# Exponential-Hodge theory

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# The Riemann existence theorem

- $P(z, \partial_z) = \sum_0^d a_k(z)\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^k, \quad a_k \in \mathbb{C}[z],\ a_d \not\equiv 0$
- $S = \{z \mid a_d(z) = 0\}$ sing. set (assumed $\neq \varnothing$)
- Associated linear system

$$(*) \qquad \frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} = A(z)\begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix}, \quad A(z) \in \mathrm{End}(\mathbb{C}(z)^d)$$

- $\rightsquigarrow$ ***Monodromy*** representation of the solution vectors by analytic continuation

$$\rho : \pi_1(\mathbb{C} \setminus S, z_o) \longrightarrow \mathrm{GL}_d(\mathbb{C})$$

# The Riemann existence theorem

- $\rho \iff (T_s \in \mathrm{GL}_d(\mathbb{C}))_{s\in S}$ (and $T_\infty := \left(\prod_s T_s\right)^{-1}$)
- ***Conversely***, any $\rho$ (any finite $S$) comes from a system $(*)$ s.t. after local merom. gauge transf. at $S \cup \infty$, ***at most simple poles*** (i.e., ***regular sing.***), i.e.,
  - $\exists\, M(z-s) \in \mathrm{GL}_d\left(\mathbb{C}(\{z-s\})\right)$ s.t.

$$(z-s)\cdot\left[M^{-1}AM + M^{-1}M'_z\right] \in \mathrm{End}(\mathbb{C}\{z-s\}).$$

- Proof: Near $s \in S$, this amounts to finding $C_s \in \mathrm{End}(\mathbb{C}^d)$ s.t. $T_s = e^{-2\pi i C_s}$. Then $A = C_s/(z-s)$ has monodromy $T_s$ around $s$. Globalization: non-explicit procedure.

# Rigid irreducible representations

- Assume $\rho$ is ***irreducible*** and ***rigid***:

$$\boxed{\text{if } T'_s \sim T_s \ \forall s \in S \cup \infty, \text{ then } \rho' \sim \rho}$$

- and assume $\forall s \in S \cup \infty$,

$$\boxed{\forall \lambda \text{ eigenvalue of } T_s, \ |\lambda| = 1}$$

- $\Rightarrow$ ***More structure*** on the solution to the Riemann existence th.

# Variations of pol. Hodge structure

THEOREM (Deligne 1987, Simpson 1990):

$\exists!$ var. of polarized Hodge structure (wt. $=0$) adapted to $\rho$

- $G(z)\in\mathrm{End}(C^\infty(\mathbb{C}\smallsetminus S)^d)$: ***pos. def. Herm.*** matrix.
- ***Hodge decomp.*** ($H^{q,p}=\overline{H^{p,q}}$):
$$C^\infty(\mathbb{C}\smallsetminus S)^d=:H=\overset{\perp}{\bigoplus_{p+q=0}}H^{p,q},$$
- $H^{p,q}\simeq C^\infty(\mathbb{C}\smallsetminus S)^{h^{p,q}}$ ***not hol.*** but $F^pH:=\bigoplus_{p'\geqslant p}H^{p,q}$ ***holomorphic*** and
$$A\cdot F^p\mathscr{O}(\mathbb{C}\smallsetminus S)^d\subset F^{p-1}\mathscr{O}(\mathbb{C}\smallsetminus S)^d$$
- $\widetilde{G}$ s. t. $\widetilde{G}_{|H^{p,q}}:=(-1)^pG_{|H^{p,q}}$, then
$$\partial_z\widetilde{G}\cdot\widetilde{G}^{-1}={}^tA.$$

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- $\Rightarrow$ Numbers $f^p = \operatorname{rk} F^p \mathscr{O}(\mathbb{C} \smallsetminus S)^d$ attached to $\rho$.

- Moreover (Griffiths),

$$\mathbb{C}[z, (z-s)^{-1}_{s \in S}]^d = \mathscr{O}(\mathbb{C} \smallsetminus S)^d_{G\text{-mod. growth}}.$$

# Hypergeom. differential eqns

- Given $\begin{cases} 0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_d < 1, \\ 0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_d < 1, \end{cases} \quad \alpha_i \neq \beta_j \; \forall i, j$.

  $$P(z, \partial_z) := \prod_{i=1}^{d} \Big( z \frac{\mathrm{d}}{\mathrm{d}z} - \alpha_i \Big) - z \prod_{j=1}^{d} \Big( z \frac{\mathrm{d}}{\mathrm{d}z} - \beta_j \Big).$$

  $S = \{0, 1\}$.

- Beukers & Heckman: irreducible rigid $\rho$, with $\lambda = e^{-2\pi \mathrm{i} \alpha}$ or $e^{2\pi \mathrm{i} \beta}$.

- Set $\ell_j = \#\{i \mid \alpha_i \leqslant \beta_j\} - j$

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- Set $\ell_j = \#\{ i \mid \alpha_i \leqslant \beta_j \} - j$, e.g.
  - $\alpha_1 \leqslant \beta_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \beta_d \Rightarrow \ell_j = 0 \; \forall j$
  - $\alpha_d \leqslant \beta_1 \Rightarrow \ell_j = d - j$.

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- mixed: $F^1 = 0$, $F^0 = \mathscr{O}(\mathbb{C} \smallsetminus S)^d \Rightarrow$ unitary conn.
- unmixed: $0 = F^d \subset \cdots \subset F^0 = \mathscr{O}(\mathbb{C} \smallsetminus S)^d$.

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PROOF: relies on the Katz algorithm (middle convolution) and a formula by M. Dettweiler & CS (2013) for computing the behaviour of $f^p$ by middle convolution.

# Confluent hypergeom. diff. eqns

$$P(z, \partial_z) := \prod_{i=1}^{d'} \left( z \frac{\mathrm{d}}{\mathrm{d}z} - \alpha_i \right) - z \prod_{j=1}^{d} \left( z \frac{\mathrm{d}}{\mathrm{d}z} - \beta_j \right)$$

with $d' < d \Rightarrow S = 0$ and $0$ is an ***irreg. sing.***
($\infty =$ reg. sing).

- Riemann existence th. breaks down for irreg. sing.
- Need ***Stokes data*** to reconstruct the differential eqn from sols.
- $\rightsquigarrow$ Riemann-Hilbert-Birkhoff correspondence.

# Irregular singularity

Matrix connection $A \in \mathrm{End}(\mathbb{C}\{z\}^d)$

THEOREM (Levelt-Turrittin) (without ramif.):

$\exists \widehat{M} \in \mathrm{GL}_d(\mathbb{C}((z)))$,

$$\boxed{\widehat{B} := \widehat{M}^{-1} A \widehat{M} + \widehat{M}^{-1} \widehat{M}'_z = \mathrm{diag}_\varphi(\varphi'_z \,\mathrm{Id} + C_\varphi / z)}$$

$\varphi \in \mathbb{C}[1/z]$ without cst term, $C_\varphi =$ cst matrix.

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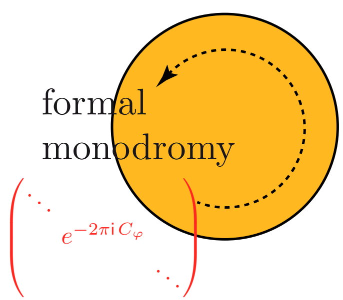

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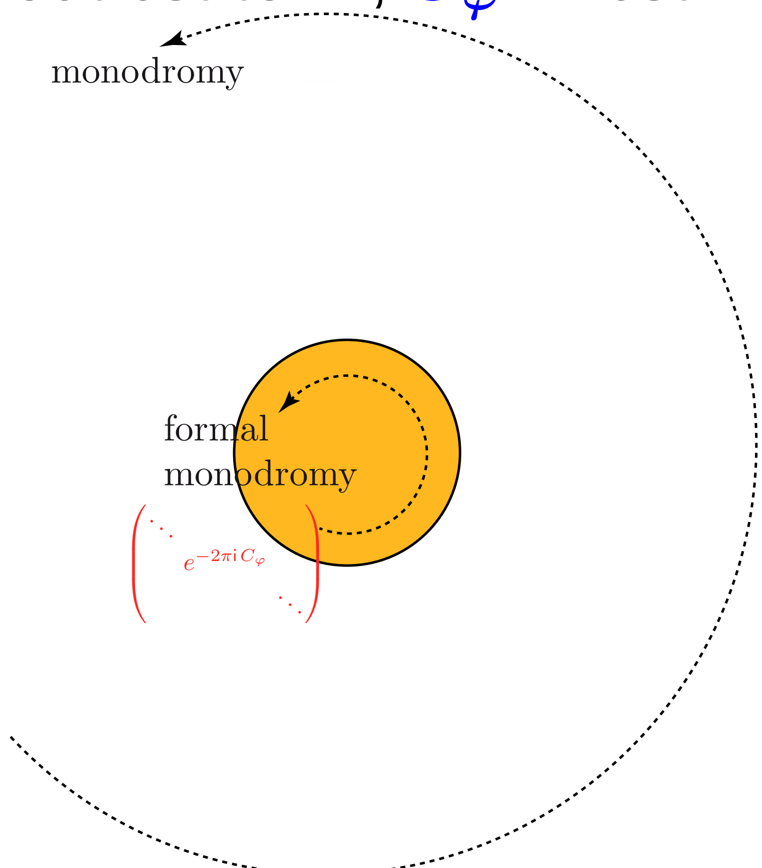

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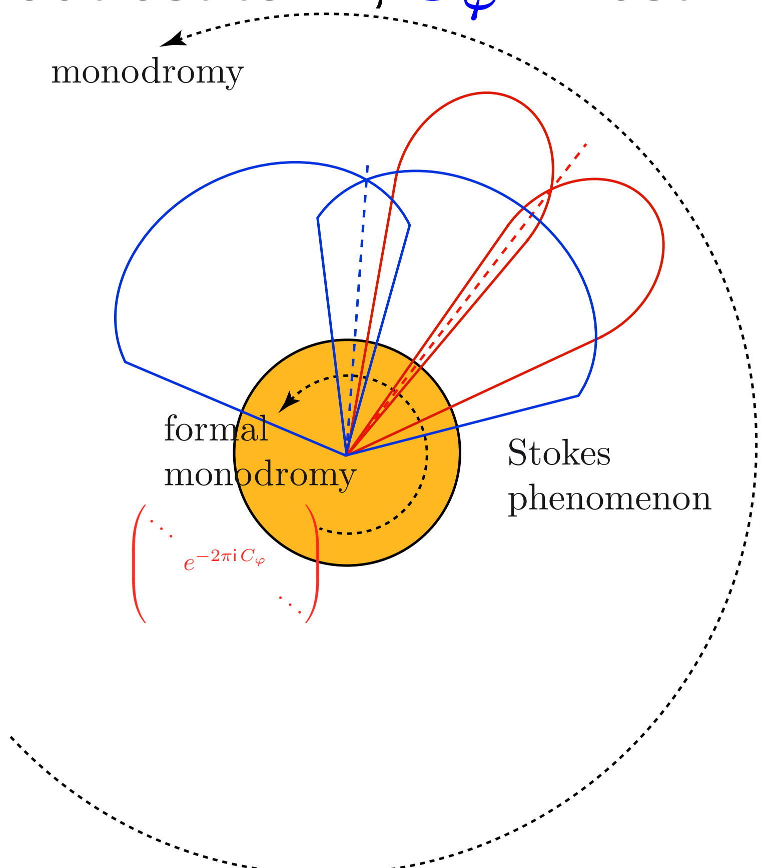

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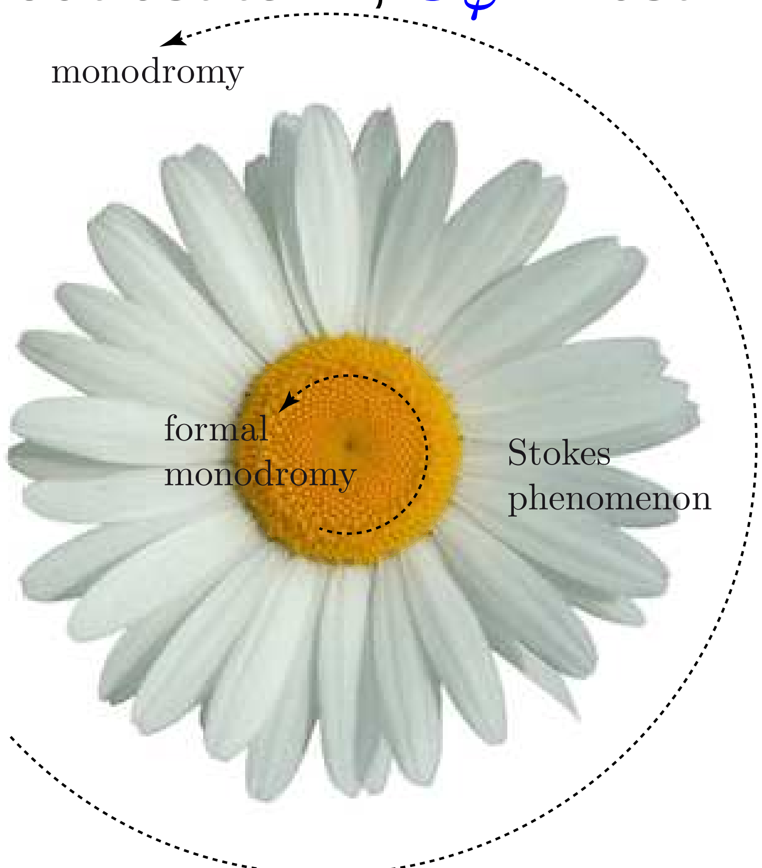

# Confluent hypergeom. diff. eqns

$$P(z,\partial_z) := \prod_{i=1}^{d'} \Big(z\frac{\mathrm{d}}{\mathrm{d}z} - \alpha_i\Big) - z \prod_{j=1}^{d} \Big(z\frac{\mathrm{d}}{\mathrm{d}z} - \beta_j\Big)$$

with $d' < d$.

- Same condition on $\alpha, \beta$'s $\Rightarrow$ ***irreducible*** and ***rigid***:

isomorphic formal struct. at $S \cup \infty \Rightarrow \exists\, \mathbb{C}(z)$ gauge transf.

- $\Rightarrow$ ***Cannot find*** a var. of polarized Hodge structure s.t. the sol. to Birkhoff existence th. is given by $\mathscr{O}(\mathbb{C} \smallsetminus S)^d_{G\text{-mod. growth}}$.
- Is there any similar struct. for Birkhoff existence th.?

# Harmonic metrics

- Diff. operator $\frac{\mathrm{d}}{\mathrm{d}z} + A(z)$, $A(z) \in \mathrm{End}(\mathbb{C}(z)^d)$, pole set $= S \subset \mathbb{C}$.
  $\iff$ $\mathbb{C}(z)$-vect. space with connection $\Rightarrow$ notion of ***irreducibility***.
- $G(z) \in \mathrm{End}(C^\infty(\mathbb{C} \smallsetminus S)^d)$: pos. def. Herm. matrix.
- $\exists!\ A'_G(z), A''_G(z) \in \mathrm{End}(C^\infty(\mathbb{C} \smallsetminus S)^d)$ s.t.

(compatibility with $G$)

$$\partial_z G(z) = {}^t\!A'_G G + G\overline{A''_G}$$

$$\overline{\partial}_z G(z) = {}^t\!A''_G G + G\overline{A'_G}$$

$$\underbrace{-A''_G}_{\theta''} = (\underbrace{A - A'_G}_{\theta'})^*.$$

- $G$ is ***harmonic w.r.t. $A$*** if

$$\boxed{\overline{\partial}_z \theta' + [\theta', \theta'^*] = 0}$$

# Harmonic metrics

THEOREM (Simpson 1990, CS 1998, Biquard-Boalch 2004, T. Mochizuki 2011):

- If $A$ is ***irreducible***, $\exists!$ harmonic metric $G$ w.r.t. $A$ s.t.
  - Coefs of $\mathrm{Char}\,\theta'$ have ***mod. growth*** at $S \cup \infty$,
  - $\mathbb{C}[z, ((z-s)^{-1})_{s\in S}]^d = (\mathscr{O}(\mathbb{C} \smallsetminus S)^d)_{G\text{-mod. growth}}$.

- E.g., the Hodge metric of a var. pol. Hodge structure is harmonic w.r.t. the reg. sing. conn. $A$.
- If $A$ is ***irreg.***, what about ***rigid*** irreducible $A$?

# The irregular Hodge filtration

Deligne (2007):

"The analogy between vector bundles with integrable connection having irregular singularities at infinity on a complex algebraic variety $U$ and $\ell$-adic sheaves with wild ramification at infinity on an algebraic variety of characteristic $p$, leads one to ask how such a vector bundle with integrable connection can be part of a system of realizations analogous to what furnishes a family of motives parametrized by $U$...

In the 'motivic' case, any de Rham cohomology group has a natural Hodge filtration. Can we hope for one on $H^i_{\mathrm{dR}}(U, \nabla)$ for some classes of $(V, \nabla)$ with irregular singularities?"

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“The reader may ask for the usefulness of a “Hodge filtration” not giving rise to a Hodge structure. I hope that it forces bounds to $p$-adic valuations of Frobenius eigenvalues. That the cohomology of ‘$e^{-z}z^{\alpha}$’ ($0 < \alpha < 1$) has Hodge degree $1 - \alpha$ is anlogous to formulas giving the $p$-adic valuation of Gauss sums.”

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- Ex. 1: $U=\mathbb{C}$, $f : z \mapsto -z^2$, $\nabla = \mathrm{d} + \mathrm{d}f = e^{z^2}\cdot \mathrm{d} \cdot e^{-z^2}$

$$\begin{array}{ccccc}
\mathbb{C}[z] & \xrightarrow{\nabla} & \mathbb{C}[z]\cdot \mathrm{d}z & \longrightarrow & H^1_{\mathrm{dR}}(U,\nabla) \\
{\scriptstyle e^{-z^2}}\downarrow\wr & & \wr\uparrow{\scriptstyle e^{z^2}} & & \wr\uparrow \\
e^{-z^2}\mathbb{C}[z] & \xrightarrow{\mathrm{d}} & e^{-z^2}\mathbb{C}[z]\cdot \mathrm{d}z & \longrightarrow & \mathbb{C}\cdot [e^{-z^2}\mathrm{d}z]
\end{array}$$

period: $\int_{\mathbb{R}} e^{-z^2}\mathrm{d}z = \pi^{1/2} \overset{?}{\Rightarrow} [e^{-z^2}\mathrm{d}z] \in F^{1/2}H^1_{\mathrm{dR}}(U,\nabla)$.

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- Ex. 2: $U = \mathbb{C}^*$, $f : z \mapsto -z$, $\nabla = \mathrm{d} + \mathrm{d}f + \alpha \mathrm{d}z/z$

$$\begin{array}{ccccc}
\mathbb{C}[z,z^{-1}] & \xrightarrow{\ \nabla\ } & \mathbb{C}[z,z^{-1}]\cdot \mathrm{d}z/z & \longrightarrow & H^1_{\mathrm{dR}}(U,\nabla) \\
{\scriptstyle e^{-z}z^{\alpha}}\downarrow\wr & & \wr\uparrow{\scriptstyle e^{z}z^{-\alpha}} & & \wr\uparrow \\
e^{-z}z^{\alpha}\mathbb{C}[z,z^{-1}] & \xrightarrow{\ \mathrm{d}\ } & e^{-z}z^{\alpha}\mathbb{C}[z,z^{-1}]\cdot \mathrm{d}z/z & \longrightarrow & \mathbb{C}\cdot[e^{-z}z^{\alpha}\mathrm{d}z/z]
\end{array}$$

period: $\displaystyle\int_0^{\infty} e^{-z}z^{\alpha}\mathrm{d}z/z = \Gamma(\alpha) \overset{?}{\Rightarrow} [e^{-z}z^{\alpha}\mathrm{d}z] \in F^{1-\alpha}H^1_{\mathrm{dR}}(U,\nabla)$.

# The Hodge filtration in dim $\geqslant 1$

- Setting:
  - $X$: smooth cplx proj. variety,
  - $D$: reduced divisor with normal crossings in $X$ locally, $D = \{x_1 \cdots x_\ell = 0\}$
  - $U = X \smallsetminus D$.

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- THEOREM (Deligne 1972):

$$H^k(U, \mathbb{C}) \simeq H^k\big(X, (\Omega_X^\bullet(\log D), \mathrm{d})\big)$$

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$$\left[\begin{aligned} &\Omega_X^1(\log D) \overset{\text{loc.}}{=} \textstyle\sum_{i=1}^{\ell} \mathscr{O}_X \dfrac{\mathrm{d}x_i}{x_i} + \sum_{j>\ell} \mathscr{O}_X \, \mathrm{d}x_j, \\ &\Omega_X^k(\log D) = \overset{k}{\wedge} \Omega_X^1(\log D) \end{aligned}\right]$$

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and $\forall p$,

$$H^k\big(X, \sigma^{\geqslant p}(\Omega_X^\bullet(\log D), \mathrm{d})\big) \longrightarrow H^k\big(X, (\Omega_X^\bullet(\log D), \mathrm{d})\big)$$

is ***injective***, its image defining the Hodge filtration $F^p H^k(U, \mathbb{C})$.

  - $\rightsquigarrow$ Mixed Hodge structure on $H^k(U, \mathbb{C})$.

# The Kontsevich complex

- Setting:
  - $X$: smooth cplx proj. variety, $D$: ncd in $X$, $U = X \setminus D$,
  - $f : X \longrightarrow \mathbb{P}^1 = \mathbb{C} \cup \infty$ hol. map, $f^{-1}(\infty) \subset D$, hence $f : U \to \mathbb{C}$. $P := f^*(\infty)$.

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- For $\alpha \in [0,1) \cap \mathbb{Q}$,

$$\Omega_f^k(\alpha) := \{\omega \in \Omega_X^k(\log D)([\alpha P]) \mid \mathrm{d}f \wedge \omega \in \Omega_X^k(\log D)([\alpha P])\}$$

- Significant $\alpha$'s: $\ell/m$, $m =$ mult. of a component of $P$, $\ell = 0, \ldots, m-1$.
- Then $\boxed{\mathrm{d} + \mathrm{d}f : \Omega_f^k(\alpha) \to \Omega_f^{k+1}(\alpha)}$

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- Significant $\alpha$'s: $\ell/m$, $m =$ mult. of a component of $P$, $\ell = 0, \dots, m-1$.
- Then $\boxed{\mathrm{d} + \mathrm{d}f : \Omega_f^k(\alpha) \to \Omega_f^{k+1}(\alpha)}$
- $\rightsquigarrow$ Kontsevich complex $(\Omega_f^\bullet(\alpha), \mathrm{d} + \mathrm{d}f)$.
- $\boxed{H^k(X, (\Omega_f^\bullet(\alpha), \mathrm{d} + \mathrm{d}f)) \simeq H^k_{\mathrm{dR}}(U, \mathrm{d} + \mathrm{d}f)}$

# The irreg. Hodge filtration in dim $\geqslant 1$

THEOREM (Kontsevich, Esnault-CS-Yu 2014, M. Saito 2014, T. Mochizuki 2015):

- $\forall p$,

$$H^k\big(X, \sigma^{\geqslant p}(\Omega^\bullet_f(\alpha), \mathrm{d}+\mathrm{d}f)\big) \longrightarrow H^k\big(X, (\Omega^\bullet_f(\alpha), \mathrm{d}+\mathrm{d}f)\big)$$

  is ***injective***, its image defining the irregular Hodge filtration $F^{p-\alpha}H^k_{\mathrm{dR}}(U, \mathrm{d}+\mathrm{d}f)$.

- $\lambda \geqslant \mu \in \mathbb{Q} \Rightarrow$

$$\boxed{F^\lambda H^k_{\mathrm{dR}}(U, \mathrm{d}+\mathrm{d}f) \subset F^\mu H^k_{\mathrm{dR}}(U, \mathrm{d}+\mathrm{d}f)}$$

- Jumps at most at $\lambda = \ell/m + p$, $p \in \mathbb{Z}$, $\ell = 0, \dots, m-1$, $m =$ mult. component of $P$.

# History of the result, dim. one

- Deligne (1984, IHÉS seminar notes).

$A \in \mathrm{GL}_d(\mathbb{C}(z))$ with ***reg sing***. on $S \cup \infty$, and ***unitary***.
$f \in \mathbb{C}(z)$. Defines a filtr. ($\lambda \in \mathbb{R}$)

$$F^{\lambda}\mathbb{C}[z,(z-s)_{s\in S}]^d \xrightarrow{\mathrm{d}+A+\mathrm{d}f} F^{\lambda-1}\mathbb{C}[z,(z-s)_{s\in S}]^d \mathrm{d}z$$

an proves $E_1$-degeneration.

- Deligne (2006). Adds more explanations and publication in the volume “Correspondance Deligne-Malgrange-Ramis” (SMF 2007).
- CS (2008). Same as Deligne, with $A$ underlying a ***pol. var. of Hodge structure***. Uses harmonic metrics through the theory of var. of twistor structures (Simpson, Mochizuki, CS).

# History of the result, dim $> 1$

- J.-D. Yu (2012): defines $F^\lambda H^k_{\mathrm{dR}}(U, \mathrm{d} + \mathrm{d}f) +$ many properties and $E_1$-degeneration in some cases.
- Esnault-CS-Yu (2013): $E_1$-degeneration by reducing to (CS, 2008) (push-forward by $f$).
- Kontsevich (2012), letters to Katzarkov and Pantev, arXiv 2014: defines the Kontsevich complex and proves $E_1$-degeneration if $P = P_{\mathrm{red}}$, by the method of Deligne & Illusie (reduction to char. $p$). ***Does not extend if*** $P \neq P_{\mathrm{red}}$. Motivated by mirror symmetry of Fano manifolds.
- M. Saito (2013): $E_1$-degeneration by comparing with limit mixed Hodge structure of $f$ at $\infty$.
- T. Mochizuki (2015): $E_1$-degeneration by using mixed twistor theory (without reducing to dim. $1$).

# Rigid irreducible diff. eqns

- Diff. operator $\frac{\mathrm{d}}{\mathrm{d}z} + A(z)$, $A(z) \in \mathrm{End}(\mathbb{C}(z)^d)$, pole set $= S \subset \mathbb{C}$.
- Assume it is ***irreducible*** and ***rigid***.
- Assume eigenvalues $\lambda$ of $\widehat{T}_s$ ($s \in S \cup \infty$) s.t. $|\lambda| = 1$.
- THEOREM (CS 2015):
  $\exists$ ***canonical*** filtration $F^p\mathbb{C}[z, ((z-s)^{-1})_{s\in S}]^d$ ($p \in \mathbb{R}$) by free $\mathbb{C}[z, ((z-s)^{-1})_{s\in S}]$-modules.
- Needs the construction of a category of ***Irregular Hodge modules*** between the category of mixed Hodge modules (M. Saito) and that of mixed twistor $D$-modules (T. Mochizuki). Use of the Arinkin-Deligne's algorithm similar to Katz' algorithm.

# Rigid irreducible diff. eqns

- QUESTIONS: For confluent hypergeom. eqns, how to compute the ***jumping indices*** and the ***rank*** of the Hodge bundles?