A SURVEY ON SEMI-SIMPLE LOCAL SYSTEMS ON ALGEBRAIC MANIFOLDS (PRELIMINARY VERSION)

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Introduction

I will review[(1)] some results concerning semi-simple representations of the fundamental group of an algebraic manifold. These representations are in one-to-one correspondence with semi-simple holomorphic vector bundles with a *flat* holomorphic connection:

Given a representation $\rho : \pi_1(X, \star) \to \mathrm{GL}_d(\mathbb{C})$, associate to it a local system $\mathcal{L}$ (or locally constant sheaf of $\mathbb{C}$-vector spaces) of rank d. Then consider the holomorphic vector bundle (*i.e.* locally free $\mathcal{O}_X$-module of rank d) $V = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{L}$ with connection $\nabla(f \otimes s) = df \otimes s$, so that $\operatorname{Ker} \nabla : V \to \Omega^1_X \otimes V$ is $\mathcal{L}$.

Conversely, given (V, ∇) with ∇ flat, put $\mathcal{L} = \operatorname{Ker} \nabla$.

It will be useful to consider the associated C^∞-bundle:

$$H = \mathcal{C}^\infty_X \otimes_{\mathcal{O}_X} V$$

Connection $D_V = D'_V + D''_V$ on H:

$$D'_V = d' \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla : H = \mathcal{C}^\infty_X \otimes_{\mathcal{O}_X} V \longrightarrow \mathcal{E}^{(1,0)}_X \otimes_{\mathcal{C}^\infty_X} H = \mathcal{E}^{(1,0)}_X \otimes_{\mathcal{O}_X} V$$

$$D''_V = d'' \otimes \mathrm{Id} : H \longrightarrow \mathcal{E}^{(0,1)}_X \otimes_{\mathcal{C}^\infty_X} H.$$

We therefore have $V = \operatorname{Ker} D''_V$ and $\nabla = D'_V\big|_V$ Moreover, D_V has curvature 0.

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[(1)]In this article, vector bundle will mean complex vector bundle, and metric will mean Hermitian metric.

Conversely, given a C^∞-bundle H with a *flat* connection D_V, we have $(D''_V)^2 = 0$, hence, by a standard result, $V \overset{\text{def}}{=} \operatorname{Ker} D''_V$ is a holomorphic subbundle of H. As $D'_V D''_V + D''_V D'_V = 0$, the $(1,0)$-part D'_V of D_V induces a holomorphic connection on V, which is flat because $(D'_V)^2 = 0$.

I will mainly explain some analytic tools concerning these objects, mainly the notion of a *harmonic metric*.

1. Harmonic metrics

We would like to compute the cohomology $H^*(X,\mathcal{L})$ of X with twisted coefficients in $\mathcal{L}$ with harmonic forms. In order to define the Laplacian, one needs a metric to measure the length of sections of $\mathcal{L}$.

1.a. Unitary representations. The best situation is when there exists a metric h on the C^∞-bundle $H = \mathcal{C}^\infty_X \otimes_{\mathcal{O}_X} V$ associated to V so that the connection D_V is the *Chern connection*, *i.e.* D''_V defines the holomorphic structure and, for any local sections u, v of H,

$$d'h(u,v) = h(D'_V u, v) + h(u, D''_V v) \quad \text{and} \quad d''h(u,v) = h(D''_V u, v) + h(u, D'_V v).$$

This means that the metric h is *flat*. The existence of such a metric is equivalent to the fact that the representation ρ is conjugate to a *unitary* representation $\rho' : \pi_1(X,\star) \to \mathrm{U}(d,\mathbb{C})$.

Let ω be a positive $(1,1)$-form on X (*i.e.* a metric on TX). If X is compact, then $H^{p,q}(X,H) = \mathrm{Harm}^{p,q}(H)$ and, if moreover (X,ω) is Kähler, we have the Hodge decomposition $H^k(X,\mathcal{L}) = \oplus_{p\geqslant 0} H^{p,q}(X,H)$ and the Hard Lefschetz Theorem which says that, for any $k \geqslant 1$, $\wedge^k\omega : H^{\dim X - k}(X,\mathcal{L}) \to H^{\dim X + k}(X,\mathcal{L})$ is an isomorphism.

1.b. Definition of a harmonic metric. Let (H, D_V) be a C^∞-bundle on X with a *flat* connection D_V. If the associated representation $\rho : \pi_1(X,\star) \to \mathrm{GL}_d(\mathbb{C})$ is not unitary, there does not exist a metric on H for which D_V is the Chern connection. We wish to find a metric which is as good as possible for D_V.

Let h be any Hermitian metric on H.

Lemma 1.1. *There exists a unique metric connection $D_E = D'_E + D''_E$ on H such that, if we put*

$$\theta'_E = D'_V - D'_E \quad ((1,0)\text{-form with values in } \operatorname{End}(H))$$
$$\theta''_E = D''_V - D''_E \quad ((0,1)\text{-form with values in } \operatorname{End}(H)),$$

then θ''_E is the h-adjoint of θ'_E, i.e. *for any local sections u, v of H, $h(\theta'_E u, v) = h(u, \theta''_E v)$.*

Proof. Easy. □

We have the following relations:

$$d'h(u,v) = h(D'_E u, v) + h(u, D''_E v),$$
$$d''h(u,v) = h(D''_E u, v) + h(u, D'_E v),$$
$$h(\theta' u, v) = h(u, \theta'' v),$$
$$D'_V = D'_E + \theta'_E, \quad D''_V = D''_E + \theta''_E.$$

Notice that, by applying d' or d'' to each of the first three lines above, we see that D''^2_E is adjoint to D'^2_E, $D''(\theta')$ is adjoint to $D'(\theta'')$ and $D'_E D''_E + D''_E D'_E$ is selfadjoint with respect to h.

Definition 1.2. The triple (H, D_V, h) (or (V, ∇, h), or simply h, if (V, ∇) is fixed) is said to be *harmonic* if the operator $D''_E + \theta'_E$ has square 0, *i.e.* the *pseudo-curvature* $R_h = (D''_E + \theta'_E)^2$ vanishes.

By looking at types, this is equivalent to

$$D''^2_E = 0, \quad D''_E(\theta'_E) = 0, \quad \theta'_E \wedge \theta'_E = 0.$$

By adjunction, this implies

$$D'^2_E = 0, \quad D'(\theta'') = 0, \quad \theta''_E \wedge \theta''_E = 0.$$

Moreover, the flatness of D_V implies then

$$D'_E(\theta'_E) = 0, \quad D''_E(\theta''_E) = 0, \quad D'_E D''_E + D''_E D'_E = -(\theta'_E\theta''_E + \theta''_E\theta'_E).$$

Let $E = \operatorname{Ker} D''_E : H \to H$. This is a holomorphic vector bundle equipped with a holomorphic $\operatorname{End}(E)$-valued 1-form θ'_E satisfying $\theta'_E \wedge \theta'_E = 0$. It is called a *Higgs bundle* and θ'_E is its *associated Higgs field*.

Examples 1.3

(1) If the metric h is flat and D_V is the Chern connection, then $D_V = D_E$, $V = E$ and $\theta_E = 0$.

(2) Let $H = \oplus_{p\in\mathbb{Z}} H^{p,w-p}$ be a C^∞ vector bundle on X, where $w \in \mathbb{Z}$ is fixed, equipped with a flat connection $D_V = D'_V + D''_V$ and a flat nondegenerate Hermitian bilinear form k such that the direct sum decomposition of H is k-orthogonal, $(-1)^p i^{-w} k$ is a metric on $H^{p,w-p}$, *i.e.* $(-1)^p i^{-w} k$ is positive definite on the fibers of $H^{p,w-p}$ for each p, and

$$\begin{aligned} D'_V(H^{p,w-p}) &\subset \big(H^{p,w-p} \oplus H^{p-1,w-p+1}\big) \otimes_{\mathcal{C}^\infty_X} \mathcal{E}^{(1,0)}_X \\ D''_V(H^{p,w-p}) &\subset \big(H^{p,w-p} \oplus H^{p+1,w-p-1}\big) \otimes_{\mathcal{C}^\infty_X} \mathcal{E}^{(0,1)}_X. \end{aligned}$$

Denote by $D'_V = D'_E + \theta'_E$ and $D''_V = D''_E + \theta''_E$ the corresponding decomposition. Then the metric h defined as $(-1)^p i^{-w} k$ on $H^{p,w-p}$ and such that the direct sum decomposition of H is h-orthogonal is a harmonic metric and the objects D'_E, D''_E, θ'_E and θ''_E are the one associated with (h, D_V) by Lemma 1.1.

1.c. Harmonic theory for harmonic metrics. If (X, ω) is a *Kähler manifold* of dimension n, one may develop harmonic theory for a harmonic metric. Let (H, D_V, h) be a harmonic bundle on X as above, with associated operators D'_E, D''_E, θ'_E and θ''_E. Put $\mathfrak{D}_\infty = D'_E + \theta''_E$ and $\mathfrak{D}_0 = D''_E + \theta'_E$, so that $D_V = \mathfrak{D}_\infty + \mathfrak{D}_0$. The main observation of C. Simpson [**6**, § 2] is that the Kähler identities

$$\Delta_{D_V} = 2\Delta_{\mathfrak{D}_\infty} = 2\Delta_{\mathfrak{D}_0},$$

are satisfied for the Laplacian, and that the Lefschetz operator $L = \omega\wedge$ commutes with these Laplacians. However, as $\mathfrak{D}_\infty, \mathfrak{D}_0$ are not of pure type $(1, 0)$ or $(0, 1)$, one does not have a Hodge decomposition, in general.

It follows from classical Hodge Theory that, when X is compact, $H^k(X, \mathcal{L})$ is equal to the space of harmonic sections $\operatorname{Harm}^k(H)$ and that Hard Lefschetz Theorem holds:

$$\omega^k \wedge\, : H^{n-k}(X, \mathcal{L}) \xrightarrow{\ \sim\ } H^{n+k}(X, \mathcal{L}).$$

If X is noncompact, the space of harmonic forms computes a L^2-cohomology space. If the metric h is well controlled at infinity on X, this will be the intersection cohomology $IH^*(\overline{X}, \mathcal{L})$ for some compactification $\overline{X}$ of X.

2. Existence of a harmonic metric

The main problem concerns the existence of a harmonic metric on a flat holomorphic bundle.

2.a. The theorem of K. Corlette (compact case). Let X be a compact Kähler manifold and let (V,∇) be a holomorphic bundle equipped with a flat connection. Define H and D_V as above.

Theorem 2.1 **(K. Corlette [2], C. Simpson [6])**. *There exists a harmonic metric on (H, D_V) if and only if the representation associated to (V,∇) is semisimple. In such a case, the harmonic metric is essentially unique.*

Brief indication of proof. Given any metric h, consider the associated operators D_E, θ_E. Given any C^∞ automorphism φ of H, there is an associated connection $\varphi \circ D_V \circ \varphi^{-1}$ and metric h_φ. One can construct operators $D_E^{(\varphi)}$ and $\theta_E^{(\varphi)}$.

The first observation is that the metric is harmonic if and only if the energy map $\varphi \mapsto \|\theta_E^{(\varphi)}\|^2$ has a critical point at $\varphi = \mathrm{Id}$.

If ψ is an endomorphism of H which is selfadjoint with respect to h, consider the family of automorphisms $\varphi_t = e^{t\psi}$, for $t \in \mathbb{R}$, and put $f_\psi(t) = \|\theta_E^{(\varphi_t)}\|^2$. Then, if $f'_\psi(0) = 0$, Corlette shows that $f''_\psi(t) > 0$ for all $t \in \mathbb{R}$, *i.e.* the function f_ψ is strictly convex.

Assume that (V, D_V) is not semisimple and that h is a harmonic metric. There exists an exact sequence

$$0 \longrightarrow (V_1,\nabla) \longrightarrow (V,\nabla) \longrightarrow (V_2,\nabla) \longrightarrow 0$$

which is *not* split. Put $d_1 = \operatorname{rk} V_1$, $d_2 = \operatorname{rk} V_2$. Consider the endomorphism ψ of H equal to $d_2 \cdot p_1 - d_1 \cdot p_2$, where p_1, p_2 are the h-orthogonal projections on H_1 and $H_1^\perp$. Write

$$D_V = \begin{pmatrix} D_{V_1} & \eta \\ 0 & D_{V_2} \end{pmatrix}$$

with $\eta \neq 0$ a one-form with values in $\operatorname{Hom}(V_2, V_1)$. Then a simple computation gives

$$\theta_E^{(\varphi_t)} = \begin{pmatrix} \theta_{E_1} & \frac{1}{2} e^{t(d_1+d_2)}\eta \\ \frac{1}{2} e^{t(d_1+d_2)}\eta^* & \theta_{E_2} \end{pmatrix}$$

and has a finite limit when $t \to -\infty$. By strict convexity, $f_\psi(t)$ cannot have a critical point, a contradiction.

The converse result, namely the existence of a harmonic metric when (V,∇) is semisimple, is far more difficult. The strict convexity property above also gives the uniqueness. □

Corollary 2.2. *Let $f : Y \to X$ be a holomorphic mapping between compact Kähler manifolds. If $\mathcal{L}$ is a semisimple local system on X, then $f^*\mathcal{L}$ is a semisimple local system on Y.*

This result mean that, if $\rho : \pi_1(X, \star) \to \mathrm{GL}_d(\mathbb{C})$ is an irreducible representation, then $\rho \circ f_*\pi_1(Y, \star) \to \mathrm{GL}_d(\mathbb{C})$ is still semisimple. In some sense, from the point of view of representations, $\pi_1(Y, \star)$ is "bigger" than $\pi_1(X, \star)$.

Proof. Indeed, if h be a metric on (V,∇), then the metric f^*h on f^*V has pseudo-curvature f^*R_h. Consequently, if h is harmonic, then f^*h also, so f^*V is semisimple. □

2.b. A theorem of C. Simpson and its generalizations (quasi-projective case)

The first result is due to C. Simpson and concerns vector bundle with connections on a Riemann surface. We need to fix a model metric near the singularities.

Let $\widetilde{X}$ be a compact Riemann surface, and let X be the complement of a finite set Σ of points. Let $\widetilde{V}$ be a holomorphic vector bundle on $\widetilde{X}$, put $V = \widetilde{V}_{|X}$ and $\nabla : \widetilde{V} \to \Omega^1_{\widetilde{X}} \otimes_{\mathcal{O}_{\widetilde{X}}} \widetilde{V}$ be a connection with logarithmic poles. Assume that, near each singular point, with local coordinate t, there exists a basis of $\widetilde{V}$ for which the matrix of ∇ is $A dt/t$ where A is constant. Put A into Jordan normal form and let $\mathcal{H}$ be the corresponding diagonal weight matrix: it is decomposed in diagonal blocks following the decomposition of A and for a Jordan block of A of size $k+1$, the corresponding block of $\mathcal{H}$ is $\operatorname{diag}(k, k-2, \cdots, -k+2, -k)$. Near the singular point, a basis vector of $\widetilde{V}$ which is an eigenvector of A with eigenvalue $\alpha \in \mathbb{C}$ and weight $w \in \mathbb{Z}$ should have norm $|t|^{\operatorname{Re}\alpha} \left|\log t\bar{t}\right|^{w/2}$ in the local model metric.

***Theorem 2.3* (C. Simpson [5]).** *If* $(\widetilde{V}, \nabla)$ *is irreducible (or semisimple),* i.e. *if the representation* $\pi_1(X, \star) \to \mathrm{GL}_d(\mathbb{C})$ *defined by* (V, ∇) *is irreducible (or semisimple), then there exists a harmonic metric on* (V, ∇)*, which is comparable to the local model metric near each singularity.*

This result has been extended in various directions:

(1) for local systems on the complement of a smooth divisor on a Kähler manifold by O. Biquard [**1**];

(2) for certain kind of local systems on the complement of a divisor with normal crossings on a Kähler manifold by Jost/Zuo [**3**];

(3) for connections with possible irregular singularities on a Riemann surface by C. Sabbah [**4**]; here, it is assumed that the bundle $(\widetilde{V}, \nabla)$ is irreducible (or semisimple), which is implied by, but *not* equivalent to (due to irregular singularities) the irreducibility or semisimplicity of the associated representation.

2.c. A nonisomonodromic deformation associated with a harmonic metric. Let us go back to the general situation. One can consider, for any nonzero complex number $\hbar$, the two operators $D''_E + \hbar\theta''_E$ and $\hbar D'_E + \theta'_E$ on the C^∞ vector bundle H associated to V. If h is a harmonic metric on (V, ∇), the various relations given after Definition 1.2 imply that $D''_E + \hbar\theta''_E$ has square zero, hence, by a classical integrability result, defines a new holomorphic structure on H. We get therefore a new holomorphic vector bundle $V_\hbar$ so that $V_1 = V$, and $\hbar D'_E + \theta'_E$ induces a flat holomorphic connection $\nabla_\hbar$ on it.

In general, this one-parameter deformation of (V, ∇) is nonconstant:

***Lemma 2.4*.** *This deformation of* (V, ∇) *is constant if and only if* (V, ∇, h) *is like in example 1.3*(2). □

Moreover, this deformation is not isomonodromic in general. In the situation of Theorem 2.3, Simpson also shows that each $V_\hbar$ can be extended as a bundle $\widetilde{V}_\hbar$ on $\widetilde{X}$ and $\nabla_\hbar$ is logarithmic with respect to this extension. Results of O. Biquard in [**1**] allow to control very precisely the local behaviour of $\nabla_\hbar$ near the singularities. In particular, the deformation $(V_\hbar, \nabla_\hbar)_{\hbar \in \mathbb{C}^*}$ is not locally isomonodromic near a singularity: if $e^{2i\pi\alpha}$ is an eigenvalue of the local monodromy of (V, ∇) with $\alpha = \alpha' + i\alpha'' \in \mathbb{C}$, then $\exp\big(2i\pi(\hbar\alpha' + i(\hbar^2+1)\alpha''/2)\big)$ is an eigenvalue of $(V_\hbar, \nabla_\hbar)$.

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