Hermitian metrics on Frobenius manifolds

Claude Sabbah

Centre de Mathématiques Laurent Schwartz

UMR 7640 du CNRS

École polytechnique, Palaiseau, France

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- t : the coordinate on the affine chart $\mathbb{A}^1 = \mathbb{P}^1 \smallsetminus \{\infty\}$

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  - with a connection $\widehat{\nabla}$ having a ***regular singularity*** at 0 but usually ***not*** at ∞, and no other pole.

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  What kind of a structure underlies the Laplace transform $(\widehat{V}, \widehat{\nabla})$ of (V, ∇)?

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- $(V, \nabla) = (\ker D'', D')$,
- Hodge filtration $F^pH = \oplus_{q\geqslant p} H^q$, $F^pV = F^pH \cap V$.

Theorem. *In general, $(\widehat{V}, \widehat{\nabla})$ underlies an* ***integrable variation of polarized twistor structure*** *of weight 0, which is tame at $\tau = 0$.*

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New complex variable z.
Closed unit disc $D_0 = \{|z| \leqslant 1\}$, $S = \partial D_0 = \{|z| = 1\}$.

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  - C induces, on the global sections of this bundle, a ***positive definite*** Hermitian form.

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Remark. Such a notion also appears in tt^* structures (Cecotti-Vafa 1991, Dubrovin 1993, Hertling 2003).

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- ***Constraints***.

Theorem (A. Douai, C.S.). *Let f be any Laurent polynomial on a torus*

$$U = (\mathbb{C}^*)^n = \operatorname{Spec} \mathbb{C}[u_1, u_1^{-1}, \ldots, u_n, u_n^{-1}],$$

which is ***convenient and nondegenerate*** *with respect to its Newton polyhedron; in particular,*

$$\mu := \dim \mathbb{C}[\underline{u}, \underline{u}^{-1}]/(\underline{u}\partial f/\partial \underline{u}) < +\infty.$$

Choose a family $\varphi_0 = 1, \varphi_1, \ldots, \varphi_{\mu-1}$ inducing a basis of this vector space. Then there exists a ***canonical Frobenius structure*** *locally on the space M of parameters $x_0, \ldots, x_{\mu-1}$ of the unfolding $F = f + \sum x_i \varphi_i$.*

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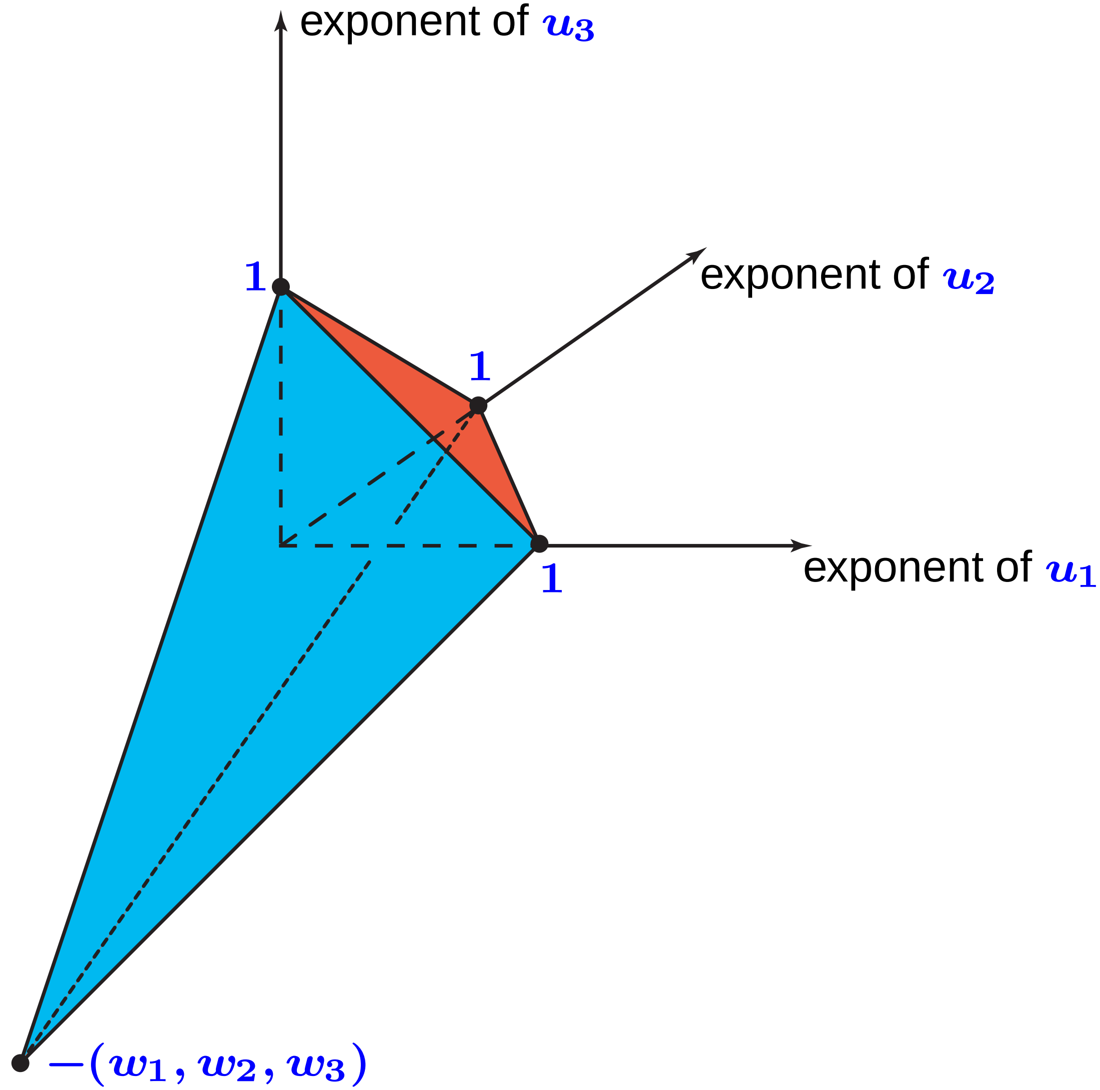

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Remark. It is expected (Étienne Mann) that the canonical Frobenius structure attached to f is isomorphic to the orbifold quantum cohomology of the weighted projective space $\mathbb{P}(1, w_1, \ldots, w_n)$.

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$\triangledown$: Levi-Civita connection of the bilinear form g on TM

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$\mathfrak{E}$: Euler vector field

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- The bilinear form g is also lifted to π^*TM as a pairing between π^*TM_z and π^*TM_{-z}.

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Definition (Cecotti-Vafa, Dubrovin, Hertling). A tt^* structure consists of a the supplementary datum of $C_z : \pi^*TM_z \otimes \overline{\pi^*TM}_{-z} \longrightarrow \mathscr{C}^\infty_M$ depending analytically on $z \in S^1$, such that $(\pi^*TM, \boldsymbol{\nabla}, C)$ is an integrable variation of polarized twistor structure of weight 0 parametrized by M.

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- identifying the corresponding bundle with $\pi^* TM$ through an ***infinitesimal period mapping***.

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Φ_z : the family of closed sets in U on which $\mathrm{Re}(f(u_1, \ldots, u_n)/z) \leqslant c < 0$.

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There is a natural intersection pairing (made sesquilinear)

$$\widehat{P}_z : H^n_{\Phi_z}(U,\mathbb{C}) \otimes \overline{H^n_{\Phi_{-z}}(U,\mathbb{C})} \longrightarrow \mathbb{C}.$$

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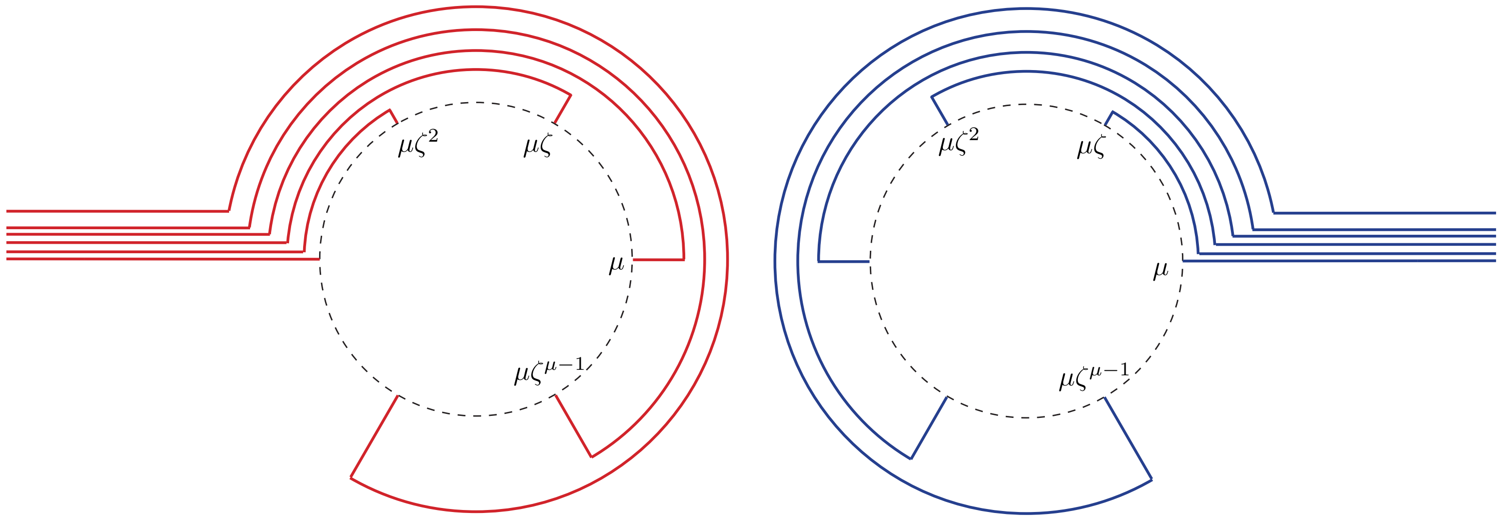

Theorem (conjectured by C. Hertling). $(\pi^* T_0 M, \nabla, C)$ *with* $C_z = \frac{(-1)^{(n-1)n/2}}{(2i\pi)^n} \widehat{P}_z$ *gives a* tt^* *structure on* M *near* 0*.*

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- The Gauss-Manin connection attached to f defines a variation of polarized ***mixed*** Hodge structures (variable t).
- The graded pieces with weight $\neq$ the expected weight are ***constant*** Hodge structures which are killed after Laplace transform and restriction to $|\tau| = 1$.
- Apply the first theorem to the right graded piece (V, ∇) of the Gauss-Manin connection.

Theorem. *In general,* $(\widehat{V}, \widehat{\nabla})$ *underlies an* ***integrable variation of polarized twistor structure*** $(\widehat{\mathscr{V}}, \widehat{\nabla}_z, C_z)$ *of weight* 0*, which is tame at* $\tau = 0$.

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- Restrict $(\widehat{\mathscr{V}}, \widehat{\nabla}_z, C_z)$ to $\tau = 1$ to get $(\pi^* T_0 M, \nabla, C)$ with its ***integrable polarized twistor structure***.