WILD GEOMETRY

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Abstract. Linear differential equations of one variable in the complex domain lead to the Stokes phenomenon and generalized monodromy data. Recent results of T. Mochizuki and K. Kedlaya on vector bundles with meromorphic connection having irregular singularities make it possible to develop the Stokes phenomenon in higher dimensions. The proposed talk will rapidly survey these results and propose tentative results for the underlying geometry, called "wild geometry" in analogy with the wild ramification in arithmetic. Some examples of Stokes-perverse sheaves will be given, which mix usual perverse sheaves in complex analytic geometry together with real constructible sheaves on the boundary of real blow-up spaces of a manifold along a divisor.

Introduction

In the usual "tame complex algebraic geometry",

- the underlying spaces are complex algebraic varieties (or complex analytic spaces),
- The monodromy phenomenon is treated sheaf-theoretically with local systems,
- introducing singularities in these local systems leads to $\mathbb{C}$-constructible sheaves, and then to perverse sheaves,
- one can realize each perverse sheaf as the sheaf of solutions of a system of holonomic differential equations with regular singularities (the connection matrix can be reduced to a normal form with logarithmic poles along a normal crossing divisor),
- Hodge theory extends in this setting (pure or mixed Hodge $\mathscr{D}$-modules of M. Saito).
- Moreover (Griffiths-Schmid), Hodge theory implies tameness (the natural extension of a variation of Hodge structures defines a meromorphic connection with regular singularities).
- Usual systems of differential equations in algebraic geometry (Gauss-Manin systems) have regular singularities (i.e., are tame).

Wild geometry addresses the question of extending these properties to differential equations having possibly irregular singularities (the matrix of the connection cannot be reduced to a matrix having logarithmic poles). The word "wild" is given with analogy to "wild ramification" in arithmetic. What is the usefulness for algebraic geometry?

• The classical theory of oscillating integrals produces such wild objects. If $F : X \to \mathbb{A}^1$ is a morphism from a smooth quasi-projective variety to the affine line, the function $I(\tau) = \int_X e^{-\tau F}\omega$, for some algebraic differential form of maximal degree on X, satisfies a differential equation which has an irregular singularity at infinity.

• Katzarkov, Kontsevich and Pantev have introduced the notion of non-commutative Hodge structure (and variations of such) as a model for the quantum cohomology of the projective space (Iritani also showed that this can be extended to the quantum cohomology of Fano toric varieties or orbifolds). This is strongly related to the notion of TERP structure of Hertling.

1. Tame versus wild in complex analytic geometry: dimension one

• Δ disc, coord. z	• $\widetilde{\Delta} = S^1 \times [0,1)$ real blow-up coord. $(e^{i\theta}, \rho)$
• $\mathscr{O}_\Delta(*0)$	• $\mathscr{A}_{\widetilde{\Delta}}^{\mathrm{mod}\,0}$
$\mathscr{L}$ local system on Δ^* (monodromy data)	$(\mathscr{L}, \mathscr{L}_\bullet)$ Stokes-filtered local system on $\widetilde{\Delta}$ (Stokes data)
• Vect. bdle with merom. connection ∇ log. poles	• Vect. bdle with merom. connection ∇ arbitrary poles
• Normal form $A dz/z$, A constant	• *Formal* normal form $(D_k/z^k + \cdots + D_1/z + A)dz/z$ all D_j constant diagonal, A constant
• R-H correspondence $\nabla \mapsto \ker \nabla_{\Delta^*} = \mathscr{L}$	• R-H correspondence $\nabla \mapsto \ker \nabla_{\Delta^*} = \mathscr{L}$ $\mathscr{L}_{\leqslant\varphi} = \ker \nabla$ acting on sections with coef. in $e^\varphi \mathscr{A}_{\widetilde{\Delta}}^{\mathrm{mod}\,0}$
• Equiv. of categories	• Equiv. of categories (Deligne)

Explanations

• $\Gamma(\widetilde{U}, \mathscr{A}_{\widetilde{\Delta}}^{\mathrm{mod}\,0})$ is the space of holomorphic functions on U^* which have moderate growth on any compact set of $\widetilde{U}$.

• $\mathfrak{I}$ is the constant sheaf with fibre $\mathscr{O}_\Delta(*0)/\mathscr{O}_\Delta$ on $S^1 \times \{0\}$. At each $\theta_o \in S^1$ one can order its germ $\mathscr{O}_\Delta(*0)/\mathscr{O}_\Delta$: $\varphi \leqslant_{\theta_o} \psi$ iff $\varphi - \psi \leqslant_{\theta_o} 0$ iff $e^{\varphi-\psi} \in \Gamma(\widetilde{U}, \mathscr{A}_{\widetilde{\Delta}}^{\mathrm{mod}\,0})$ for some neighbourhood $\widetilde{U}$ of θ_o. Write $\varphi - \psi = a_k/z^k + \cdots + a_1/z$, $a_k \neq 0$. Then $\varphi \leqslant_{\theta_o} \varphi$ iff $\mathrm{Re}(a_k/z^k) < 0$ in some neighbourhood of θ_o, that is, iff $k\theta_o - \arg a_k \in (-\pi/2, \pi/2) \bmod 2\pi$.

• $\mathscr{L}_{\leqslant\varphi}$ ($\varphi \in \mathscr{O}_\Delta(*0)/\mathscr{O}_\Delta$) is a family of subsheaves of $\mathscr{L}_{|S^1}$ such that, for each $\theta_o \in S^1$, $\varphi \leqslant_{\theta_o} \psi \Rightarrow \mathscr{L}_{\leqslant\varphi,\theta_o} \subset \mathscr{L}_{\leqslant\psi,\theta_o}$. This filtration at θ_o should also be *locally* *graded*: $\mathscr{L}_{\theta_o} \simeq \bigoplus_\varphi \mathrm{gr}_\varphi \mathscr{L}_{\theta_o}$ and each graded sheaf $\mathrm{gr}_\varphi \mathscr{L}$ should be a local system on S^1.

Note: This description is simplified, as I did not mention the ramification question (i.e., one should replace z with $z^{1/q}$). But the presentation of the Stokes filtration extends in a straightforward way to the ramified case.

• The formal normal form in the wild case is called the Levelt-Turrittin theorem.

• The equivalence of categories in the case of regular singularities is well-known. In the irregular case, it is due to Deligne in 1978, although another version, using Stokes matrices, is due to Malgrange-Sibuya.

2. Dimension two and higher: the question of a formal normal form

The work of Deligne in 1970 has made clear that the one-dimensional tame theory extends to higher dimension with similar properties. What about the wild case? The question of finding a formal normal form for a flat meromorphic connection ∇ on a vector bundle E on Δ^2 with poles on $z_1 = 0$ or $z_1 z_2 = 0$ has been solved only very recently. I had conjectured in 2000 that a formal normal form analogous to the one-dimensional case should exist, maybe only after a sequence of point blowing-ups, along the pull-back divisor, and I had proved this in some particular cases. Recently, there have been two proofs of this conjecture of very different flavour.

Proof of T. Mochizuki. It applies to connections on smooth projective surfaces X with poles along a divisor D. Algebraicity is needed to use characteristic p arguments. The main idea is to replace the flat meromorphic connection with a meromorphic Higgs field ($E \to \Omega^1_X(*D) \otimes E$ which is $\mathscr{O}_X$-linear, so that a normal form for the Higgs filed (easy) gives a formal normal form for the connection. This is not a priori possible, but the idea is to go to characteristic p (p large prime number) and use the p-curvature of the corresponding connection as a substitute for the Higgs field.

Proof of K. Kedlaya. It applies to the local case, and even the formal case. It uses a completely different kind of ideas, related to p-adic differential equations. It relies on valuative arguments, in the sense of Riemann-Zariski, and more accurately in the sense of Berkovich.

Extension to higher dimension. T. Mochizuki has used the existence of a formal normal form in dimension two in the algebraic case to prove a similar result in higher dimension, with complex analytic methods, but using the same idea of reducing to the study of Higgs fields. It is not exaggerated to say that the proof is a "tour de force".

K. Kedlaya on the other hand is now trying to extend his method to higher dimensions, but the valuative argument becomes much more complicated.

3. Stokes filtration on local systems and the R-H correspondence

• X smooth complex projective variety (or complex analytic manifold if the result of Kedlaya is proved).

• D is a divisor with simple normal crossings.

• $\widetilde{X}$ is the real blow-up of the components of D (local polar coordinates). $\widetilde{X}$ is a topological manifold with boundary $\partial\widetilde{X}$, locally diffeomorphic to $(S^1)^\ell \times [0,1)^\ell \times \mathbb{C}^{n-\ell}$.

• Sheaf $\mathscr{A}_{\widetilde{X}}^{\mathrm{mod}\,D}$.

• Sheaf $\mathcal{I}$: Pull-back on $\partial\widetilde{X}$ of $\mathscr{O}_X(*D)/\mathscr{O}_X$. Local order at each point $\widetilde{x} \in \partial\widetilde{X}$ by the moderate growth of $e^{\varphi-\psi}$. (Local ramifications around D are forgotten during the talk for the sake of simplicity).

• $\mathscr{L}$ local system on $\partial\widetilde{X}$. How to define the family of subsheaves $\mathscr{L}_{\leqslant\varphi}$?

Problem: the subsheaves $\mathscr{L}_{\leqslant\varphi}$ are defined only over the open subset over which φ is defined. *Consequence*: the true topological space on which the Stokes filtration is defined is the *étale space* $\mathcal{I}^{\text{ét}}$ of the sheaf $\mathcal{I}$. This is not a Hausdorff space. But one can develop the notion of Stokes filtration.

• E is a vector bundle on X with meromorphic connection having poles of arbitrary order along D.

• R-H correspondence defined as in dimension one.

• Proving that the R-H correspondence is an equivalence of categories needs asymptotic analysis. It cannot be done with the previous definition of normal form, which is not strong enough. One needs *good normal forms* which can be obtained after more complex blowing-ups.

Example: $z = (z_1, z')$, $D = \{z_1 = 0\}$ and the matrix of the connection has coefficient on dz_1 of the form $[D_k(z')/z_1^k + \cdots + D_1(z')/z_1 + A]dz_1/z_1$, with D_j diagonals and holomorphic w.r.t. z'. Goodness: the difference of the eigenvalues of the matrix, which take the form $a_\ell(z')/z_1^\ell + \cdots + a_1(z')/z_1$, are such that $a_\ell(z')$ does not vanish at $z' = 0$.

Theorem. *In dimension two, the R-H correspondence induces an equivalence between meromorphic connections having a good formal normal form and good Stokes-filtered local systems on* $\partial\widetilde{X}$.

This is essentially proved in arbitrary dimension by T. Mochizuki.

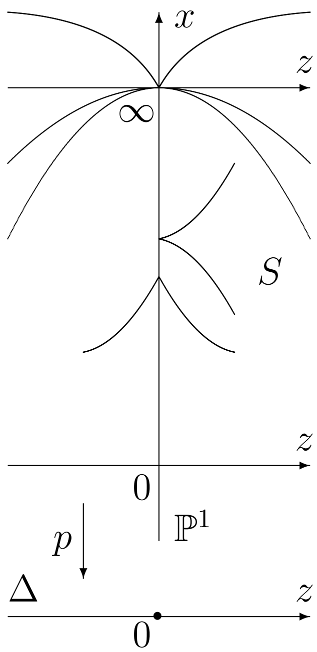
4. Example of computation of a Stokes filtration by direct image

One finds the following in an article by two physicists (Schwarz & Shapiro, Nucl.Phys.B809:547-560,2009, arXiv: `0809.0086`.

DEFINITION (A. Schwarz & I. Shapiro):

- Physics is a part of mathematics devoted to the calculation of integrals of the form $\int h(x)e^{g(x)}dx$.
- Different branches of physics are distinguished by the range of the variable x and by the names used for $h(x)$, $g(x)$ and for the integral.
- Of course this is a joke, physics is not a part of mathematics. However, it is true that the main mathematical problem of physics is the calculation of integrals of the form $\int h(x)e^{g(x)}dx$.

The setting is as on the following picture:



- (E,∇) free $\mathscr{O}_\Delta[x]$-module, merom. conn. on $\Delta\times\mathbb{A}^1$ (coord. (z,x))
- Log. poles on $S\cup(\Delta\times\{\infty\})$
- $p:\Delta\times\mathbb{A}^1\to\Delta$ the projection
- $N=\operatorname{coker}\Big(E(*S)\xrightarrow{\nabla_{\partial_x}+\mathrm{Id}}E(*S)\Big)$
- Connection induced by ∇_{∂_z}
- Corresponds to $\displaystyle\int_p f(z,x)e^x dx$
- $N=$ Gauss-Manin system of $(E(*S),\nabla+dx)$ rel. p.

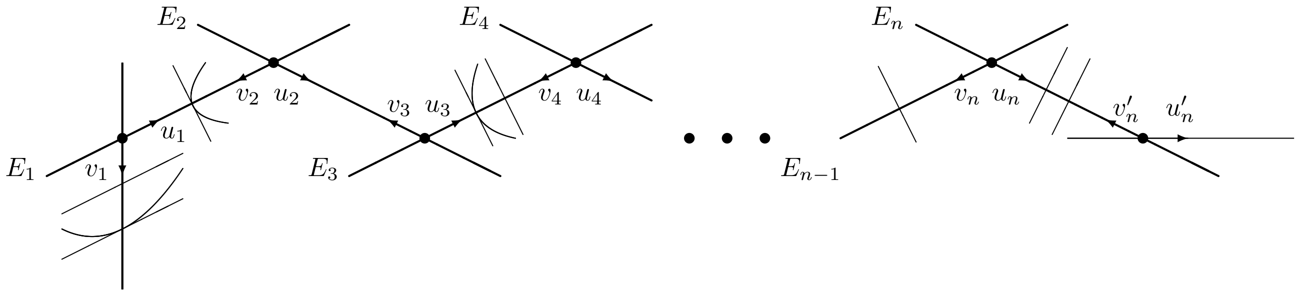
Question. *To compute the formal normal form of N at $z=0$ and the Stokes filtration in terms of the local system* $\ker\nabla$ *on* $(\Delta\times\mathbb{A}^1)\smallsetminus S$.

Results. After ramification w.r.t. z (this was neglected in the general presentation of Section **??**, but needed here), can assume that the components S_i of S going through $(0,\infty)$ have eqn. $(1/x)=z^{q_i}u_i(z)$, $u_i=$ unit. Set $\varphi_i(z)=1/[z^{q_i}u_i(z)]$.

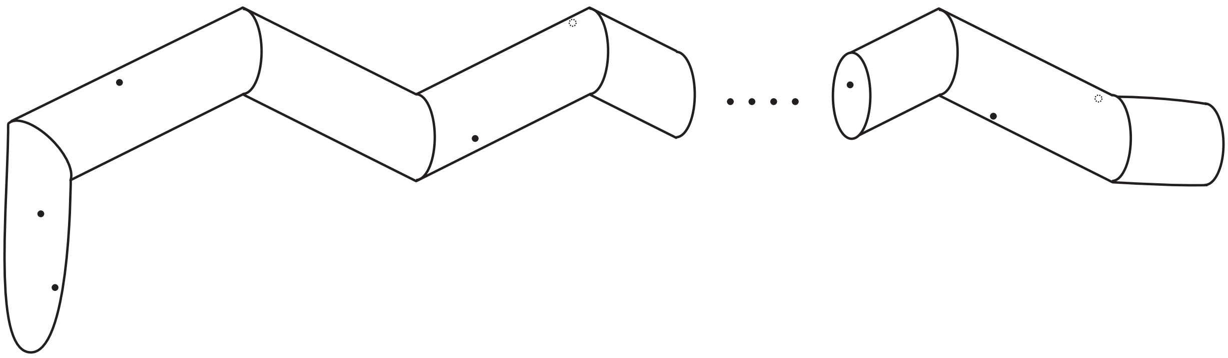
Theorem **(C. Roucairol).** *The formal normal form of N is (up to ramification) a direct sum of terms $\nabla_i+d\varphi_i$, where ∇_i has log. poles. Each ∇_i acts on a* *vect. bdle of rank equal to* $\operatorname{rk} E$ *and has monodromy whose char. pol. equals that of the monodromy of the nearby cycles of* (E, ∇) *along* S_i. $\square$

***Theorem* (C.S.).** *The Stokes-filtered local system attached to* N *is obtained by direct image from the (2-dimensional) Stokes-filtered local system attached to* $(E(*S), \nabla + dx)$.

Proof. We will compute the $\leqslant 0$ of the Stokes filtration (i.e., compute moderate growth section, $\varphi = 0$). In order to compute the Stokes-filtered local system attached to $(E(*S), \nabla + dx)$, one has to blow-up the point $(0, \infty)$ in order to separate the curves S_i from the crossing point of the divisor. In this example, it is enough to blow-up successively the intersection point of the exceptional divisor and the strict transform of $\Delta \times \{\infty\}$. We get a map $\pi : X \to \Delta$.

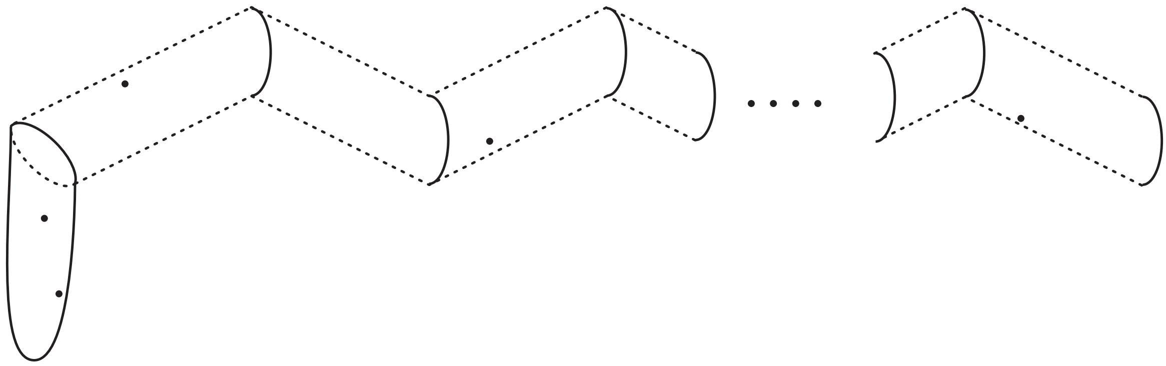


We now consider the real blow-up of the components of the normal crossing divisor. The map π lifts as a map $\widetilde{\pi} : \widetilde{X} \to \widetilde{\Delta}$. Fix $\theta \in \partial\widetilde{\Delta}$ where one wants to compute the Stokes filtration. Its fibre by $\widetilde{\pi}$ looks like a pipe:

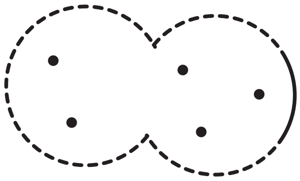


There are some holes in the pipe, corresponding to the intersection with S_i, hence we get a *leaky pipe*.

Considering the subset of the pipe in the neighbourhood of which the function $e^{x \circ e}$ has moderate growth will cut out a *leaky half-pipe*. The way it is cut out depends on θ, and in particular some holes disappear for some values of θ.



This can simply be drawn like this:



And one proves that the $\mathrm{DR}^{\mathrm{mod}\,D}(e^{+}(E(*S),\nabla+dx))$ restricted to $\widetilde{\pi}^{-1}(\theta)$ is a perverse sheaf on the open disc, with singularities at the holes, and extended by 0 at the dashed boundary. The computation of its hypercohomology gives the expected dimension given by the theorem of C. Roucairol, from which on deduces the theorem. □

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