A}_{S^1}^{\mathsf{rd}\,0}\longrightarrow \mathscr{A}_{S^1}[1/z]\longrightarrow arpi^{-1}\mathbb{C}(\!(z)\!)\longrightarrow 0$$

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Problem. To describe a stratif. of \widetilde{X} adapted to this complex.

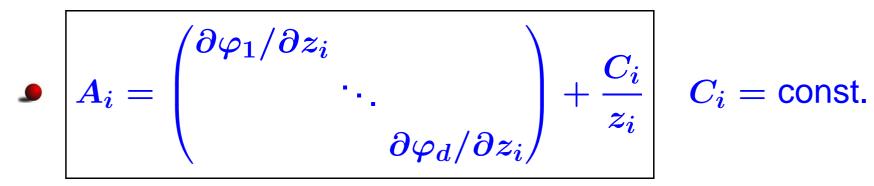
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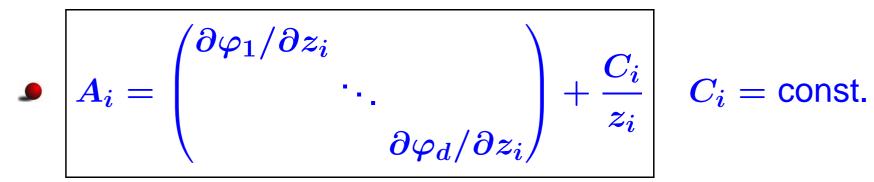
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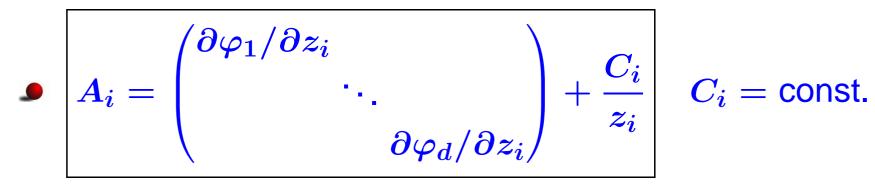


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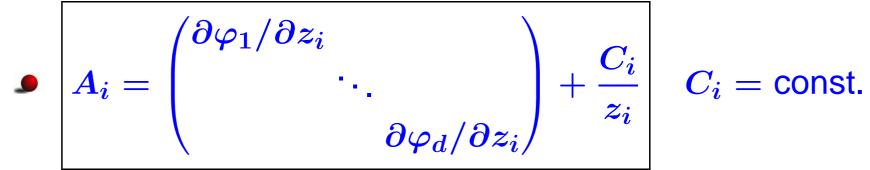
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- Corollary. If dim X = 2, and any (E, ∇) , then the complex of rapid decay sols of ∇ is real constructible on \widetilde{X} .