HERMITIAN METRICS ON FROBENIUS MANIFOLDS

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1. Laplace transform of variations of polarized complex Hodge structures

Let V be a holomorphic vector bundle on the Riemann sphere $\mathbb{P}^1$ equipped with a connection ∇ having logarithmic poles at a finite set $P = \{p_1, \dots, p_r, p_{r+1} = \infty\}$.

Let t denote the coordinate on the affine chart $\mathbb{A}^1 = \mathbb{P}^1 \smallsetminus \{\infty\}$. One can find a relation of the form

$$P(t, \nabla_{\partial_t}) := a_d(t)(\nabla_{\partial_t})^d + \cdots + a_0(t) = 0, \qquad a_0(t), \dots, a_d(t) \in \mathbb{C}[t].$$

This is the *Picard-Fuchs equation* associated to (V, ∇).

In this equation, replace ∇_{∂_t} with τ and t with $-\widehat{\nabla}_{\partial_\tau}$, and use the commutation relation

$$t \cdot \nabla_{\partial_t} = \nabla_{\partial_t} \cdot t - 1, \quad \text{that is} \quad \widehat{\nabla}_{\partial_\tau} \cdot \tau = \tau \cdot \widehat{\nabla}_{\partial_\tau} + 1,$$

to write P as

$$P(t, \nabla_{\partial_t}) = \widehat{P}(\tau, \nabla_{\partial_\tau}) = b_{\widehat{d}}(\tau)(\widehat{\nabla}_{\partial_\tau})^{\widehat{d}} + \cdots + b_0(\tau).$$

The operator $\widehat{P}$ is called the *Laplace transform* of P. It is known that, with the previous assumption on V, $b_{\widehat{d}}(\tau) = c\tau^d$ and $\widehat{d} = \deg_t a_d(t)$. It defines a bundle $\widehat{V}$ on the complex line $\widehat{\mathbb{A}}^1$ (τ-coordinate) of rank $\widehat{d}$, with a connection $\widehat{\nabla}$ having a regular singularity at 0 but usually not at ∞, and no other pole.

Assume now that (V, ∇) underlies a variation of polarized complex Hodge structure of some weight $w \in \mathbb{Z}$. We address the following

Question. What kind of a structure underlies the Laplace transform $(\widehat{V}, \widehat{\nabla})$ of (V, ∇)?

Notation. $X = \mathbb{P}^1$, $X^* = \mathbb{P}^1 \smallsetminus P$. We work with the weight $w = 0$.

We consider on X^* a variation of complex Hodge structure of weight 0, which is polarized. It consists of the datum of a C^∞ vector bundle H on X^* equipped with a flat connection D, a decomposition $H = \oplus_{p\in\mathbb{Z}} H^p$ (H^p is usually written as $H^{p,-p}$ as the weight is 0) and a Hermitian metric h on H, satisfying the following properties:

– the decomposition is orthogonal with respect to h and the nondegenerate Hermitian form $k = \oplus_p (-1)^p h_{|H^p}$ is D-flat,

– (Griffiths' transversality)

$$
\begin{aligned}
D'(H^p) &\subset (H^p \oplus H^{p-1}) \otimes_{\mathscr{O}_{X^*}} \Omega^1_{X^*} \\
D''(H^p) &\subset (H^p \oplus H^{p+1}) \otimes_{\mathscr{O}_{\overline{X}^*}} \Omega^1_{\overline{X}^*}.
\end{aligned}
\tag{1.1}
$$

We define the Hodge filtration $F^\bullet H$ as

$$
F^p H = \oplus_{q\geqslant p} H^q,
$$

so that $D'F^pH \subset F^{p-1}H \otimes_{\mathscr{O}_{X^*}} \Omega^1_{X^*}$.

We denote by (V,∇) the holomorphic bundle with connection $(\operatorname{Ker} D'', D')$ and we put $F^pV = F^pH \cap V$. We have $\nabla F^pV \subset F^{p-1}V \otimes_{\mathscr{O}_{X^*}} \Omega^1_{X^*}$.

Remarks

(1) The variation of complex Hodge structure provides V with a filtration $F^\bullet V$. In general, there is no way to get from it a filtration on the Laplace transform $\widehat{V}$. Therefore $(\widehat{V},\widehat{\nabla})$ is unlikely to naturally underlie a variation of polarized complex Hodge structure in the classical sense. This is also prevented by the irregular singularity at infinity, according to the regularity theorem of Griffiths and Schmid.

(2) The answer to the arithmetic analogue of this question is known, by the work of Deligne, Katz and Laumon (Fourier-Deligne transform of ℓ-adic sheaves) and enters in the proof of Weil conjectures.

Theorem 1. *In general, $(\widehat{V},\widehat{\nabla})$ underlies an integrable variation of polarized twistor structure of weight w, which is tame at $\tau = 0$.*

Polarized twistor structure of weight 0

The following notions have been introduced by C. Simpson.

Let $\widehat{\mathscr{V}}$ be a vector bundle on the closed unit disc $\boldsymbol{D}_0$ of the new variable z. Denote by $\sigma : \boldsymbol{D}_0 \to \overline{\boldsymbol{D}_\infty}$ the antiholomorphic involution $z \mapsto -1/\overline{z}$.

A polarized twistor structure consists of the datum of $\widehat{\mathscr{V}}$ as above and of a sesquilinear pairing

$$
C : \widehat{\mathscr{V}}_{|\boldsymbol{S}} \otimes_{\mathscr{O}_{\boldsymbol{S}}} \sigma^* \overline{\widehat{\mathscr{V}}}_{|\boldsymbol{S}} \longrightarrow \mathscr{O}_{\boldsymbol{S}} \qquad (\boldsymbol{S} = \partial \boldsymbol{D}_0 = \partial \boldsymbol{D}_\infty)
$$

such that

(1) C is *nondegenerate*, *i.e.*, defines a gluing between $\widehat{\mathscr{V}}^\vee$ and $\overline{\widehat{\mathscr{V}}}$,

(2) the bundle on $\mathbb{P}^1$ obtained by gluing $\widehat{\mathscr{V}}^\vee$ and $\overline{\widehat{\mathscr{V}}}$ along $\boldsymbol{S}$ with C is *trivial*,

(3) C induces, on the global sections of this bundle, a *positive definite* Hermitian form.

Variation of polarized twistor structure of weight 0

It consists of the datum of a vector bundle $\widehat{\mathscr{V}}$ on $\widehat{\mathbb{A}}^{1*} \times \boldsymbol{D}_0$ equipped with a z-connection $\widehat{\nabla}_z$, *i.e.*, a meromorphic connection relative to τ only, with pole of order one at most along $z = 0$, and of a sesquilinear pairing

$$C : \widehat{\mathscr{V}}_{|\widehat{\mathbb{A}}^{1*}\times \boldsymbol{S}} \otimes_{\mathscr{O}_{\boldsymbol{S}}} \overline{\widehat{\mathscr{V}}}_{|\widehat{\mathbb{A}}^{1*}\times \boldsymbol{S}} \longrightarrow \mathscr{C}^{\infty,\mathrm{an}}_{|\widehat{\mathbb{A}}^{1*}\times \boldsymbol{S}}$$

such that

– C is compatible with $\widehat{\nabla}_z$,

– when restricted to any point $\tau_o \in \widehat{\mathbb{A}}^{1*}$, we get a polarized twistor structure of weight 0.

Holomorphic vector bundle with connection (V, ∇) and a *harmonic metric* $h \Longleftrightarrow$ Variation of polarized twistor structure of weight 0.

Integrability

The variation $(\widehat{\mathscr{V}}, C)$ is *integrable* if the z-connection $\widehat{\nabla}_z$ comes from an *integrable absolute meromorphic connection of Poincaré rank one* compatible with C.

Through the correspondence above, integrability implies that each fibre V_{τ_o} is equipped with a polarized Hodge structure. However, it is also equipped with other operators, and the variation is not a variation of Hodge structure in general.

Remark. Such a notion also appears in tt^* structures (Cecotti-Vafa 1991, Dubrovin 1993, Hertling 2003).

2. Frobenius structure (Dubrovin) or flat structure (K. Saito)

Data :

(0) M : complex manifold of dimension μ

(1) $\star$: *commutative associative product* with *unit* e on vector fields, depending holomorphically on the point in M,

(2) g : *nondegenerate bilinear form* on vector fields, depending holomorphically on the point in M.

(3) $\mathfrak{E}$: homogeneity (Euler) holomorphic vector field on M,

Constraints :

– *Symmetry* of the 4-tensor $(\xi_1, \xi_2, \xi_3, \xi_4) \longmapsto \triangledown_{\xi_1} g(\xi_2 \star \xi_3, \xi_4)$;

– *Flatness* of $\triangledown$ (*torsionless* connection associated to g) and $\triangledown e = 0$;

– *Homogeneity*: $\mathscr{L}_{\mathfrak{E}} e = -e$, $\mathscr{L}_{\mathfrak{E}}(\star) = \star$, $\mathscr{L}_{\mathfrak{E}}(g) = Dg$ for some $D \in \mathbb{C}$.

***Theorem* (joint work with Antoine Douai (Nice))**

Let f be any Laurent polynomial on a torus

$$U = (\mathbb{C}^*)^n = \operatorname{Spec} \mathbb{C}[u_1, u_1^{-1}, \dots, u_n, u_n^{-1}],$$

which is convenient and nondegenerate *with respect to its Newton polyhedron; in particular,*

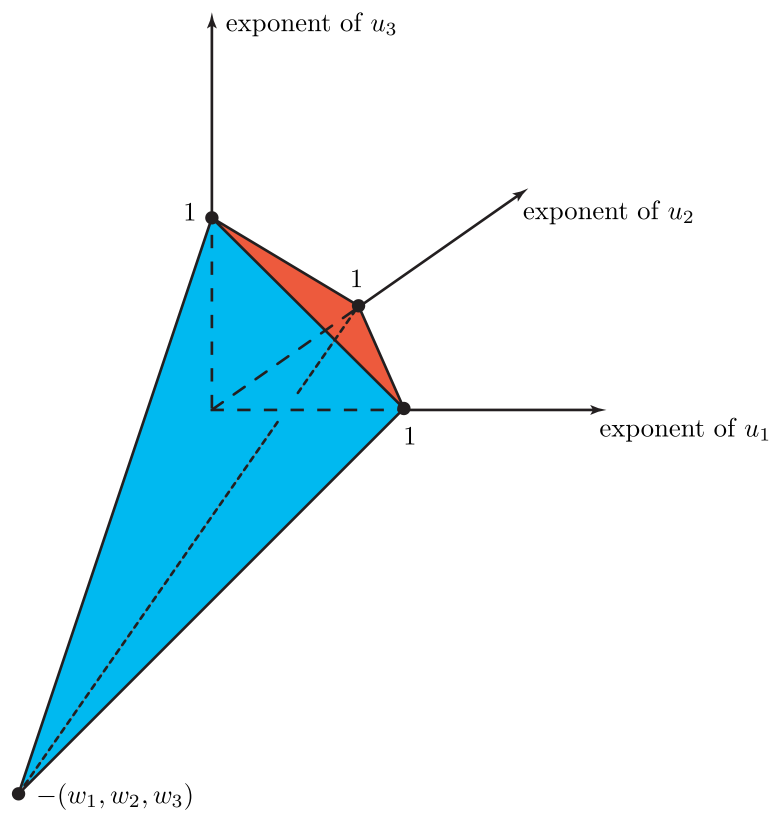
$$\mu := \dim \mathbb{C}[u_1, u_1^{-1}, \dots, u_n, u_n^{-1}]/(u_1\partial f/\partial u_1, \dots, u_n\partial f/\partial u_n) < +\infty.$$

Choose a family $\varphi_0 = 1, \varphi_1, \dots, \varphi_{\mu-1}$ *inducing a basis of this vector space. Then there exists a* canonical Frobenius structure *locally on the space of parameters* $x_0, \dots, x_{\mu-1}$ *of the unfolding* $F = f + \sum x_i\varphi_i$.

An example.

- $w_1, \dots, w_n$ ($n \geqslant 1$): positive integers.
- $f(u_1, \dots, u_n) = u_1 + \cdots + u_n + \dfrac{1}{u_1^{w_1} \cdots u_n^{w_n}}$.
- $\mu = 1 + w_1 + \cdots + w_n$, f has μ simple critical points.
- Critical values : $\mu e^{2i\pi k/\mu}$ ($k = 0, \dots, \mu - 1$).

It is expected (Étienne Mann) that the canonical Frobenius structure attached to f is isomorphic to the orbifold quantum cohomology of the weighted projective space $\mathbb{P}(1, w_1, \dots, w_n)$.



3. tt^* Structure

Other presentation of the constraints

Add a new variable z, hence $\pi : M \times \mathbb{C} \to M$. The constraints can be read on the pull-back bundle π^*TM: *flatness* of the connection

$$\nabla_\xi \eta = \triangledown_\xi \eta - \frac{\xi \star \eta}{z}$$
$$\nabla_{\partial_z} \eta = \mathfrak{E} \star \eta \cdot \frac{1}{z^2} - \triangledown_\eta \mathfrak{E} \cdot \frac{1}{z}.$$

The bilinear form g is also lifted to π^*TM as a pairing between π^*TM_z and π^*TM_{-z}.

***Definition* (Cecotti-Vafa, Dubrovin, Hertling).** A tt^* structure consists of a the supplementary datum of $C_z : \pi^*TM_z \otimes \overline{\pi^*TM}_{-z} \to \mathscr{C}^\infty_M$ depending analytically on $z \in S^1$, such that (π^*TM, ∇, C) is an integrable variation of polarized twistor structure of weight 0.

***Remark*.** This produces a *harmonic* Hermitian metric and a real structure on TM.

4. tt^* Structure for Laurent polynomials

The connection ∇ on π^*TM is obtained by

– considering the Laplace transform of the Gauss-Manin connection with respect to the unfolding F,

– identifying the corresponding bundle with π^*TM through a *infinitesimal period mapping.*

Let us restrict to $0 \in M$ corresponding to the function f. When restricted to a neighbourhood of S^1, the bundle (π^*T_0M, ∇) is identified to the holomorphic bundle with connection associated to the locally constant sheaf

$$H^n_{\Phi_z}(U, \mathbb{Q}),$$

where Φ_z denotes the family of closed sets in U on which $\mathrm{Re}(f(u_1, \dots, u_n)/z) \leqslant c < 0$.

There is a natural intersection pairing (made sesquilinear)

$$\widehat{P}_z : H^n_{\Phi_z}(U, \mathbb{C}) \otimes \overline{H^n_{\Phi_{-z}}(U, \mathbb{C})} \longrightarrow \mathbb{C}.$$

***Theorem 2*.** *(π^*T_0M, ∇, C) with $C_z = \dfrac{(-1)^{(n-1)n/2}}{(2i\pi)^n} \widehat{P}_z$ gives a tt^* structure on M near 0.*

The proof uses Theorem 1.

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