

# Introduction to Stokes structures

## I: dimension one

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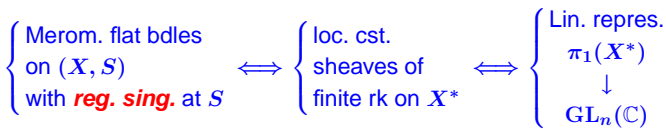
# Ubiquity of the Stokes phenomenon

Various places where the Stokes phenomenon occurs.

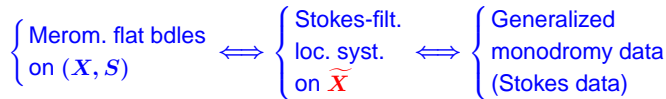
- Asympt. behaviour of sols of the **Airy** Eqn (**Stokes...**).
- Global behaviour of vanishing cycles of functions  $X \rightarrow \mathbb{C}$  in alg. geom. (**Pham, Berry...**)
- Analogy with the theory of wild ramification in Arithmetic (**Deligne...**).
- Frobenius manifolds and quantum cohomology (**Dubrovin...**).
- tt\* geometry (**Cecotti & Vafa...**).
- Geometric Langlands correspondence with wild ramification (**Frenkel & B. Gross...**).
- Wild character varieties (**Boalch...**).
- Similarities with the theory of stability conditions on some Abelian categories (**Bridgeland, Kontsevich...**).

## Aim: RH corresp. for merom. ODE's

- **Riemann-Hilbert** corresp. (categorical) on a punctured Riemann surf.  $X^* = X \setminus S$ :



- **Riemann-Hilbert-Birkhoff** corresp. (categorical) on a punctured Riemann surf.  $X^* = X \setminus S$ :



## Other approaches

- Explicit computation of sols (integral formulas)
- realizing Stokes data with effective solutions ( $\rightsquigarrow$  theory of multisummation)
- Constructing moduli spaces of diff. eqns and realizing the RHB corresp. by a **map** between moduli spaces.
- replacing the group  $\text{GL}_n(\mathbb{C})$  with other reductive algebraic groups.
- Extending the categorical approach to the Tannakian aspect ( $\rightsquigarrow$  Differential Galois theory).

## Stokes phenomenon in dim. one

- $\Delta =$  complex disc, complex coord.  $z$ .
- Linear cplx diff. eqn.  $df/dz = A(z) \cdot f$ ,
- $A(z)$  matrix of size  $d \times d$ , merom. pole at  $z = 0$ .
- Gauge equiv.:  $P \in \text{GL}_d(\mathbb{C}(\{z\}))$ ,

$$A \sim B = P[A] := P^{-1}AP + P^{-1}P'$$

- **Norm. form:**  $B = \begin{pmatrix} \varphi'_1 & & \\ & \ddots & \\ & & \varphi'_d \end{pmatrix} + \frac{C}{z}$   $\varphi_k \in \frac{1}{z}\mathbb{C}[\frac{1}{z}]$   
 $C = \text{const. non reson.}$

**Theorem (Levelt-Turrittin).** Given  $A$ ,  $\exists$  a **formal** gauge transf.  $\hat{P} \in \text{GL}_d(\mathbb{C}(\{z^{1/p}\}))$  s.t.  $B = \hat{P}[A]$  is a normal form.

## Asympt. analysis in dim. one

- **Real or. blow-up:**  $\tilde{\Delta} = [0, \varepsilon) \times S^1$ , coord.  $\rho, e^{i\theta}$ .

$$\varpi : \begin{cases} \tilde{\Delta} \rightarrow \Delta \\ S^1 \rightarrow 0 \end{cases} \quad (\rho, e^{i\theta}) \mapsto z = \rho e^{i\theta}$$

- Sheaf  $\mathcal{A}_{\tilde{\Delta}} = \ker \bar{z}\partial_{\bar{z}} : \mathcal{C}_{\tilde{\Delta}}^{\infty} \rightarrow \mathcal{C}_{\tilde{\Delta}}^{\infty}$   
( $\mathcal{A}_{\tilde{\Delta}^*} = \mathcal{O}_{\tilde{\Delta}^*}$ )
- Sheaves  $\mathcal{A}_{S^1}^{\text{rd}0} \subset \mathcal{A}_{S^1} \subset \mathcal{A}_{S^1}^{\text{mod}0}$ .
- **Basic exact sequence:**

$$0 \rightarrow \mathcal{A}_{S^1}^{\text{rd}0} \rightarrow \mathcal{A}_{S^1} \rightarrow \varpi^{-1}\mathbb{C}[z] \rightarrow 0$$

## Asympt. analysis in dim. one

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- Sheaves  $\mathcal{A}_{S^1}^{\text{rd}0} \subset \mathcal{A}_{S^1} \subset \mathcal{A}_{S^1}^{\text{mod}0}$ .
- **Example:**  $\varphi = u(z)/z^q$  s.t.  $u(z) \in \mathbb{C}[z]$ ,  $q \geq 1$ , and  $u(0) \neq 0$  or  $u(z) \equiv 0$ . Then  $\forall \alpha \in \mathbb{C}$  and  $\forall e^{i\theta_0} \in S^1$

$$z^{\alpha} e^{\varphi} \in \begin{cases} \mathcal{A}_{\theta_0}^{\text{rd}0} & \iff \text{Re}(u(0)e^{-ik\theta_0}) < 0, \\ \mathcal{A}_{\theta_0}^{\text{mod}0} & \iff \text{idem or } u(z) \equiv 0. \end{cases}$$

## Asympt. analysis in dim. one

**Theorem (Hukuhara-Turrittin).**

Locally on  $S^1$ ,  $\exists$  a lifting  $\tilde{P} \in \text{GL}_d(\mathcal{A}_{S^1}[1/z])$  of  $\hat{P}$  s.t.  $\tilde{P}[A] = \hat{P}[A] = B$  normal form.

**Corollary.** The sheaf on  $\tilde{\Delta}$  of sols of

$$df/dz = A(z) \cdot f$$

having entries in  $\mathcal{A}_{\tilde{\Delta}}^{\text{rd}0}$ , resp. in  $\mathcal{A}_{\tilde{\Delta}}^{\text{mod}0}$ , is a real constr. sheaf, constant on any open interval  $I$  of  $S^1$  s.t.  $\forall k, \text{Re}(\varphi_k)$  does not vanish.

**Example.**  $\varphi = z^{-q}u(z)$ ,  $u(0) \neq 0$ ,

$$\text{On } S^1, \quad \text{Re } \varphi = 0 \iff \theta = \frac{1}{q}(\arg u(0) + \pi/2) \pmod{\mathbb{Z} \cdot \pi/q}.$$

## The Malgrange-Sibuya theorem

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Fix a norm. form (**irregular type**), e.g. non-ramified:

$$B = \text{diag}(\varphi'_1, \dots, \varphi'_d) + \frac{C}{z}$$

**B-marked connections** ( $\sim$ : merom. gauge equiv.):

$$\text{Iso}(B) = \left\{ (A, \hat{P}) \mid B = \hat{P}[A] \right\} / \sim$$

**Stokes sheaf**  $\text{St}(B)$  on  $S^1$ :

$$\text{St}(B)_\theta = \left\{ \text{Id} + Q \mid Q \in \text{End}(\mathcal{A}_\theta^{\text{rd}0}), (\text{Id} + Q)[B] = B \right\}$$

**Theorem** (Malgrange-Sibuya).

$$\text{Iso}(B) \simeq H^1(S^1, \text{St}(B))$$

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## Stokes-filtered loc. syst. (non-ramif. case)

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- Aim:** To specify the struct. of sol. space of a merom. ODE without
  - making explicit the realization as functions,
  - fixing the normal form.
- The local system  $\mathcal{L}$  on  $S^1$ : Sols of  $df/dz = A(z)f$  on  $\Delta^*$ , extended to  $\tilde{\Delta} = [0, \varepsilon) \times S^1$  and restricted to  $\{0\} \times S^1$ . Hence  $\mathcal{L} \iff$  **monodromy of sols.**
- For every  $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ , a pair of nested subsheaves  $\mathcal{L}_{<\varphi} \subset \mathcal{L}_{\leq\varphi}$  of  $\mathcal{L}$ .

$$\begin{aligned} \mathcal{L}_{<\varphi, \theta} &= \{f_\theta \mid e^{-\varphi} f(z) \in \mathcal{A}_\theta^{\text{mod}0}\} \\ \mathcal{L}_{\leq\varphi, \theta} &= \{f_\theta \mid e^{-\varphi} f(z) \in \mathcal{A}_\theta^{\text{rd}0}\} \end{aligned}$$

- Hukuhara-Turrittin**  $\Rightarrow \mathcal{L}_{<\varphi} = \mathcal{L}_{\leq\varphi}$  except if  $\varphi = \varphi_k$  for some  $k = 1, \dots, d$ .

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## Stokes-filtered loc. syst. (non-ramif. case)

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- Aim:** To give an intrinsic characterization of the category of Stokes-filtered local systems.
- Definition.** A (non-ramif.) Stokes-filt. loc. syst. on  $S^1$ :
  - A loc. syst.  $\mathcal{L}$  on  $S^1$ ,
  - $\forall \varphi \in z^{-1}\mathbb{C}[z^{-1}]$ , an  $\mathbb{R}$ -const. subsheaf  $\mathcal{L}_{\leq\varphi} \subset \mathcal{L}$
 s.t.
  - $\forall \theta \in S^1, \mathcal{L}_{\leq\psi, \theta} \subset \mathcal{L}_{\leq\varphi, \theta} \iff \psi = \varphi, \text{ or } \text{Re}(\psi - \varphi) < 0 \text{ near } \theta,$
  - setting  $\forall \theta, \mathcal{L}_{<\varphi, \theta} = \sum_{\psi < \varphi} \mathcal{L}_{\leq\psi, \theta}$ $\rightsquigarrow \mathcal{L}_{<\varphi}$  and  $\text{gr}_\varphi \mathcal{L} := \mathcal{L}_{\leq\varphi} / \mathcal{L}_{<\varphi}$
- one asks that  $\forall \varphi$ ,
  - $\text{gr}_\varphi \mathcal{L}$  is a **local system** on  $S^1$ ,
  - $\forall \theta, \dim \mathcal{L}_{\leq\varphi, \theta} = \sum_{\psi \leq \varphi} \text{rk gr}_\psi \mathcal{L}$ , exhaust. filt..
- Remark:** can define  $(\mathcal{L}, \mathcal{L}_*)$  over  $\mathbb{Z}, \mathbb{Q}, \dots$

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## Stokes-filtered loc. syst. (non-ramif. case)

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Let  $(\mathcal{L}, \mathcal{L}_*)$  be a non-ramif. Stokes-filt. loc. syst.

- $\Phi := \{\varphi \mid \text{rk gr}_\varphi \mathcal{L} \neq 0\}$  is **finite** and  $\sum_{\varphi \in \Phi} \text{rk gr}_\varphi \mathcal{L} = \text{rk } \mathcal{L}$ .
- $\forall \varphi \in \Phi, \forall \theta, \mathcal{L}_{\leq\varphi, \theta} \stackrel{(*)}{\simeq} \bigoplus_{\psi \leq \varphi} \text{gr}_\psi \mathcal{L}_\theta$ .
- Level structure**
  - Levels of  $B$  (hence  $A$ ):  $\{q_1 < \dots < q_r\}$
  - $q_i :=$  pole ord. of some  $\psi - \varphi, \varphi \neq \psi \in \Phi$ .
- $\# \text{Levels}(A) = 1$ .  $\rightsquigarrow$  theory of summability.  $2q$  **Stokes directions** for **each**  $(\varphi, \psi)$ .
- $\# \text{Levels}(A) > 1$ .  $\rightsquigarrow$  theory of multisummability. **Principal** and **Secondary** Stokes directions.

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## Stokes-filtered loc. syst. (non-ramif. case)

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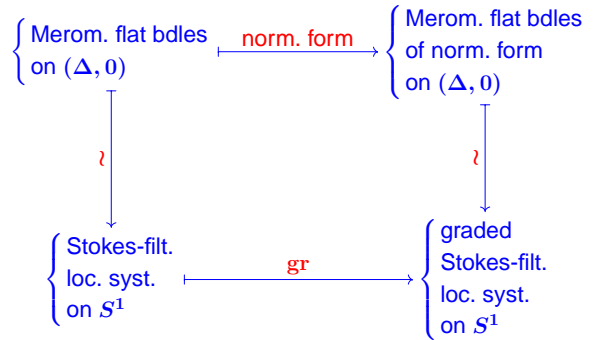
- Theorem**
  - $\forall$  open  $I \subset S^1$  which  $\ni$  at most one Stokes dir.  $\forall$  pair in  $\Phi$ , then **(\*)** holds on  $I$  (e.g.  $|I| \leq \pi/q_r + \varepsilon$ ).
  - Any morphism  $\lambda : (\mathcal{L}, \mathcal{L}_*) \rightarrow (\mathcal{L}', \mathcal{L}'_*)$  **graded** on  $I$  w.r.t. some iso **(\*)** and **(\*)'**, hence is **strict**, i.e.,  $\forall \varphi, \lambda(\mathcal{L}_{\leq\varphi}) = \mathcal{L}'_{\leq\varphi} \cap \lambda(\mathcal{L})$ .
  - Uniqueness** of the splitting if  $\# \text{Level}(A) = 1$  and moreover  $|I| = \pi/q + \varepsilon$ .
- Duality.**
  - The exact sequences
 
$$\begin{aligned} 0 \rightarrow \mathcal{L}_{\leq\varphi} \rightarrow \mathcal{L} \rightarrow \mathcal{L}^{>\varphi} \rightarrow 0 \\ 0 \rightarrow \mathcal{L}_{<-\varphi} \rightarrow \mathcal{L} \rightarrow \mathcal{L}^{>-\varphi} \rightarrow 0 \end{aligned}$$
 are switched by duality  $\mathcal{H}om_{\mathbb{C}}(\cdot, \mathbb{C})$ .  
 $\Rightarrow \text{gr}_\varphi(\mathcal{L}^\vee) \simeq (\text{gr}_{-\varphi} \mathcal{L})^\vee. \text{Ext}^k(\cdot, \mathbb{C}) = 0$  if  $k \geq 1$ .

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## Deligne's RH correspondence

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**Theorem** (Deligne's RH corresp.).



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## Stokes data (non-ramif. case, pure level)

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- Case  $\# \text{Level}(A) = 1$  (level =  $q$ )**
- Stokes data**
  - $(L_\ell)_{\ell \in \mathbb{Z}/2q\mathbb{Z}}$ :  $\mathbb{C}$ -vect. spaces,
  - Isoms  $S_\ell^{\ell+1} : L_\ell \xrightarrow{\sim} L_{\ell+1}$
  - Exhaustive filtrations  $\left\{ \begin{array}{l} F_\bullet L_{2\mu} \nearrow \\ F^\bullet L_{2\mu+1} \searrow \end{array} \right.$
- Opposedness property:**

$$L_{2\mu} = \bigoplus_k F_k L_{2\mu} \cap S_{2\mu-1}^{2\mu}(F^k L_{2\mu-1})$$

$$L_{2\mu+1} = \bigoplus_k F^k L_{2\mu+1} \cap S_{2\mu}^{2\mu+1}(F^k L_{2\mu})$$

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## Stokes data (non-ramif. case, pure level)

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- Case  $\# \text{Level}(A) = 1$  (level =  $q$ )**
- Opposed filtrations  $\Rightarrow$  **unique** splittings
 
$$\tau_{2\mu} : L_{2\mu} \xrightarrow{\sim} \text{gr}^F L_{2\mu} = \bigoplus_k \text{gr}_k^F L_{2\mu}$$

$$\tau_{2\mu+1} : L_{2\mu+1} \xrightarrow{\sim} \text{gr}^F L_{2\mu+1} = \bigoplus_k \text{gr}_k^F L_{2\mu+1}$$
- $\rightsquigarrow$  **Stokes multipliers**

$$\Sigma_\ell^{\ell+1} := \tau_{\ell+1} \circ S_\ell^{\ell+1} \circ \tau_\ell^{-1} : \text{gr}_F L_\ell \rightarrow \text{gr}_F L_{\ell+1}$$
  - $\Sigma_\ell^{\ell+1}$  block lower/upper triangular,
  - diag. blocks  $(\Sigma_\ell^{\ell+1})_{jj}$  are isos.

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## Stokes data (non-ramif. case, pure level) 17

$(\mathcal{L}, \mathcal{L}_\bullet)$  Stokes-filt. loc. syst. pure level  $q$

$\xleftrightarrow{\theta_0}$  Stokes data of pure level  $q$ .

- Fix  $\theta_0 \in S^1$  not a Stokes dir.  $\Rightarrow$  numbering of  $\Phi$  s.t.

$$\varphi_1 <_{\theta_0} \cdots <_{\theta_0} \varphi_r$$

- $2q$  generic dirs  $(\theta_\ell := \theta_0 + \ell\pi/q)_{\ell \in \mathbb{Z}/2q\mathbb{Z}}$  on  $S^1$ .

$$\Rightarrow \begin{cases} \varphi_1 <_{\theta_{2\mu}} \cdots <_{\theta_{2\mu}} \varphi_r \\ \varphi_r <_{\theta_{2\mu+1}} \cdots <_{\theta_{2\mu+1}} \varphi_1 \end{cases}$$

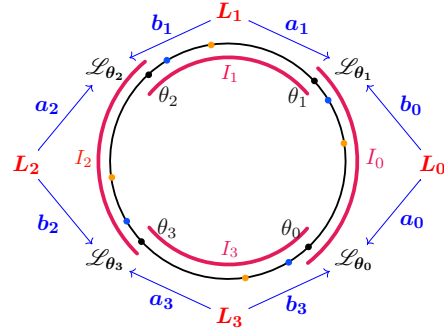
- $\Rightarrow (\mathcal{L}_{\leq \varphi_j, \theta_\ell})_j$ :  $\begin{cases} \text{filt. } \nearrow & \text{if } \ell = 2\mu \\ \text{filt. } \searrow & \text{if } \ell = 2\mu + 1 \end{cases}$

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## Stokes data (non-ramif. case, pure level) 18

$(\mathcal{L}, \mathcal{L}_\bullet)$  Stokes-filt. loc. syst. pure level  $q$  (e.g.  $q = 2$ )

$\xleftrightarrow{\theta_0}$  Stokes data of pure level  $q$ .  $L_i = \Gamma(I_i, \mathcal{L})$

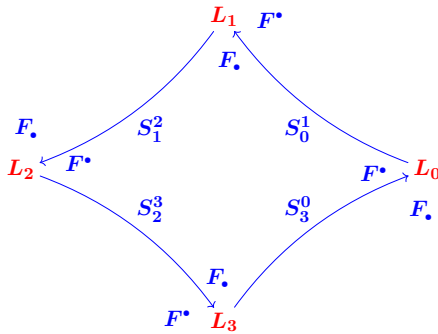


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## Stokes data (non-ramif. case, pure level) 19

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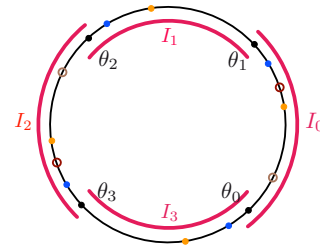


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## Stokes data (non-ramif. case) 20

$(\mathcal{L}, \mathcal{L}_\bullet)$  Stokes-filt. loc. syst. max level  $q_r$

(e.g.  $q_1 = 1, q_2 = 2$ )



$\rightarrow$  more difficult to describe the char. properties of Stokes data  $(L_\ell, S_\ell^{\ell+1})$

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## Examples 21

- How** to compute Stokes data?
- Use of the Fourier transf. (Marco's talk): more complicated  $\Leftarrow$  simpler.
- Explicit procedures (... , Mochizuki), but **difficult to obtain closed formulas for Stokes data**.
  - e.g. Airy diff. eq. (ramified)

$$(\partial_y^2 - y)u = 0 \quad z = 1/y \quad \rightsquigarrow \quad A(z) = -\frac{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}{z^4} + \frac{\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}}{z}$$

- Unramified case,  $q = 1$ : Fourier transform of a reg. sing. diff. eq. on  $\mathbb{P}^1$ .
- Gaussian type: irreg. sing. at  $\infty$ , unramif.,  $q = 2$ .
- FT of  $\mathcal{E}^\varphi$ : irreg. sing. at  $\infty$ , possibly ramif.

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## Dubrovin's conjecture 22

- $X$ : smooth proj. Fano var. which admits a **full exceptional collection**:

- $(E_1, \dots, E_m) \in \mathbf{D}^b(\text{Coh}(X))$  generate as a triang. category,
  - for  $i \neq j$ ,  $\text{Ext}^k(E_i, E_j) = 0$  except  $i < j$  and  $k = k(i, j)$ ,
  - $\text{Ext}^k(E_i, E_i) = 0$  except  $k = 0$ ,  $\text{Hom} = \mathbb{C}$ .
- $\rightsquigarrow S_X = (S_{ij}) \in \mathbf{M}_m(\mathbb{Z})$ ,

$$S_{ij} = \chi(E_i, E_j) := \sum_k (-1)^k \dim \text{Ext}^k(E_i, E_j).$$

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## Dubrovin's conjecture 23

- $f : U \rightarrow \mathbb{C}$  a **tame** reg. fnct. on a smooth affine var.
- $\rightsquigarrow$  finite # of crit. pts.
- GM syst.:

$$\text{GM} := \left( \frac{\Omega^{\max}(U)[z, z^{-1}]}{(zd + df)\Omega^{\max-1}(U)[z, z^{-1}]}, \quad z^2 \partial_z - f \right)$$

- GM at  $z = 0$ : one level,  $q = 1$ , Stokes matr.  $S_f^\pm$ , can be chosen with entries in  $\mathbb{Z}$ .  
Diag. blocks  $\leftrightarrow$  monodr. of vanishing cycles of  $f$ .

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## Dubrovin's conjecture 24

### Dubrovin's conjecture.

If  $f$  is the Landau-Ginzburg potential mirror to  $X$  good Fano, then  $\exists$  choices a bases of vanishing cycles of  $f$  s.t.

$$S_f^+ = S_X$$

- $X = \mathbb{P}^n$  (Dubrovin, Guzzetti)
- ...

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