Wild ramification in complex algebraic geometry

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Programme SEDIGA ANR-08-BLAN-0317-01

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- Conversely, Hodge Theory ***requires*** tame sing. (Griffiths-Schmid) and $\mathbb{C}$-Alg. Geom. produces tame sing. (Gauss-Manin systems)

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  - Better analogy with constr. $\overline{\mathbb{Q}}_\ell$-sheaves on $X_{\mathbb{F}_q}$.

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- THEOREM (C. Hertling, H. Iritani, Reichelt-Sevenheck, C.S.): Quantum cohom. of Fano toric varieties underlies a var. of polarized ***nc.*** $\mathbb{Q}$-Hodge structure on a Zariski dense open set of the Kähler moduli space.

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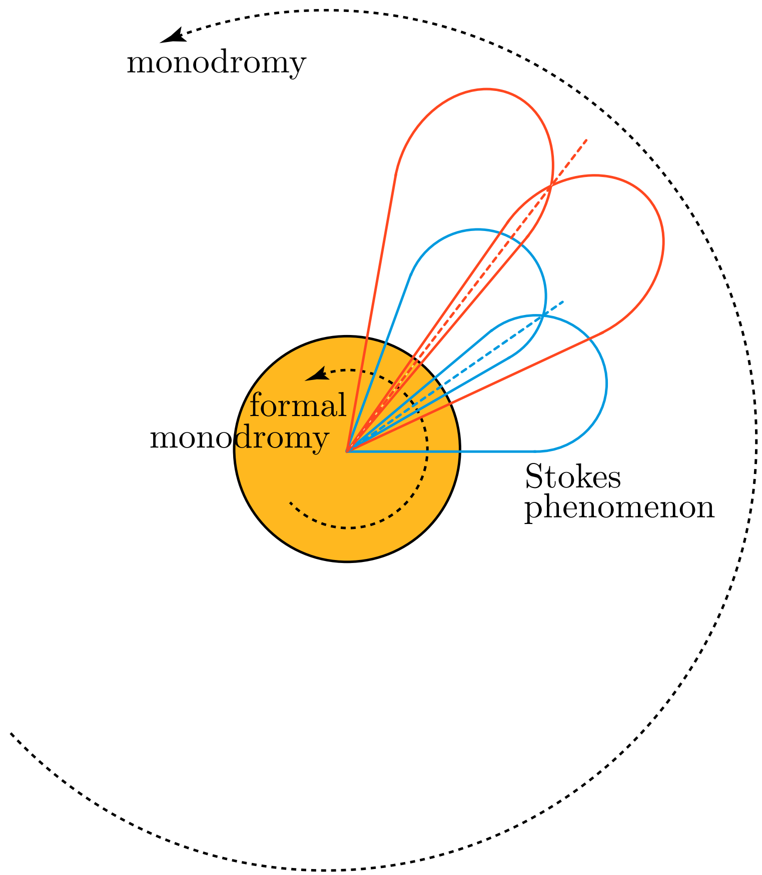

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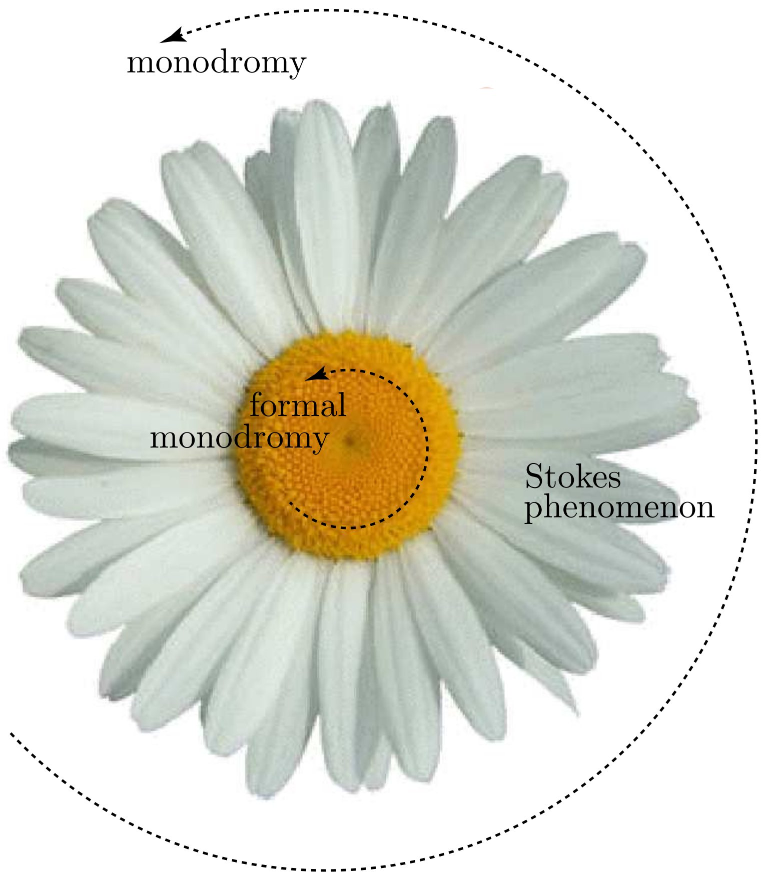

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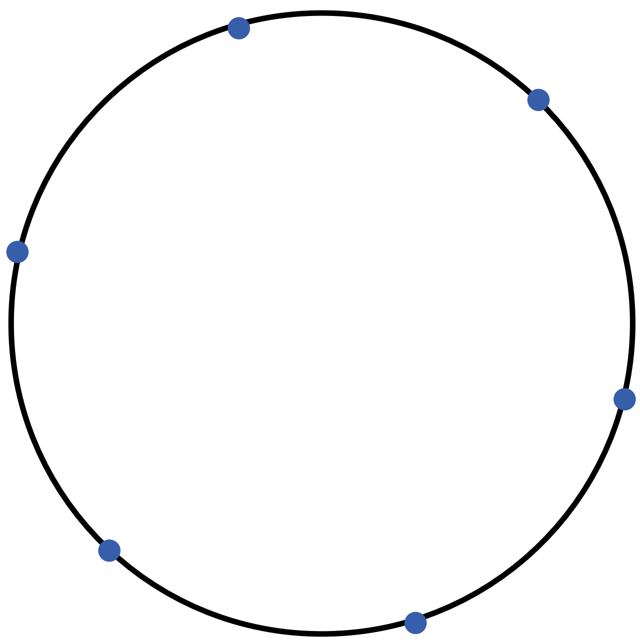

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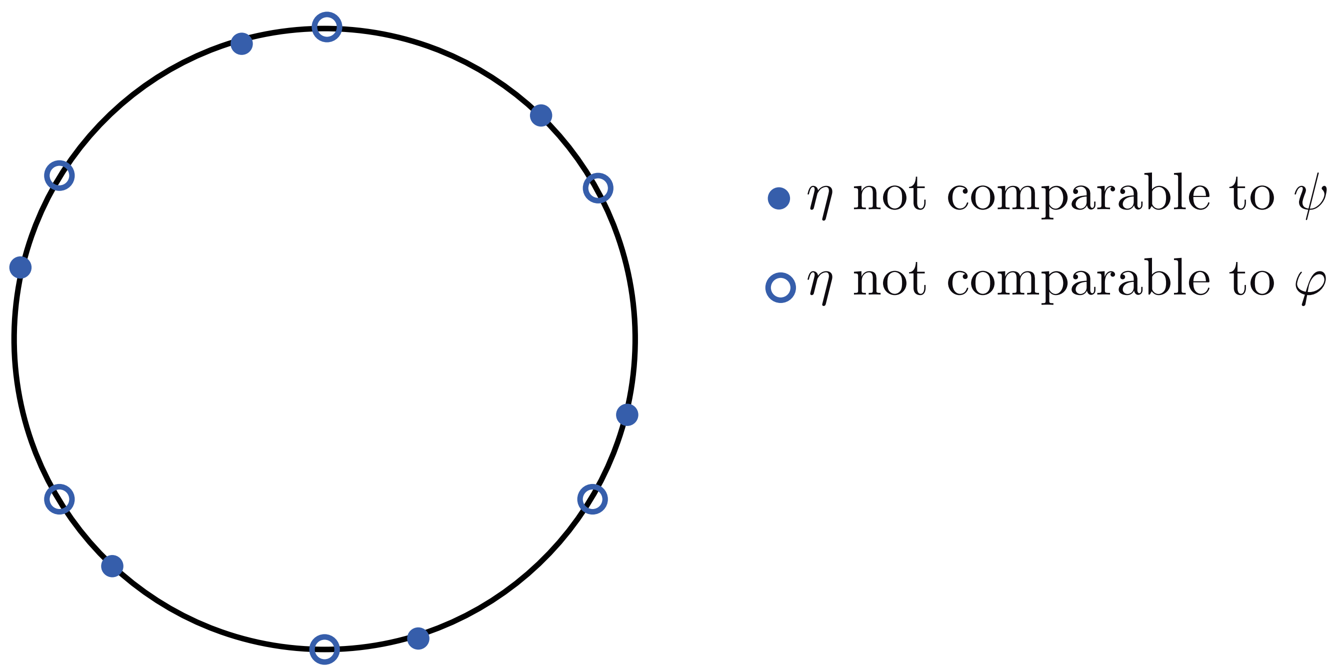

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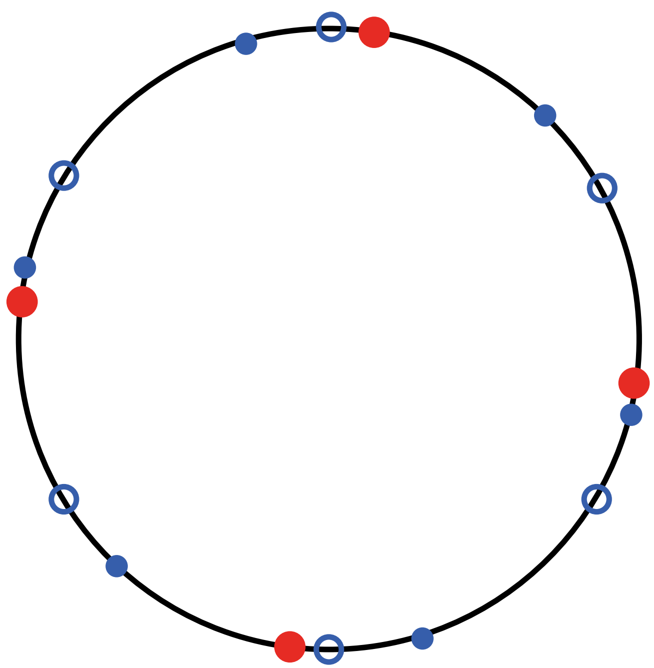

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  (should take “ramified polar parts” instead)

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Levelt-Turrittin in dim. $\geqslant 2$

QUESTION (2 variables):
$(\widehat{\mathscr{M}}, \widehat{\nabla})$: free $\mathbb{C}[\![x_1, x_2]\!][x_1^{-1}, x_2^{-1}]$-module with ***flat*** connect.
$\exists$? (maybe after ramif. $x_1 = y_1^{r_1}$, $x_2 = y_2^{r_2}$) a decomp.

$$(\widehat{\mathscr{M}}, \widehat{\nabla}) \simeq \bigoplus_\eta (\widehat{\mathscr{E}}^\eta \otimes \widehat{\mathscr{R}}_\eta)$$

$\eta \in \Phi \subset \mathbb{C}[\![x_1, x_2]\!][x_1^{-1}, x_2^{-1}]/\mathbb{C}[\![x_1, x_2]\!]$,
$\widehat{\mathscr{E}}^\eta = (\mathbb{C}[\![x_1, x_2]\!][x_1^{-1}, x_2^{-1}], \mathrm{d} + \mathrm{d}\eta)$,
$\widehat{\mathscr{R}}_\eta$ ***reg. sing.*** along $D = \{x_1 x_2 = 0\}$.

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(***good*** formal structure, import. for ***asympt. analysis***)

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APPLICATION TO ASYMPT. ANALYSIS: Y. Sibuya (70's), H. Majima (1984), C.S. (1993, 2000): $\dim X = 2$, T. Mochizuki (2010): $\dim X \geqslant 2$.

Stokes-filtered loc. syst. (dim. $\geqslant$ 2)

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  (should take “ramified polar parts” instead)

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THEOREM: Fix a ***good*** $\widetilde{\Sigma} \subset \mathfrak{I}^{\text{ét}}$. Then the category of $\mathscr{F}_{\leqslant}$ with support $\subset \widetilde{\Sigma}$ is ***abelian*** and every morphism is ***strict***.

Riemann-Hilbert corr. (global case)

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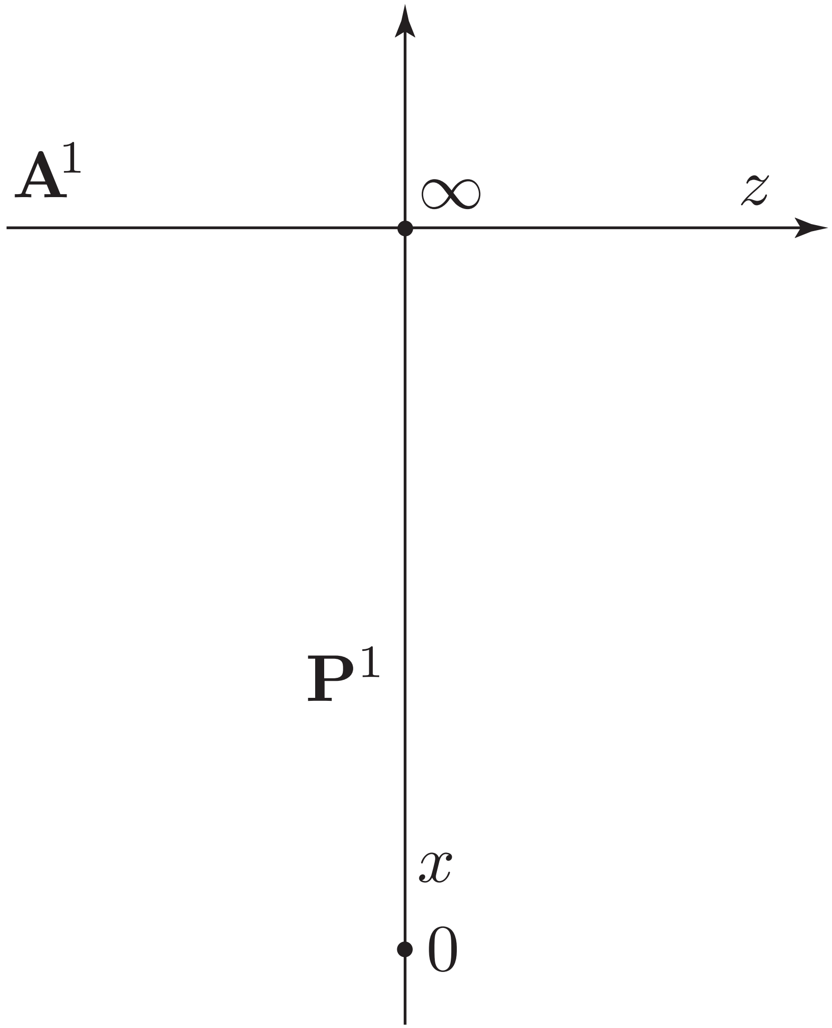
Deligne, 2007: "La théorie des structures de Stokes fournit une notion de structure de Betti si $\dim(X) = 1$. On voudrait une définition en toute dimension, et une stabilité par les six opérations ($Rf_*, Rf_!, f^*, Rf^!, \otimes^L$, *RHom*). On est loin du compte."

Example

Example

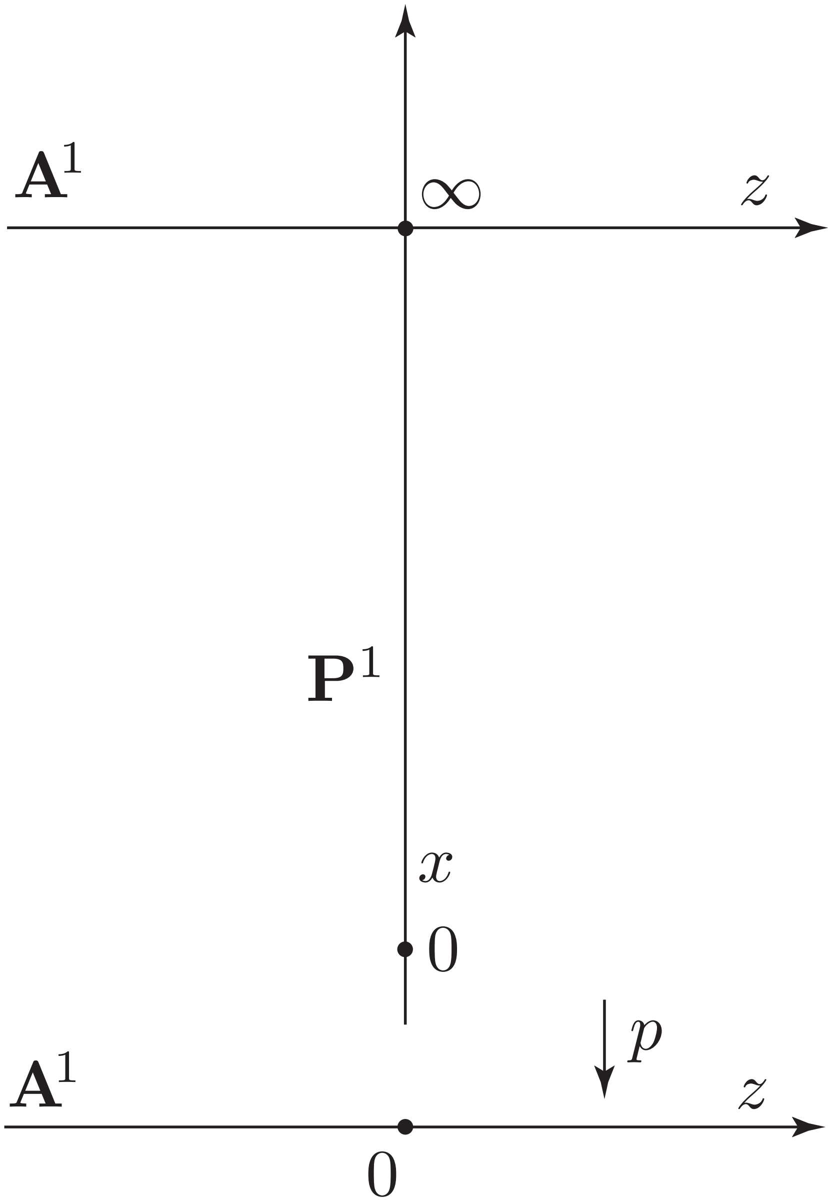
$\mathbb{P}^{\mathbf{1}}$

Example



$\mathbb{P}^{\mathbf{1}} \times \mathbb{A}^{\mathbf{1}}$

Example



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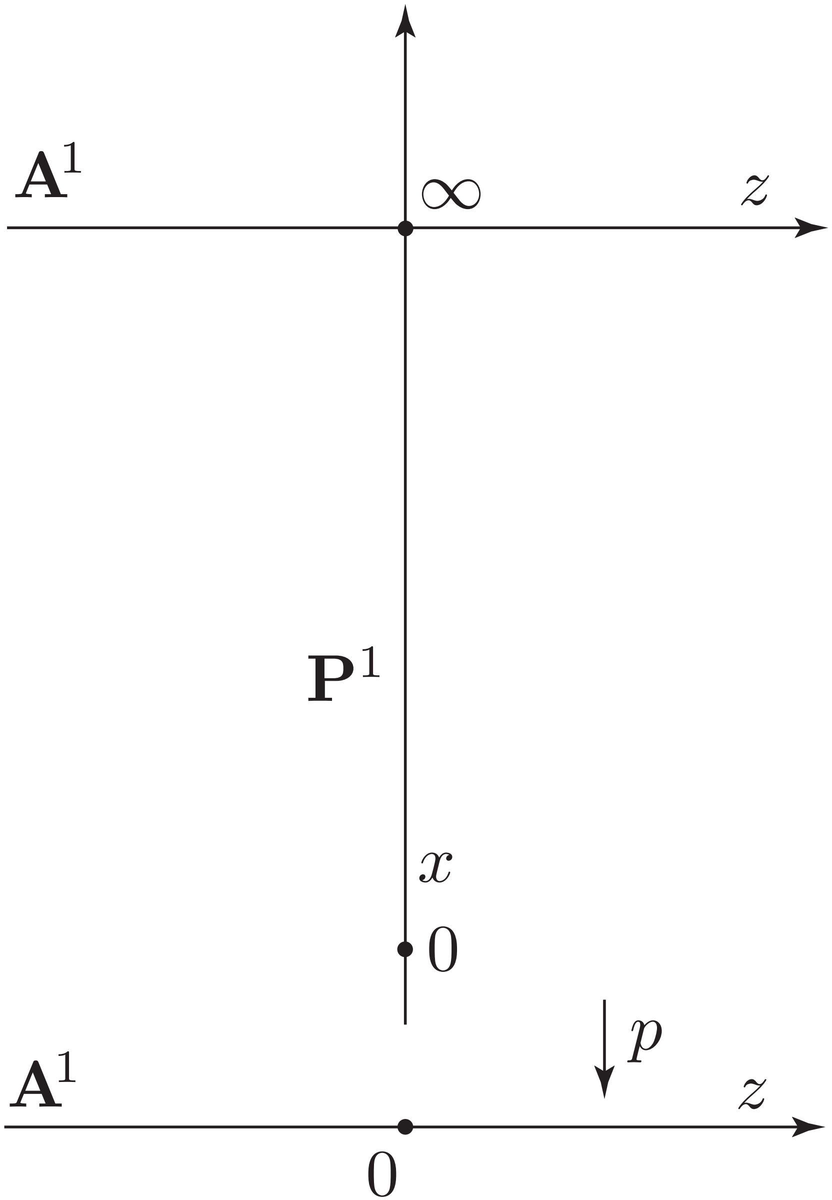

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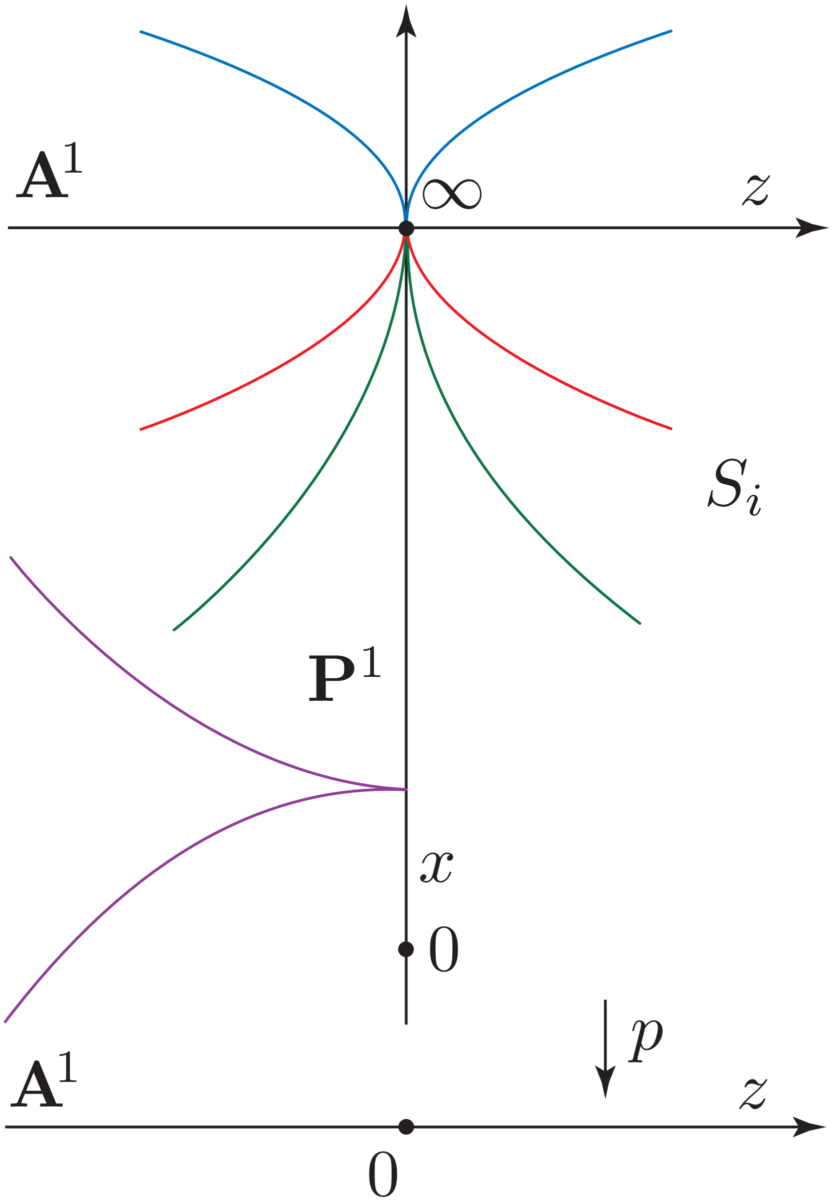

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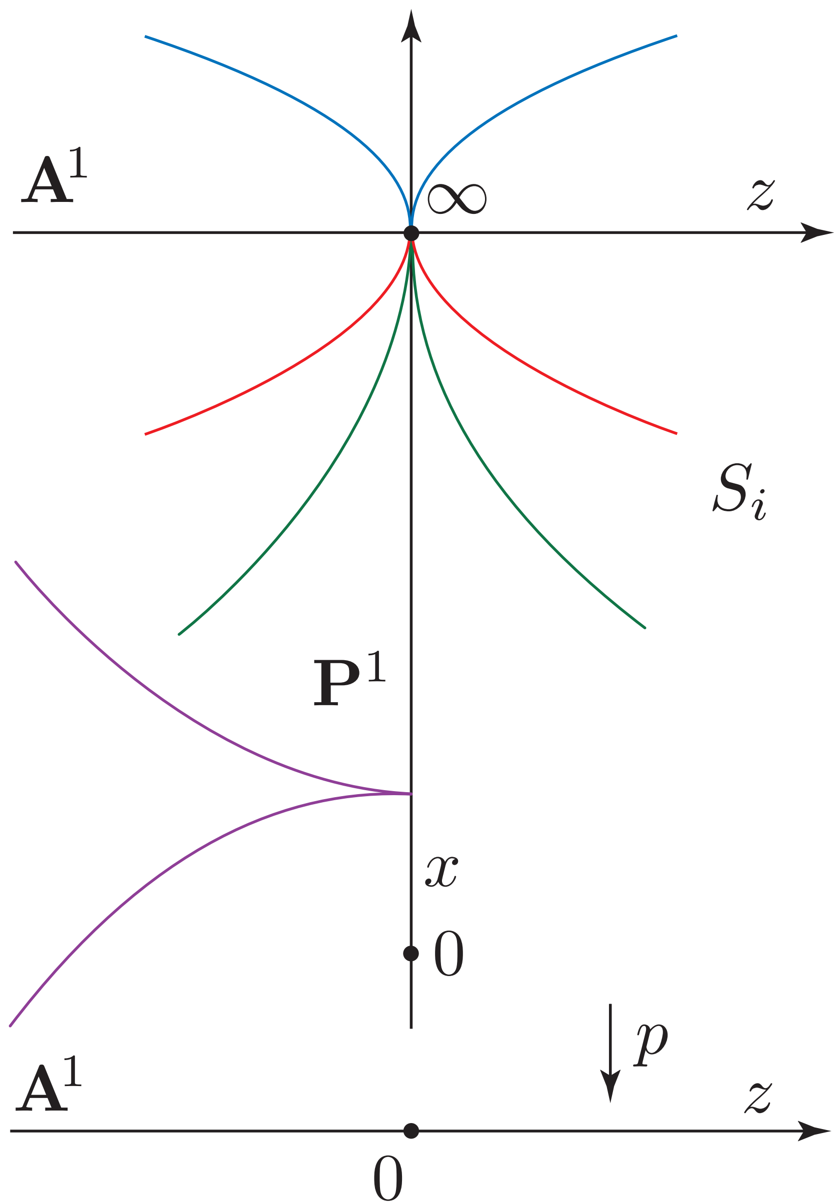

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i.e. diff. eqn for $\int_{\gamma_z} f(x, z) e^x \, dx$

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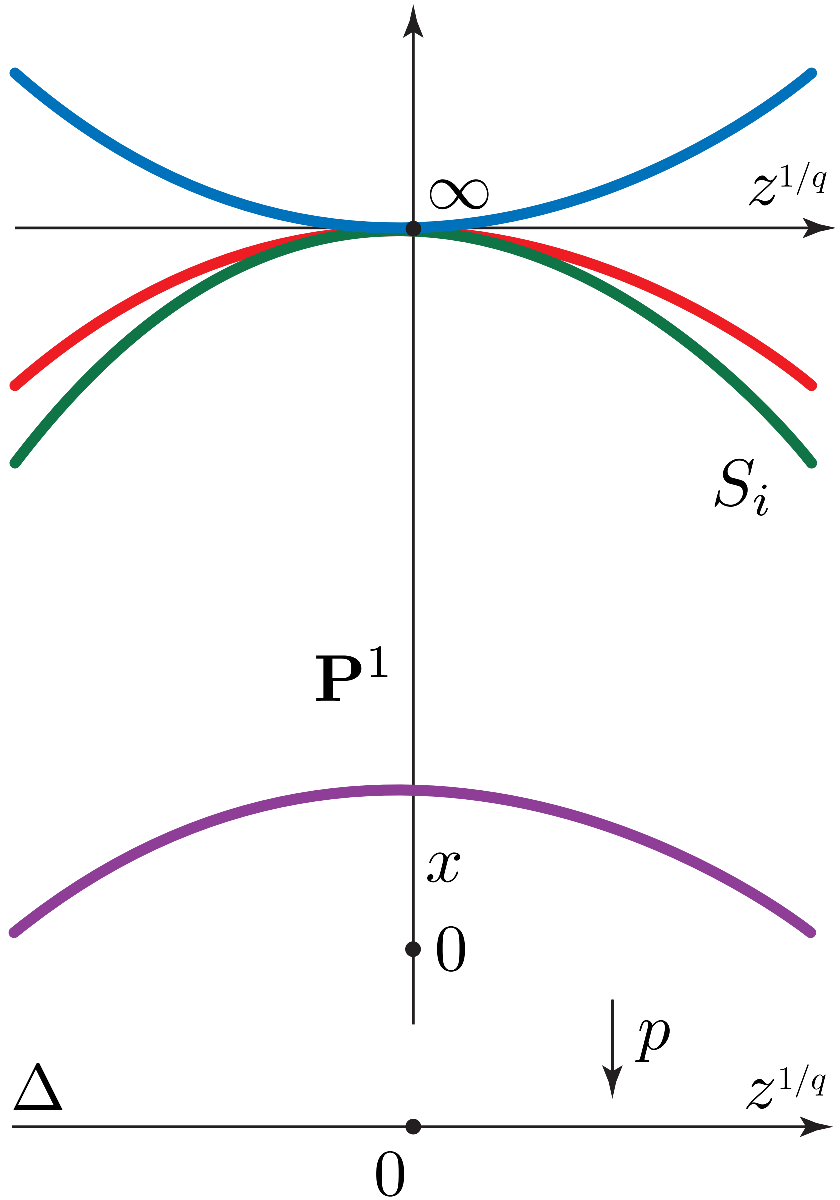

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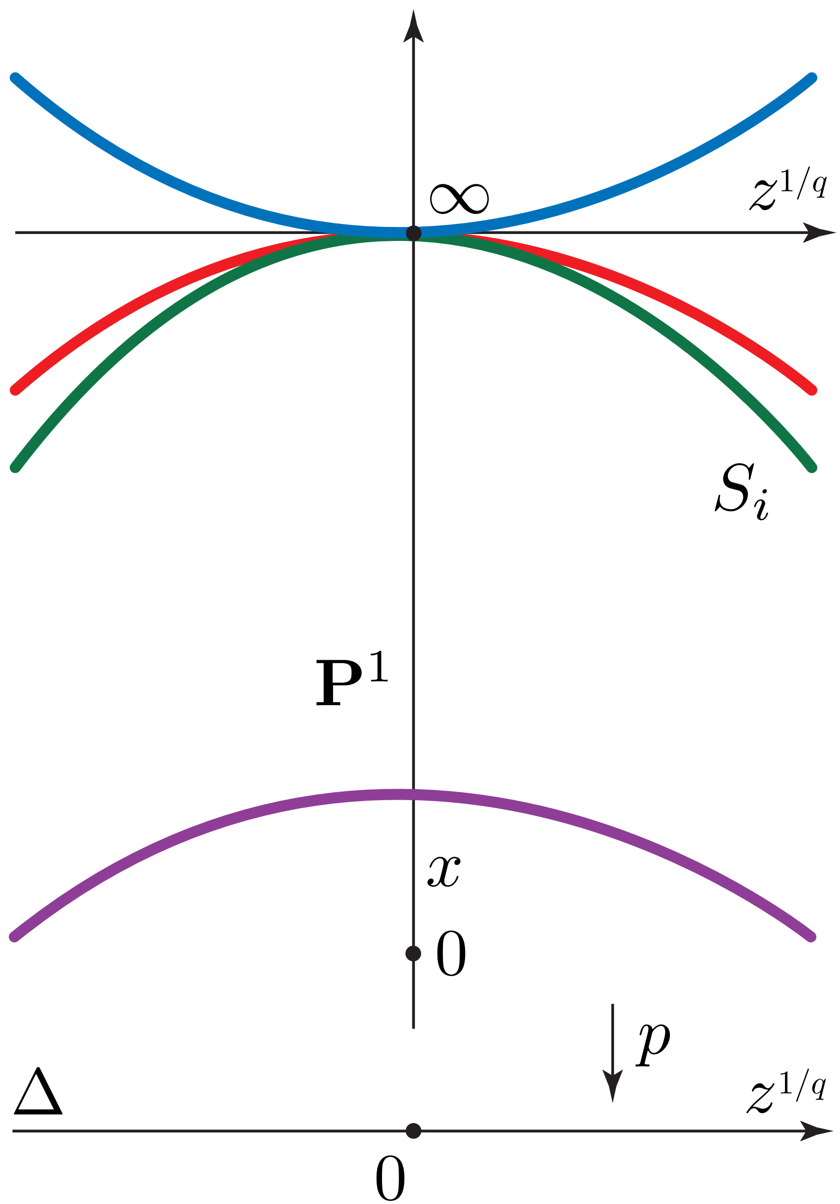
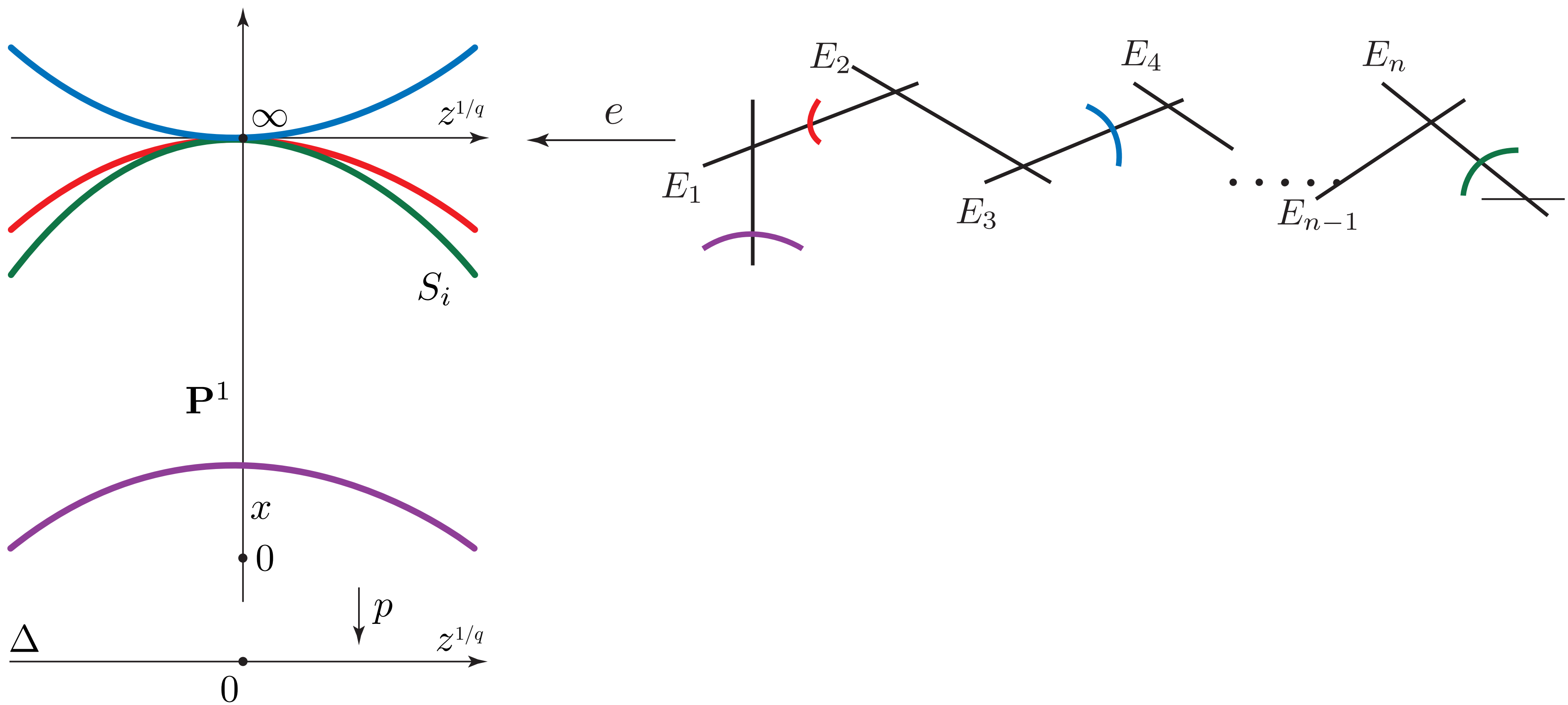


THM (C. Roucairol, 2007):
$\widehat{\mathscr{N}} = \bigoplus_i (\widehat{\mathscr{E}^{\eta_i}} \otimes \widehat{\mathscr{R}_i})$
$\eta_i(z)$ = pol. part of $x(z)_{|S_i}$
$\mathscr{R}_i$ = ***vanishing cycle*** module of $\mathscr{M}$ along S_i.

Example



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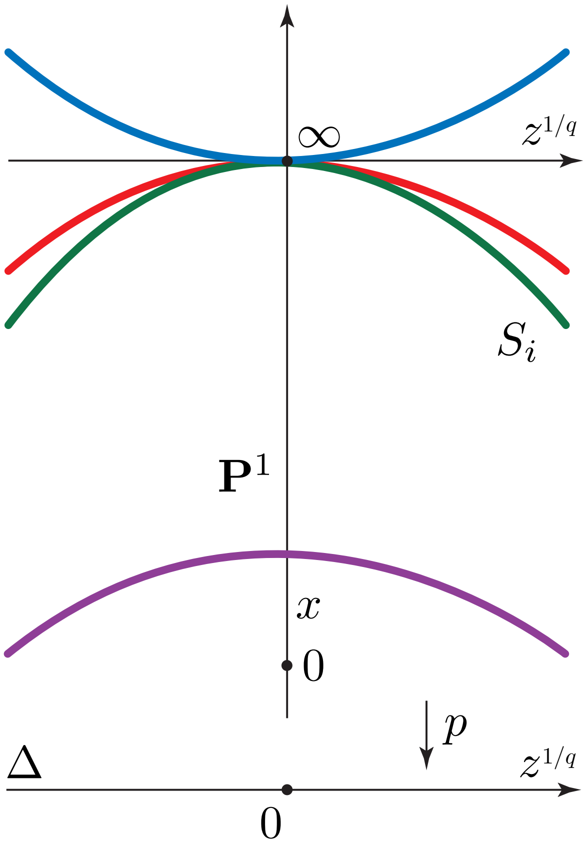
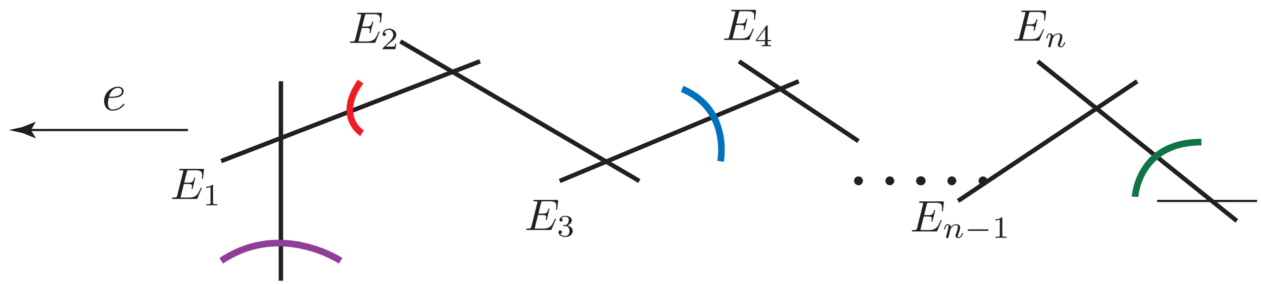
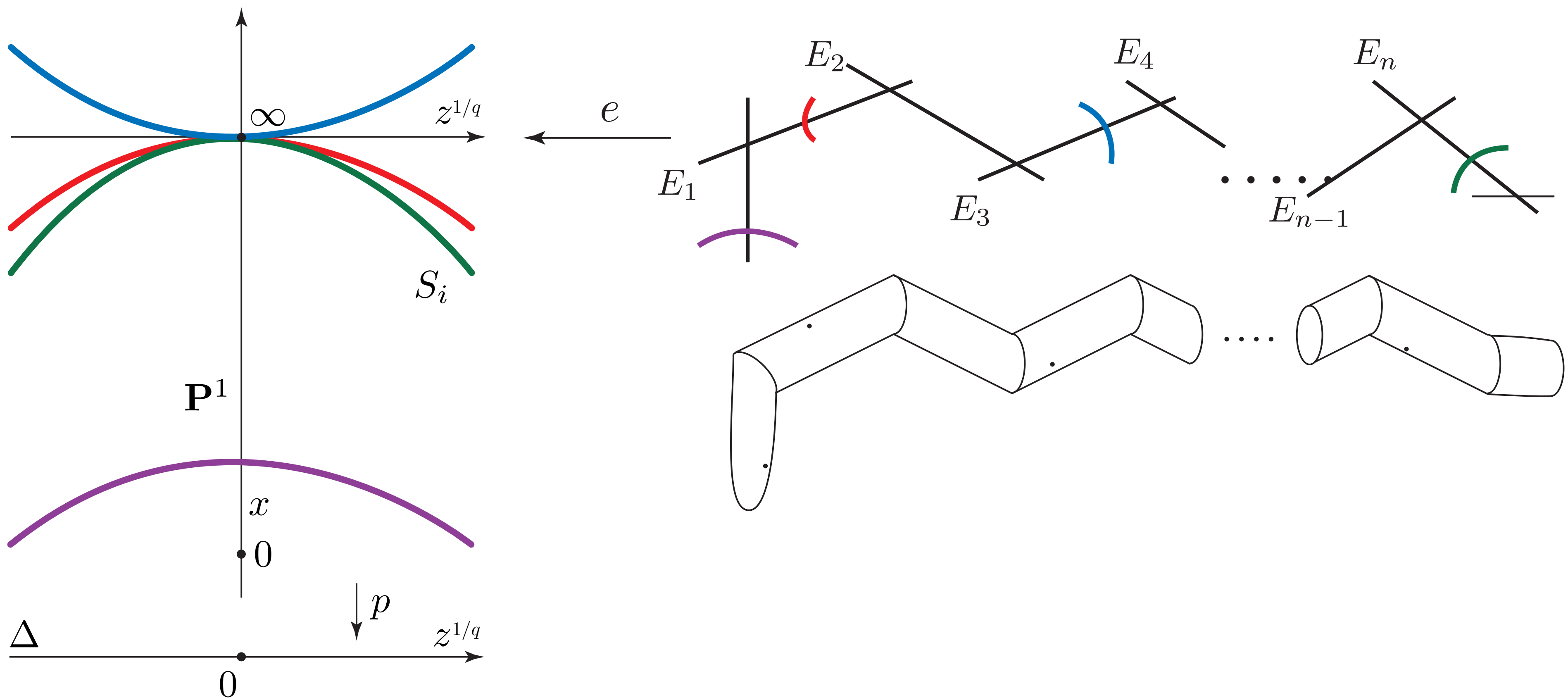




THM (C.S.):
Stokes filtr. of $\mathscr{N} = p_*(\mathscr{E}^x \otimes \mathscr{M})$
$=$ ***push-forward*** of the
Stokes filtr. of $e^*(\mathscr{E}^x \otimes \mathscr{M})$.

Example



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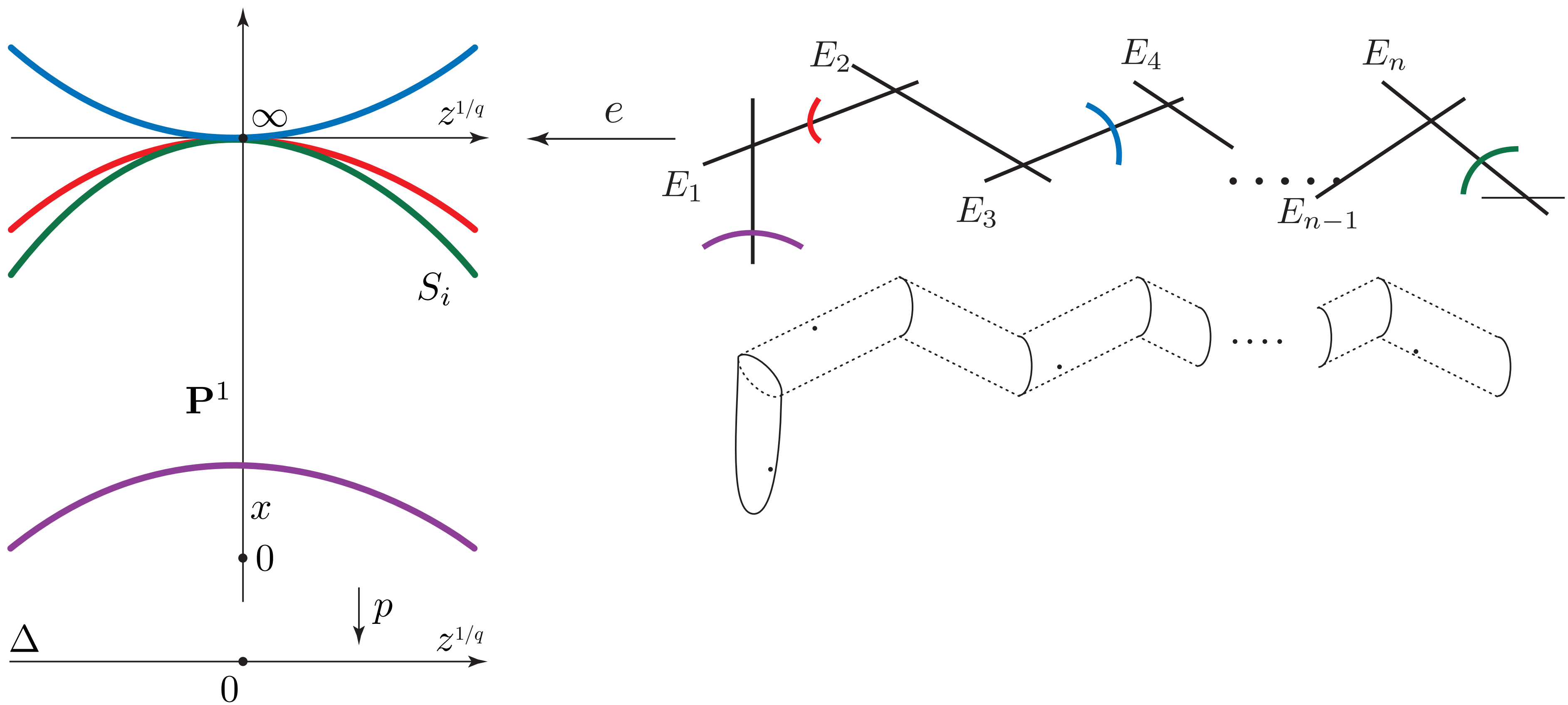

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