SEMISIMPLE PERVERSE SHEAVES AND A CONJECTURE OF KASHIWARA

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Contents

1. Harmonic flat bundles

Let X be a compact Kähler manifold. A theorem of Corlette [**2**] associates to any irreducible representation of the fundamental group $\pi_1(X,*)$ of X, that is, to any irreducible flat holomorphic bundle (V,∇), a unique (up to multiplicative constant) Hermitian metric which is *harmonic*. If the representation is only assumed to be semisimple, the associated bundle also admits such a metric, but the uniqueness statement has to be adapted in an obvious way.

A striking aspect of this correspondence is that a global property is characterized by the existence of an object defined by local properties. From a physical point of view, the simplicity is like a long distance interaction and the harmonic metric is like a field uniquely determined from the interaction, which propagates it.

This theorem has many consequences. Let me quote some of them.

(1) If Y is any compact complex submanifold in X, then a semisimple representation of $\pi_1(X)$ restricts to a semisimple representation of $\pi_1(Y)$ [because the harmonicity property restricts to submanifolds].

(2) One can compute cohomology $H^k(X, \operatorname{Ker}\nabla)$ with harmonic forms, as if the representation were unitary, and one has a Hard Lefschetz theorem for the cohomology with coefficients in the local system $\operatorname{Ker}\nabla$.

(3) If $f : X \to Y$ is a smooth projective morphism between compact Kähler manifolds and (V, ∇) is a semisimple flat holomorphic bundle on X, then for any k, $H^k(f^{-1}(y), \operatorname{Ker}\nabla)$ form a semisimple local system on Y. Moreover, a relative Hard Lefschetz theorem holds and, as a consequence, the complex $\boldsymbol{R}f_* \operatorname{Ker}\nabla$ decomposes as the direct sum $\oplus_k R^k f_* \operatorname{Ker}\nabla[-k]$.

2. The conjecture of Kashiwara

These results have led Kashiwara to conjecture that the smoothness of X or of f is not necessary, provided that one replaces "local systems" like $\operatorname{Ker}\nabla$ with "$\mathbb{C}$-perverse sheaves". This conjecture is now proved, in two different ways:

- in an arithmetic way, by Drinfeld [**3**], who uses a conjecture by de Jong now proved by Gaitsgory [**4**] and Boeckle-Khare [**1**],
- in an analytic way, using harmonic bundles on quasiprojective manifold, by T. Mochizuki [**5, 6**] and C.S. [**7**].

The purpose of this talk is to give an idea of the tools used in the analytic proof.

Note however that the story does not end there. Indeed, the original conjecture of Kashiwara concerns also semisimple *holonomic $\mathcal{D}$-modules* with possible *irregular singularities.* This raises the question of constructing over $\mathbb{C}$ a category similar to that of pure ℓ-adic perverse sheaves with wild ramification.

3. Notion of weight and polarization

Before giving a rough sketch of the analytic proof of the conjecture, let me raise some elementary questions. An important issue in the proof of this theorem is to define in a correct way a notion of weight, *i.e.* a category of objects with weights, which behave well by direct images. This good behaviour is usually associated with the existence of a polarization.

Let me consider the cohomology $H^j(X, \operatorname{Ker}\nabla)$, where (V, ∇) corresponds to a finite dimensional linear representation of the fundamental group of a compact Kähler manifold X. We have a priori neither a natural Hodge decomposition, nor a Hermitan metric, which could be positive definite on the primitive classes.

When the representation is irreducible, we can use the canonically defined harmonic metric to get a Laplacian, with respect to which we can identify $H^j(X, \operatorname{Ker}\nabla)$ with harmonic forms with values in V, and, as the Hard

Lefschetz theorem holds, everything can be recovered from primitive classes $PH^j(X, \operatorname{Ker}\nabla)$, on which the L^2-metric multiplied by i^{-j} is positive definite.

In dimension 0, polarized pure objects of weight $w \in \mathbb{Z}$ should therefore be

- complex vector spaces H with a Hermitian metric h such that $i^{-w}h$ is positive definite.

What about variations of such objects? If we have a Hodge structure $H = \oplus_{p+q=w} H^{p,q}$, we know that, in a variation, the Hermitian form h does not vary in a flat way. What varies in a flat way is the Hermitian form k defined as $(-1)^p h$ on $H^{p,q}$. Usually, k varies flatly because it is associated to Poincaré duality. What is the analogue of k?

4. Sketch of the analytic proof

Generalization of flat bundles with harmonic metrics (C.S.). One introduces the category of *polarizable regular twistor $\mathcal{D}$-modules* of weight w. At this point, the process is analogous to that of M. Saito, constructing a category of polarizable Hodge $\mathcal{D}$-modules.

One proves then the following two statements:

Theorem 4.1 **(C.S.)**

(1) *Kashiwara's conjecture (decomposition theorem) in this category;*

(2) *smooth polarizable twistor $\mathcal{D}$-modules of weight* 0 *correspond to harmonic flat bundles.*

Comparison result on a punctured Riemann surface. This mainly follows from Simpson's results [**8**]:

(C.S.) Polarized regular twistor $\mathcal{D}$-modules of weight 0 on a *compact* Riemann surface X correspond to *tame harmonic flat bundles* on $X^* = X \smallsetminus$singular points.

(T.M.) Polarized regular twistor $\mathcal{D}$-modules of weight 0 on an open disc X with singularity at the center correspond to *harmonic flat bundles* on X^* which are tame at the center of the disc.

Comparison result on the complement of a normal crossing divisor (T.M.). Let X be a product of open discs and let D be a union of coordinate hyperplanes. T. Mochizuki introduces the notion of harmonic flat bundle on $X^* = X \smallsetminus D$ which is *tame along D*.

Theorem **(Mochizuki).** *Polarized regular twistor $\mathcal{D}$-modules of weight* 0 *on X with singularities along D correspond to* harmonic flat bundles *on X^* which are tame along D.*

Global theory

Theorem. *Let X be a projective complex manifold and let D be a normal crossing divisor in X. Set $X^* = X \smallsetminus D$.*

$$\text{\textit{Semisimple representations of }}\pi_1(X^*)$$
$$\updownarrow$$
$$\text{\textit{harmonic flat bundles on }}X^*\text{\textit{ which are tame along }}D.$$

Proof

(Corlette 88): The case $D = \varnothing$.
(Simpson 90): The case $\dim X = 1$.
(Mochizuki 04 using results of Jost-Zuo): The general case. □

***Corollary 4.2* (C.S. 4.1(1) and Mochizuki 03).** *Let Z be an irreducible projective variety over $\mathbb{C}$. We have then an equivalence:*

Semisimple representations of $\pi_1(Z^o)$
(Z^o some smooth Zariski non empty open subset of Z)

$$\updownarrow$$

Semisimple perverse sheaves strictly supported by Z
($= \mathrm{IC}_Z(\mathcal{L})$, $\mathcal{L}$ semisimple local system on some Z^o)

$$\updownarrow$$

Polarized regular twistor $\mathcal{D}$-modules of weight 0 strictly supported by Z
($=$ tame harmonic flat bundle on some Z^o)

[Note that the upper equivalence is topological and known since the 80's.]

Proof of the conjecture of Kashiwara. Let $\mathcal{F}$ be a simple perverse sheaf on X. Then there exists an irreducible closed subvariety $Z \subset X$, a Zariski dense smooth open set Z^o and a local system $\mathcal{L}$ on Z^o such that $\mathcal{F} = \mathrm{IC}(\mathcal{L})$.

Let $f : X \to Y$ be a projective mapping. Let $\mathcal{T}$ be a polarized regular twistor $\mathcal{D}$-module on X corresponding to $\mathcal{F}$ (Cor. 4.2). Its direct image by f decomposes as the direct sum of polariezd regular twistor $\mathcal{D}$-modules (*cf.* 4.1(1)). Hence $Rf_*\mathcal{F}$ decomposes as the direct sum of (shifted) simple perverse sheaves. □

5. Variation of twistor structures and harmonic metrics

In the analytic proof of Kashiwara's conjecture, expressing the harmonic property of the metric in terms of *variation of twsitor structures* has proved extremely useful in order to handle with singularities. This language of twistor structures, suggested by Deligne and introduced in a systematic way by Simpson, enables one to mimick variations of Hodge structures, and thus to apply techniques of algebraic geometry to those C^∞ objects like flat bundles with harmonic metric.

The construction that I will explain now is local. So, let X be any complex manifold and let (V,∇) be a flat holomorphic bundle. Let H be the associated C^∞ bundle. We will have to consider various distinct holomorphic structures on H, one of them being V. We denote by D_V the flat C^∞-connection on H such that $V = \operatorname{Ker} D''_V$ and $\nabla = D'_V$ restricted to $\operatorname{Ker} D''_V$.

Given moreover a Hermitian metric h on H, there exist a unique *metric* connection D_E and a unique 1-form valued endomorphism θ_E such that $D_V = D_E + \theta_E$ and θ''_E is the h-adjoint of θ'_E.

Of course, if the metric is compatible with the flat connection, then $D_E = D_V$ is the Chern connection of the metric and $\theta_E = 0$.

The metrized flat bundle (H, D_V, h) is said to be *harmonic* if the curvature of the operator $D''_E + \theta'_E$ is zero. Combined with the flatness of D_V and the selfadjunction of θ_E, this leads to a bunch of relations which can be summarized as follows:

(1) For any complex number z, the d''-operator $\mathcal{D}''_z \overset{\text{def}}{=} D''_E + z\theta''_E$ is a holomorphic structure on H, *i.e.* $\operatorname{Ker}\mathcal{D}''_z$ is a holomorphic vector bundle. If $z = 1$, we recover V, and if $z = 0$, we denote it by E. For any $z \neq 0$, I denote it by V_z.

(2) For any z, the operator $\mathcal{D}'_z \overset{\text{def}}{=} zD'_E + \theta'_E$ has square 0 and commutes with $\mathcal{D}''_z$. Therefore,

- if $z \neq 0$, $D'_E + z^{-1}\theta'_E$ is a flat holomorphic connection on V_z,
- if $z = 0$, $\mathcal{D}'_0 = \theta'_E$ is a Higgs field on E.

If X is Kähler, we have the Kähler identities for the Laplacian attached to the operator $\mathcal{D}_z$:

$$\Delta_z = (1 + |z|^2)\Delta_0.$$

6. Rank one on a disc

It is instructive to consider the simplest case. Let X be a disc in $\mathbb{C}$ with coordinate t and let X^* be the punctured disc, which will be the base manifold for the bundle. Assume that (V,∇) has rank one, with harmonic metric h. Notice that V is holomorphically trivial.

Let v' be some holomorphic frame of V, with $\|v'\| = e^{\varphi}$, and

$$\nabla v' = \Big(2\partial_t g + \frac{\beta_1}{t}\Big) v'\, dt$$

with g holomorphic on X^* and $\beta_1 \in \mathbb{C}$. Consider the orthonormal frame $\varepsilon = e^{-\varphi}v'$ of H. Then

$$D'_V\varepsilon = \Big(2\partial_t g + \frac{\beta_1}{t} - \partial_t\varphi\Big)\varepsilon\, dt, \quad D''_V\varepsilon = -\partial_{\bar t}\varphi\varepsilon\, d\bar t,$$

and we have

$$\theta'_E\varepsilon = \Big(\partial_t g + \frac{\beta_1}{2t} - \partial_t\varphi\Big)\varepsilon\, dt, \qquad \theta''_E\varepsilon = \overline{\Big(\partial_t g + \frac{\beta_1}{2t} - \partial_t\varphi\Big)}\varepsilon\, d\bar t,$$
$$D'_E\varepsilon = \Big(\partial_t g + \frac{\beta_1}{2t}\Big)\varepsilon\, dt, \qquad D''_E\varepsilon = -\overline{\Big(\partial_t g + \frac{\beta_1}{2t}\Big)}\varepsilon\, d\bar t.$$

Therefore, the condition $D''_E(\theta'_E) = 0$ is equivalent to φ being harmonic.

If φ is harmonic on X^*, there exists (by working on the universal cover of X^*) a holomorphic function ψ on X^* and a real number c, such that $\varphi(t) = 2(\operatorname{Re}\psi(t) + c\log|t|)$. We have $\partial_t\varphi = \partial_t\psi + c/t$ and $\partial_{\bar t}\varphi = \overline{\partial_t\varphi}$. Then as above, we put $v = e^{-\psi}v'$, and g is replaced with $g - \psi$. So we can assume from the beginning that $\varphi = b_1\log|t|$, $b_1 \in \mathbb{R}$.

Let us consider the formulas

$$\begin{aligned}\mathfrak{b}(z) &= b_1 z + \overline{\beta}_1(1-z),\\ b_z &= \operatorname{Re}(\mathfrak{b}(z)),\\ \beta_z &= \frac{\beta_1 - b_1}{2} + \operatorname{Re}\beta_1 z - \frac{\overline{\beta}_1 - b_1}{2}z^2\end{aligned}$$

Let us put $v_z = e^{(\bar z-1)g-(z-1)\bar g}|t|^{\mathfrak{b}(z)}\varepsilon$. Then, for any z, v_z is a holomorphic frame of V_z and, in this frame,

$$\mathcal{D}'_z v_z = \Big((1+|z|^2)\partial_t g + \frac{\beta_z}{t}\Big)v_z\, dt.$$

Conclusion. *Let (V, ∇, h) be any harmonic flat bundle of rank one on X^*. Then*

(1) *the metric is moderate, i.e. the holomorphic sections of V with h-norm having a moderate growth at 0 form a locally free $\mathcal{O}_X[1/t]$-module $\widetilde V$ of rank one;*

(2) *we get locally free $\mathcal{O}_X[1/t]$-modules $\widetilde V_z$ for any $z \in \mathbb{C}$, with z-connection $\mathcal{D}_z$; moreover, the filtration by the order of growth is a filtration by free $\mathcal{O}_X$-modules;*

(3) *∇ extends as a meromorphic connection to $\widetilde V$ $\Leftrightarrow$ g is meromorphic $\Leftrightarrow$ $\exists\, z \in \mathbb{C}$ such that $\mathcal{D}'_z$ extends as a meromorphic z-connection on $\widetilde V_z$; for a given $z \in \mathbb{C}$, the jumps of this filtration take place at $b_z + \mathbb{Z}$;*

(4) *∇ has regular singularity (in rank one, $\Leftrightarrow$ pole of order one) $\Leftrightarrow$ g is holomorphic $\Leftrightarrow$ $\exists\, z \in \mathbb{C}$ such that $\mathcal{D}'_z$ has regular singularity $\Leftrightarrow$ the harmonic bundle is tame.*

7. Twistor conjugation

Let us introduce the following involution, called *twistor conjugation*. For $f(z)$ defined on $\mathbb{C}^*$, put $\overline{f(z)} = \overline{f(-1/\overline{z})}$, *i.e.* , the twistor conjugate of $\sum_{n\in\mathbb{Z}} f_n z^n$ is $\sum_n (-1)^n \overline{f}_n z^{-n}$. For instance, $\overline{z} = -1/z$.

For such a function f, its twistor real part "Re"(f) is defined as usual as $\frac{1}{2}(f+\overline{f})$.

Then the formula for β_z can also be written, if $z \neq 0$, as

$$\frac{\beta_z}{z} = \text{“Re”}(\mathfrak{b}(z)).$$

Conclusion. *The set of isomorphism classes of harmonic flat bundle on the punctured disc is in one-to-one correspondence with the set of pairs $(g, \mathfrak{b})$, with $g \in \mathcal{O}_{X^*,0}$ and $\mathfrak{b}$ is an equivalence class of complex affine forms $z \mapsto uz + v$ modulo addition of $\mathbb{Z} + i\mathbb{R}$.*

8. Variation of polarized twistor structure

It is possible to give z the role of a true coordinate, by considering the product $X \times \mathbb{C}$ and the projection $\pi : X \times \mathbb{C} \to \mathbb{C}$. Then the C^∞ bundle $\pi^* H$ comes equipped with a d''-operator $\overline{\partial}_z + \mathcal{D}''_z$, and its kernel $\mathcal{H}'$ is a holomorphic bundle, equipped with a flat z-connection $\mathcal{D}'_z$. A priori, there is no covariant derivative defined along the vector field ∂_z.

I could have abruptly begun this talk in a axiomatic style by the following

***Definition* ([9]).** A *twistor structure* is a holomorphic bundle on $\mathbb{P}^1$. It is pure of weight w if the bundle is semistable of slope w, that is, isomorphic to a power of $\mathcal{O}_{\mathbb{P}^1}(w)$.

The category of pure twistor structures (the morphisms are all morphisms of vector bundles) share two properties with Hodge structures:

- the full subcategory of twistor structures of a given weight is abelian,
- there is no nonzero morphism from weight w to weight $w' < w$.

However, I would like to emphasize a more precise definition, which includes an analogue of the Hermitian form k which will behave flatly in famililes.

I consider $\mathbb{P}^1$ covered by two charts Ω_0, Ω_∞, z being the affine coordinate on Ω_0. I regard the twistor conjugation a a functor which transforms a vector bundle on Ω_0 into a vector bundle on Ω_∞.

Definition 8.1. A pure twistor structure of weight w consists of a triple $\mathcal{T} = (\mathcal{H}', \mathcal{H}'', C)$, where $\mathcal{H}', \mathcal{H}''$ are holomorphic vector bundles on Ω_0 and C is a sesquilinear pairing $\mathcal{H}' \otimes_{\mathcal{O}_\mathbf{S}} \overline{\mathcal{H}''} \to \mathcal{O}_\mathbf{S}$ such that

(1) C is nondegenerate, *i.e.* defines a gluing $\mathcal{H}'^{\vee}$ with $\overline{\mathcal{H}''}$, giving rise to a vector bundle $\widetilde{\mathcal{H}}$ on $\mathbb{P}^1$,

(2) the vector bundle $\widetilde{\mathcal{H}}$ is semistable with slope w.

The Tate twist is defined for any $k \in \frac{1}{2}\mathbb{Z}$ by

$$\mathcal{T}(k) = (\mathcal{H}', \mathcal{H}'', (iz)^{-2k}C).$$

If $\mathcal{T}$ is pure of weight w, then $\mathcal{T}(k)$ is pure of weight $w - 2k$.

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