HODGE THEORY OF THE MIDDLE CONVOLUTION
(JOINT WORK WITH MICHAEL DETTWEILER)

MANNHEIM, NOVEMBER 8, 2012

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Abstract. We compute the behaviour of Hodge data by tensor product with a unitary rank-one local system and middle convolution by a Kummer unitary rank-one local system for an irreducible variation of polarized complex Hodge structure on a punctured complex affine line.

1. Introduction

Let $\mathscr{V}$ be a $\mathbb{C}$-local system of rank n on $X := \mathbb{P}^1 \smallsetminus \{x_1, \dots, x_r, x_{r+1} = \infty\}$. We say that $\mathscr{V}$ is *physically rigid* if any local system $\mathscr{V}'$ which is *locally* isomorphic to $\mathscr{V}$ is also *globally* isomorphic to $\mathscr{V}$. In other words, given a family of matrices $T_1, \dots, T_r$ in $\mathrm{GL}_n(\mathbb{C})$, and setting $T_{r+1} = (T_1 \cdots T_r)^{-1}$, we say that it is physically rigid if for any other family $(T'_1, \dots, T'_r)$ such that T'_i is conjugate to T_i for each $i = 1, \dots, r+1$, there exists a common conjugation matrix C such that $T'_i = CT_iC^{-1}$.

N. Katz [**?**] has introduced a way to measuring the distance to physical rigidity, called *index of rigidity*, defined as follows:

$$\operatorname{rig} \mathscr{V} = \chi(\mathbb{P}^1, j_* \mathscr{E}nd(\mathscr{V})) \quad j : X \hookrightarrow \mathbb{P}^1,$$

$$\operatorname{rig}(T_1, \dots, T_r) = \sum_{i=1}^{r} \dim Z(T_i) + \dim Z(T_\infty) - (r-1)n^2,$$

where $Z(T_i)$ is the centralizer of T_i, that is, the space of matrices which commute with T_i. Let me recall:

***Theorem* (N. Katz [?]).** *Let $\mathscr{V}$ be an irreducible local system on X. Then* $\operatorname{rig} \mathscr{V}$ *is an even integer $\leqslant 2$, and $\mathscr{V}$ is rigid if and only if* $\operatorname{rig} \mathscr{V} = 2$.

There are two basic operations on local systems that one considers:

- The twist by a rank-one local system defined by its local monodromy data $\lambda_1, \dots, \lambda_r, \lambda_{r+1} = 1/\lambda_1 \cdots \lambda_r$.
- The middle convolution MC_χ by a rank-one local system on $\mathbb{C}^*$, with monodromy χ (to be explained later).

Katz has shown that they both preserve irreducibility and do not change rig (this is obvious for the first one), and he has reduced the study of irreducible and rigid local systems to rank one local systems (which are also irreducible and rigid in a tautological way) by the so-called "Katz algorithm".

***Theorem* (N. Katz [?]).** *Let $\mathscr{V}$ be an irreducible local system such that T_{r+1} is scalar. Then $\mathscr{V}$ is rigid if and only if there exists a suitable sequence of middle convolutions and twists by rank-one local systems such that, after applying in a suitable way these operations to $\mathscr{V}$, one finds a rank-one local system.*

Concerning Hodge theory, we have the following results:

Theorem.

(1) *(Deligne, [?]) Any irreducible local system $\mathscr{V}$ on X underlies at most one (up to a shift of the Hodge filtration) variation of polarized complex Hodge structure.*

(2) *(Simpson, [?]) An irreducible and rigid local system on X underlies a variation of polarized complex Hodge structure (unique up to a shift, according to (1)) if and only if the eigenvalues of the monodromies $T_1, \dots, T_{r+1}$ have absolute value equal to one.*

What is the relation of Simpson's theorem with the Katz algorithm? Firstly, the χ's and λ_i's entering in the Katz algorithms are nothing but monomials in the eigenvalues of $T_1, \dots, T_{r+1}$. Therefore, that the eigenvalues have absolute value equal to one is preserved all along the Katz algorithm. If the eigenvalues of the T_i's have absolute value equal to one, the rank-one local system obtained as the abutment of the algorithm is a *unitary local system*, in particular a variation of polarized complex Hodge structure. It is thus tempting to follow the behaviour of the VHS all along the algorithm and give precise formulas for various Hodge-type invariants.

2. The numerical Hodge invariants

Let $\mathscr{V}$ be a local system on X, let (V, ∇) be the associated holomorphic (flat) bundle, and let $(H, D) = (\mathscr{C}^\infty \otimes V, \nabla + \overline{\partial})$ be the associated C^∞ flat bundle. A variation of polarized complex Hodge structure on (H, D) (of weight 0) consists of a C^∞ decomposition $H = \oplus_p H^p$ and a Hermitian metric h on H such that

(1) $F^pH := \oplus_{p' \geqslant p} H^{p}$ is a holomorphic subbundle (that is, $\overline{\partial} F^pH \subset F^pH \otimes \mathscr{A}_X^{0,1}$) and the corresponding holomorphic bundle $F^pV := F^pH \cap \operatorname{Ker} \overline{\partial}$ satisfies $\nabla F^pV \subset F^{p-1}V$,

(2) the decomposition is h-orthogonal,

(3) the sesquilinear form $S = h(C\bullet, \bullet)$, $C =$ Weil operator $(-1)^p$ on H^p, is Hermitian non degenerate and D-flat.

Local Hodge data

- $h^p(V) = \operatorname{rk} H^p = \operatorname{rk} \operatorname{gr}_F^p V$,
- (Nearby cycles) $\nu^p_{x_i,\lambda}(V) = \dim \operatorname{gr}_F^p \psi_{x_i,\lambda}(V)$,
- (Vanishing cycles with eigenvalue $\lambda \neq 1$) $\mu^p_{x_i,\lambda}(V) = \nu^p_{x_i,\lambda}(V)$,
- (Vanishing cycles with eigenvalue $\lambda = 1$) $\mu^p_{x_i,1}(V) = \dim \mathrm{N}(\operatorname{gr}_F^p \psi_{x_i,1}(V))$, $\mathrm{N} =$ nilp. part of monodr. on $\psi_{x_i,1}(V)$.

The vanishing cycles correspond to those of the intermediate extension $j_*\mathscr{V}$.

2.1. Global Hodge data. According to Schmid, one can extend each F^pV as a vector bundle F^p on $\mathbb{P}^1$ by setting $F^p = j_*F^pV \cap V^0$, where V^0 is the Deligne logarithmic extension on which the eigenvalues of the residue of the connection belong to $[0, 1)$.

- $\delta^p(V) = \deg F^p/F^{p+1}$.

3. The theorem

Let $\mathscr{V}$ be an irreducible non constant local system on X, which underlies a variation of polarized complex Hodge structure.

Proposition. *With these assumptions,* $\mathrm{MC}_\chi(\mathscr{V})$ *underlies a natural variation of polarizable complex Hodge structure, and if* $\chi, \chi' \neq 1$ *and* $\chi = e^{-2\pi i\alpha_o}$ *with* $\alpha_o \in (0, 1)$ *(and similarly for* χ'*),*

$$\mathrm{MC}_{\chi'}\,\mathrm{MC}_\chi(\mathscr{V}) \simeq \begin{cases} \mathrm{MC}_{\chi'\chi}(\mathscr{V})(-1) & \text{if } \alpha_o + \alpha'_o \in (0, 1], \\ \mathrm{MC}_{\chi'\chi}(\mathscr{V}) & \text{if } \alpha_o + \alpha'_o \in (1, 2). \end{cases}$$

Theorem. *Assume moreover that* $T_\infty = \lambda_o \operatorname{Id}$*, and choose* $\chi = \lambda_o$*. Then*

(1) $h^p(\mathrm{MC}_\chi(\mathscr{V})) = \delta^{p-1}(V) - \delta^p(V) - h^p(V) + \sum_{i=1}^r \big(\mu^{p-1}_{x_i,\neq 1}(V) + \mu^p_{x_i,1}(M)\big)$,

(2) *Set* $\lambda_o = e^{-2\pi i\alpha_o}$ *with* $\alpha_o \in (0, 1)$*. For* $i = 1, \dots, r$ *and* $\lambda = e^{-2\pi i\alpha} \in S^1$ *we have*

$$\mu^p_{x_i,\lambda}(\mathrm{MC}_\chi(\mathscr{V})) = \begin{cases} \mu^{p-1}_{x_i,\lambda/\lambda_o}(\mathscr{V}) & \text{if } \alpha \in (\alpha_o, 1) \cup \{0\}, \\ \mu^p_{x_i,\lambda/\lambda_o}(\mathscr{V}) & \text{if } \alpha \in (0, \alpha_o]. \end{cases}$$

(3) *With the same assumptions, we have*

$$\delta^p(\mathrm{MC}_\chi(\mathscr{V})) = \delta^p(\mathscr{V}) + h^p(\mathscr{V}) - \sum_{i=1}^r \Big(\mu^p_{x_i,1}(\mathscr{V}) + \sum_{\alpha \in (0,1-\alpha_o)} \mu^{p-1}_{x_i,\lambda}(\mathscr{V})\Big).$$

4. Applications to G_2 examples

We choose $r = 3$ (and we work with $r + 1 = 4$). M. Dettweiler and S. Reiter [?] have classified irreducible rigid local systems of rank 7 such that the Zariski closure of the group generated by T_1, T_2, T_3 in GL_7 is the group G_2 (defined as the subgroup leaving invariant a trilinear form called the Dickson form and a nondegenerate bilinear form). Here is an example, obtained by the Katz algorithm: By applying to the trivial family $(1, 1, 1, 1)$ of size one, the following sequence ($\mathrm{j} = \sqrt[3]{1}$)

$$\begin{aligned}&\mathrm{H}_{-1,1,\bar{\mathrm{j}},-\mathrm{j}} \circ \mathrm{MC}_{-\mathrm{j}} \circ \mathrm{H}_{1,-\bar{\mathrm{j}},-\mathrm{j},1} \circ \mathrm{MC}_{-\bar{\mathrm{j}}} \circ \mathrm{H}_{-1,1,-\mathrm{j},\bar{\mathrm{j}}} \circ \mathrm{MC}_{-1} \circ \\ &\qquad \circ \mathrm{H}_{1,-\bar{\mathrm{j}},-\mathrm{j},1} \circ \mathrm{MC}_{-1} \circ \mathrm{H}_{1,-1,-1,1} \circ \mathrm{MC}_{-\mathrm{j}} \circ \mathrm{H}_{1,-1,-1,1} \circ \mathrm{MC}_{-\bar{\mathrm{j}}} \circ \mathrm{H}_{-1,-\bar{\mathrm{j}},-\bar{\mathrm{j}},-\bar{\mathrm{j}}},\end{aligned}$$

they get such an example.

Then the algorithm gives the Hodge invariants. The variation of Hodge structure is real. In particular the Hodge filtration has length three, which is the minimal possible among the variation of polarized real Hodge structure for an irreducible G_2-example.

In fact, the possible lengths of the Hodge filtration of an irreducible G_2-example have been classified by Green, Griffiths and Kerr (cf. [?, ?]).

In particular there are irreducible G_2-example underlying a variation of polarized $\mathbb{Q}$-Hodge structure whose Hodge filtration has maximal length seven. However, such an example, which is motivic, is not rigid, but orthogonally rigid.

5. Definition of the middle convolution

A sheaf-theoretic definition, [?]**.** Consider the diagram

$$\begin{tikzcd} & X \times \mathbb{C}^* \arrow[r, "s"] \arrow[dl, "p"'] \arrow[dr, "p'"] & \mathbb{C} \\ X & & \mathbb{C}^* \end{tikzcd}$$

where s is the sum map $s(x, x') = x + x'$, and let $\mathscr{L}_\chi$ be the local system on $\mathbb{C}^*$ with monodromy χ around the origin.We have two convolutions:

$$\mathscr{V} \star_! \mathscr{L}_\chi := \boldsymbol{R}s_!(p^{-1}\mathscr{V} \otimes p'^{-1}\mathscr{L}_\chi), \quad \mathscr{V} \star_* \mathscr{L}_\chi := \boldsymbol{R}s_*(p^{-1}\mathscr{V} \otimes p'^{-1}\mathscr{L}_\chi)$$

and a natural morphism $\mathscr{V} \star_! \mathscr{L}_\chi \to \mathscr{V} \star_* \mathscr{L}_\chi$. The middle convolution $\mathrm{MC}_\chi(\mathscr{V})$ is the cone of this morphism. one can prove that this is a perverse sheaf (when suitably shifted) on $\mathbb{C}$. This presentation is used to handle Hodge structures, through a Thom-Sebastiani theorem of M. Saito [?].

5.1. A naive definition, [?]. Given $T_1, \dots, T_r \in \mathrm{GL}_n(\mathbb{C})$ and $\chi \in \mathbb{C}^*$, we define invertible matrices $S_i^{(\chi)} \in \mathrm{GL}_{rn}(\mathbb{C})$ as

$$S_i^{(\chi)} = \begin{pmatrix} \mathrm{Id}_n & 0 & \dots & \dots & \dots & 0 \\ & \ddots & & & & \\ T_1 - \mathrm{Id}_n & \dots & \chi T_i & \chi(T_{i+1} - \mathrm{Id}_n) & \dots & \chi(T_r - \mathrm{Id}_n) \\ & & & \ddots & & \\ & & & & \ddots & \\ 0 & & \dots & & & \mathrm{Id}_n \end{pmatrix}$$

and two subspaces of $\mathbb{C}^{rn}$

$$\mathscr{K} = \begin{pmatrix} \mathrm{Ker}(T_1 - \mathrm{Id}_n) \\ \vdots \\ \mathrm{Ker}(T_r - \mathrm{Id}_n) \end{pmatrix}, \qquad \mathscr{L}_\chi = \bigcap_{i=1}^{r} \mathrm{Ker}(S_i^{(\chi)} - \mathrm{Id}_{rn}).$$

Both subspaces are preserved under $S_i^{(\chi)}$, hence $S_i^{(\chi)}$ induce automorphisms of $\mathbb{C}^{rn}/(\mathscr{K} + \mathscr{L}_\chi)$ denoted by $T_i^{(\chi)}$. The family $(T_1^{(\chi)}, \dots, T_r^{(\chi)})$ is called *middle convolution of* $(T_1, \dots, T_r)$ *with* χ, and denoted by $\mathrm{MC}_\chi(T_1, \dots, T_r)$