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IRREGULAR HODGE THEORY AND PERIODS

LUMINY, NOVEMBER 28 – DECEMBER 2, 2022

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Abstract. These notes explain a series of joint works with Javier Fresán and Jeng-Daw Yu [**7, 8, 9**], motivated by conjectures made by Broadhurst and Roberts on arithmetic properties of moments of Bessel functions [**1, 4, 2, 3, 10, 5, 6**]. The purpose is to introduce the notion of irregular Hodge filtration, in the special case of an exponential mixed Hodge structure, and to illustrate the interest of considering this notion for computing Hodge filtrations of mixed Hodge structures related with Bessel moments. A Betti variant of this method is also introduced, in order to compute explicitly a period matrix of a pure motive associated to Bessel moments.

CONTENTS

LECTURE 1

INTRODUCTION, MOTIVATIONS, RESULTS

1.1. Periods and quadratic relations

Let X be a smooth complex quasi-projective variety of dimension n. The *comparison isomorphism* (in the middle dimension, say) is

$$\mathrm{H}^n_{\mathrm{dR}}(X) \overset{\text{comp}}{\simeq} \mathrm{H}^n(X,\mathbb{Q})\otimes\mathbb{C}.$$

If X is defined over $\mathbb{Q}$, $\mathrm{H}^n_{\mathrm{dR}}(X)=\mathrm{H}^n_{\mathrm{dR}}(X_{\mathbb{Q}})\otimes\mathbb{C}$. By choosing $\mathbb{Q}$-bases of $\mathrm{H}^n_{\mathrm{dR}}(X_{\mathbb{Q}})$ and $\mathrm{H}^n(X,\mathbb{Q})$, the matrix P of the period isomorphism has transcendental entries. Let the index mid denote the image from the cohomology with compact support to the cohomology (in dimension n). Then $\mathrm{H}^n_{\mathrm{dR,mid}}(X_{\mathbb{Q}})$ and $\mathrm{H}^n_{\mathrm{mid}}(X,\mathbb{Q})$ are self-dual by de Rham or Poincaré duality, and this leads to "quadratic relations":

$$(2\pi\mathsf{i})^n I_{\mathrm{B}} = P\cdot I_{\mathrm{dR}}^{-1}\cdot {}^tP,$$

where I_B and I_{dR} are respectively the Betti and the de Rham intersection matrices (the latter uses the isomorphism $\mathrm{tr}_X=(1/2\pi\mathsf{i})^n\int_X:\mathrm{H}^{2n}_{\mathrm{dR,c}}(X)\to\mathbb{C}$). If X is defined over $\mathbb{Q}$, these are polynomial relations of degree 2 with coefficients in $\mathbb{Q}$ between the entries of the period matrix.
If X is affine, the entries of P can be expressed as integrals $\int_\gamma\omega$, where γ is an n-cycle and ω an algebraic n-form on X.

1.2. Bessel functions and their moments

The modified Bessel differential equation on the unknown $u(t)$ ("modified" because of the minus sign, instead of $+$)

$$(t\partial_t)^2u-t^2u=0$$

has two independent solutions (I_0 entire, K_0 multivalued)

$$I_0(t)=\frac{1}{2\pi\mathsf{i}}\oint\exp\Big(-\frac{t}{2}(y+1/y)\Big)\frac{\mathrm{d}y}{y},$$
$$K_0(t)=\frac12\int_0^\infty\exp\Big(-\frac{t}{2}(y+1/y)\Big)\frac{\mathrm{d}y}{y}\qquad(|\arg t|<\pi/2).$$

Note: setting $z=(t/2)^2$ and $x=(t/2)y$, we have $(t/2)(y+1/y)=x+z/x$. The modified Bessel differential operator reads, in the variable z, as $(z\partial_z)^2-z$: this is the Kloosterman differential operator.

Two physicists, Broadhurst and Roberts, were interested in *Bessel moments*, e.g. for $k = 2\ell + 1$ odd

$$\mathrm{BM}_k(i,j) = \mathrm{cst}_{i,j} \int_0^\infty I_0(t)^i K_0(t)^{k-i} t^{2j-1} \mathrm{d}t, \quad i,j \in [1,\ell]$$

and have cooked up by experimental computation a conjectural quadratic relation

$$(-2\pi\mathsf{i})^k \mathcal{B}_k = \mathrm{BM}_k \cdot D_k \cdot {}^t\mathrm{BM}_k,$$

where $\mathcal{B}_k$ is explicit has entries of the form: product of factorials times a Bernoulli number, while D_k is defined over $\mathbb{Q}$ by a complicated induction on k.

***Theorem A* (F-S-Y, Zhu).** *There exists such a relation.*

Proof by means of a geometric interpretation to relate with classical period matrices. That such a geometric interpretation should exist is suggested by the following relation was found by P. Vanhove:

$$\int_0^\infty I_0(t) K_0(t)^{\ell+1} t \,\mathrm{d}t = \frac{1}{2^\ell} \int_{x_i \geqslant 0} \frac{1}{(1+\sum_{i=1}^\ell x_i)(1+\sum_{i=1}^\ell 1/x_i) - 1} \prod_{i=1}^\ell \frac{\mathrm{d}x_i}{x_i}.$$

However, the (modified) Bessel differential equation is *not* a Picard-Fuchs equation because it has an irregular singularity at infinity. Hence the notion of "geometric interpretation" has to be made precise.

1.3. Bessel moments and arithmetic

B-R also knew that the Kloosterman differential equation is a heuristic analogue of the *Kloosterman ℓ-adic sheaves* related to Kloosterman exponential sums.
p: a prime number, q: power of p, $\mathbb{F}_p \subset \mathbb{F}_q \subset \overline{\mathbb{F}}_p$, $\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \to \mathbb{F}_p$: trace map.
Kloosterman sum: $\forall\, z \in \mathbb{F}_q^\times$, the real number

$$\mathrm{Kl}_2(z;q) = \sum_{x \in \mathbb{F}_q^\times} \exp\bigl[2\pi\mathsf{i}(\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(x + z/x))/p\bigr].$$

Weil: $\exists\, \alpha_z \in \overline{\mathbb{Q}}$, $|\alpha_z| = \sqrt{q}$ s.t. $\mathrm{Kl}_2(z;q) = -(\alpha_z + q/\alpha_z)$.
$\forall\, k \geqslant 1$, the k-th symmetric powers of Kloosterman sums

$$\mathrm{Kl}_2^{\mathrm{Sym}^k}(z;q) = \sum_{i=0}^k \alpha_z^i (q/\alpha_z)^{k-i}$$

and the *moments*

$$m_2^k(q) = \sum_{z \in \mathbb{F}_q^\times} \mathrm{Kl}_2^{\mathrm{Sym}^k}(z;q).$$

Generating series:

$$Z_k(p;T) = \exp\Bigl(\sum_{n=1}^{\infty} m_2^k(p^n)\frac{T^n}{n}\Bigr),$$

In fact, $Z_k(p;T) = (1-T)M_k(p;T)$, $M_k(p;T) \in \mathbb{Z}[T]$, $|\text{roots}| = p^{-(k+1)/2}$.
Complete L-function (e.g. k odd, $k = 2\ell+1$):

$$\Lambda_k(s) = L_{k,\infty}(s)\cdot L_k(s) = \pi^{-\ell s/2}\prod_{j=1}^{\ell}\Gamma\Bigl(\frac{s-j}{2}\Bigr)\cdot(\mathfrak{N}_k)^{s/2}\prod_{p \text{ prime}}\frac{1}{M_k(p;p^{-s})}.$$

***Theorem* (Conj. of Broadhurst & Roberts, F-S-Y).** *Assume k is odd (similar result for even k). The function $\Lambda_k(s)$ admits a meromorphic continuation to the complex plane and satisfies the functional equation*

$$\Lambda_k(s) = \Lambda_k(k+2-s).$$

***Remark* (Arithmetic and Bessel moments).** Deligne conjectures the value of the function $L_k(s)$ at some integers called "critical integers" (i.e., neither a pole of $s \mapsto L_{k,\infty}(s)$ nor of $s \mapsto L_{k,\infty}(k+2-s)$). We prove that these conjectural values are equal to some explicit sub-determinants of the matrix BM_k.

1.4. The geometry behind

We consider the Laurent polynomial

$$g_1(y) = y + 1/y$$

as a regular function on $\mathbb{C}^*$ and its k-Thom-Sebastiani relative ($k \geqslant 1$)

$$g_k : (\mathbb{C}^*)^k \longrightarrow \mathbb{C}, \quad g_k(y^{(1)},\dots,y^{(k)}) = g_1(y^{(1)}) + \cdots + g_1(y^{(k)}).$$

The group μ_2 acts diagonally on $(\mathbb{C}^*)^k$ by $(y^{(1)},\dots,y^{(k)}) \mapsto \pm(y^{(1)},\dots,y^{(k)})$, and the group $\mathfrak{S}_k$ acts by permutation $y^{(j)} \mapsto y^{(\sigma(j))}$.
Then $\mathcal{H}_k = g_k^{-1}(0)$ is preserved by the action of $\mathfrak{S}_k \times \mu_2$. It has at most isolated singularities. Let $\chi : \mathfrak{S}_k \times \mu_2 \to \{\pm 1\}$ defined by $\chi(\sigma,\varepsilon) = \mathrm{sgn}(\sigma)$.
The cohomology

$$\mathrm{H}^{k-1}_{\mathrm{mid}}(\mathcal{H}_k,\mathbb{C})^{\mathfrak{S}_k\times\mu_2,\chi} := \mathrm{Im}\Bigl[\mathrm{H}^{k-1}_{\mathrm{c}}\mathcal{H}_k,\mathbb{C})^{\mathfrak{S}_k\times\mu_2,\chi} \longrightarrow \mathrm{H}^{k-1}(\mathcal{H}_k,\mathbb{C})^{\mathfrak{S}_k\times\mu_2,\chi}\Bigr]$$

is pure of weight $k-1$.

***Theorem B* (F-S-Y).** *The nonzero Hodge numbers of $\mathrm{H}^{k-1}_{\mathrm{mid}}(\mathcal{H}_k,\mathbb{C})^{\mathfrak{S}_k\times\mu_2,\chi}$ are all equal to one ($\exists$ a precise formula).*

Arithmetic methods + Thm of Patrikis-Taylor on potential automorphy $\Longrightarrow$ the functional equation *if the property of Theorem* B *is fulfilled.*

LECTURE 2

COMPUTATION OF HODGE NUMBERS

In this chapter, we explain a method for computing the Hodge numbers of $\mathrm{H}^{k-1}_{\mathrm{mid}}(\mathcal{H}_k, \mathbb{C})^{\mathfrak{S}_k \times \mu_2, \chi}$.

2.1. Computation of Hodge number via Laplace transformation

It is well-known in analysis that the Fourier transformation can be helpful for computing various integrals. In Hodge theory, the Laplace transformation cannot be used similarly because

- the Laplace transform of a vector bundle with regular meromorphic connection on the affine line is in general a vector bundle with an *irregular* meromorphic connection,
- and a theorem of Griffiths asserts the regularity of the connection underlying a polarizable variation of Hodge structure,

so Laplace transformation does not preserve the category of pVHS. We will see that, in some way, one can overcome this difficulty.

Let $g : Y \to \mathbb{A}^1$ be a regular function on a smooth quasi-projective variety Y. Set $\mathcal{H} = g^{-1}(0)$, $X = \mathbb{A}^1_t \times Y$ and $f = tg : X \to \mathbb{A}^1$. Consider the *twisted de Rham complex*

$$(\Omega^\bullet_X, \mathrm{d} + \mathrm{d}f).$$

Proposition. *There are isomorphisms*

$$(*) \qquad \begin{cases} \mathrm{H}^r_{\mathrm{dR,c}}(X, f) := \mathrm{H}^r_{\mathrm{c}}(X, (\Omega^\bullet_X, \mathrm{d} + \mathrm{d}f)) \simeq \mathrm{H}^r_{\mathrm{dR,c}}(\mathbb{A}^1_t \times \mathcal{H}) \simeq \mathrm{H}^{r-2}_{\mathrm{dR,c}}(\mathcal{H}), \\ \mathrm{H}^r_{\mathrm{dR}}(X, f) := \mathrm{H}^r(X, (\Omega^\bullet_X, \mathrm{d} + \mathrm{d}f)) \simeq \mathrm{H}^r_{\mathcal{H}}(Y) = \mathrm{H}^r(Y, Y \smallsetminus \mathcal{H}). \end{cases}$$

On noting that $\mathrm{d} + \mathrm{d}f = \mathrm{e}^{-tg} \circ \mathrm{d} \circ \mathrm{e}^{tg}$, the first line is an analogue of the property that the Fourier transform of the constant function on $\mathbb{R}$ is the Dirac distribution at the origin.

In 1984, Deligne emphasized that, when X is a curve and f is any regular function on it, the twisted cohomology $\mathrm{H}^1(X, (\Omega^\bullet_X, \mathrm{d} + \mathrm{d}f))$ can be equipped canonically with a decreasing filtration $F^\bullet_{\mathrm{irr}}$ (the irregular Hodge filtration), that is in general indexed by $\mathbb{Q}$. More recently J.-D. Yu has extended this construction to any pair (X,f), yielding a filtration $F^\bullet_{\mathrm{irr}}\mathrm{H}^r_*(X,f)$.

Computation of $\mathrm{H}^r_{\mathrm{dR}}(X,f)$ ***and*** $\mathrm{H}^r_{\mathrm{dR,c}}(X,f)$ ***and their irregular Hodge filtration***

Let $\overline{f}:\overline{X}\to\mathbb{P}^1$ a projectivization of f s.t. $D:=\overline{X}\smallsetminus X$ is a sncd. Set $P=\overline{f}^*(\infty)$ the pole divisor of $\overline{f}$.

Recall: $\mathrm{H}^r_{\mathrm{dR}}(X)$ is the hypercohomology of the log-dR complex $(\Omega^\bullet_{\overline{X}}(\log D),\mathrm{d})$ and $\mathrm{H}^r_{\mathrm{dR,c}}(X)$ that of $(\Omega^\bullet_{\overline{X}}(\log D)(-D),\mathrm{d})$. The Hodge filtration is obtained by the "stupid" truncation of these complexes.

Similarly, $\mathrm{H}^r_{\mathrm{dR}}(X,f)$ is the hypercohomology of the complex

$$\mathcal{O}_{\overline{X}}\xrightarrow{\ \mathrm{d}+\mathrm{d}f\ }\Omega^1_{\overline{X}}(\log D)(P)\longrightarrow\cdots\longrightarrow\Omega^n_{\overline{X}}(\log D)(nP),$$

and similarly for $\mathrm{H}^r_{\mathrm{dR,c}}(X,f)$. The irregular Hodge filtration $F^p_{\mathrm{irr}}\mathrm{H}^r_{\mathrm{dR}}(X,f)$ is also defined by the stupid truncation of this complex. However, one can extend it so that it is indexed by $\mathbb{Q}$: For $\alpha\in(0,1)\cap\mathbb{Q}$,

$$\begin{aligned}&F^{p-\alpha}_{\mathrm{irr}}(\Omega^\bullet_{\overline{X}}(*D),\mathrm{d}+\mathrm{d}\overline{f})\\ &\quad=\Bigl\{0\longrightarrow\cdots\longrightarrow\Omega^p_{\overline{X}}(\log D)([\alpha P])\xrightarrow{\ \mathrm{d}+\mathrm{d}\overline{f}\ }\Omega^p_{\overline{X}}(\log D)([(\alpha+1)P])\longrightarrow\cdots\Bigr\}\end{aligned}$$

and an analogous formula for $F^{p-\alpha}_{\mathrm{irr}}\mathrm{H}^r_{\mathrm{c}}(X,f)$. Clearly, the jumps are governed by the multiplicities of the components of P.

Theorem. *Under the isomorphism* $(*)$, *we have*

$$\boxed{\begin{cases}F^\bullet\bigl(\mathrm{H}^{r-2}_{\mathrm{c}}(\mathcal{H})(-1)\bigr)\simeq F^\bullet\mathrm{H}^r_{\mathrm{c}}(\mathbb{A}^1_t\times\mathcal{H})\simeq F^\bullet_{\mathrm{irr}}\mathrm{H}^\bullet_{\mathrm{c}}(X,f),\\ F^\bullet\mathrm{H}^r_{\mathcal{H}}(Y)\simeq F^\bullet_{\mathrm{irr}}\mathrm{H}^\bullet(X,f).\end{cases}}$$

In particular, the jumps of the irregular Hodge filtration are integers.

2.2. The Kloosterman connection and its symmetric powers

On $\mathbb{G}_{\mathrm{m},z}$, diff. eqn $(z\partial_z)^2-z\longleftrightarrow$ Kloosterman connection:

$$\mathrm{Kl}_2=\Bigl(\mathcal{O}^2_{\mathbb{G}_{\mathrm{m}}},\mathrm{d}+\begin{pmatrix}0&z\\1&0\end{pmatrix}\frac{\mathrm{d}z}{z}\Bigr),\quad\text{basis }v_0,v_1:\begin{cases}z\partial_zv_0=v_1\\ z\partial_zv_1=zv_0.\end{cases}$$

$z=(t/2)^2$, $\rightsquigarrow$ modified Bessel eqn $(t\partial_t)^2-t^2$ and corresponding $\widetilde{\mathrm{Kl}}_2$ on $\mathbb{G}_{\mathrm{m},t}$.

Observation. Recall $g_k = \sum_{i=1}^k (y^{(i)} + 1/y^{(i)})$ on $\mathbb{G}_\mathrm{m}^k$, set $f_k = tg_k$ on $\mathbb{A}_t^1 \times \mathbb{G}_\mathrm{m}^k$. Then, pushing forward by the projection $\mathbb{G}_{\mathrm{m},t} \times \mathbb{G}_\mathrm{m}^k \to \mathbb{G}_{\mathrm{m},t}$ one finds ($? =!, *$)

$$\mathrm{H}^1_{\mathrm{dR},?}(\mathbb{G}_{\mathrm{m},t}, \bigotimes^k \widetilde{\mathrm{Kl}}_2) \simeq \mathrm{H}^{k+1}_?(\mathbb{G}_{\mathrm{m},t} \times \mathbb{G}_\mathrm{m}^k, f_k),$$

and thus

$$\mathrm{H}^1_{\mathrm{dR},?}(\mathbb{G}_{\mathrm{m},z}, \mathrm{Sym}^k \mathrm{Kl}_2) \simeq \mathrm{H}^{k+1}_?(\mathbb{G}_{\mathrm{m},t} \times \mathbb{G}_\mathrm{m}^k, f_k)^{\mathfrak{S}_k \times \mu_2, \chi}.$$

After a little work, one finally finds

$$\boxed{\mathrm{H}^{k-1}_{\mathrm{dR,mid}}(\mathcal{H}_k, \mathbb{C})^{\mathfrak{S}_k \times \mu_2, \chi} \simeq \mathrm{H}^{k+1}_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m},t} \times \mathbb{G}_\mathrm{m}^k, f_k)^{\mathfrak{S}_k \times \mu_2, \chi} \simeq \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m},z}, \mathrm{Sym}^k \mathrm{Kl}_2)}$$

2.3. Sketch of the proof of Theorem B

The proof goes as follows:

(1) Find a natural basis of $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m},z}, \mathrm{Sym}^k \mathrm{Kl}_2)$: rather easy as the base has dimension 1 and the connection matrix of $\mathrm{Sym}^k \mathrm{Kl}_2$ is rather simple. Explicitly:

$$\{w_j = v_0^k z^j \mathrm{d}z/z \mid j = 1, \dots, \lfloor (k-1)/2 \rfloor\} \quad \text{if } k \text{ is not a multiple of 4},$$

and a variant of this formula if $4 \mid k$. This defines a filtration $G^\bullet$ of $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m},z}, \mathrm{Sym}^k \mathrm{Kl}_2)$ such that the graded quotients have dimension 0 or 1. Explicitly (if $4 \nmid k$):

$$G^p \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m},z}, \mathrm{Sym}^k \mathrm{Kl}_2) = \Big\langle w_j \;\Big|\; 1 \leqslant j \leqslant \lfloor (k+1-p)/2 \rfloor \Big\rangle.$$

(2) Lift each w_j as the $(k+1)$-differential form

$$\omega_j = t^{2j} \frac{\mathrm{d}t}{t} \cdot \frac{\mathrm{d}y_1}{y_1} \cdots \frac{\mathrm{d}y_k}{y_k}$$

on $\mathbb{G}_{\mathrm{m},t} \times \mathbb{G}_\mathrm{m}^k$ and compute the order of its poles in a suitable good compactification of $\mathbb{G}_{\mathrm{m},t} \times \mathbb{G}_\mathrm{m}^k$. Deduce, via the above isomorphisms,

$$G^p \subset F^p_{\mathrm{irr}} = F^p \mathrm{H}^{k-1}_{\mathrm{dR,mid}}(\mathcal{H}_k, \mathbb{C})^{\mathfrak{S}_k \times \mu_2, \chi}(-1).$$

(3) The middle cohomology of $\mathcal{H}_k$ is pure of weight $k-1$, hence F^p satisfies a symmetry property. On the other hand, a simple calculation shows that G^p satisfies the same symmetry property.

(4) One concludes that $F^p = G^p$, and thus the nonzero Hodge numbers of $\mathrm{H}^{k-1}_{\mathrm{dR,mid}}(\mathcal{H}_k, \mathbb{C})^{\mathfrak{S}_k \times \mu_2, \chi}(-1)$ are all equal to 1.

Final remark. This programs works well if k is odd. If k is even, various problems occur:

- The hypersurface $\mathcal{H}_k$ has some isolated singularities,
- the Laurent polynomial f_k on $\mathbb{G}_{\mathrm{m},t} \times \mathbb{G}_{\mathrm{m}}^k$ is degenerate,
- the definition of the basis has to be a little modified if $4 \mid k$.

Solving these problems require some other technical tools involving more information from the Laplace transform of $\widetilde{\mathrm{Kl}}_2$.

LECTURE 3

PERIODS AND LAPLACE TRANSFORMATION

What is the Betti (i.e., topological) analogue of the isomorphism $(*)$ between de Rham cohomologies?

3.1. Moderate and rapid decay cohomologies

We consider the general setting of a regular function $f : X \to \mathbb{A}^1$ (X smooth quasi-projective). We will describe the Betti analogues of $\mathrm{H}^r_{\mathrm{dR,c}}(X, f)$ and $\mathrm{H}^r_{\mathrm{dR}}(X, f)$ and the corresponding period isomorphism.

- Choose $\overline{f} : \overline{X} \to \mathbb{P}^1$ extending f with $\overline{X}$ smooth projective and $D = \overline{X}$ a sncd.
- Set $D = H \cup P$, $P = \overline{f}^{-1}(\infty)$.
- $\varpi : \widetilde{X}(D) = \widetilde{X} \to \overline{X}$: real oriented blowing up of the components of D.

Loc.: coord. $(x_1, \dots, x_n) \in \mathbb{C}^n$, $D = \{x_1 \cdots x_\ell = 0\}$,
  - $\widetilde{\mathbb{C}}^n = (\mathbb{R}_+)^\ell \times (S^1)^\ell \times \mathbb{C}^{n-\ell}$,
  - coord.: $(r\mathrm{e}^{\mathrm{i}\theta}, x') = (r_j, \mathrm{e}^{\mathrm{i}\theta_j})_{j=1,\dots,\ell}, x_{\ell+1}, \dots, x_n$,
  - $\varpi : (r_j, \mathrm{e}^{\mathrm{i}\theta_j}) \mapsto r_j\mathrm{e}^{\mathrm{i}\theta_j} = x_j$.
- $\exists\ \widetilde{f} : \widetilde{X} \to \widetilde{\mathbb{P}}^1(\infty)$. Locally:
  - $\overline{f} = u(x)/x_1^{m_1} \cdots x_\ell^{m_\ell}$, $m_j > 0$, u holom. invertible if $\ell \geqslant 1$, but possibly $\ell = 0$.
  - $|t|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{P}^1$.
  - $\widetilde{f}$: $|t| = |\overline{f}|$ and $\theta = \arg u(x) - \sum_j m_j\theta_j$.
- Commutative diagram:

$$\begin{tikzcd}
X \arrow[dr, hook] \arrow[drr, hook] \arrow[dd, "f"'] & & \\
& \overline{X} \arrow[dd, "\overline{f}"] & \widetilde{X} \arrow[l, "\varpi"] \arrow[dd, "\widetilde{f}"] \\
\mathbb{A}^1 \arrow[dr, hook] \arrow[drr, hook] & & \\
& \mathbb{P}^1 & \widetilde{\mathbb{P}}^1 \arrow[l]
\end{tikzcd}$$

- Subsets $\boxed{\widetilde{X}_{\mathrm{rd}} \subset \widetilde{X}_{\mathrm{mod}} \subset \widetilde{X}}$: e^{-f} has rapid decay resp. moderate growth.

Locally:

- Above $\mathbb{A}^1$: $\widetilde{X}_{\mathrm{rd}|\mathbb{A}^1} = X$, $\widetilde{X}_{\mathrm{mod}|\mathbb{A}^1} = \widetilde{X}_{|\mathbb{A}^1}$,
- Above ∞: $\varpi^{-1}(P)_{\mathrm{rd}} = \varpi^{-1}(P)_{\mathrm{mod}}$. Locally:

$$\arg u(x) - \sum_j m_j\theta_j \in (-\pi/2, \pi/2) \bmod 2\pi.$$

Theorem.

$$\boxed{\mathrm{H}^r_{\mathrm{dR,c}}(X,f) \overset{\mathrm{comp}}{\simeq} \mathrm{H}^r_{\mathrm{c}}(\widetilde{X}_{\mathrm{rd}}, \mathbb{C}), \quad \mathrm{H}^r_{\mathrm{dR}}(X,f) \overset{\mathrm{comp}}{\simeq} \mathrm{H}^r_{\mathrm{c}}(\widetilde{X}_{\mathrm{mod}}, \mathbb{C})}$$

The space $\widetilde{X}$ is a complex manifold with C^∞ corners. It is equipped with a sheaf of "holomorphic functions" $\mathcal{A}_{\widetilde{X}}$: this is the sheaf of C^∞ functions on $\widetilde{X}$ annihilated by the Cauchy-Riemann operator (if h is a local reduced equation of D, then the action of $h\overline{\partial}$ on $\mathcal{C}^\infty_X$ can be lifted to $\mathcal{C}^\infty_{\widetilde{X}}$). This sheaf $\mathcal{A}_{\widetilde{X}}$ is however not coherent. It has two companions:

- the sub-sheaf $\mathcal{A}^{\mathrm{rd}}_{\widetilde{X}}$ consisting of those functions whose Taylor expansion vanishes identically on $\partial\widetilde{X} = \varpi^{-1}(D)$ (rapid decay),
- the sup-sheaf $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}}$ consisting of functions whose restrictions to $X \smallsetminus D$ has moderate growth along D.

These sheaves satisfy the following properties:

- They are flat over $\varpi^{-1}\mathcal{O}_X$ (T. Mochizuki).
- $\varpi_*\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}} = \mathcal{O}_X(*D)$ and $R^j\varpi_*\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}} = 0$ for $j \geqslant 1$.
- $(\varpi_*\mathcal{A}^{\mathrm{rd}}_{\widetilde{X}})_{|D} = 0$ and $(R^1\varpi_*\mathcal{A}^{\mathrm{rd}}_{\widetilde{X}})_{|D} \simeq \mathcal{O}_{\widehat{D}}/(\mathcal{O}_X)_{|D}$.

Sketch of proof of the theorem. Let us consider the mod case. The twisted de Rham complex $\mathrm{DR}(E^f) = (\Omega^\bullet_{\overline{X}}(*D), \mathrm{d} + \mathrm{d}\overline{f})$ can be lifted to the *moderate twisted de Rham complex* $\mathrm{DR}^{\mathrm{mod}}(E^f)$ on $\widetilde{X}$: the p-th term is $\mathcal{A}^{\mathrm{mod}}_{\widetilde{X}} \otimes_{\varpi^{-1}\mathcal{O}_{\overline{X}}} \Omega^j_{\widetilde{X}}(*D)$, and the differential is the lift of $\mathrm{d} + \mathrm{d}\overline{f}$. Then, from the properties above one obtains $R\varpi_* \mathrm{DR}^{\mathrm{mod}}(E^f) \simeq \mathrm{DR}(E^f)$, and thus

$$\mathrm{H}^r_{\mathrm{dR}}(X,f) \simeq \mathrm{H}^r(\widetilde{X}, \mathrm{DR}^{\mathrm{mod}}(E^f)).$$

It is not difficult to compute that $\mathcal{H}^0 \mathrm{DR}^{\mathrm{mod}}(E^f)$ is the constant sheaf on $\widetilde{X}_{\mathrm{mod}}$ extended by zero to $\widetilde{X}$. It is less obvious (but follows from asymptotic analysis) that $\mathcal{H}^j \mathrm{DR}^{\mathrm{mod}}(E^f) = 0$ for $j \geqslant 1$. This concludes the proof. $\square$

Let $g : Y \to \mathbb{A}^1$, $\mathcal{H} = g^{-1}(0)$ and $f = tg : X = \mathbb{A}^1_t \times Y \to \mathbb{A}^1$.

Corollary. *The isomorphism* $(*)$ *extends as an isomorphism of period structures*

$$\big(\mathrm{H}^{r-2}_{\mathrm{dR,c}}(\mathcal{H}), \mathrm{H}^{r-2}_{\mathrm{c}}(\mathcal{H}, \mathbb{Q}), \mathrm{comp}\big) \simeq \big(\mathrm{H}^r_{\mathrm{dR,c}}(X,f), \mathrm{H}^r_{\mathrm{c}}(\widetilde{X}_{\mathrm{rd}}, \mathbb{Q}), \mathrm{comp}\big).$$

3.2. Period structure of $\mathrm{Sym}^k \mathrm{Kl}_2$ and quadratic relations

The advantage of considering the period structure of $\mathrm{Sym}^k \mathrm{Kl}_2$ is that it lives in dimension one, so that the geometry is very simple.

De Rham $\mathbb{Q}$-structure and pairing. The bundle with connection (Kl_2, ∇) has a natural $\mathcal{O}_{\mathbb{G}_\mathrm{m}}$-basis (v_0, v_1) and a natural non-degenerate skew-symmetric pairing such that $\langle v_0, v_1\rangle = 1$.

- $\rightsquigarrow$ Natural non-degenerate $(-1)^k$-symmetric pairing on $(\mathrm{Sym}^k \mathrm{Kl}_2, \nabla)$.
- $\rightsquigarrow$ Natural non-degenerate $(-1)^{k+1}$-symmetric pairing on $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)$.
- Computation shows that the matrix S_k of this pairing in the basis (w_j) has rational entries.
- $\rightsquigarrow$ Define the de Rham $\mathbb{Q}$-structure on $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)$ by means of (w_j).

Betti $\mathbb{Q}$-structure and pairing. The sheaf of horizontal section Kl_2^∇ on $\mathbb{C}^*$ is naturally endowed with a $\mathbb{Q}$-structure. Bloch and Esnault have introduced the notion of rapid-decay (or moderate growth) cycle with respect to a vector bundle with connection.
We obtain the vector spaces (that are naturally equipped with a $\mathbb{Q}$-structure and a $\mathbb{Q}$-basis)

$$\mathrm{H}_1^{\mathrm{rd}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2), \quad \mathrm{H}_1^{\mathrm{mod}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2), \quad \mathrm{H}_1^{\mathrm{mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2).$$

It is a straightforward calculation to obtain the 'middle' period matrix $(\mathsf{P}_{i,j})$ with respect to this basis and the basis (w_j). Lifting the bases first to the setting of $(\mathbb{G}_\mathrm{m} \times \mathbb{G}_\mathrm{m}^k, f_k)$ and then to that of $\mathcal{H}_k$, we obtain:

***Theorem* (F-S-Y).** *Assume k odd (there is a modified statement for k even). There exist bases of $\mathrm{H}^{\mathrm{mid}}_{k-1}(\mathcal{H}_k, \mathbb{Q})^{\mathfrak{S}_k \times \mu_2, \chi}$ and of $\mathrm{H}^{k-1}_{\mathrm{dR,mid}}(\mathcal{H}_{k,\mathbb{Q}})^{\mathfrak{S}_k \times \mu_2, \chi}$ such that the corresponding period matrix $\mathsf{P} = (\mathsf{P}_{i,j})$ is the matrix of Bessel moments $(i, j = 1, \dots, (k-1)/2)$*

$$\mathsf{P}_{i,j} = \mathrm{cst}_{i,j} \int_0^\infty I_0(t)^i K_0(t)^{k-i}\, t^{2j}\, \frac{\mathrm{d}t}{t}.$$

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