A conjecture by M. Kashiwara

Conjecture

*The properties of polarizable Hodge $\mathcal{D}$-modules which do not explicitly involve the Hodge filtration remain valid when one replaces "**polarizable Hodge $\mathcal{D}$-module**" with "**semisimple holonomic $\mathcal{D}$-module**".*

Main Theorem

*Let $\mathcal{F}$ be a **semisimple** locally constant sheaf on a smooth projective manifold X, and let $f : X \longrightarrow S$ be a holomorphic map with values in a compact Riemann surface S. Then, the direct image complex $Rf_*\mathcal{F}$ decomposes, in the derived category, in a direct sum of irreducible perverse sheaves (with some shifts) on S.*

3 main sources of ideas:

- the theory of ***polarizable Hodge modules*** of M. Saito,
- the notion of a ***variation of twistor structure***, introduced by C. Simpson (after a suggestion of Deligne),
- the use of ***distributions*** and of ***Hermitian forms*** on $\mathcal{D}$-modules, after D. Barlet et M. Kashiwara.

Harmonic metrics and Higgs bundles

Theorem **(K. Corlette 1988, C. Simpson)**
Let (V, ∇) *be a holomorphic vector bundle equipped with a flat connection on a compact Kähler manifold* X. *Then,* (V, ∇) *has a harmonic metric* h *if and only if the locally constant sheaf* $\mathcal{F}$ *of its horizontal sections is* ***semisimple***.

$$D_V = D'_V + D''_V$$

the flat connection on the bundle

$$H = \mathcal{C}^\infty_X \bigotimes_{\mathcal{O}_X} V$$

obtained from ∇, so that

$$(V, \nabla) = (\mathrm{Ker}\, D''_V, D'_V).$$

Let h be a metric on H.

One may then find a unique connection

$$D_E = D'_E + D''_E \quad \text{on} \quad H$$

which preserves the metric h in such a way that, if one puts

$$\theta'_E = D'_E - D'_V, \qquad \theta''_E = D''_E - D''_V,$$

the $(0, 1)$-form θ''_E with values in $\mathrm{End}(H)$ is the h-adjoint of the $(1, 0)$-form θ'_E.

Definition

The metric h *is* ***harmonic*** *relatively to the flat holomorphic bundle* (V, ∇) *if*

$$(D''_E + \theta'_E)^2 = 0$$

that is,

$$D''^2_E = 0, \qquad D''_E(\theta'_E) = 0, \qquad \theta'_E \wedge \theta'_E = 0.$$

$$E = \mathrm{Ker}\, D''_E : H \to H$$

E is a holomorphic bundle equipped with a 1-form θ'_E with values in $\mathrm{End}(E)$, which satisfies

$$\theta'_E \wedge \theta'_E = 0$$

θ'_E is a ***Higgs field*** for E.

The flatness of D_V also imposes relations as

$$D'^2_E = 0, \qquad D'_E(\theta'_E) = 0, \qquad D''_E(\theta''_E) = 0.$$

Consequently, for all $z_o \in \mathbb{C}$, the operator

$$D''_E + z_o\theta''_E$$

is a complex structure on H.

The associated holomorphic bundle

$$V_{z_o} = \mathrm{Ker}\, D''_E + z_o \theta''_E$$

is equipped, if $z_o \neq 0$, with a flat holomorphic connection

$$\nabla_{z_o} = D'_E + \frac{1}{z_o}\theta'_E.$$

For $z_o = 1$ one recovers (V, ∇).

If h is harmonic, the identities of Kähler geometry apply to the mixed operators

$$\mathcal{D}_\infty = D'_E + \theta''_E, \quad \mathcal{D}_0 = D''_E + \theta'_E$$

$$D_V = \mathcal{D}_\infty + \mathcal{D}_0$$

$$\Delta_{D_V} = 2\Delta_{\mathcal{D}_\infty} = 2\Delta_{\mathcal{D}_0}.$$

Simpson deduces from them the ***Hard Lefschetz Theorem***.

Family of flat bundles (V_{z_o}, ∇_{z_o}) for $z_o \neq 0$.
One also has operators which satisfy the identities of Kähler geometry:

$$\mathcal{D}_{z_o} = (z_o D'_E + \theta'_E) + (D''_E + z_o \theta''_E) = z_o \mathcal{D}_\infty + \mathcal{D}_0$$
$$\Delta_{z_o} = (1 + |z_o|^2)\Delta_{D_V}.$$

$\Longrightarrow$ all locally constant sheaves $\mathcal{F}_{z_o}$, $z_o \neq 0$, have the same cohomology.

Variations of polarized Hodge structures

$\boldsymbol{H}$ a vector bundle $\boldsymbol{C^\infty}$ equipped with a decomposition

$$\boldsymbol{H} = \bigoplus_{p\in\mathbb{Z}} \boldsymbol{H}^{p,w-p} \qquad (w \in \mathbb{Z}),$$

with a flat connection

$$D_V = D'_V + D''_V$$

and with a nondegenerate Hermitian form $\boldsymbol{k}$ such that

– the decomposition is ***k-orthogonal***,

– $(-1)^p k$ on $H^{p,w-p}$ is ***positive definite***, and (Griffiths' ***transversality relations***)

$$D'_V(H^{p,w-p}) \subset \left(H^{p,w-p} \oplus H^{p-1,w-p+1}\right) \otimes_{\mathcal{O}_X} \Omega^1_X$$
$$D''_V(H^{p,w-p}) \subset \left(H^{p,w-p} \oplus H^{p+1,w-p-1}\right) \otimes_{\mathcal{O}_{\overline{X}}} \Omega^1_{\overline{X}}$$

where $\overline{X}$ is the complex conjugate manifold. One says that this is a ***variation of polarized complex Hodge structures of weight*** w.

$$h = (-1)^p k \quad \text{on} \quad H^{p,w-p}$$

Decomposition

$$D'_V = D'_E + \theta'_E, \qquad D''_V = D''_E + \theta''_E.$$

The metric h is ***harmonic***.

Variations of polarized twistor structures

C. Simpson presents this notion by stating the

Meta theorem **(C. Simpson)**
If the words ***"Hodge structure"*** *are replaced with* ***"twistor structure"*** *in the assumptions and conclusions of any theorem in Hodge theory, one still gets a true statement, the proof of which is analogous to that of its model.*

The notion of a twistor structure is a

“deshomogeneization”

of that of a Hodge structure, which has a notion of ***weight***.

The Hodge graduation on a bundle on $\boldsymbol{X}$ is replaced with the extension of this bundle as a bundle on $\boldsymbol{X} \times \mathbb{P}^1$.

The conjugation $\boldsymbol{H}^{q,p} = \overline{\boldsymbol{H}^{p,q}}$ is replaced with a geometric conjugation.

Geometric conjugation

Let $f(x)$ be a holomorphic function on an open set of X. Its conjugate

$$\overline{f}(x) \stackrel{\text{def}}{=} \overline{f(x)}$$

is a holomorphic function on the ***complex conjugate manifold*** $\overline{X}$.

Let $g(z)$ be a holomorphic function on an open set U of $\mathbb{P}^1$. Its "conjugate"

$$\overline{g}(z) \stackrel{\text{def}}{=} g(-1/z)$$

is a holomorphic function on the ***"complex conjugate set"*** $\overline{U}$.

Mix these two notions to define $\overline{f}(x, z)$.

If $\mathcal{F}$ is a $\mathcal{O}_{X \times U}$-module, then $\overline{\mathcal{F}}$ is a $\mathcal{O}_{\overline{X} \times \overline{U}}$-module.

Polarized twistor structure

A ***twistor structure*** $\mathcal{T} = (\mathcal{H}', \mathcal{H}'', C)$ ***of weight*** w ***on*** H consists of
– two $\mathcal{O}_{U_0}$-modules $\mathcal{H}', \mathcal{H}''$, locally free of rank d
– and of a ***glueing*** between $\mathcal{H}'^*$ and $\overline{\mathcal{H}}''$ on an annulus $\mathbf{A}$ invariant under ***geometric conjugation***

$$C : \Gamma(\mathbf{A}, \mathcal{H}') \underset{\mathcal{O}(\mathbf{A})}{\otimes} \overline{\Gamma(\mathbf{A}, \mathcal{H}'')} \longrightarrow \mathcal{O}(\mathbf{A}).$$

The glueing defines a bundle $\widetilde{\mathcal{H}}$ on $\mathbb{P}^1$ ***isomorphic to*** $\mathcal{O}_{\mathbb{P}^1}(w)^d$ such that $\overline{H} = \Gamma(\mathbb{P}^1, \widetilde{\mathcal{H}}(-w))$.

Weil operator: $\widetilde{\mathcal{T}} = (\mathcal{H}', \mathcal{H}'', (iz)^{-w}C)$ has weight $\mathbf{0}$.

Tate twist: $\mathcal{T}(k) = (\mathcal{H}', \mathcal{H}'', (-z^2)^{-k}C)$.

Hermitian duality:

$$\mathcal{T}^* = (\mathcal{H}'', \mathcal{H}', C^*) \quad \text{with} \quad C^*(x, \overline{y}) = \overline{C(y, \overline{x})}.$$

$$w(\mathcal{T}^*) = -w(\mathcal{T}), \quad w(\mathcal{T}(k)) = w(\mathcal{T}) - 2k,$$
$$\mathcal{T}(k)^* = \mathcal{T}^*(-k).$$

Polarization:

If $\mathcal{T}$ has weight $\mathbf{0}$, a polarization is an isomorphism

$$S : \mathcal{H}'' \xrightarrow{\sim} \mathcal{H}'$$

such that

$$C \circ (S \otimes \mathrm{Id}) : \Gamma(\mathrm{A}, \mathcal{H}'') \underset{\mathcal{O}(\mathrm{A})}{\otimes} \overline{\Gamma(\mathrm{A}, \mathcal{H}'')} \longrightarrow \mathcal{O}(\mathrm{A})$$

induces a ***positive definite Hermitian form*** h on

$$H \underset{\mathbb{C}}{\otimes} \overline{H}.$$

In general,

$$\mathcal{S} : \mathcal{T} \longrightarrow \mathcal{T}^*(-w),$$
$$\mathcal{S} = (S', S''), \quad S', S'' : \mathcal{H}'' \longrightarrow \mathcal{H}'$$

such that

$$\mathcal{S}^* = (-1)^w \mathcal{S}, \quad S' = (-1)^w S''$$

and S'' is a polarization of $\widetilde{\mathcal{T}} = (\mathcal{H}', \mathcal{H}'', (iz)^{-w} C)$.

Variations of polarized twistor structures

– Two locally free $\mathcal{O}_{X\times U_0}$-modules $\mathcal{H}', \mathcal{H}''$ of rank d equipped with a ***flat connection***

$$\nabla : \mathcal{H}' \longrightarrow \mathcal{H}' \underset{\mathcal{O}_{X\times U_0}}{\otimes} \frac{1}{z}\,\Omega^1_{X\times U_0}$$

– and a ***glueing***

$$C : \pi_{\mathrm{A}*}\mathcal{H}' \underset{\mathcal{O}(\mathrm{A})}{\otimes} \overline{\pi_{\mathrm{A}*}\mathcal{H}''} \longrightarrow \mathcal{C}^\infty_X(\mathcal{O}(\mathrm{A}))$$

compatible with the connection, so that the restriction to any fibre is a ***twistor structure of weight*** w.

A ***polarization*** is an isomorphism $S : \mathcal{H}'' \xrightarrow{\sim} \mathcal{H}'$ ***compatible with the connection*** and which induces a polarization on each fibre.

The bundle $\widetilde{\mathcal{H}}$ on $\mathbb{P}^1$ obtained by glueing $\mathcal{H}'^*$ and $\overline{\mathcal{H}''}$ is C^∞ with respect to variables of X and holomorphic with respect to the variable of $\mathbb{P}^1$. If $\overline{H}$ is the bundle $\pi_*\widetilde{\mathcal{H}}(-w)$, the polarization defines a ***metric*** h on H.

The bundle $\mathcal{H}'_{|z=1}$ is a holomorphic subbundle of H equipped with a ***flat holomorphic connection***.

The bundle $\mathcal{H}'_{|z=0}$ is a holomorphic subbundle of H equipped with a ***Higgs field***.

***Theorem* (C. Simpson)**

Such a metric is ***harmonic****. Conversely, any harmonic metric on* $\boldsymbol{H}$ *is obtained in this way.*

Corollary

If $\boldsymbol{X}$ *is compact Kähler, The restriction to* $z = 1$ *induces an equivalence between the category of variations of polarized twistor structures of weight* $\mathbf{0}$ *and that of semisimple representations of* $\pi_1(X)$.

Hodge-Simpson Theorem

If X *is compact Kähler,* L *the Lefschetz operator, and if*

$$\mathcal{T} = (\mathcal{H}', \mathcal{H}'', C)$$

is a variation of polarized twistor structure of weight w *on* X, *then for any* $k \geqslant 0$, *the primitive part of the* k*th de Rham cohomology is a polarized twistor structure of weight* $w + k$.

$\mathcal{R}$-modules

$$\mathcal{R}_{X\times U_0} \simeq \mathcal{O}_{X\times U_0}\langle z\partial_{x_1},\dots,z\partial_{x_n}\rangle \quad \textit{locally}$$

It is the Rees ring associated with $\mathcal{D}_X$ and its filtration $F_\bullet\mathcal{D}_X$ by the order of operators

$$R_F\mathcal{D}_X = \bigoplus_{k\in\mathbb{N}} F_k(\mathcal{D}_X)\cdot z^k$$

when one forgets the grading, *i.e.*

$$\mathcal{R}_{X\times U_0} = \mathcal{O}_{X\times U_0} \bigotimes_{\mathbb{C}[z]} R_F\mathcal{D}_X.$$

$\mathcal{R}_{X\times U_0}$-modules $\Leftrightarrow$ $\mathcal{O}_{X\times U_0}$-modules with a flat connection

A $\mathcal{R}_{X\times U_0}$-module is ***strict*** if it has no $\mathcal{O}_{U_0}$-torsion.

Example

A $R_F\mathcal{D}_X$-module $\mathcal{M}$ is strict iff $\mathcal{M} = R_F M$ for some $\mathcal{D}_X$-module M equipped with a filtration F compatible with that of $\mathcal{D}_X$.

Meromorphic distributions

Replace the sheaf of functions C^∞ on X with values in $\mathcal{O}(\mathrm{A})$ with that of distributions on X with values in $\mathcal{O}(\mathrm{A})$.

If α is a complex number, put

$$\begin{aligned}\alpha \star z &= z\,\mathrm{Re}\,\alpha + i(z^2+1)\,\mathrm{Im}\,\alpha/2 \\ &= z\Big(\mathrm{Re}\,\alpha + \frac{i}{2}\,(z+1/z)\,\mathrm{Im}\,\alpha\Big).\end{aligned}$$

The function

$$0 \neq z \longmapsto \frac{\alpha \star z}{z}$$

is "***real***", *i.e.* invariant under

$$i \longleftrightarrow -i \quad \text{and} \quad z \longleftrightarrow \overline{z} = -1/z.$$

One wants to consider the one-variable distribution:

$$|t|^{2(\alpha\star z)/z}\,(\log|t|)^{\ell}.$$

This distribution takes values in $\mathcal{O}(\mathbf{A})$ when multiplied with

$$\Gamma(1+(\alpha\star z)/z)^{-(\ell+1)}.$$

The poles with respect to z are ***purely imaginary***.

The sheaf $\mathfrak{Db}_{X_{\mathbb{R}}}^{(\mathbf{A},*i\mathbb{R})}$

Twistor structure of weight w on a $\mathcal{D}_X$-module

– Two ***strict holonomic*** $\mathcal{R}_{X\times U_0}$-modules $\mathcal{M}', \mathcal{M}''$,

$$\mathcal{M}'_{|z=1} = \mathcal{M}''_{|z=1} = M,$$

– and a $\mathcal{R} \otimes_{\mathcal{O}(\mathrm{A})} \overline{\mathcal{R}}$-linear pairing

$$C : \pi_{\mathrm{A}*}\mathcal{M}' \underset{\mathcal{O}(\mathrm{A})}{\otimes} \overline{\pi_{\mathrm{A}*}\mathcal{M}''} \longrightarrow \mathfrak{Db}_{X_{\mathbb{R}}}^{(\mathrm{A},*i\mathbb{R})}.$$

Moreover, $\mathcal{M}', \mathcal{M}''$ are ***specializable*** along any germ of holomorphic function and the specialized modules are objects of the same kind.

In dimension 0 one gets a ***twistor structure of weight*** $w + \cdots$.

Specialization of C:

$$\psi_{t,\alpha}C : \psi_{t,\alpha}\mathcal{M}'_{\mathrm{A}} \underset{\mathcal{O}(\mathrm{A})}{\otimes} \overline{\psi_{t,\alpha}\mathcal{M}''_{\mathrm{A}}} \longrightarrow \mathfrak{C}^{(\mathrm{A},*i\mathbb{R})}_{X_{0,\mathbb{R}}}$$

$$(m, \overline{\mu}) \longmapsto$$

$$\left[\varphi \mapsto \left(\frac{i}{2\pi}\right)^n \mathrm{Res}_{s=(\alpha\star z)/z} \left\langle C(m,\overline{\mu}), \varphi\, |t|^{2s}\, \chi(t)\right\rangle\right].$$

A polarization is an isomorphism of $\mathcal{R}_{X\times U_0}$-modules

$$S : \mathcal{M}'' \xrightarrow{\sim} \mathcal{M}'$$

which induces a polarization on the specialized objects.

Theorem

The category of regular holonomic $\mathcal{D}_X$-modules equipped with a polarized twistor structure is ***semisimple****.*

If $f : X \longrightarrow Y$ is a morphism between smooth projective manifolds, the direct image of a regular holonomic module equipped with a polarized twistor structure decomposes in ***direct sum*** *of its cohomology modules, which are regular holonomic $\mathcal{D}_Y$-modules equipped with a polarized twistor structure (the weight is obtained in the usual way).*

Conjecture

If X is smooth projective, the restriction functor to $z = 1$ is an equivalence between the category of regular holonomic $\mathcal{D}_X$-modules ***equipped with a polarized twistor structure of weight*** $\mathbf{0}$ *and that of* ***semisimple*** *regular holonomic $\mathcal{D}_X$-modules.*

According to C. Simpson's work, this conjecture is true in the following cases:

- **The *smooth* $\mathcal{D}_X$-modules and the locally constant sheaves.**
- X **is a compact Riemann surface.**

Conclusion

This gives the main theorem.

The general case should use the work of Jost et Zuo, which partly generalize those of Simpson in dimension bigger than one.

The case of ***irregular*** holonomic $\mathcal{D}$-modules is still largely open. Nevertheless, there are some partial results in dimension one: the construction of a harmonic metric associated with a ***irreducible irregular*** meromorphic connection on a bundle on a compact Riemann surface.