SPECIALIZATION OF $\mathcal{D}$-MODULES APPLICATION TO PURE $\mathcal{D}$-MODULES

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Claude Sabbah

Contents

Introduction

In the theory of pure Hodge Modules of M. Saito, and in generalizations of it, the *nearby and vanishing cycle functors* play a fundamental role, aside from the other Grothendieck *six operations*. Indeed, it is used in the very definition of a pure Module and is a convenient way to be sure that the properties that one expects for a limit mixed Hodge structure are satisfied.

At the level of $\mathcal{D}$-modules, these functors are introduced as the result of grading with respect to the so-called *Malgrange-Kashiwara filtration*, also called the V-filtration. We still call the graded $\mathcal{D}$-modules the *nearby or vanishing cycles* of the original $\mathcal{D}$-module, even if there are no cycles in the landscape, to keep in mind the correspondence with the topological situation.

Pure $\mathcal{D}$-modules are holonomic module with a supplementary structure, *e.g.* a "Hodge" filtration, a polarization, a rational or real structure, a twistor structure, a Hermitian form, *etc.*, and one of the main points in the proofs concerning purity (Decomposition Theorem, Local Invariant Cycle Theorem, *etc.*) consists in proving, under certain assumptions, that this structure is compatible with taking nearby or vanishing cycles.

At the topological level, an essential result concerning the nearby and vanishing cycle functors is that they preserve perversity, when suitably shifted. At the level of $\mathcal{D}$-modules, this property is translated by the fact that the nearby or vanishing cycle object associated to a $\mathcal{D}$-module is a single $\mathcal{D}$-module. Nevertheless, purity is not, in general, preserved by this functor: starting from a pure object, its associated nearby or vanishing cycle object is usually mixed; however, the weight filtration is well understood, being the *monodromy filtration*.

The underlying $\mathcal{D}$-module of a "pure" $\mathcal{D}$-module has regular singularities. Nevertheless, the V-filtration exists without this assumption, and may still be useful, *e.g.* when studying the Fourier-Laplace transform of a pure $\mathcal{D}$-module. However, it may give very few information for a irregular $\mathcal{D}$-module (it only gives information concerning the formal nearby or vanishing cycles, or the formal monodromy). Another construction, due to Malgrange, may then be more useful, and is related to what is called a *parabolic filtration* in the theory of vector bundles.

1. The Malgrange-Kashiwara filtration and its properties

1.1. The formalism of the V-filtration. We briefly review the construction of the Malgrange-Kashiwara filtration for coherent $\mathcal{D}_X$-modules (see *e.g.* [**11**]). This filtration was introduced by M. Kashiwara [**4**] in order to generalize previous results by B. Malgrange [**7**] to arbitrary regular holonomic $\mathcal{D}$-modules. The presentation we give here comes from various published sources (*e.g.* [**11, 14**]) and from an unpublished letter of B. Malgrange to P. Deligne dated january 1984.

1.1.1. The setting. We may consider an algebraic setting or a complex analytic setting. So X denotes a complex algebraic variety and $f : X \to \mathbb{A}^1$ denotes a regular function, or X is a complex manifold and $f : X \to \mathbb{C}$ is a holomorphic function. According to Kashiwara's equivalence for coherent $\mathcal{D}$-modules, we may (and do) assume that f is smooth, by replacing X with $X \times \mathbb{A}^1$ and f by the projection to $\mathbb{A}^1$. We replace the original $\mathcal{D}$-module with its direct image by the graph inclusion $i_f : X \hookrightarrow X \times \mathbb{A}^1$. We now denote by t the coordinate on $\mathbb{A}^1$.

Let $t : X \to \mathbb{A}^1$ be a smooth regular function and put $X_0 = t^{-1}(0)$. Denote by $V_\bullet\mathcal{D}_X$ the increasing filtration indexed by $\mathbb{Z}$ associated with t: in any local coordinate system $(t, x_2, \ldots, x_n) = (t, x')$ of X, the germ $P \in \mathcal{D}_X$ is in $V_k\mathcal{D}_X$ if

- $P = \sum_{\boldsymbol{j}=(j_1,\boldsymbol{j}')} a_{\boldsymbol{j}}(t, x')(t\partial_t)^{j_1}\partial_{x'}^{\boldsymbol{j}'}$, if $k = 0$:
- $P = t^{|k|}Q$ with $Q \in V_0\mathcal{D}_X$, if $k \in -\mathbb{N}$;
- $P = \sum_{0\leqslant j\leqslant k} Q_j\partial_t^j$ with $Q_j \in V_0\mathcal{D}_X$, if $k \in \mathbb{N}$.

In other words, if we denote by $\mathcal{I}$ the ideal (t), we have

$$P \in V_k\mathcal{D}_X \iff \forall\,\ell \in \mathbb{N},\ P \cdot \mathcal{I}^\ell \subset \mathcal{I}^{\ell-k}.$$

The sheaf $V_0\mathcal{D}_X$ is also called the sheaf of logarithmic differential operators along the smooth divisor $t = 0$. We have the following properties (see *e.g.* [**11**]):

- $V_k\mathcal{D}_X \cdot V_\ell\mathcal{D}_X \subset V_{k+\ell}\mathcal{D}_X$ with equality for $k, \ell \leqslant 0$ or $k, \ell \geqslant 0$.
- $V_k\mathcal{D}_{X\smallsetminus X_0} = \mathcal{D}_{X\smallsetminus X_0}$ for any $k \in \mathbb{Z}$.
- $(\cap_k V_k\mathcal{D}_X)_{|X_0} = \{0\}$.

1.1.2. Good V-filtrations. Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module equipped with an exhaustive increasing filtration $U_\bullet\mathcal{M}$ indexed by $\mathbb{Z}$ such that $V_k\mathcal{D}_X \cdot U_\ell\mathcal{M} \subset U_{k+\ell}\mathcal{M}$ for any $k, \ell \in \mathbb{Z}$. The filtration is *good* if, for any compact set $K \subset X$, there exists $k_0 \geqslant 0$ such that, in a neighbourhood of K, we have for all $k \geqslant k_0$

$$U_{-k}\mathcal{M} = t^{k-k_0}U_{-k_0}\mathcal{M} \quad \text{and} \quad U_k\mathcal{M} = \sum_{0\leqslant j\leqslant k-k_0} \partial_t^j U_{k_0}\mathcal{M},$$

and each $U_\ell\mathcal{M}$ is $V_0\mathcal{D}_X$-coherent.

Introduce the Rees ring $R_V\mathcal{D}_X = \oplus_k V_k\mathcal{D}_X \cdot q^k$, where q is a new variable. The filtration $U_\bullet\mathcal{M}$ is good if and only if the Rees module $\oplus_k U_k\mathcal{M} \cdot q^k$ is coherent over $R_V\mathcal{D}_X$. Equivalently, there should exist locally a presentation $\mathcal{D}_X^b \to \mathcal{D}_X^a \to \mathcal{M} \to 0$, inducing for each $k \in \mathbb{Z}$ a presentation $U_k\mathcal{D}_X^b \to U_k\mathcal{D}_X^a \to U_k\mathcal{M} \to 0$, where the filtration on the free modules $\mathcal{D}_X^a, \mathcal{D}_X^b$ are obtained by suitably shifting $V_\bullet\mathcal{D}_X$ on each factor.

***Proposition* (Artin-Rees).** *If $\mathcal{N}$ is a coherent $\mathcal{D}_X$-submodule of $\mathcal{M}$ and $U_\bullet\mathcal{M}$ is a good filtration of $\mathcal{M}$, then $U_\bullet\mathcal{N} \overset{\text{def}}{=} \mathcal{N} \cap U_\bullet\mathcal{M}$ is also good.* □

1.1.3. Specializable $\mathcal{D}$-modules. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is said to be *specializable* along $\{t = 0\}$ if any local section m of $\mathcal{M}$ has a Bernstein polynomial $b_m(s) \in \mathbb{C}[s] \smallsetminus \{0\}$ such that $b_m(-\partial_t t)m \in V_{-1}(\mathcal{D}_X) \cdot m$. If b_m is the minimal such polynomial, we define the order of m as $\max\{\alpha \mid b_m(\alpha) = 0\}$.

For a coherent $\mathcal{D}_X$-module, to be specializable is equivalent to the local existence of a good V-filtration $U_\bullet\mathcal{M}$ such that there exists a *monic* polynomial $b(s) \in \mathbb{C}[s]$ such that

$$(*) \qquad b(-(\partial_t t + k)) \cdot \mathrm{gr}_k^U \mathcal{M} = 0 \quad \text{for all } k \in \mathbb{Z}.$$

Remarks

(1) It is straightforward to develop the theory below in the case of right $\mathcal{D}_X$-modules. If $U_\bullet(\mathcal{M})$ is a V-filtration of the left module $\mathcal{M}$, then $U_\bullet(\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) \overset{\text{def}}{=} \omega_X \otimes_{\mathcal{O}_X} U_\bullet(\mathcal{M})$ is the corresponding filtration of the corresponding right module. This correspondence is compatible with taking the graded object with respect to $U_\bullet$. The operator $-\partial_t t$ (acting on the left) corresponds to $t\partial_t$ (acting on the right).

(2) Given an increasing filtration $U_\bullet$ (lower indices), we define the associated decreasing filtration (upper indices) by $U^k = U_{-k-1}$. If $b(-(\partial_t t + k)) \cdot \mathrm{gr}_k^U \mathcal{M} = 0$ for all $k \in \mathbb{Z}$, we have $b'(t\partial_t - \ell) \cdot \mathrm{gr}_U^\ell \mathcal{M} = 0$ for all $\ell \in \mathbb{Z}$, if we put $b'(s) = b(-s)$.

(3) It may happen that the constant filtration $U_k\mathcal{M} = \mathcal{M}$ satisfies $(*)$. In such a case, the theory below is not very interesting, because all $\psi_{t,\alpha}\mathcal{M}$ identically vanish. For holonomic $\mathcal{D}_X$-modules, this may happen when the module has irregular singularities along $t = 0$.

Fix a total order $\leqslant$ on $\mathbb{C}$, which induces the usual order on $\mathbb{R}$ and such that $\alpha + a < \beta + a \Leftrightarrow \alpha < \beta$ for any $a \in \mathbb{R}$ (this is not really necessary, but is convenient). We may now define the increasing filtration $V_\bullet\mathcal{M}$ by the order, indexed by $\mathbb{C}$ (in fact by a discrete subset $A + \mathbb{Z}$, where $A \subset \mathbb{C}$ is finite). It is globally defined along X_0.

Notice that, in most interesting cases, the set A is already contained in $\mathbb{R}$ or even in $\mathbb{Q}$ (quai-unipotence of monodromy).

If $\mathcal{M}$ is specializable along $\{t = 0\}$, then the filtration by the order $V_\bullet\mathcal{M}$ is a good V-filtration of $\mathcal{M}$ indexed by a discrete subset of $\mathbb{C}$. Each graded piece $\psi_{t,\alpha}\mathcal{M} \overset{\text{def}}{=} \mathrm{gr}_\alpha^V \mathcal{M} = V_\alpha\mathcal{M}/V_{<\alpha}\mathcal{M}$ is a coherent $\mathcal{D}_{X_0}$-module, on which the endomorphism N induced by $-\partial_t t + \alpha$ is nilpotent. Put $T = \exp(2i\pi\alpha \,\mathrm{Id} + N) : \psi_{t,\alpha}\mathcal{M} \to \psi_{t,\alpha}\mathcal{M}$. Then T is invertible on $\psi_{t,\alpha}\mathcal{M}$.

For any $\alpha \in \mathbb{C}$, there are $\mathcal{D}_{X_0}$-linear morphisms

$$t : \psi_{t,\alpha}\mathcal{M} \longrightarrow \psi_{t,\alpha-1}\mathcal{M} \quad \text{and} \quad -\partial_t : \psi_{t,\alpha}\mathcal{M} \longrightarrow \psi_{t,\alpha+1}\mathcal{M}$$

The first one is an isomorphism if $\alpha \neq 0$ and the second one if $\alpha \neq -1$. We denote by $\mathrm{Can} : \psi_{t,-1}\mathcal{M} \to \psi_{t,0}\mathcal{M}$ the morphism induced by $-\partial_t$ and by $\mathrm{var} : \psi_{t,0}\mathcal{M} \to \psi_{t,-1}\mathcal{M}$ the morphism induced by t. We have $\mathrm{var} \circ \mathrm{Can} = N$ (acting on $\psi_{t,-1}\mathcal{M}$) and $\mathrm{Can}\circ\mathrm{var} = N$ (acting on $\psi_{t,0}\mathcal{M}$). It is also convenient to introduce $\mathrm{can} = -\partial_t \sum_{n\geqslant 1} \frac{(2i\pi)^n}{n!}\cdot(-t\partial_t)^{n-1}$, so that $\mathrm{can}\circ\mathrm{var} = T - \mathrm{Id}_{\psi_{t,0}\mathcal{M}}$ and $\mathrm{var}\circ\mathrm{can} = T - \mathrm{Id}_{\psi_{t,-1}\mathcal{M}}$.

Any morphism between specializable $\mathcal{D}_X$-modules is *strictly compatible* with the filtration $V_\bullet$. Any coherent sub or quotient module of a specializable $\mathcal{D}_X$-module is so. For a specializable $\mathcal{D}_X$-module,

(1) $\mathrm{Can}: \psi_{t,-1}\mathcal{M} \to \psi_{t,0}\mathcal{M}$ is onto iff $\mathcal{M}$ has no coherent quotient $\mathcal{D}_X$-module supported on X_0,

(2) $\mathrm{var}: \psi_{t,0}\mathcal{M} \to \psi_{t,-1}\mathcal{M}$ is injective iff $\mathcal{M}$ has no coherent sub $\mathcal{D}_X$-module supported on X_0,

(3) $\psi_{t,0}\mathcal{M} = \mathrm{Im}\,\mathrm{Can}\oplus\mathrm{Ker}\,\mathrm{var}$ iff $\mathcal{M} = \mathcal{M}'\oplus\mathcal{M}''$ with $\mathcal{M}'$ satisfying (1) and (2) and $\mathcal{M}''$ supported on X_0.

The nilpotent endomorphism

$$-(\partial_t t+\alpha): \psi_{t,\alpha}\mathcal{M} \longrightarrow \psi_{t,\alpha}\mathcal{M}$$

is denoted by N. There exists a unique increasing filtration $M(N)_\bullet$ of $\psi_{t,\alpha}\mathcal{M}$ by $\mathcal{D}_{X_0}$-submodules, indexed by $\mathbb{Z}$, such that, for any $\ell\geqslant 0$, N maps M_k into M_{k-2} for all k and N^ℓ induces an isomorphism $\mathrm{gr}^M_\ell \xrightarrow{\sim} \mathrm{gr}^M_{-\ell}$ for any $\ell\geqslant 0$. It is called the *monodromy filtration* of N (*cf.* [**3**, § 1.6]). Each $\mathrm{gr}^M\psi_{t,\alpha}\mathcal{M}$ has a *Lefschetz decomposition*, with basic pieces the *primitive parts* ($\ell\geqslant 0$)

$$P\mathrm{gr}^M_\ell\psi_{t,\alpha}\mathcal{M} \overset{\mathrm{def}}{=} \mathrm{Ker}\left[N^{\ell+1}: \mathrm{gr}^M_\ell\psi_{t,\alpha}\mathcal{M} \longrightarrow \mathrm{gr}^M_{-\ell-2}\psi_{t,\alpha}\mathcal{M}\right].$$

1.2. Some properties of the V-filtration. We state without proof the known properties of the V-filtration. Let us recall first the fundamental result of Bernstein and Kashiwara that *any holonomic $\mathcal{D}_X$-module is specializable along any hypersurface* (see a unified proof in [**10**]).

1.2.1. Algebraicity. Consider the algebraic setting, and denote by an exponent "an" the corresponding analytic object. So $\mathcal{M}^{\mathrm{an}} = \mathcal{O}_X^{\mathrm{an}}\otimes_{\mathcal{O}_X}\mathcal{M}$, *etc.* If $\mathcal{M}$ is specializable along $\{t=0\}$, then so is $\mathcal{M}^{\mathrm{an}}$ and we have, for any $\alpha\in\mathbb{C}$,

$$V_\alpha(\mathcal{M}^{\mathrm{an}}) = (V_\alpha\mathcal{M})^{\mathrm{an}} = \mathcal{O}_X^{\mathrm{an}}\otimes_{\mathcal{O}_X} V_\alpha\mathcal{M}.$$

This follows from the uniqueness of the V-filtration.

1.2.2. Direct image. Let $f: X\to Y$ be holomorphic map between complex algebraic manifolds and let $t\in\mathbb{C}$ be a new variable. Put $F = f\times\mathrm{Id}: X\times\mathbb{C}\to Y\times\mathbb{C}$. Let $\mathcal{M}$ be a right $\mathcal{D}_{X\times\mathbb{C}}$-module. Let $U_\bullet\mathcal{M}$ be a good V-filtration of $\mathcal{M}$ along $X\times\{0\}$. Define the direct image $F_+\mathcal{M}$ viewing $\mathcal{M}$ as a $\mathcal{D}_{X\times\mathbb{C}/\mathbb{C}}$-module, with a compatible ∂_t-action. One may then define $F_+U_k\mathcal{M}$ as a subcomplex of $F_+\mathcal{M}$. In such a way, one gets functorially a V-filtration

$$U_\bullet\mathcal{H}^i(F_+\mathcal{M}) = \text{image}\,[\mathcal{H}^i(U_\bullet F_+\mathcal{M}) \longrightarrow \mathcal{H}^i(F_+\mathcal{M})].$$

Assume that $\mathcal{M}$ is good and that F is proper on the support of $\mathcal{M}$. Then the V-filtration above on the $\mathcal{D}_{\mathcal{Y}\times\mathbb{C}}$-modules $\mathcal{H}^i(F_+\mathcal{M})$ is good.

If moreover $\mathcal{M}$ is specializable along $X\times\{0\}$, then $\mathcal{H}^i(F_+\mathcal{M})$ are specializable along $Y\times\{0\}$. Moreover, for any α, we have a canonical and functorial isomorphism

$$\psi_{t,\alpha}\mathcal{H}^i(F_+\mathcal{M}) = \mathcal{H}^i(f_+\psi_{t,\alpha}\mathcal{M}).$$

1.2.3. Duality. Let D the duality functor, from the category of left holonomic $\mathcal{D}_X$-modules to itself.

***Theorem* (*cf.* [11, 14, 15]).** *There exist natural isomorphisms of functors from* $\mathrm{Mod}_h(\mathcal{D}_X)$ *to* $\mathrm{Mod}_h(\mathcal{D}_Z)$

$$\begin{aligned}
\delta_{X,\alpha} &: \psi_{t,\alpha}\circ D_X \longrightarrow D_Z\circ\psi_{t,-1-\alpha}, \quad (\alpha\in\,]-1,0[)\\
\delta_{X,-1} &: \psi_{t,-1}\circ D_X \longrightarrow D_Z\circ\psi_{t,-1}\\
\delta_{X,0} &: \psi_{t,0}\circ D_X \longrightarrow D_Z\circ\psi_{t,0}
\end{aligned}$$

which satisfy the following properties, putting $\delta_X=\delta_{X,\alpha}$*:*

- $\delta_X = D_Z\circ\delta_X\circ D_X$*;*
- $\delta_X\circ N = D_Z(N)\circ\delta_X$*;*
- $\delta_{X,0}\circ\mathrm{Can} = D_Z(\mathrm{var})\circ\delta_{X,-1}$ *and* $\delta_{X,-1}\circ\mathrm{var} = D_Z(\mathrm{Can})\circ\delta_{X,0}$.

1.2.4. Hermitian duality. Let $\mathfrak{Db}_{X_\mathbb{R}}$ (also denoted by $\mathfrak{Db}_X$ for short) be the sheaf of distributions on $X_\mathbb{R}$. It acts on the sheaf C^∞-forms φ with compact support of maximal degree, which is a right $\mathcal{D}_X$ and $\mathcal{D}_{\overline{X}}$-module. Then $\mathfrak{Db}_X$ is a left $\mathcal{D}_X$ and $\mathcal{D}_{\overline{X}}$-module by the formula $(P\overline{Q}\mu)(\varphi)=\mu(\varphi\cdot P\overline{Q})$. The sheaf $\mathfrak{C}_{X_\mathbb{R}}=\mathfrak{Db}_X^{(n,n)}$ of currents of maximal degree is a right $\mathcal{D}_X$ and $\mathcal{D}_{\overline{X}}$-module obtained from $\mathfrak{Db}_X$ by "going from left to right".

Denote by C_X the Hermitian duality functor[(1)]. Recall that C_X is a contravariant functor from the derived category $D^-(\mathcal{D}_X)$ to the category $D^+(\mathcal{D}_{\overline{X}})$ defined as

$$C_X(\mathcal{M}^\bullet) = \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet,\mathfrak{Db}_X).$$

It restricts as a functor from the full subcategory $D^b_{hr}(\mathcal{D}_X)$ of bounded complexes with regular holonomic cohomology to $D^b_{hr}(\mathcal{D}_{\overline{X}})$ and is equal to the functor $\mathcal{H}om_{\mathcal{D}_X}(\bullet,\mathfrak{Db}_X)$ on the category of regular holonomic $\mathcal{D}_X$-modules (see [5],

(1) It is called improperly the "conjugation functor" in [1].

see also [**1**, Chap. VII]), defining there an anti-equivalence of categories between $\mathrm{Mod}_{hr}(\mathcal{D}_X)$ and $\mathrm{Mod}_{hr}(\mathcal{D}_{\overline{X}})$, and between $D^b_{hr}(\mathcal{D}_X)$ and $D^b_{hr}(\mathcal{D}_{\overline{X}})$, $C_{\overline{X}}$ being a quasi-inverse functor. On $D^b_{hr}(\mathcal{D}_X)$ we have

$$\mathcal{H}^k C_X \mathcal{M}^\bullet = C_X \mathcal{H}^k \mathcal{M}^\bullet.$$

***Theorem* (*cf.* [13]).** *There exist natural isomorphisms of functors from* $\mathrm{Mod}_{hr}(\mathcal{D}_X)$ *to* $\mathrm{Mod}_{hr}(\mathcal{D}_{\overline{Z}})$

$$\gamma_{X,\alpha} : \psi_{t,\alpha} \circ C_X \longrightarrow C_Z \circ \psi_{t,\alpha}, \quad (\alpha \in [-1,0])$$

which satisfy the following properties, putting $\gamma_X = \gamma_{X,\alpha}$*:*

- $\gamma_X = C_Z \circ \gamma_{\overline{X}} \circ C_X$*;*
- $\gamma_X \circ N = C_Z(N) \circ \gamma_X$*;*
- $\gamma_{X,0} \circ \mathrm{Can} = C_Z(\mathrm{var}) \circ \gamma_{X,-1}$ *and* $\gamma_{X,-1} \circ \mathrm{var} = C_Z(\mathrm{Can}) \circ \gamma_{X,0}$.

1.3. Examples. The V-filtration is mainly used when the $\mathcal{D}_X$-module is regular holonomic. Nevertheless, there exist holonomic $\mathcal{D}_X$-modules which are not regular, but are regular only along some hypersurface; the V-filtration along this hypersurface may be useful. The partial Fourier-Laplace transform of a regular holonomic $\mathcal{D}_X$-module relative to a function f on X gives such an example.

1.3.1. Comparison with the topological nearby or vanishing cycles. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Assume that it is strongly regular along $\{t=0\}$: this means that for any $\alpha \in \mathbb{C}$, the holonomic $\mathcal{D}_X$-module $\mathcal{M} \otimes t^\alpha$ is regular along $\{t=0\}$ in the sense of Mebkhout, *i.e.* its irregularity sheaf along $\{t=0\}$ is zero.

***Theorem* (Malgrange, Kashiwara).** *There are functorial isomorphisms in* $D^b_c(\mathbb{C}[T,T^{-1}])$*:*

$$\begin{aligned} {}^p\mathrm{DR}^{\mathrm{an}}(\psi_{t,\alpha}\mathcal{M}) &\xrightarrow{\sim} {}^p\psi_{t,\exp 2i\pi\alpha}{}^p\mathrm{DR}^{\mathrm{an}}(\mathcal{M}) \quad (\alpha \in [-1,0[) \\ {}^p\mathrm{DR}^{\mathrm{an}}(\psi_{t,0}\mathcal{M}) &\xrightarrow{\sim} {}^p\phi_{t,1}{}^p\mathrm{DR}^{\mathrm{an}}(\mathcal{M}), \end{aligned}$$

which are compatible with the morphisms can *and* var*.*

1.3.2. Application to the partial Fourier transform. We work here in the algebraic setting. Let $f : X \to \mathbb{A}^1$ be a function on a smooth complex quasi-projective variety X. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Its *partial Fourier transform* $\widehat{\mathcal{M}}$ with respect to f is the $\mathcal{D}_{X\times\check{\mathbb{A}}^1} = \mathcal{D}_X[y]\langle\partial_y\rangle$-module $\widehat{\mathcal{M}} = \mathcal{M}[y]e^{-yf}$, where y is the variable on $\check{\mathbb{A}}^1$. It is holonomic.

***Proposition*.** *Under these conditions, if* $\mathcal{M}$ *is regular holonomic even at infinity on* X*, then* $\widehat{\mathcal{M}}$ *is strongly regular along* $y=0$*.*

It is therefore interesting to use the V-filtration of $\widehat{\mathcal{M}}$ along $y=0$. In such a way, one may state Hodge properties concerning the monodromy of f near $f=\infty$ without using any compactification of X. Let us explain this with more details.

We continue to assume that $\mathcal{M}$ is regular even at infinity on X. Consider the direct image $f_+\mathcal{M}$. Then the cohomology modules $\mathcal{H}^i f_+\mathcal{M}$ of $f_+\mathcal{M}$ are regular holonomic (even at infinity) on the Weyl algebra $\mathbb{C}[t]\langle\partial_t\rangle$, if t denotes the coordinate on the affine line (see *e.g.* . [**2**]). The Fourier transform $\widehat{\mathcal{H}^i f_+\mathcal{M}}$ obtained by "doing $y=\partial_t$ and $\partial_y=-t$" is therefore a holonomic $\mathbb{C}[y]\langle\partial_y\rangle$-module with a regular singularity at $y=0$ and possibly an irregular singularity at $y=\infty$. It has no other singularity, so that $\mathbb{C}[y,y^{-1}]\otimes_{\mathbb{C}[y]}\widehat{\mathcal{H}^i f_+\mathcal{M}}$ is a free $\mathbb{C}[y,y^{-1}]$-module of rank μ_i. Hence this number μ_i is equal to $\dim\widehat{\mathcal{H}^i f_+\mathcal{M}}/(y-1)\widehat{\mathcal{H}^i f_+\mathcal{M}}$ (one proves this first for $\mathbb{C}[t]\langle\partial_t\rangle$-modules of the form $\mathbb{C}[t]\langle\partial_t\rangle/(P)$ with $P\in\mathbb{C}[t]\langle\partial_t\rangle$ nonzero and regular even at infinity, then, by an extension argument, for any regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module; see for instance [**8**, Chap. V]).

Moreover, the monodromy of $\widehat{\mathcal{H}^i f_+\mathcal{M}}$ around $y=0$ (more precisely, the monodromy on the *vanishing cycles* $\psi_{y,\alpha}$ for $\alpha\neq-1$) is identified with the monodromy at $t=\infty$ of $\mathcal{H}^i f_+\mathcal{M}$.

Consequently, we may compute the latter without using any compactification of X.

Assume for instance that X is affine and that f behaves well at infinity along fibers (*cf.* last part of N. Katz book [**6**]). For instance, consider a polynomial $f:\mathbb{A}^n\to\mathbb{A}^1$ which is convenient and nondegenerate with respect to its Newton polyhedron at infinity: using this technique, one may relate the Newton filtration that one naturally gets on $\Omega_{\mathbb{A}^n}$ with the Hodge filtration on "$\lim_{t\to\infty}H^{n-1}(f^{-1}(t))$" at $f=\infty$.

2. Strictness

Strictness is a basic property of pure or mixed objects. Its preservation under various functor is usually nontrivial. It is the generalization of the degeneration at E_1 of the Hodge-Frölicher (also called Hodge $\Rightarrow$ de Rham) spectral sequence.

2.1. $R_F\mathcal{D}_X$ and $\mathcal{R}_{\mathcal{X}}$-modules. Let $F_\bullet\mathcal{D}_X$ the increasing filtration of $\mathcal{D}_X$ by the order of differential operators, and let $R_F\mathcal{D}_X=\oplus_{k\in\mathbb{N}}F_k\mathcal{D}_X\cdot z^k$ be the corresponding Rees ring, where z is a new variable. The Rees construction

$$(\mathcal{M},F_\bullet)\longmapsto\bigoplus_{k\in\mathbb{Z}}F_k\mathcal{M}\cdot z^k$$

gives a one-to-one correspondence between pairs $(\mathcal{M}, F_\bullet)$ of $\mathcal{D}_X$-modules equipped with a good filtration $F_\bullet\mathcal{M}$, and graded $R_F\mathcal{D}_X$-modules which have no $\mathbb{C}[z]$-torsion.

More generally, one may consider the sheaf $\mathcal{R}_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_X[z]} R_F\mathcal{D}_X$ on $\mathcal{X} = X \times \mathbb{C}$, where $\mathcal{O}_{\mathcal{X}}$ denotes the sheaf of analytic functions. It is written in local coordinates as $\mathcal{O}_{\mathcal{X}}\langle \eth_{x_1}, \dots, \eth_{x_n}\rangle$, where $\eth_{x_i} = z\partial_{x_i}$.

These rings have the same coherence properties as $\mathcal{D}_X$ has. The direct image functor is defined for any morphism $f : X \to Y$, and coherence is preserved if f is proper.

Say that a $\mathcal{R}_{\mathcal{X}}$-module is strict if it has no $\mathcal{O}_{\mathbb{C}}$-torsion (no nontrivial section is killed by a function of z only).

Example. Considering non-graded $R_F\mathcal{D}_X$-modules, and more generally sheaves of $\mathcal{R}_{\mathcal{X}}$-modules may be useful. For instance, a $\mathcal{R}_{\mathcal{X}}$-module $\mathcal{M}$ which is $\mathcal{O}_{\mathcal{X}}$-locally free has two associated interesting locally free $\mathcal{O}_X$-modules: its restriction to $z = 1$ is a $\mathcal{D}_X$-module, *i.e.* a vector bundle with a *flat connection*; its restriction to $z = 0$ is a (non-graded) $\mathrm{gr}_F\mathcal{D}_X$-module, *i.e.* a holomorphic vector bundle with a *Higgs field*.

Usually, strictness is *not* preserved by proper direct image. Nevertheless, one has an important result in M. Saito's theory of mixed Hodge Modules (*cf.* [**14, 16**]):

Theorem. *If $(\mathcal{M}, F_\bullet)$ is the filtered $\mathcal{D}_X$-module underlying a mixed Hodge Module and if $f : X \to Y$ is proper, then the direct image complex $f_+R_F\mathcal{M}$ is strict,* i.e. *its cohomology modules are strict.*

This statement has, for instance, the following consequence (*cf.* [**12**]):

Theorem. *Let X be a smooth complex quasi-projective variety and let $f : X \to \mathbb{A}^1$ be a proper morphism. Let D be a normal crossing divisor in X. Then the hypercohomology of the complexes $(\Omega^\bullet_X\langle\log D\rangle, d - df\wedge\,)$ and $(\Omega^\bullet_X\langle\log D\rangle, df\wedge\,)$ have the same (finite) dimension.*

More generally,

Theorem. *Let (M, F) be a mixed Hodge Module on X. The hypercohomology spaces on X of the complexes $(\Omega^\bullet_X \otimes_{\mathcal{O}_X} M, \nabla - df\wedge)$ and $(\Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathrm{gr}^F M, \mathrm{gr}^F\nabla - df\wedge)$ have the same (finite) dimension.*

Proof. Let (M,F) be a well-filtered coherent $\mathcal{D}_X$-module. Let y be a new variable. One puts on $M\otimes_{\mathbb{C}}\mathbb{C}[y,y^{-1}]$ the filtration

$$G_k\left(M\otimes_{\mathbb{C}}\mathbb{C}[y,y^{-1}]\right)=\bigoplus_j F_{j+k}My^{-j}$$

so that $G_k=y^kG_0$ and

$$\mathrm{gr}_0^G(M\otimes_{\mathbb{C}}\mathbb{C}[y,y^{-1}])=G_0/y^{-1}G_0\simeq\mathrm{gr}^FM\quad\text{and}\quad G_0/(y-1)G_0=M.$$

The proof of the theorem relies on the

Proposition. *Assume that the cohomology of the direct image f_+M is holonomic and regular at infinity, and that $f_+(M,F)$ is strict. Then, for all $i\in\mathbb{Z}$, the $\mathbb{C}[y^{-1}]$-module*

$$E_i=\boldsymbol{H}^i\left(X,(\Omega_X^\bullet\underset{\mathcal{O}_X}{\otimes}G_0(M[y,y^{-1}]),y^{-1}\nabla-df\wedge)\right)$$

is free of finite rank.

Let us now end the proof of the theorem. First, one consider the localized Fourier transform:

$$\begin{aligned}\mathbb{C}[y,y^{-1}]\underset{\mathbb{C}[y]}{\otimes}\widehat{\mathcal{H}^if_+M}&=\boldsymbol{H}^i\left(X,(\Omega_X^{\bullet+\dim X}\underset{\mathcal{O}_X}{\otimes}M[y,y^{-1}],\nabla-ydf\wedge)\right)\\&=\boldsymbol{H}^i\left(X,(\Omega_X^{\bullet+\dim X}\underset{\mathcal{O}_X}{\otimes}M[y,y^{-1}],y^{-1}\nabla-df\wedge)\right)\end{aligned}$$

is a free $\mathbb{C}[y,y^{-1}]$-module of rank μ_i. One interprets the free $\mathbb{C}[y^{-1}]$-module E_i as a bundle on the chart with coordinate y^{-1}, such that $\mathbb{C}[y,y^{-1}]\otimes_{\mathbb{C}[y^{-1}]}E_i$ is the localized Fourier transform above. Therefore, μ_i is the rank of E_i as a free $\mathbb{C}[y^{-1}]$-module. Consequently, the fibre of E_i at $y^{-1}=1$ or at $y^{-1}=0$ have the same rank μ_i. □

Remark. This number μ_i can be computed in terms of local analytic data of f: this is the total number of vanishing cocycles of f in degree i, with respect to a suitable constructible sheaf on X^{an}.

2.2. V-filtration and strictness. One may introduce the filtration V on the rings $R_F\mathcal{D}_X$ or $\mathcal{R}_{\mathcal{X}}$ with $\mathcal{X}=X\times\mathbb{C}$, and therefore define the notion of specializable module. the Bernstein relation takes the form

$$b_m(-\eth_tt)\cdot m\in V_{-1}\mathcal{R}_{\mathcal{X}}\cdot m,$$

with

$$b_m(s)=\prod_{\alpha\in A}\prod_{\ell\in\mathbb{Z}}(s-(\alpha+\ell)z)^{\nu_\alpha}.$$

It is important to find a good V-filtration such that the various graded pieces are strict.

***Definitions* (Strict specializability)**

(1) A specializable $\mathcal{R}_{\mathcal{X}}$-module is said to be *strictly specializable* along $\mathcal{X}_0$ if one can find, locally near any point $(x_o, z_o) \in \mathcal{X}_0$, a good filtration $V_\bullet \mathcal{M}$ satisfying the analogue of $(*)$ and such that moreover

(a) for every $\alpha \in \mathbb{R}$, $\psi_{t,\alpha}\mathcal{M}$ is a strict $\mathcal{R}_{\mathcal{X}_0}$-module (hence $V_\bullet\mathcal{M}$ is its Malgrange-Kashiwara filtration);

(b) $t : \psi_{t,\alpha}\mathcal{M} \to \psi_{t,\alpha-1}\mathcal{M}$ is an isomorphism for $\alpha < 0$;

(c) $\eth_t : \psi_{t,\alpha}\mathcal{M} \to \psi_{t,\alpha+1}\mathcal{M}$ is isomorphism for $\alpha > -1$.

(2) A morphism $\varphi : \mathcal{M} \to \mathcal{N}$ between two strictly specializable $\mathcal{R}_{\mathcal{X}}$-modules is *strictly specializable* if, for any $\alpha \in A + \mathbb{Z}$, the morphisms $\psi_{t,\alpha}\varphi$ are strict.

(3) Let $f : X \to \mathbb{C}$ be an analytic function and let $\mathcal{M}$ be a $\mathcal{R}_{\mathcal{X}}$-module. Denote by $i_f : X \hookrightarrow X \times \mathbb{C}$ the graph inclusion. We say that $\mathcal{M}$ is strictly specializable along $f = 0$ if $i_{f,+}\mathcal{M}$ is strictly specializable along $X \times \{0\}$. We then set $\psi_{f,\alpha}\mathcal{M} = \psi_{t,\alpha}(i_{f,+}\mathcal{M})$ for $\alpha \neq 0$. These are coherent $\mathcal{R}_{\mathcal{X}}$-modules. If $f = t$ is smooth, we have, by an easy verification, $\psi_{t,\alpha}(i_{f,+}\mathcal{M}) = i_{f,+}(\psi_{f,\alpha}\mathcal{M})$ for any α.

***Theorem* (a criterion for strictness of the direct image, [14])**

Let $f : X \to Y$ be holomorphic map between complex analytic or algebraic manifolds and let $t \in \mathbb{C}$ be a new variable. Put $F = f \times \mathrm{Id} : X \times \mathbb{C} \to Y \times \mathbb{C}$. Let $\mathcal{M}$ be a right $\mathcal{R}_{\mathcal{X}\times\mathbb{C}}$-module. Assume that $\mathcal{M}$ is good, strictly specializable and regular along $X \times \{0\}$, and that F is proper on the support of $\mathcal{M}$. Assume moreover that, for any $\alpha \in [-1, 0]$, the complexes $f_+\psi_{t,\alpha}\mathcal{M}$ are strict. Then the $\mathcal{R}_{\mathcal{Y}\times\mathbb{C}}$-modules $\mathcal{H}^i(F_+\mathcal{M})$ are strictly specializable and regular along $Y \times \{0\}$, hence strict in a neighbourhood of $Y \times \{0\}$. Moreover, for any α, we have a canonical and functorial isomorphism

$$\psi_{t,\alpha}\mathcal{H}^i(F_+\mathcal{M}) = \mathcal{H}^i(f_+\psi_{t,\alpha}\mathcal{M}).$$

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C. Sabbah, UMR 7640 du CNRS, Centre de Mathématiques, École polytechnique, F–91128 Palaiseau cedex, France • *E-mail :* sabbah@math.polytechnique.fr
Url : http://www.math.polytechnique.fr/cmat/sabbah/sabbah.html