Introduction to Stokes structures

II: dimension $\geqslant 2$

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Connections in dim. $\geqslant 2$

Local approach.

- Δ^n, coord. $z = (z_1, \dots, z_n)$.
- Linear diff. system: $\dfrac{\mathrm{d}f}{\mathrm{d}z_i} = A_i(z) \cdot f, \quad i = 1, \dots, n$.
- $A_i(z)$ matrix of size d, merom., pole along a div. D.
- Gauge equiv.: $P \in \mathrm{GL}_d(\mathbb{C}\{z\}(*D))$,

$$\boxed{B_i := P[A_i] = P^{-1}A_iP + P^{-1}\partial P/\partial z_i}$$

- ***Integrability cond.***:

$$\boxed{\frac{\partial A_i}{\partial z_j} - \frac{\partial A_j}{\partial z_i} = [A_i, A_j]} \qquad \forall i, j.$$

Connections in dim. $\geqslant 2$

Global approach.

- X = cplx. manifold, D = hypersurface.
- Linear diff. system:
  - holom. vect. bdle E on X,
  - merom. connection $\nabla : E \to \Omega^1_X(*D) \otimes E$
- ***Integrability cond.***: $\boxed{\nabla^2 = 0}$
- In local coord. $(z_1, \ldots, z_n)$ and in a local basis of E,

$$\nabla = \mathrm{d} + \sum_{i=1}^{n} A_i(z)\mathrm{d}z_i, \quad A_i \in \mathrm{Mat}_d(\mathbb{C}\{z\}(*D)).$$

$$\nabla^2 = 0 \iff \frac{\partial A_i}{\partial z_j} - \frac{\partial A_j}{\partial z_i} = -[A_i, A_j]$$

Connections in dim. $\geqslant 2$

Global approach.

- $X =$ cplx. manifold, $D =$ hypersurface.
- Linear diff. system:
  - merom. vect. bdle M on X: coh. $\mathscr{O}_X(*D)$-mod.
  - connection $\nabla : M \to \Omega^1_X \otimes M$
- ***Integrability cond.***: $\boxed{\nabla^2 = 0}$
- In local coord. $(z_1, \ldots, z_n)$ and in a local basis of E,

$$\nabla = \mathrm{d} + \sum_{i=1}^{n} A_i(z)\mathrm{d}z_i, \quad A_i \in \mathrm{Mat}_d(\mathbb{C}\{z\}(*D)).$$

$$\nabla^2 = 0 \iff \frac{\partial A_i}{\partial z_j} - \frac{\partial A_j}{\partial z_i} = -[A_i, A_j]$$

Integrable deformations

- (M^o, ∇^o) on Δ:
  - $M^o := \mathscr{O}_\Delta^d(*0)$,
  - $\nabla^o := \mathrm{d} + A^o(z)\mathrm{d}z$.
- ***Integrable deform. param. by*** (X, x^o):
  - (M, ∇) on $\Delta \times X$, poles on $\{0\} \times X$,
  - ∇ ***integrable***,

  s.t. $\boxed{(M, \nabla)_{|x^o} = (M^o, \nabla^o)}$
- $\nabla_{|x^o}$?

$$\begin{array}{c} \nabla : M \to \Omega^1_{\Delta \times X} \otimes M \\ \downarrow \\ \nabla^{\mathrm{rel}} : M \to \Omega^1_{\Delta \times X/X} \otimes M \\ \Downarrow \\ \boxed{\nabla_{|x^o} := (\nabla^{\mathrm{rel}})_{|x^o}} \end{array}$$

Integrable deformations

- Example:
  - $M^o = \mathscr{O}_\Delta(*0)$,
  - $\nabla^o = \mathrm{d}$,
  - $X = \mathbb{C}$,
  - $M = \mathscr{O}_{\Delta\times X}(*(0 \times X))$,
  - $\nabla = \mathrm{d} + \mathrm{d}(x/z)$,
  - $\nabla^{\mathrm{rel}} = \mathrm{d} - x\,\mathrm{d}z/z^2$.

  Then $(M, \nabla)_{|x}$
  - regular at $x = 0$,
  - irregular for any $x \neq 0$.

  $\Longrightarrow$ bad example, should impose more properties.

Isomonodromic deformations

- Assume (M^o, ∇^o) ***regular***.
- (M, ∇) integr. deform. of (M^o, ∇^o) is ***isomonodromic*** if
  - any $(M, \nabla)_{|x}$ is reg. on $(\Delta, 0)$.
- ***Rigidity***: If X is 1-connected, $\exists!!$ isomono. deform. (M, ∇) of (M^o, ∇^o) on X.
  - $\Longleftarrow \pi_1(\Delta, 1) = \pi_1\big(\Delta \times X, (1, x_o)\big)$.

Iso-level deformations

- $B^o(z)$: (non-ramified) normal form:

$$B^o \mathrm{d}z = \begin{pmatrix} \mathrm{d}\varphi_1^o & & \\ & \ddots & \\ & & \mathrm{d}\varphi_d^o \end{pmatrix} + C^o \frac{\mathrm{d}z}{z}$$

$\varphi_k^o \in \frac{1}{z}\mathbb{C}[\frac{1}{z}]$
$C^o =$ const.
non reson.

- ***Iso-level deformation*** on $\Delta \times X$:

$$\nabla = \mathrm{d} + \begin{pmatrix} \mathrm{d}\varphi_1 & & \\ & \ddots & \\ & & \mathrm{d}\varphi_d \end{pmatrix} + C^o \frac{\mathrm{d}z}{z}$$

s.t.

  - $\varphi_k(z,x) \in \Gamma(X, z^{-1}\mathscr{O}_X[z^{-1}])$,
  - pole order of $z \mapsto \varphi_k(z,x)$ cst.,
  - pole order of $z \mapsto (\varphi_k(z,x) - \varphi_j(z,x))$ cst.

Iso-level deformations

- ***Theorem*** (Ueno 1980, Jimbo-Miwa-Ueno, Malgrange, Mochizuki). Given:
  - $(M^o, \nabla^o) = (\mathscr{O}_\Delta(*0)^d, \mathrm{d} + A^o \mathrm{d}z)$ with formal normal form $\mathrm{d} + B^o \mathrm{d}z$.
  - $(\mathscr{O}_{\Delta \times X}(*(0 \times X))^d, \mathrm{d} + B\, \mathrm{d}z)$: iso-level integr. deform. of $\mathrm{d} + B^o \mathrm{d}z$.

  $+$ assume X 1-connected.

  $\Longrightarrow \exists!!$ integr. deform. (M, ∇) s.t.

  $\forall x \in X, \quad (M, \nabla)_{|x}$ has norm. form $\mathrm{d} + B_x\, \mathrm{d}z$.

- $\rightsquigarrow$ ***"isomonodromic" deformation*** of irreg. sings.

Isomonodromic deformations

- ***Proofs***.
  - Show that $\mathrm{St}(B_x)$ is loc. cst. w.r.t. x, or
  - show that $(\mathscr{L}, \mathscr{L}_\bullet) \mapsto (\mathscr{L}, \mathscr{L}_\bullet)_{|x^o}$ equiv. of categ. [this proof extends to more gen. cases.]
- ***Applications***.
  - $\#\mathrm{Level}(B^o) = 1 \rightsquigarrow$ ***universal*** isomono. deform.
  - used in Frobenius mflds.
- What about ***degenerations***? Example:
  - $B^o = \mathrm{diag}(x_1^o/z^2, \dots, x_d^o/z^2) + C^o/z$, $x_i^o \neq x_j^o$.
  - Univ. isomono. deform. param. by $(X, x^o) =$ univ. cover. of $(\mathbb{C}^d \smallsetminus \text{diags}, x^o)$.
  - Degenerations along diags?

Normal form in dim. $\geqslant 2$

- $X = \Delta^n$, coord. $z = (z_1, \ldots, z_n)$,
- $D = \{z_1 \cdots z_\ell = 0\}$, n.c.d.
- $\varphi_1, \ldots, \varphi_d \in \mathscr{O}_X[(z_1 \cdots z_\ell)^{-1}]/\mathscr{O}_X$,
- $B_i(z)$ matrix of size d, merom., pole along D.
- $$\boxed{B_i = \begin{pmatrix} \partial\varphi_1/\partial z_i & & \\ & \ddots & \\ & & \partial\varphi_d/\partial z_i \end{pmatrix} + \frac{C_i}{z_i}} \quad \begin{array}{l} i = 1, \ldots, n \\ C_i = \text{const.} \end{array}$$
- $+$ ***Integrability cond.***: $[C_i, C_j] = 0$.
- New condition: ***goodness***.

$$\forall j, k \quad \varphi_j - \varphi_k \begin{cases} = z^{-m_{jk}} \cdot \text{unit}, \quad m_{jk} \in \mathbb{N}^\ell \setminus \{0\}, \text{ or} \\ \equiv 0 \end{cases}$$

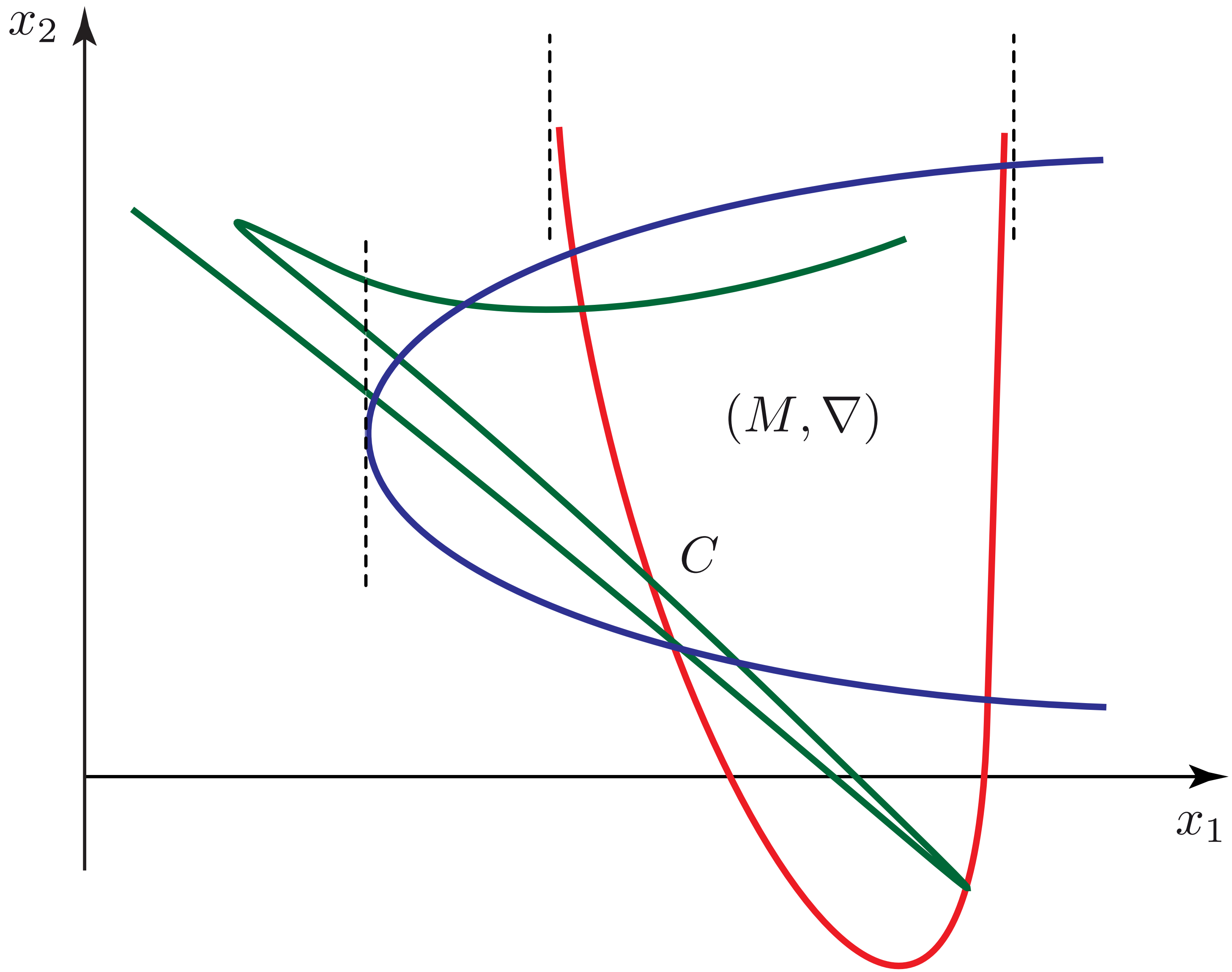
Example: partial Laplace transf.

- $\mathbb{A}^2$, coord. (x_1, x_2),
- Curve $C = \bigcup_i C_i$,
- $\rho : \pi_1(\mathbb{A}^2 \smallsetminus C, \star) \to \mathrm{GL}_n(\mathbb{C})$,
- $\iff$ local syst. $\mathscr{L}$ on $\mathbb{A}^2 \smallsetminus C$,
- $\overset{\text{RH}}{\iff}$ merom. flat bdle on $\mathbb{C}^2$ with reg. sing.
- $\overset{\text{alg.}}{\iff}$ $\mathbb{C}[x_1, x_2](*C)$-mod. of finite type (M, ∇) with flat connection having reg. sing. along C and ∞,
- $\Longrightarrow \mathbb{C}[x_1, x_2]\langle \partial_{x_1}, \partial_{x_2} \rangle$-mod. of finite type.
- Partial Laplace transf.:

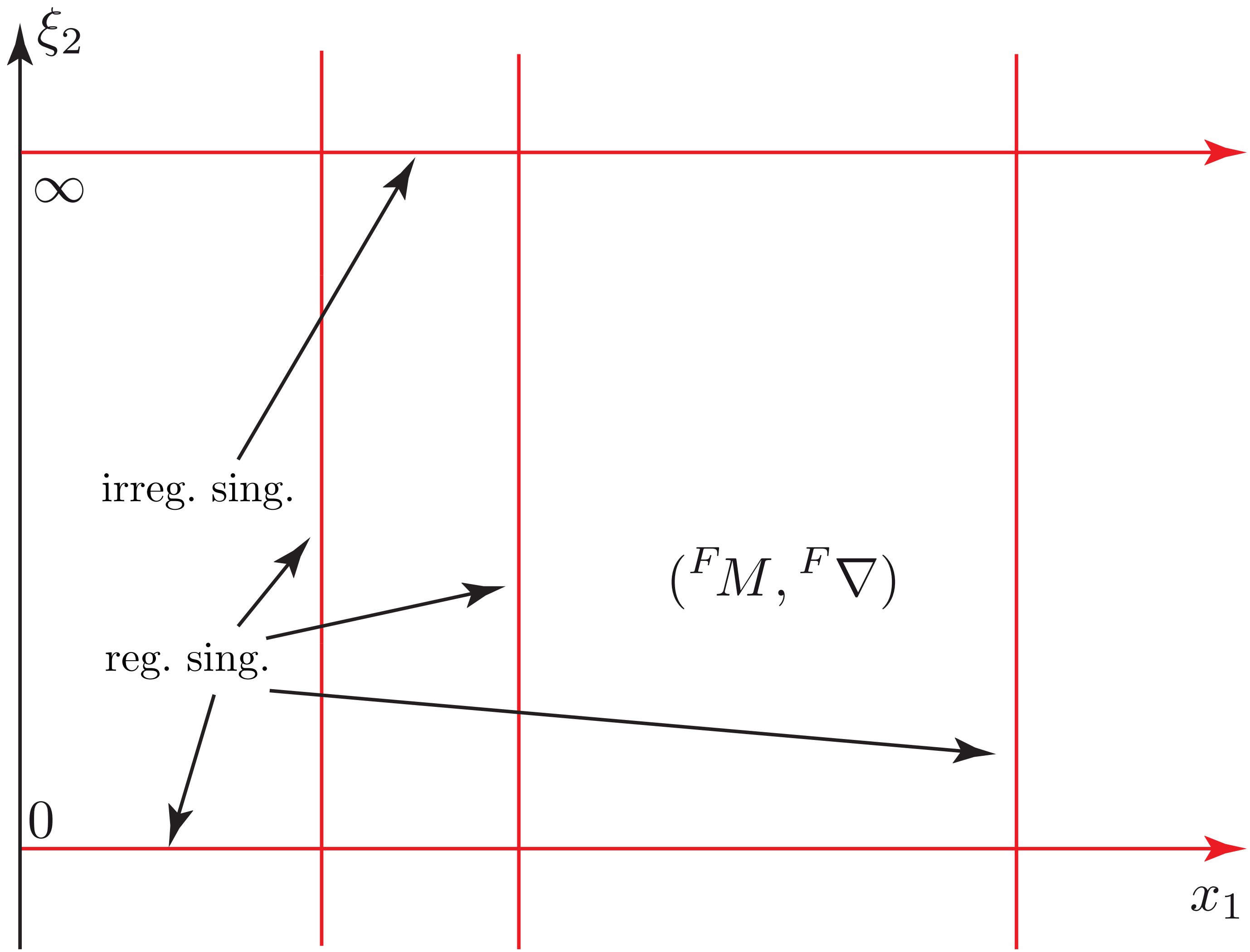
$${}^F\!M := M \text{ as a } \mathbb{C}[x_1, \xi_2]\langle \partial_{x_1}, \partial_{\xi_2} \rangle \text{ -mod.: } \begin{cases} \xi_2 = \partial_{x_2} \\ \partial_{\xi_2} = -x_2 \end{cases}$$

- $\mathrm{Sing}\, {}^F\!M$: $\{\xi_2 = 0\} \cup \{\xi_2 = \infty\} \cup \bigcup_k \{x_1 = x_{1,k}\}$

Example: partial Laplace transf.



Example: partial Laplace transf.



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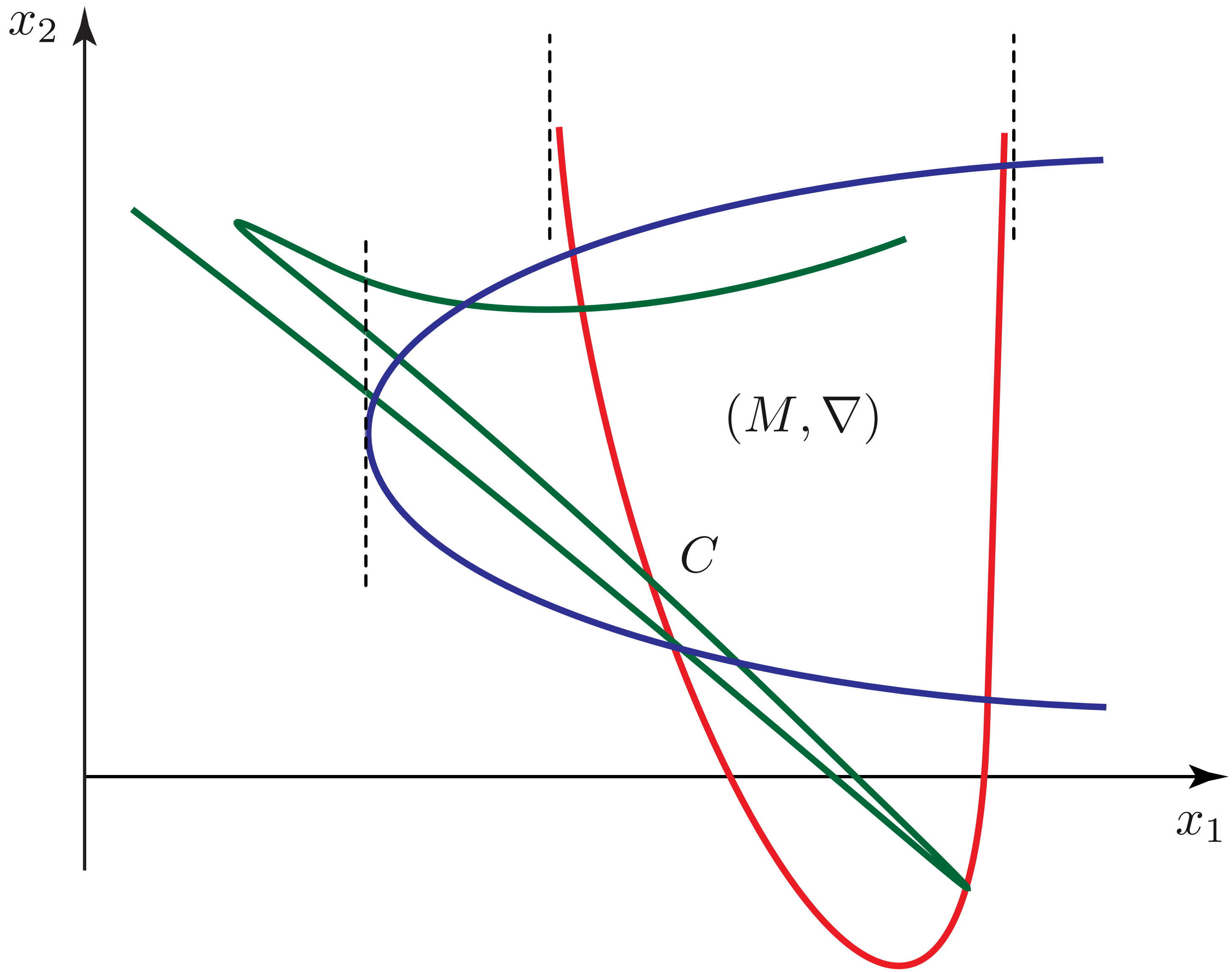

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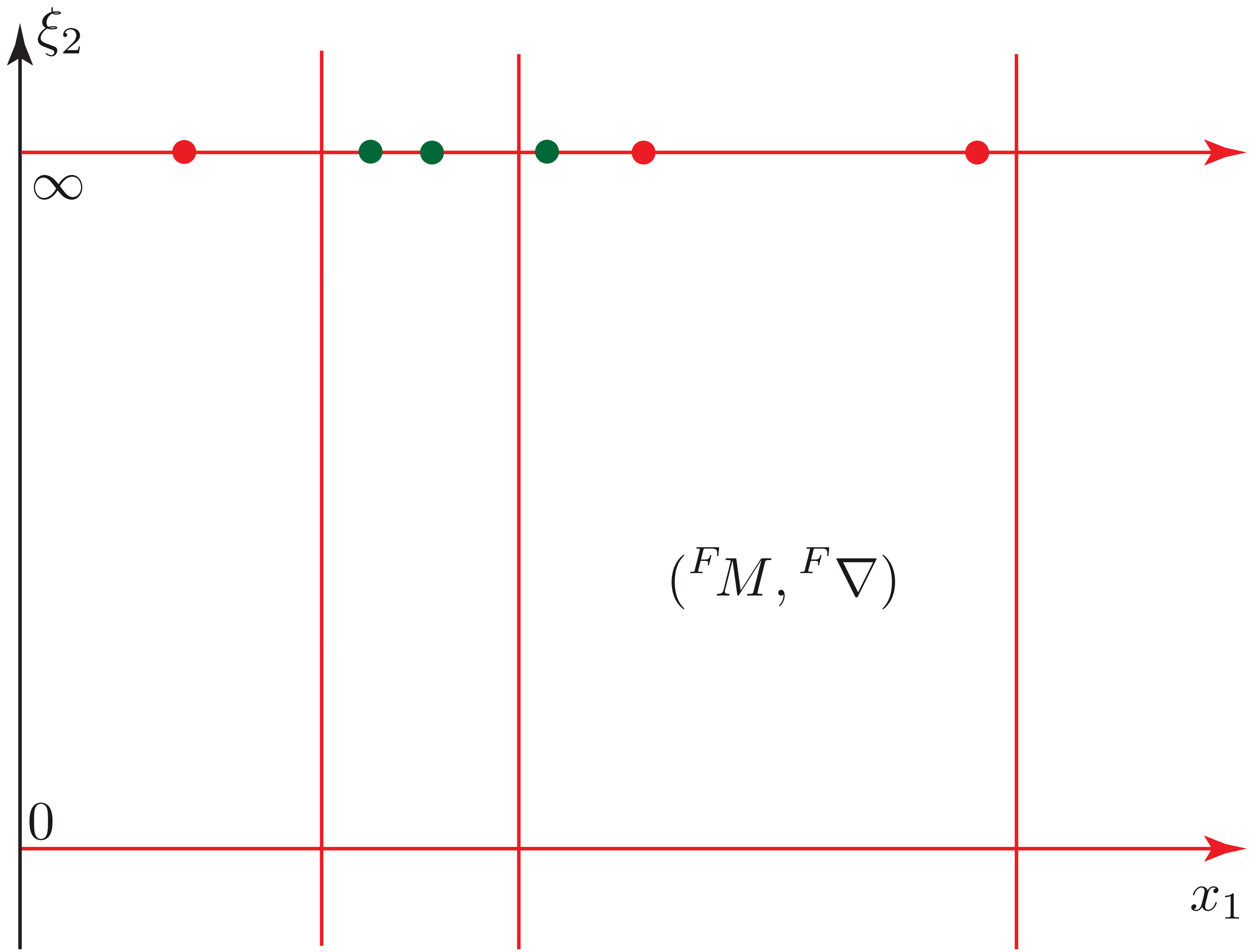
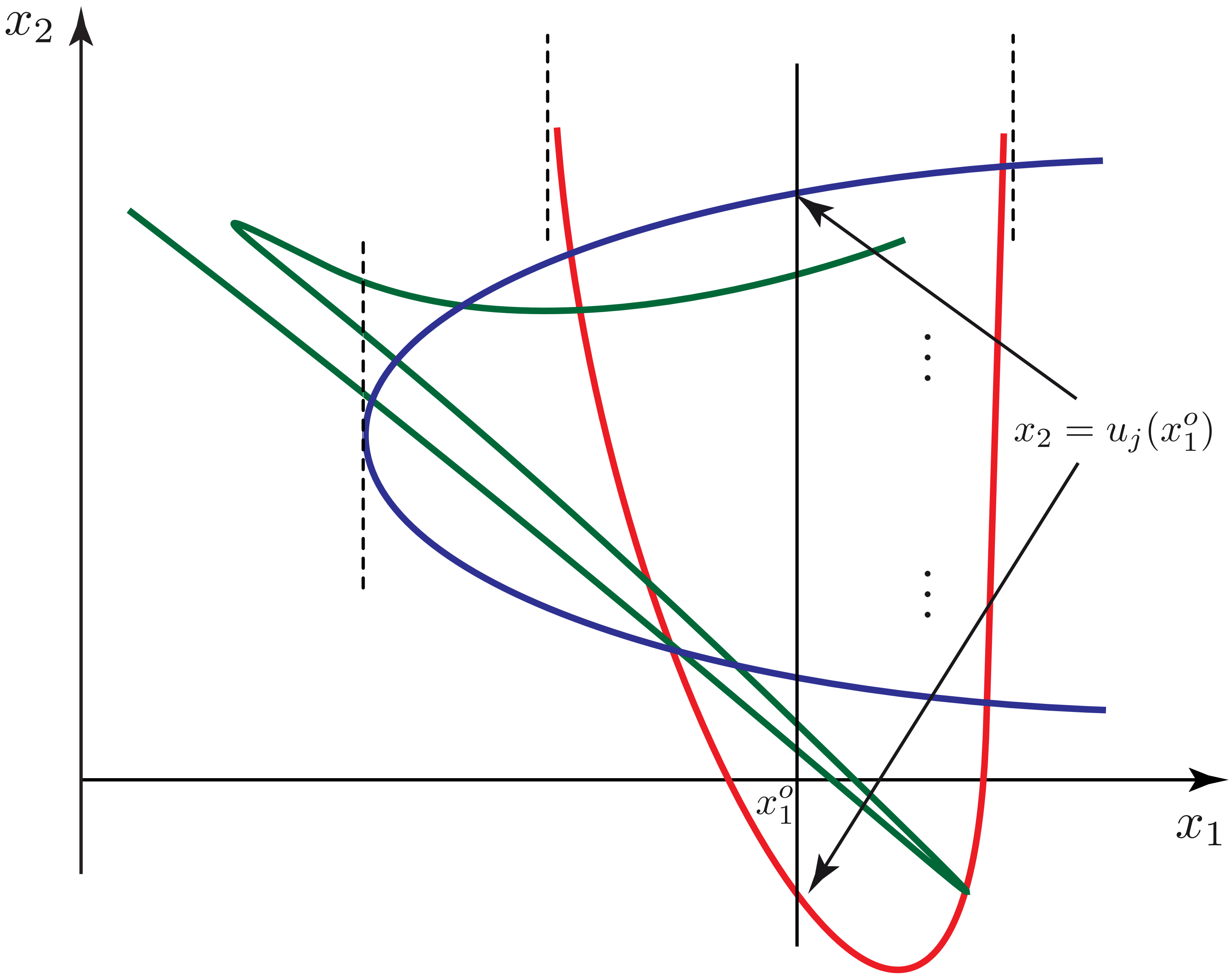
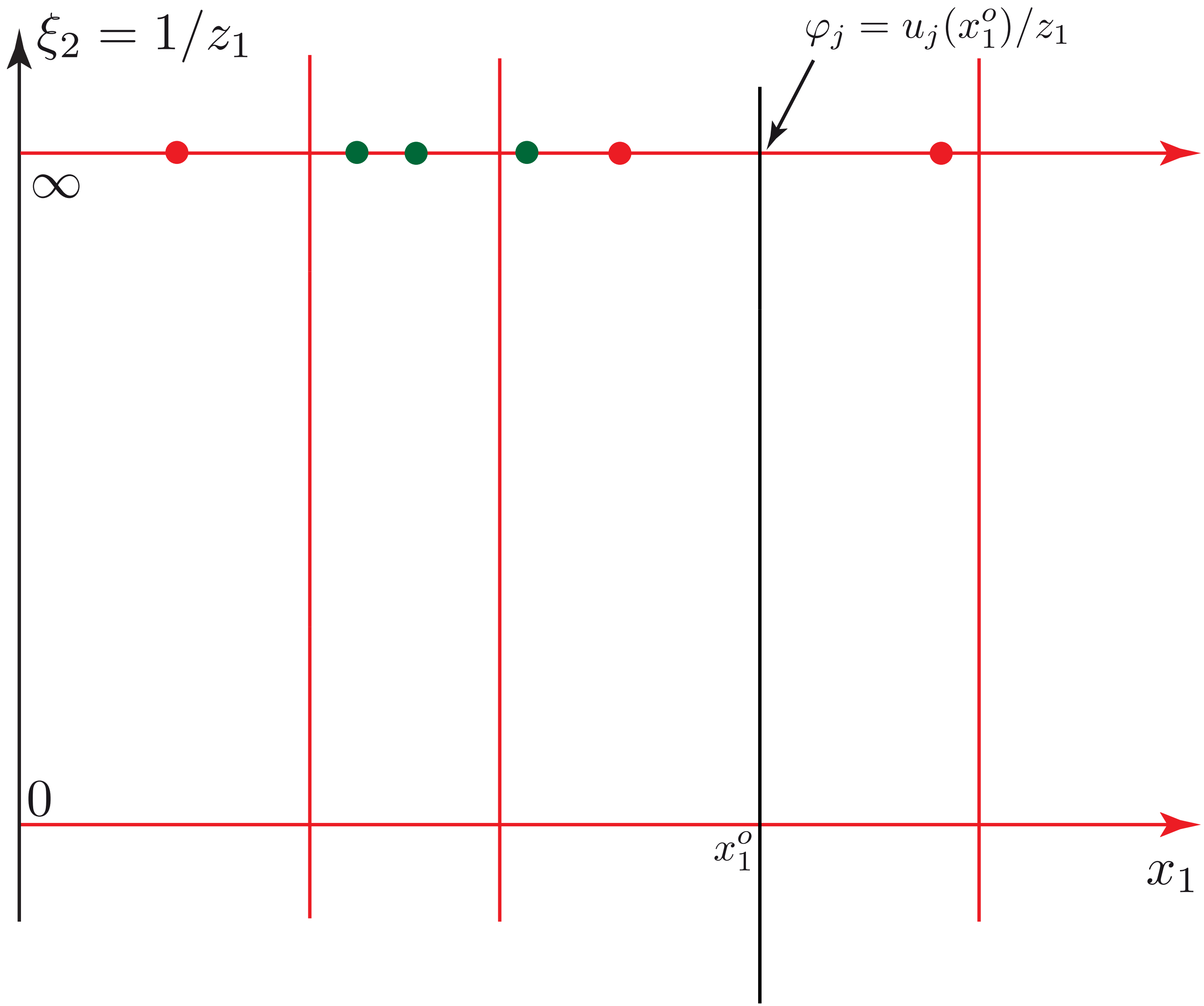


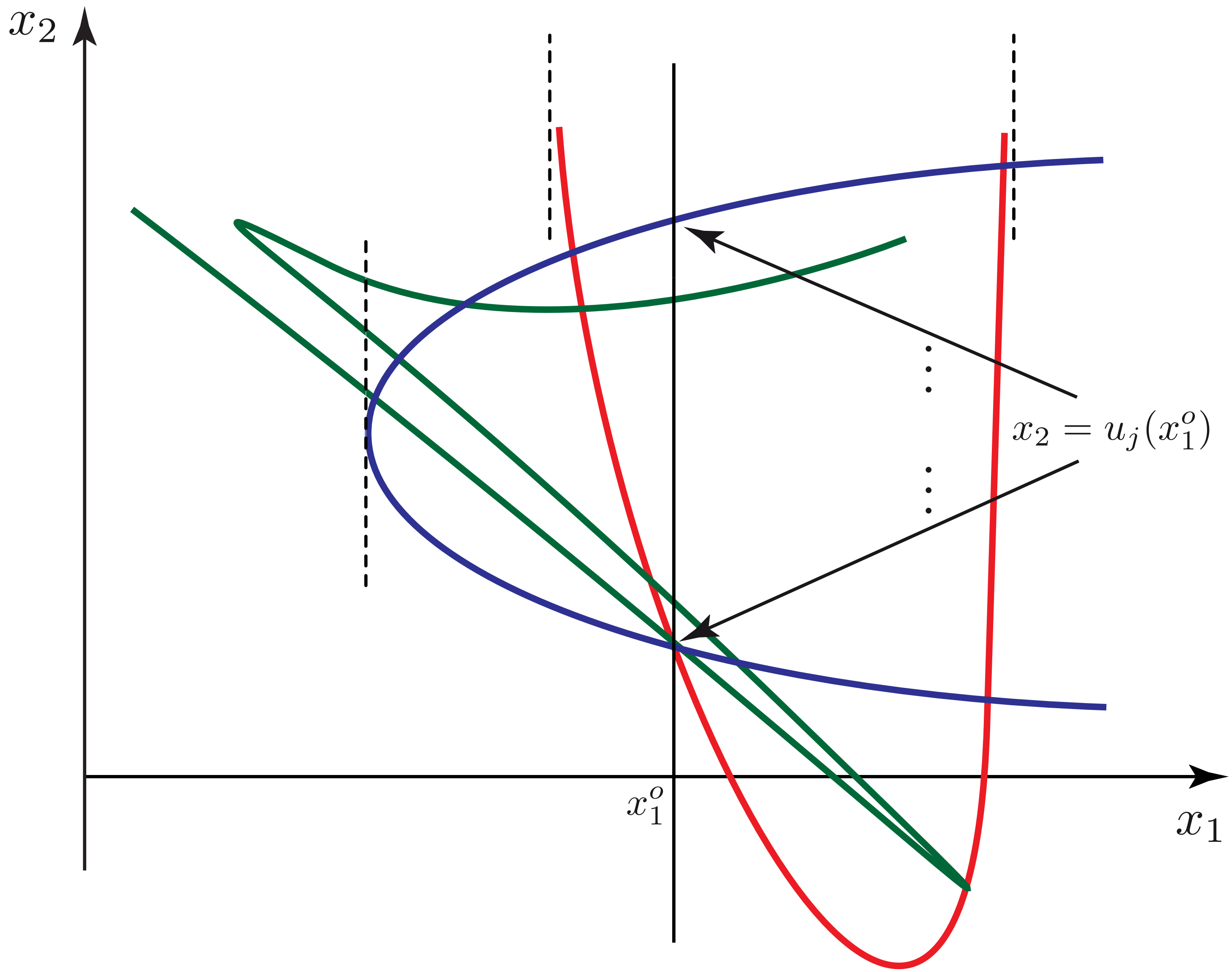
Example: partial Laplace transf.



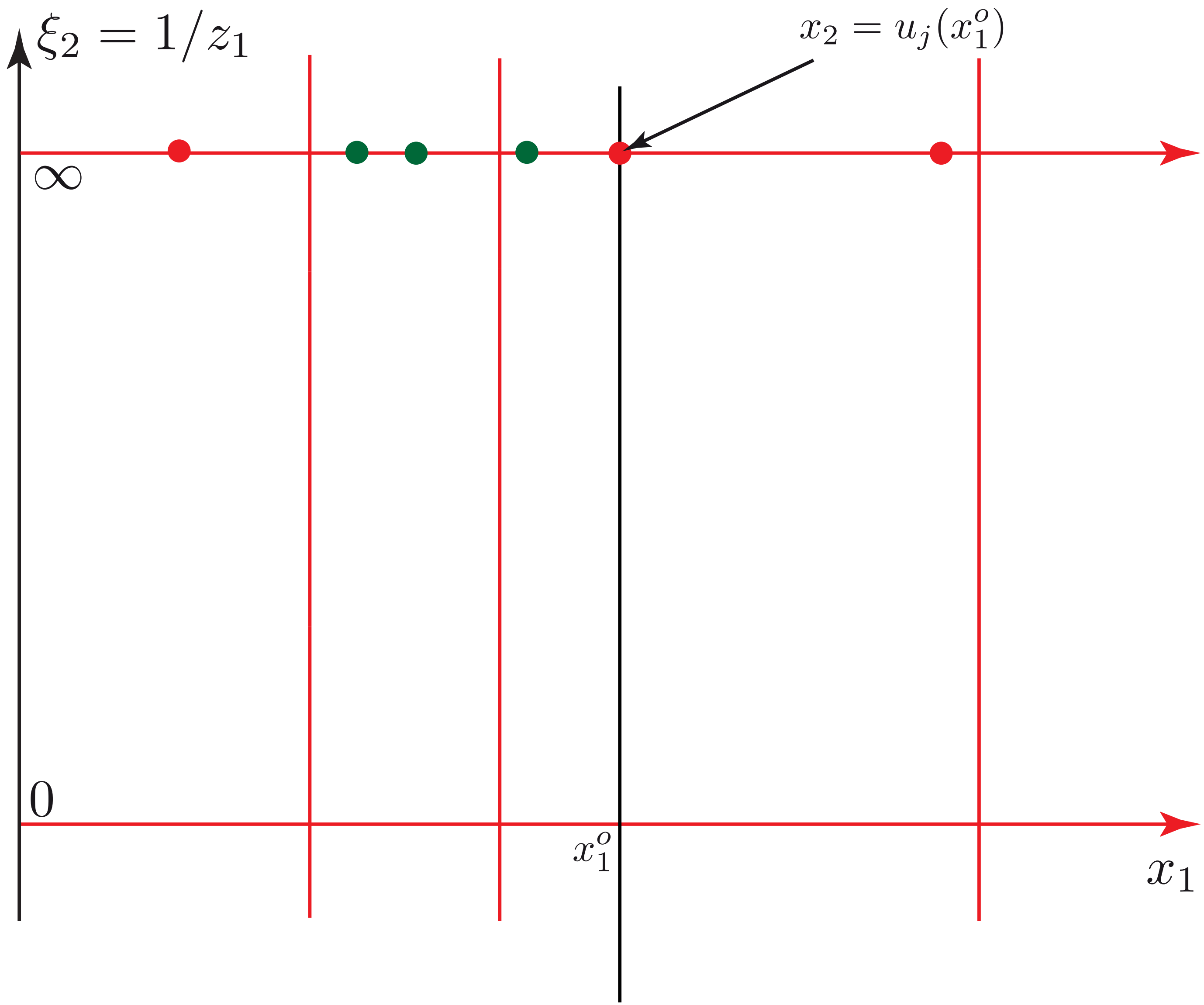
Example: partial Laplace transf.



Example: partial Laplace transf.



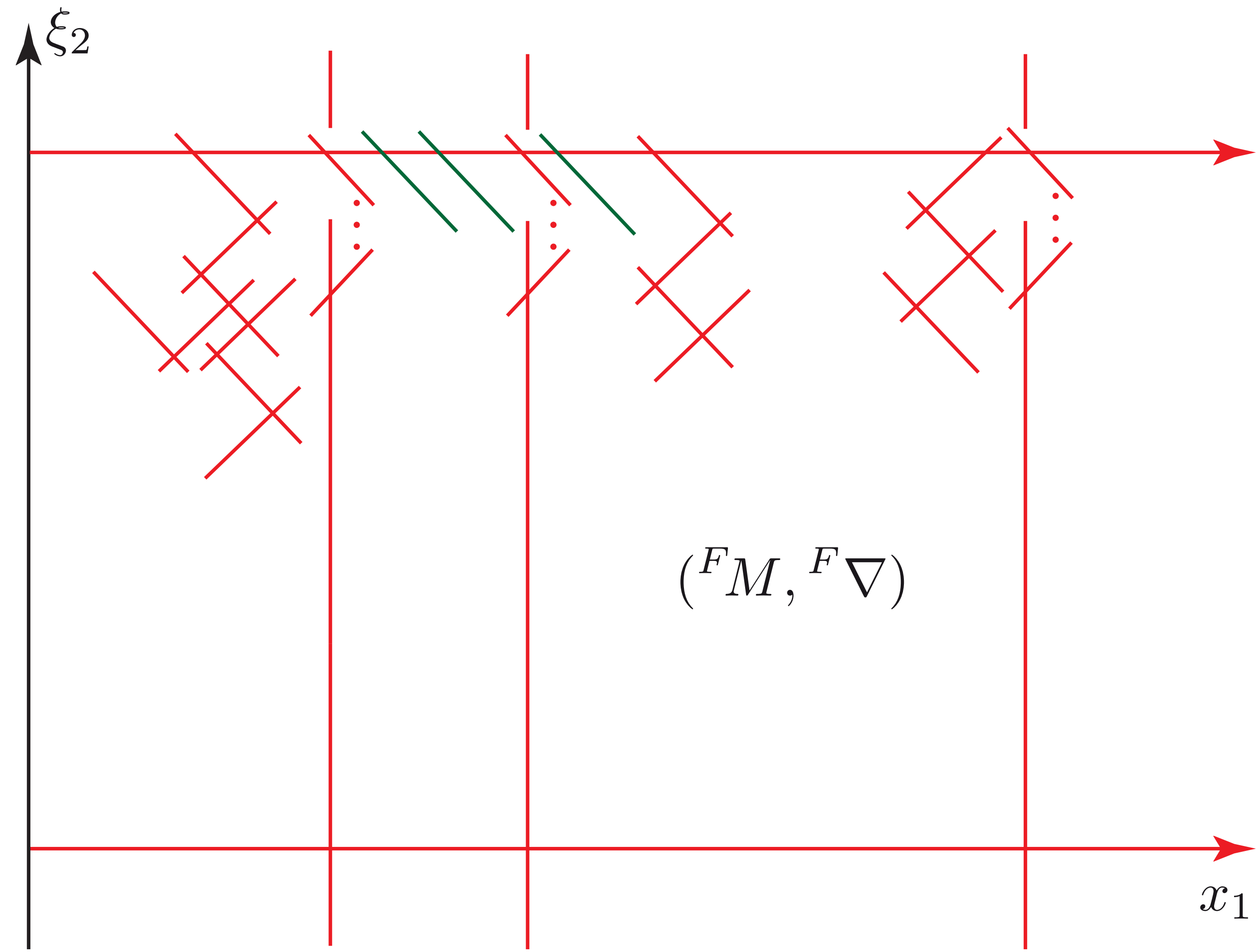
Example: partial Laplace transf.



Normal form for the Laplace transf.

Theorem (CS 00).
Assume (M, ∇) regular on $\mathbb{A}^2$. $\exists$ a sequence of cplx blowing-ups at each turning point of ${}^F M$ s.t. the pull-back of $({}^F M, {}^F\nabla)$ has good ***formal*** normal form at every point of the pull-back of $\xi_2 = \infty$.

Normal form for the Laplace transf.



Normal form in dim. $\geqslant 2$

Theorem (T. Mochizuki, K. Kedlaya).
Given (M, ∇) on X with poles along D,

- $\exists$ projective modif. $\pi : (X', D') \to (X, D)$ s.t.
  - $D' = \pi^{-1}(D)$ is a n.c.d.,
  - $\forall x'_o \in D'$, after local ramif. around D',

$$\exists \widehat{P} \in \mathrm{GL}_d(\mathscr{O}_{\widehat{X'|x'_o}}(*D')), \quad \widehat{P}[A_i] = B_i \ \ \forall i = 1, \dots, n.$$

$(B_1, \dots, B_n)$: ***good*** normal form at x'_o.

Remarks.

- Conj. by C.S. in 2000 and proved in particular cases in dim. 2.
- Proved by T. Mochizuki, if X, M, ∇ are algebraic.
- Proved by K. Kedlaya in the local (formal) setting.

Asympt. analysis in dim. $\geqslant 2$

- X cplx manifold, D n.c.d., $D \overset{\text{loc}}{=} \{z_1 \cdots z_\ell = 0\}$
- Strata:

$$D_I^\circ := \bigcap_{i \in I} D_i \smallsetminus \bigcup_{j \notin I} D_j, \qquad D_I^\circ \overset{\text{loc}}{=} \{z_1 = \cdots = z_\ell = 0\}$$

- $\varpi : \widetilde{X} \to X$: oriented real blow-up of X along the components of D.
- Loc. coord. on $\widetilde{X}$:

$$(\rho_1, \ldots, \rho_\ell, e^{i\theta_1}, \ldots, e^{i\theta_\ell}, z_{\ell+1}, \ldots, z_n).$$

Locally:

$$\widetilde{X} = [0, \varepsilon)^\ell \times (S^1)^\ell \times \Delta^{n-\ell} \quad \text{PL manifold.}$$

$$\partial\widetilde{X} := \varpi^{-1}(D) = \partial[0, \varepsilon)^\ell \times (S^1)^\ell \times \Delta^{n-\ell}$$

$$\partial\widetilde{X}_I^\circ := \varpi^{-1}(D_I^\circ) = \{0\} \times (S^1)^\ell \times \Delta^{n-\ell}$$

Asympt. analysis in dim. $\geqslant 2$

- Sheaves $\mathscr{A}^{\mathrm{rd}\,D}_{\widetilde{X}} \subset \mathscr{A}_{\widetilde{X}} \subset \mathscr{C}^{\infty}_{\widetilde{X}}$.

$$\mathscr{A}_{\widetilde{X}} := \bigcap_{i=1}^{\ell} \ker(\overline{z}_i \partial_{\overline{z}_i}) \cap \bigcap_{j=\ell+1}^{n} \ker \partial_{\overline{z}_j}.$$

$$\mathscr{A}^{\mathrm{rd}\,D}_{\widetilde{X}} \subset \mathscr{A}_{\widetilde{X}} \subset \mathscr{A}^{\mathrm{mod}\,D}_{\widetilde{X}}.$$

Theorem (Hukuhara-Turrittin, H. Majima '84, C.S. '00, Mochizuki '11).
Assume (M, ∇) ***good*** along D.
$\Rightarrow$ Locally on $\partial\widetilde{X}$, $\exists$ a lifting $\widetilde{P} \in \mathrm{GL}_d(\mathscr{A}_{\widetilde{X}}(*D))$ of $\widehat{P}$ s.t. $\widetilde{P}[A_i] = B_i \; \forall i$.

Good coverings

- Fix a stratum D_I° of D.
- $\forall x_o \in D_I^\circ$, $\Phi_{x_o} \subset \mathscr{O}_{X,x_o}(*D)/\mathscr{O}_{X,x_o}$ (neglect ramif.).
- $\bigsqcup_{x \in D_I^\circ} \Phi_x$: can be endowed with a natural topology (sheaf space).
- $\rightsquigarrow$ finite covering, which is ***good***:

$$\Sigma_I^\circ \longrightarrow D_I^\circ$$

- Lift Σ_I° to $\widetilde{X}$:

$$\begin{array}{ccc} \widetilde{\Sigma}_I^\circ & \longrightarrow & \partial \widetilde{X}_I^\circ \\ \downarrow & \square & \downarrow \varpi \\ \Sigma_I^\circ & \longrightarrow & D_I^\circ \end{array}$$

- $\forall \widetilde{x}_o \in \partial \widetilde{X}_I^\circ \overset{\text{loc}}{=} (S^1)^\ell \times \Delta^{n-\ell}$, order on

$$(\widetilde{\Sigma}_I^\circ)_{\widetilde{x}_o} = \Phi_{x_o}.$$

Stokes-filtered loc. syst. on $\partial\widetilde{X}_I^\circ$

- Embeddings:

$$\begin{array}{ccccccc} & & \widetilde{X} & \hookleftarrow & \partial\widetilde{X} & \hookleftarrow & \partial\widetilde{X}_I^\circ \\ & \overset{\widetilde{\jmath}}{\nearrow} & \downarrow \varpi & & \downarrow & & \downarrow \\ X \smallsetminus D & \hookrightarrow & X & \hookleftarrow & D & \hookleftarrow & D_I^\circ \end{array}$$

- $\ker \nabla_{|X\smallsetminus D}$: loc. syst. on $X \smallsetminus D$.
- ***Lemma***: $R\widetilde{\jmath}_* \ker \nabla_{|X\smallsetminus D} = \widetilde{\jmath}_* \ker \nabla_{|X\smallsetminus D}$: loc. syst. on $\widetilde{X}$.
- $\mathscr{L} = (\widetilde{\jmath}_* \ker \nabla_{|X\smallsetminus D})_{|\partial\widetilde{X}}$: loc. syst. on $\partial\widetilde{X}$.
- $\mathscr{L}_I^\circ := \mathscr{L}_{|\partial\widetilde{X}_I^\circ}$: loc. syst. on $\partial\widetilde{X}_I^\circ$.

Stokes-filtered loc. syst. on $\partial\widetilde{X}_I^\circ$

- $\forall \widetilde{x}_o \in \partial\widetilde{X}_I^\circ$, $\forall\varphi \in \Phi_{x_o}$, nested subsp. of $\mathscr{L}^\circ_{I,\widetilde{x}_o}$:

$$\mathscr{L}^\circ_{I,\leqslant\varphi,\widetilde{x}_o} = \{f_{\widetilde{x}_o} \mid e^{-\varphi} f(z) \in \mathscr{A}^{\mathrm{mod}\,D}_{\widetilde{x}_o}\}$$

$$\mathscr{L}^\circ_{I,<\varphi,\widetilde{x}_o} = \{f_{\widetilde{x}_o} \mid \exists i \in I,\ e^{-\varphi} f(z) \in \mathscr{A}^{\mathrm{rd}\,D_i}_{\widetilde{x}_o}\}$$

Stokes-filtered loc. syst. on $\partial\widetilde{X}_I^\circ$

- $\forall \widetilde{x}_o \in \partial\widetilde{X}_I^\circ$, $\forall\varphi \in \Phi_{x_o}$, nested subsp. of $\mathscr{L}^\circ_{I,\widetilde{x}_o}$:

$$\boxed{\begin{aligned}\mathscr{L}^\circ_{I,\leqslant\varphi,\widetilde{x}_o} &= \big\{f_{\widetilde{x}_o} \mid e^{-\varphi}f(z) \in \mathscr{A}^{\mathrm{mod}\,D}_{\widetilde{x}_o}\big\}\\ \mathscr{L}^\circ_{I,<\varphi,\widetilde{x}_o} &= \big\{f_{\widetilde{x}_o} \mid \exists i \in I,\ e^{-\varphi}f(z) \in \mathscr{A}^{\mathrm{rd}\,D_i}_{\widetilde{x}_o}\big\}\end{aligned}}$$

⚠

$$\mathrm{rd}\,D \not\Longleftrightarrow (\mathrm{mod}\,D\,\&\ \neq 0),$$

e.g. $D = \{z_1z_2 = 0\}$, $x_o = (0,0)$,

$$\mathrm{Re}(1/z_1) < 0 \text{ at } \widetilde{x}_o \Longrightarrow e^{1/z_1}\begin{cases}\in \mathscr{A}^{\mathrm{mod}\,D}_{\widetilde{x}_o},\\ \in \mathscr{A}^{\mathrm{rd}\,D_1}_{\widetilde{x}_o},\\ \notin \mathscr{A}^{\mathrm{rd}\,D}_{\widetilde{x}_o}.\end{cases}$$

Stokes-filtered loc. syst. on $\partial\widetilde{X}_I^\circ$

- $\forall \widetilde{x}_o \in \partial\widetilde{X}_I^\circ$, $\forall \varphi \in \Phi_{x_o}$, nested subsp. of $\mathscr{L}_{I,\widetilde{x}_o}^\circ$:

$$\boxed{\begin{aligned}\mathscr{L}_{I,\leqslant\varphi,\widetilde{x}_o}^\circ &= \{f_{\widetilde{x}_o} \mid e^{-\varphi} f(z) \in \mathscr{A}_{\widetilde{x}_o}^{\mathrm{mod}\,D}\}\\ \mathscr{L}_{I,<\varphi,\widetilde{x}_o}^\circ &= \{f_{\widetilde{x}_o} \mid \exists i \in I,\ e^{-\varphi} f(z) \in \mathscr{A}_{\widetilde{x}_o}^{\mathrm{rd}\,D_i}\}\end{aligned}}$$

- $\rightsquigarrow$ pair of nested subsheaves $\mathscr{L}_{I,<}^\circ \subset \mathscr{L}_{I,\leqslant}^\circ$ on $\widetilde{\Sigma}_I^\circ$.
- $\rightsquigarrow$ notion of Stokes-filtered loc. syst. $(\mathscr{L}_I^\circ, \mathscr{L}_{I,\bullet}^\circ)$ on $(\partial\widetilde{X}_I^\circ, \widetilde{\Sigma}_I^\circ)$.

Theorem (Mochizuki 11, CS 13):
Given a good cov. Σ_I°, equiv. of categ.

Germs along D_I° of merom. flat bdles on (X, D), ass. cov. $\subset \Sigma_I^\circ$

$$\updownarrow$$

Stokes-filtered loc. syst. on $(\partial\widetilde{X}_I^\circ, \widetilde{\Sigma}_I^\circ)$

Stokes-filtered loc. syst. on $\partial\widetilde{X}$

- $\Sigma := \bigsqcup_I \Sigma_I^\circ$: top. space, but maybe not Hausdorff, e.g.
  - $\varphi_1 = 1/z_1z_2$, $\varphi_2 = 1/z_1 + 1/z_1z_2$,
  - $\varphi_1 - \varphi_2 = 0$ on $\{z_1 \neq 0, z_2 = 0\}$
- Σ & $\widetilde{\Sigma} := \bigsqcup_I \widetilde{\Sigma}_I^\circ$: good stratified coverings.
- $\rightsquigarrow$ Stokes-filt. loc. syst. $(\mathscr{L}, \mathscr{L}_\bullet)$ on $(\widetilde{X}, \widetilde{\Sigma})$.

Theorem (RHB correspondence, Mochizuki 11, CS 13):
Given a good strat. cov. Σ, equiv. of categ.

Merom. flat bdles on (X, D), ass. strat. cov. $\subset \Sigma$

$$\updownarrow$$

Stokes-filtered loc. syst. on $(\widetilde{X}, \widetilde{\Sigma})$