On the irregular Hodge filtration

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- Simplest example: $\mathscr{M} = \mathscr{O}_X$, $F_0\mathscr{M} = \mathscr{O}_X$,
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  - e.g., $\mathscr{E}^f := (\mathscr{O}_X(*P_{\mathrm{red}}), \mathrm{d} + \mathrm{d}f)$.
- General question: If $(\mathscr{M}, F_\bullet\mathscr{M})$ underlies a MHM, to define an ***irregular Hodge filtration*** $F_\bullet^{\mathrm{Del}}(\mathscr{M} \otimes \mathscr{E}^f)$, and to prove that the correspond. spectral sequence

$$H^{p+q}(X, \mathrm{gr}^p_{F_{\mathrm{Del}}} \mathrm{DR}(\mathscr{M} \otimes \mathscr{E}^f)) \Longrightarrow H^{p+q}(X, \mathrm{DR}(\mathscr{M} \otimes \mathscr{E}^f))$$

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- In gen., should expect $F_p^{\mathrm{Del}}(\mathscr{M} \otimes \mathscr{E}^f)$ with $p \in A + \mathbb{Z}$, $A \subset \mathbb{Q}$ finite.
- Motivation for $F_\bullet^{\mathrm{Del}}$: estimates for p-adic exp. sums in terms of the ***irregular Hodge polygon***.

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- pf of E_1-degeneration by using sol. of a Birkhoff pb as obtained by M. Saito from Hodge theory.

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- $\exists\ V$-filtr. along $\tau=0$: $V_\tau^\alpha(R_F\mathscr{M}[\tau]\otimes\mathscr{E}^{t\tau/z})$

$$\boxed{\frac{V_\tau^\alpha(R_F\mathscr{M}[\tau]\otimes\mathscr{E}^{t\tau/z})}{(\tau-z)V_\tau^\alpha(R_F\mathscr{M}[\tau]\otimes\mathscr{E}^{t\tau/z})}=:R_{F^{\mathrm{Del}}_{\alpha+\bullet}}(\mathscr{M}\otimes\mathscr{E}^t).}$$

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Proof: Same as C.S. (2010).

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- New: strictness for the push-forward $X \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. $\Leftarrow$ good behaviour of the V-filtr. w.r.t. projective push-forward. $\square$

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THEOREM (Kontsevich 2012): $\forall k$, $\dim H^k\big(X, (\Omega_f^\bullet, u\mathrm{d}+v\mathrm{d}f)\big)$ indept. of $u, v \in \mathbb{C}$, equal to $H^k_{\mathrm{dR}}(U, \mathrm{d} + \mathrm{d}f)$. In partic.

$$E_1^{p,q} = H^q(X, \Omega_f^p) \Longrightarrow H^{p+q}(X, (\Omega_f^\bullet, \mathrm{d})) \quad \textbf{\textit{degen. at }} E_1$$

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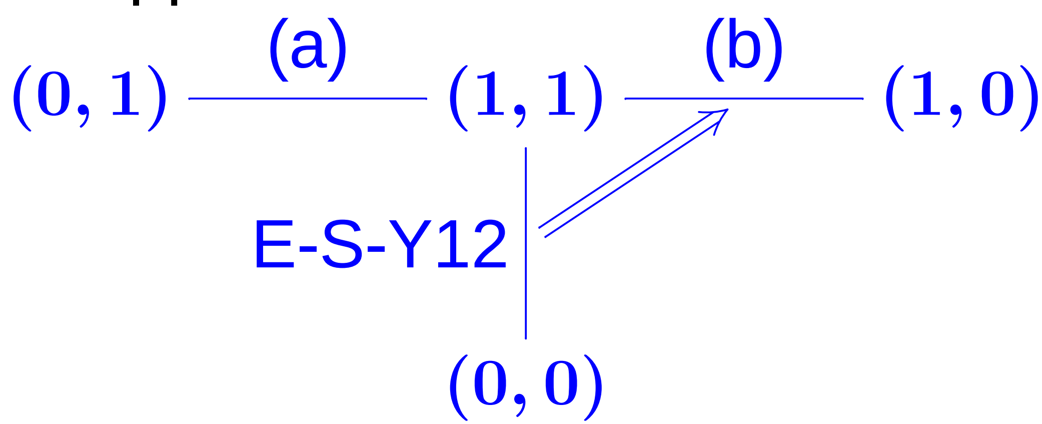

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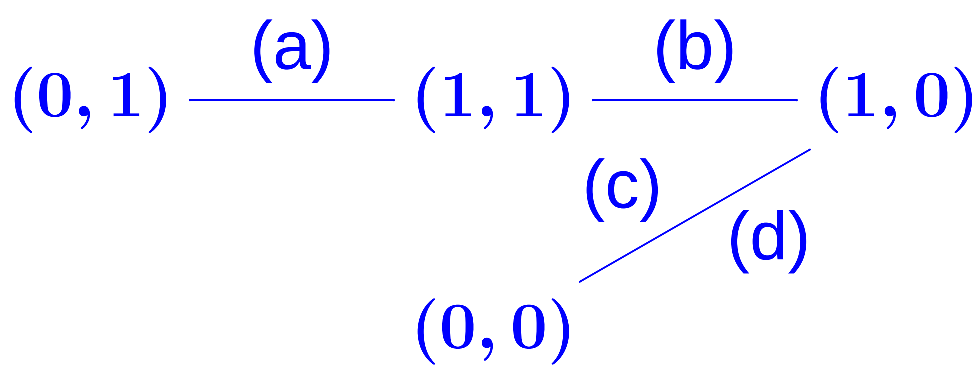

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- $[(1,1) \text{ “=” } (0,0)] \Rightarrow$(b): Semi-continuity argument.

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- (d) (M. Saito): Consider $(\Omega_X^\bullet(\log D)/\Omega_f^\bullet, \mathrm{d})$ supported on P_{red}. Apply results of Steenbrink (1976-77) for the Hodge structure on the nearby cycles of f along $f^{-1}(\infty)$.

Open question

To define a sub-category of the category of ***wild twistor $\mathscr{D}$-modules*** (C.S., T. Mochizuki) for which the associated holonomic $\mathscr{D}$-module has a good filtration like $F_\bullet^{\mathrm{Del}}$, with strictness properties (morphisms, projective push-forward).